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# UNIFIED APPROACH OF GENERALIZED INVERSE AND ITS APPLICATIONS

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ABSTRACT

This paper deals with *Unified Approach of* ***Generalized inverse (g-inverse) and its applications.*** General approaches of generalized inverses with a special convergence are discussed. Derivation of g-inverse by using minimal polynomial is shown. Mathematica codes are used in these examples. Special derivations of g-inverses using Contour integration are given. G-inverse has been applied in network theory and to optimize problems.

**Key words:** convergence, minimal polynomial, inconsistent system, network theory, optimizes.

## 1. INTRODUCTION

The inverse of a matrix was defined and its various properties were discussed by many authors. It is stated that if a matrix $A$ has an inverse, the matrix must be square and its determinant must be non-zero. Let us consider a system of linear equations

$$Ax=b.$$

If $A$ is an $n\times n$ non-singular matrix, the solution to the system in the equation $Ax=b$ exists and is unique and is given by

$$x = A^{-1}b$$

However, there are cases where $A$ is not a square matrix (i.e. rectangular matrix) and also the cases where $A$ is

$n\times n$ singular matrix; i.e when the linear equations are inconsistent. In these cases there may still be solution to the system and a unified theory to treat all cases may be desirable. One such theory involves the use of **generalized inverse of matrices**. The generalized inverse is also referred to as **Pseudo-inverse, Moore-Penrose inverses or simply g-inverse** with possible subscripting of the letter g.

**Moore** [1] first published the work on generalized inverses. **Penrose** [2] defined uniquely determined generalized inverse matrix and investigated some of its properties.

## 2. DEFINITION

***Generalized inverse (g- inverse)***

Let $A$ be $m\times n$ matrix of rank $R(A) = r \le \min(m,n)$. Then a *generalized inverse (g-inverse)* of $A$ is an $n\times m$ matrix denoted by $A^-$ such that $x = A^-b$ is a solution of the consistent set of linear equations $Ax=b$.

$A$ matrix $A^-$ satisfying $AA^-A = A$ obviously coincides with $A^{-1}$ when $A^{-1}$ exists.

## 3. DIFFERENT CLASSES OF G-INVERSES

Let $A$ be an $m\times n$ matrix over the complex field $\boldsymbol{C}$. Clearly, analogous results are obtainable when the matrices are defined over a real field.

Consider the following matrix equations:

*(i)* $AXA = A$, *(1.1)*

*(ii)* $XAX = X$, *(1.2)*

*(iii)* $(XA)^* = XA$, *(1.3)*

*(iv)* $(AX)^* = AX$, *(1.4)*

where` * `denotes the conjugate transpose.

$X$ is a **g-inverse** if equation *(1.1)* is satisfied and we denote $X = A^-$.

(a) If *(1.1)* and *(1.2)* are satisfied then $X$ is a **reflexive g-inverse** and we denote $X=A^r$.

(b) If *(1.1), (1.2)* & *(1.3)* are satisfied then $X$ is a **left weak g-inverse** and we denote $X = A^w$.

(c) If *(1.1), (1.2)* & *(1.4)* are satisfied then $X$ is **right weak g-inverse** and we denote $X = A^n$.
(d) If *(1.1), (1.2), (1.3)* & *(1.4)* are all satisfied then we call $X$ is **Pseudo -inverse** or (**Moore & Penrose generalized inverse**) and we denote $X = A^+$**.** It is also known as **M-P g-inverse.**

## 4. EXPANSIONS AND CONVERGENCE OF G-INVERSE

For applications of a constructive nature (and some theoretical purposes) it is highly desirable to have some representations of $A^+$ in terms of $A$ and $A^*$. Although **Moore** [1]**, Penrose** [2] and most recently **Graybill** [3] have given methods for determining M-P inverse, it would be highly desirable to have a representation analogous to the Neumann expansion for the inverse of a non-singular matrix.

A Neumann type series expansion of $A^+$ involving only positive power of $A^*A$ is given by the following theorem:

***Theorem*** For any *square* matrix $A \neq 0$ and a real number $\alpha$ with

$$0 < \alpha < \min_{\substack{d_{ii}\neq 0 \\ i=1,2,\ldots n}} \frac{2}{|d_{ii}|^2}, \qquad (1.5)$$

where $d_{ii}$ are the (diagonal) elements of $D$ in the representation of $A$, the series

$$\alpha \sum_{k=0}^{m} (I - \alpha A^* A)^k A^* ,$$

converges and

$$\alpha \sum_{k=0}^{m} (I - \alpha A^* A)^k A^* = A^* \quad . \qquad (1.6)$$

***Proof*** *Let* $A = WDV$ to write

$$A^+ = W^* D^+ V^* \qquad (1.7)$$

where W and $V$ are unitary matrices

$$\text{and} \qquad D^+ = (d_{ii}^+) \quad . \qquad (1.8)$$

Using $A=WDV$ we obtain

$$(I - \alpha A^* A)^k = W^* (I - \alpha D^* D)^k W$$

*k=0,1,2,…………*

$$(I - \alpha A^* A)^k A^* = W^* (I - \alpha D^* D)^k D^* V^*$$

Using *(1.5)* we get

$$\sum_{k=0}^{\infty} (1 - \alpha |d_{ii}|^2)^k d_{ii}^* = \alpha^{-1} d_{ii}^+$$

*i=1,………,n*

so that

$$\alpha \sum_{k=0}^{m} (I - \alpha A^* A)^k A^* = W^* D^+ V^* = A^+ .$$

## 5. SPECIAL REPRESENTATIONS OF G-INVERSES

***An interpolation polynomial for the Moore-Penrose inverse***

Here we express $A^+$ as a Lagrange-Sylvester interpolation polynomial in powers of $A, A^*$. For any complex square matrix $A$ let $\sigma(A)$ denote the spectrum of $A$ and $\psi(A)$ its *minimal polynomial* written as $\psi(\lambda) = \prod_{\mu \in \sigma(A)} (\lambda - \mu)^{\nu(\mu)}$,

where the root $\mu \in \sigma(A)$ is *simple* if $\nu(\mu) = 1$ and *multiple* otherwise.

For any scalar function $f(\lambda)$ which is analytic at the multiple roots of $\psi(\lambda)$ and defined at the simple roots of $\psi(\lambda)$ it is possible to construct a matric function $f(A)$ which satisfies the first four requirements:

a) $f(\lambda) = K \Rightarrow f(A) = KI$
b) $f(\lambda) = \lambda \Rightarrow f(A) = A$

c) $f(\lambda) = g(\lambda) + h(\lambda) \Rightarrow f(A) = g(A) + h(A)$
d) $f(\lambda) = g(\lambda) h(\lambda) \Rightarrow f(A) = g(A) h(A)$.

We intend to construct $A^+$ as the matric function $f(A)$ corresponding to the scalar function $f(\lambda) = \lambda^+$ and consider only the case where $\lambda = 0 \in \sigma(A)$ as otherwise A is nonsingular.

***Corollary*** If $\lambda = 0$ is a simple root, this effort to construct $A^+$ this way lead only to the satisfaction of *(1.1)* and *(1.2).*

We use therefore $A^{+} = (A^{*}A)^{+}A^{*}$ to construct $A^{+}$ by using the Lagrange-Sylvester interpolation polynomial to give explicit expressions associated with $(A^{*}A)$, as all the roots in $\sigma(A^{*}A)$ are simple.

$$A^{*}A = \sum_{\lambda\in\sigma(A^{*}A)} \lambda \frac{\prod_{\lambda\neq\mu\in\sigma(A^{*}A)}(A^{*}A-\mu I)}{\prod_{\lambda\neq\mu\in\sigma(A^{*}A)}(\lambda-\mu)} \qquad (1.9)$$

so that $A^{+} = \sum_{\lambda\in\sigma(A^{*}A)} \lambda^{+} \left(\frac{\prod_{\lambda\neq\mu\in\sigma(A^{*}A)}(A^{*}A-\mu I)}{\prod_{\lambda\neq\mu\in\sigma(A^{*}A)}(\lambda-\mu)}\right)A^{*}$

.

We call (1.9) the *Lagrange-Sylvester interpolation polynomial* for $A^{+}$.

***Example*** Let $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

Then $A^{*}A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

And $\psi(A^{*}A) = (A^{*}A)^{2} - 2(A^{*}A)$ is the minimal polynomial.

Writing $\psi(\lambda) = \lambda(\lambda-2)$

we have $(A^{*}A)^{+} = \frac{1}{2}\ \frac{(A^{*}A)}{2} = \begin{pmatrix} 1/4 & -1/4 \\ -1/4 & 1/4 \end{pmatrix}$

and

$$A^{+} = (A^{*}A)^{+}A^{*} = \begin{pmatrix} 1/4 & -1/4 \\ -1/4 & 1/4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 0 \end{pmatrix}.$$

***Example*** Let $A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

$$A^{*}A = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

The minimal polynomial of $(A^{*}A)$ is $\psi(\lambda) = \lambda(\lambda-2)(\lambda-4)$

Therefore,

$$(A^{*}A)^{+} = \frac{1}{2}\left(\frac{A^{*}A(A^{*}A-4I)}{2(2-4)}\right) + \frac{1}{4}\left(\frac{A^{*}A(A^{*}A-2I)}{4(4-2)}\right)$$

$$= \frac{14}{32}(A^{*}A) - \frac{3}{32}(A^{*}A)^{2}$$

$$= \frac{1}{16}\begin{pmatrix} 5 & -1 & -3 & -1 \\ -1 & 5 & -1 & -3 \\ -3 & -1 & 5 & -1 \\ -1 & -3 & -1 & 5 \end{pmatrix}.$$

Hence

$A^{+}$

$$= \frac{1}{16}\begin{pmatrix} 5 & -1 & -3 & -1 \\ -1 & 5 & -1 & -3 \\ -3 & -1 & 5 & -1 \\ -1 & -3 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{8} & \frac{-3}{8} & \frac{-1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} & \frac{-3}{8} & \frac{-1}{8} \\ \frac{-1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{-3}{8} \\ \frac{-3}{8} & \frac{-1}{8} & \frac{1}{8} & \frac{3}{8} \end{pmatrix}.$$

***Mathematica code***

```
a =
{{1,0,0,-1},{-1,1,0,0},{0,-1,1,0},{0,0,-1,1}}
{{1,0,0,-1},{-1,1,0,0},{0,-1,1,0},{0,0,-1,1}}
```

**b = Transpose[a]**

$$\{\{1,-1,0,0\},\{0,1,-1,0\},\{0,0,1,-1\},\{-1,0,0,1\}\}$$

**c = b.a**

$$\{\{2,-1,0,-1\},\{-1,2,-1,0\},\{0,-1,2,-1\},\{-1,0,-1,2\}\}$$

**gi = PseudoInverse[c]**

$$\left\{\left\{\frac{5}{16},-\frac{1}{16},-\frac{3}{16},-\frac{1}{16}\right\},\left\{-\frac{1}{16},\frac{5}{16},-\frac{1}{16},-\frac{3}{16}\right\},\left\{-\frac{3}{16},-\frac{1}{16},\frac{5}{16},-\frac{1}{16}\right\},\left\{-\frac{1}{16},-\frac{3}{16},-\frac{1}{16},\frac{5}{16}\right\}\right\}$$

**ginverse = gi.b**

$$\left\{\left\{\frac{3}{8},-\frac{3}{8},-\frac{1}{8},\frac{1}{8}\right\},\left\{\frac{1}{8},\frac{3}{8},-\frac{3}{8},-\frac{1}{8}\right\},\left\{-\frac{1}{8},\frac{1}{8},\frac{3}{8},-\frac{3}{8}\right\},\left\{-\frac{3}{8},-\frac{1}{8},\frac{1}{8},\frac{3}{8}\right\}\right\}$$

**% / / MatrixForm**

$$\begin{bmatrix} \frac{3}{8} & -\frac{3}{8} & -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} & -\frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{1}{8} & \frac{3}{8} & -\frac{3}{8} \\ -\frac{3}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{3}{8} \end{bmatrix}$$

***Theorem*** *If* $A$ is any $m \times n$ matrix such that $(A\ A^*)^{-1}$ exists , then

$$A^{+} = \frac{1}{2\pi i}\int_{c} A^{*}(AA^{*} - Iz)^{-1}\frac{1}{z}dz$$

where $C$ is a closed contour containing non-zero eigenvalues of $AA^*$ but not containing the zero eigenvalue of $AA^*$ in or on $C$.

***Theorem*** The M-P g-inverse of a $m \times n$ matrix $A$ of complex numbers is given by the formula

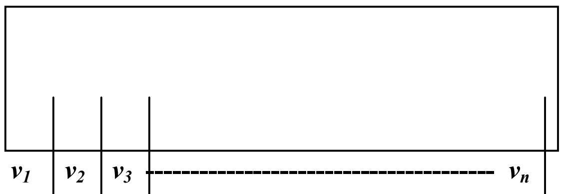
$$A^{+} = \int_{0}^{\infty} e^{-A^{*}A}A^{*}dt$$

## 6. APPLICATIONS OF G-INVERSES

***6.1 Applications in Network Theory***

Here we demonstrate the use of g-inverse in the analysis of active networks. The indefinite admittance matrix connecting the node currents and voltages in an n-terminal network plays an important role in network analysis [4]. Its singularity, however, poses difficult problems. Since singular matrices do not admit a regular inverse, special techniques had to be devised by network analysis to handle such matrices. It is our objective to show how the g-inverse can be brought into service to obtain all the results in an elegant way. The emphasis will be on the development of a suitable calculus, which we hope will be of use in general network theory, rather than on detailed examination of particular problems.

***6.2 The indefinite admittance matrix and its inverse***




Consider a n-terminal network as shown in Fig. Let currents $i_1, i_2, \ldots, i_n$ enter the terminals 1, ….., $n$ from outside and let voltages $v_1, \ldots, v_n$ be measured between these terminals and an arbitrary reference terminal *F*.

Let $i = (i_1, i_2, \ldots, i_n)'$ denote the current vector and $v = (v_1, \ldots, v_n)'$ the voltages vector. Such a network is defined by a linear relationship.

$$i = Yv \qquad (1.10)$$

where the matrix $Y$ is known as the *indefinite admittance matrix*.

By applying Kirchoff's current law and the relativity law of potentials one finds that the matrix $Y$ is constrained by the relations $Ye = 0,\ e'Y = 0'$, where $e = (1,1, \ldots, 1)'$ , i.e , the sum of the elements in each row and in each column of $Y$ is zero. Such a matrix is said to be *doubly centered.* Thus $Y$ is singular and the relationship between $i$ and $v$ induced by $Y$ is not one to one , so that the inverse relationship (such as $v = Y^{-1}i$, when $Y$ is nonsingular) cannot be uniquely deduced from *(1.10)* alone .From the equation $i = Yv$ we find that $v = Y^{-1}i$ provides an inverse relationship for some g-inverse $Y^{-}$.

By using Kirchoff's voltage law and the relativity law of currents, one finds that, in a relationship such as ,$v = Zi$, the matrix $Z$ is also doubly centered, that is, $Ze = 0$, $e'Z = 0'$. Then the problem may be posed as that of finding a g-inverse $Z$ of $Y$, which also be doubly centered.

We shall consider the case where the network is fully connected, i.e $R(Y) = n-1$, so that if B is matrix such that $YB = 0$, then there exists a vector such that $B = e't$.

***Theorem*** *The* unique doubly centered inverse of Y is $Y^+$, the Moore-Penrose inverse.

***Proof*** Let $Z$ be a g-inverse of $Y$, that is,

$$YZY=Y \Leftrightarrow Y(I-ZY)=0 \Leftrightarrow I-ZY=et' \qquad (1.11)$$

If $Z$ is doubly centered, then
$I-ZY = et' \Rightarrow e'(I-ZY) = e'et'$
$e' = eet' = nt'$
that is , $e = nt$ and $ZY = I - n^{-1}ee'$ or $ZY$ is symmetrical. Similarly, $Yz$ is symmetrical.
Also from (*1.11*), $R(Z) \geq R(Y)$, but $Ze' = 0$ and hence
$R(Z) = n-1 = R(Y)$
But we have $ZYZ = Z$.

Thus, if $Z$ is doubly centered, then it satisfies all the four conditions of M-P g-inverse i.e $Z= Y^+$, which is unique.

On the other hand if $Z= Y^+$, then column space of $Y^+$ =column space $Y'$ which implies that $Y^+e = 0$.Similarly, $e'Y^+ = 0$ so that $Y^+$ is doubly centered.

### 6.3 *On the applications of g-inverse on optimizing technique*

***Proposition 1*** Let $AX=Y$ be a system of consistent equations (linear constraints of a programming problem)[5]. Our object is to minimize

$Q=X^*NX$ where

$(N)_{m\times m}$ is a positive matrix subject to the satisfaction of the constraints.

***Proof*** $A^{g1}$ is a g-inverse of $A \ni$

$$AA^{g1}A = A$$
$$(A^{g1}A)^*N=N(A^{g1}A)$$

then $X=A^{g1}Y$

is a solution of the quadratic programming problem with linear constraints.
Further a choice $A^{g1}$ is given by

$A^{g1}=N^{-1}A^*(AN^{-1}A^*)^g$ and $A^{g1}$ is called the *minimum norm generalized inverse.*
Note that if $N=I$ then $X=A^+Y$
*is the minimum norm best solution* of $AX=Y$.

***Proposition 2*** Let $AX=Y$ be possibly a set of inconsistent equations (constraints of a programming problem) our object is to minimize

$$Q=(Y-AX)^*N(Y-AX)$$

Subject to the condition $X=A^{g2}Y$ is the best in the class approximate solution of

$$AX=Y.$$

***Proof*** $X=A^{g2}Y$ is the solution
Then $AA^{g2}A=A$, $(AA^{g2})M=MAA^{g2}$
and a choice of $A^{g2}$ is given by

$$A^{g2}=(A^*MA)^gA^*M$$

$A^{g2}$ *is called the least squares generalized inverse.*

## 7. CONCLUSION

The aim of this paper is to find out the inverse of a matrix when it is rectangular or square but singular. Such inverse is known as *Generalized Inverse (g – inverse)*. It gives general approach of g-inverse and its applications with their analytical and numerical prospects. This paper introduces some new concepts about the numerical results with Mathematica code ***PseudoInverse.*** We have shown uniform convergence of g-inverse by a special type of expansion. Generalized inverse is determined here by using minimal polynomial. An application of generalized inverse in Network theory is given.

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