Nonperturbative Phenomena in Field Theory and Gravity, and use of Borel Summation to Capture Nonperturbative Information

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Declaration

I, Md Ashiq Rahman, declare that this thesis submitted in partial fulfillment of the requirements for the conferral of the degree Bachelor of Science in Physics, from BRAC University, is wholly my own work unless otherwise referenced or acknowledged. This document has not been submitted for qualifications at any other academic institution.

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Abstract

This thesis addresses non-perturbative phenomena and techniques to analyze non-perturbative problems. First, we discuss the simplest non-perturbative example in quantum gravity: blackholes, and how they non-perturbatively emit Hawking Radiation. This is directly linked to the blackhole information paradox. We then discuss non-perturbative phenomena in quantum field theory. In particular, we discuss instantons in Yang-Mills and large-N sigma models. We discuss Borel summation as a technique to capture non-perturbative terms in a perturbative expansion. We apply Borel summation to the simple Painleve-I system and solve it numerically.

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Chapter 1

Introduction

The idea of a blackhole or a spacetime singularity dates back to the eighteenth century when Mitchell and Laplace discussed the possibility of a formation of a singularity when a star collapses at the end of its life span. The theoretical existence of blackholes was first predicted in general relativity when Karl Schwarzchild first solved the Einstein equations of general relativity in vacuum. However, he did not know that the singurality that arose from his solution was merely a problem with regard to the choice of coordinates and not a real spacetime singularity. After that, many scientists like Weyl, Flam, Eddington, Finkelstein, Rindler contributed to the solution, when Kruskal and Szekeres finally found a solution that has no coordinate singularity and leads us to the spacetime singularity. The name "blackhole" was first coined by Wheeler in 1967 in a public talk that he gave.

Throughout that era, the scientific community believed that blackholes have such a mighty gravitational force that nothing can come out of it, not even light. They also believed that blackholes are eternal. This idea is partially true. Stephen Hawking revised this idea in the 1970s and showed that blackholes do emit something and that they 'evaporate' and come to an end. He discovered that blackholes emit thermal radiation and diminish in size in the process. Nowadays, through the advancement in our understanding of Quantum Field Theory, we know that the radiation escapes from the blackhole by the process of *Quantum Tunneling*.

When quantum tunneling problems are tackled with perturbation theory, we end up finding power series solutions. These power series are dependent upon a parameter that controls the strength of the perturbation. In order to find a better approximation to our non-exact solution, we add all the corrections in the higher orders.

However, most of the series that we find are only convergent upto a certain number of terms and then diverge uncontrollably. It was later understood that quantum tunneling is actually non-perturbative in nature, so we have to resort to instanton contributions and other calculation techniques to find a better approximation to our solution.

In this thesis, we begin by deriving the process of hawking radiation. After that, we move on to descibing the first clues of the existence of instantons and explain a very useful summation technique devised by Emile Borel. Then we give a brief outline of the basics of quantum chromodynamics, focusing on Yang-Mills theories and various Sigma models. We also show how instantons arise in these theories. Moreover, we explore the properties of these theories at the Large-N limit.

In the last chapter of this dissertation, we apply the Borel summation technique from the first part of the thesis to solve the Painleve-I equation. This equation is manifest in two-dimensional quantum gravity, where it gives us the all genus solution. We also discuss very briefly the *resurgence theory* in the end. This theory connects the perturbative QFT with its nonperturbative counterpart.

Chapter 2

The Quantum evaporation of Blackholes

2.1 The Free Scalar Field quantized

The Klein-Gordon equation, with the coavriant derivative D, for a real scalar field $\phi(x)$ is [1]:

$$(D^{\mu}\partial_{\mu} - m^2)\Phi(x) = 0 \tag{2.1}$$

Suppose, the solution space S is spanned by the set of solutions ϕ_α. It is assumed that there exists a Cauchy surface Σ, i.e, the spacetime is globally hyperbolic. Hence, the initial conditions on Σ directly affects the points on the space S. It has a *naturally symplectic* inner product.

$$\begin{aligned}\phi_\alpha \wedge \phi_\beta &= \int_\Sigma dS_\mu(\phi_\alpha \partial^\mu \phi_\beta - \phi_\beta \partial^\mu \phi_\alpha) \\ &= \int_\Sigma dS_\mu \phi_\alpha \overleftrightarrow{\partial^\mu} \phi_\beta \\ &= -\phi_\beta \wedge \phi_\alpha\end{aligned} \tag{2.2}$$

Here, 'natural' refers to the fact that $\wedge$ is independent of the value of Σ chosen. The indices α and β are internal space indices. Thus,

$$\begin{aligned}(\phi_\alpha \wedge \phi_\beta)_\Sigma - (\phi_\alpha \wedge \phi_\beta)_{\Sigma'} &= \int_\Sigma d^4x\sqrt{-g}D_\mu(\phi_\alpha \partial^\mu \phi_\beta - \phi_\beta \partial^\mu \phi_\alpha) \\ &= \int_\Sigma d^4\sqrt{-g}D_\mu(\phi_\alpha \overleftrightarrow{\partial^\mu} \phi_\beta)\end{aligned} \tag{2.3}$$

However,

$$\begin{aligned} D_\mu(\phi_\alpha \overleftrightarrow{\partial^\mu} \phi_\beta) &= D_\mu(\phi_\alpha \partial^\mu \phi_\beta - \phi_\beta \partial^\mu \phi_\alpha) \\ &= \phi_\alpha (D_\mu \partial^\mu \phi_\beta) - (D_\mu \partial^\mu \phi_\alpha)\phi_\beta \\ &= \phi_\alpha (m^2 \phi_\beta) - (m^2 \phi_\alpha)\phi_\beta \\ &= 0 \end{aligned} \tag{2.4}$$

Here, the Klein-Gordon equation 2.1 is used in the last step.

Using the Darboux's theorem, the antisymmetric form $\phi_\alpha \wedge \phi_\beta$ can be written as a canonical block diagonal form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Therefore, pairwise the real solutions ϕ, ϕ' to the Klein-Gordon equation can be written as (ϕ, ϕ'). Now, the norm of the complex solution $\psi = (\phi - i\phi')/\sqrt{2}$ is defined as $||\psi||$, such that $||\psi||^2 = \phi \wedge \phi' = 1$. Equivalently,

$$||\psi|| = |\psi \wedge \psi| \tag{2.5}$$

Analogous to equation 2.2, a similar complex valued symplectic form is

$$||\psi||^2 = i \int_\Sigma dS_\mu \psi^* \overleftrightarrow{\partial}^\mu \psi \tag{2.6}$$

In general, the Klein-Gordon equation can be written in terms of a solution set ψ_i of complex basis. The Hermitian inner product is defined as

$$(\psi_i, \psi_j) = i \int dS_\mu \psi_i^* \overleftrightarrow{\partial}^\mu \psi_j \tag{2.7}$$

and that $(\psi_i, \psi_j) = \delta_{ij}$. However, since $||\psi||^2 = -||\psi^*||^2$, the inner product is not positive definite. Moreover, the basis ψ_i can be taken such that

$$\begin{pmatrix} (\psi_i, \psi_j) = \delta_{ij} & (\psi_i, \psi_j^*) = 0 \\ \\ (\psi_i^*, \psi_j = 0 & (\psi_i^*, \psi_j^*) = -\delta_{ij} \end{pmatrix} \tag{2.8}$$

The complex solution $\Psi = \Sigma_i a_i \psi_i$ can be thought of as the wavefunction of a free particle. This is because when the inner product $(,)$ is confined to these solutions, it is positive definite, but fails in the presence of interactions. Only the free particle solutions have positive inner product. Also, it is only valid for *complex* scalar fields.

A real solution Φ of the K-G equation can be expressed as

$$\Phi(x) = \sum [a_i \psi_i(x) + a_i^* \psi_i^*(x)]. \tag{2.9}$$

When quantizing ϕ, $*$ becomes $\dagger$ for the a_i since the a_i become operators instead of complex numbers.

$$\Phi(x) = \sum [a_i \psi_i(x) + a_i^\dagger \psi_i^*(x)] \tag{2.10}$$

Here, a_i are operators in a Hilbert space H, having Hermitian conjugates $a_i^\dagger$, which satisfy the following commutation relations [1]

$$\begin{aligned} [a_i, a_j] &= 0, \\ [a_i, a_j^\dagger] &= \delta_{ij} \end{aligned} \tag{2.11}$$

Here, $\hbar$ is taken to be equal to 1.

This Hilbert space is chosen to be a Fock space. It is built from a 'vacuum' state $|vac\rangle$. A vacuum state is the lowest energy quantum state. Among the different classifications of vacuum states, such as the QED and QCD vacuum states, the Bunch-Davies is a convenient vacuum state in a curved spacetime, such as near a blackhole's event horizon. In curved spacetime there is a set of choices of vacuum state. Alpha vacua is an isometry invariant set of choices for the background de Sitter space. One special case of Alpha vacua is the *Bunch-Davies vacuum*, which satisfies the Hadamard condition. [2]

Now, the Hilbert space satisfies the following:

$$a_i \, |vac\rangle = 0, \ \forall_i \tag{2.12}$$

$$\langle vac|vac\rangle = 1 \tag{2.13}$$

Thus, H has the basis

$$\{|vac\rangle\,, \ a_i^\dagger \, |vac\rangle\,, \ a_i^\dagger a_j^\dagger \, |vac\rangle\,, ...\} \tag{2.14}$$

On this space, the inner product $\langle \, | \, \rangle$ is positive definite. The complex basis ψ_i of solutions for the K-G equation chosen, that satisfies 2.8, determines the choice of $|vac\rangle$, which in turn affects the Hilbert space H's basis.There are a lot of such bases.

Consider ψ_i' where

$$\psi_i' = \sum_j (A_{ij}\psi_j + B_{ij}\psi_j^*) \tag{2.15}$$

If the inner product of ψ_i satisfy equation 2.8, then the coefficients A and B

satisfy:

$$\begin{aligned} AA^\dagger - BB^\dagger &= \mathbf{1} \\ AB^T - BA^T &= 0 \end{aligned} \tag{2.16}$$

When 2.16 is inverted, the result is

$$\psi_j = \sum_k A'_{jk}\psi'_k + B'_{jk}\psi_k*' \tag{2.17}$$

where

$$A' = A^\dagger,\ \ B' = -B^T \tag{2.18}$$

Also, A' and B' must fulfill the same conditions as A and B, thus

$$A'A'^\dagger - B'B'^\dagger = \mathbf{1} \tag{2.19}$$

$$A'B'^T - B'A'^T = 0 \tag{2.20}$$

Equivalently,

$$A^\dagger A - B^T B^* = \mathbf{1} \tag{2.21}$$

$$A^\dagger B - B^T A^* = 0 \tag{2.22}$$

These conditions do not automatically follow from 2.17, 2.18. These also imply that a change of basis is invertible. [1–3]

There is no favored choice of vacuum because there is no favored choice of basis that satisfies 2.9 in general spacetime. However, the basis of *positive frequency* eigenfunctions u_i of the Killing vector k can be chosen in a stationary spacetime. Thus,

$$k^\mu \partial_\mu u_i = -i\omega_i u_i, \quad \omega_i \geq 0 \tag{2.23}$$

It is to be noted that the Klein-Gordon equation is mapped to its solutions by k. Also, k can be diagonalized with pure-imaginary eigenvalues since it is anti-hermitian.

Also, eigenfunctions whose eigenvalues are distinct are orthogonal, thus

$$(u_i, u_j^*) = 0 \tag{2.24}$$

Furthermore, u_i can be chosen so that 2.8 can be satisfied, by normalizing it such that $(u_i, u_j) = \partial_{ij}$. The functions with $\omega = 0$ are omitted.

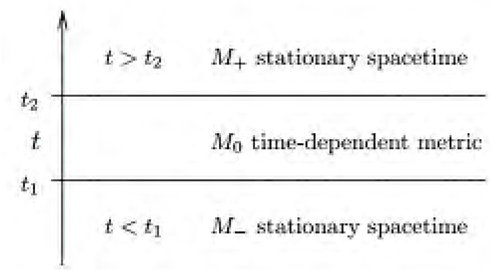


Figure 2.1: Particle production in non-stationary spacetime

The vacuum state $|vac\rangle$ is as a matter of fact the lowest energy state. The one-particle states are represented by $a_i^\dagger |vac\rangle$, the two-particle states are written as $a_i^\dagger a_j^\dagger |vac\rangle$, and so on. The number operator is defined to be

$$N = \sum_i a_i^\dagger a_i \tag{2.25}$$

2.2 Particle Creation in Non-Stationary Spacetimes

For a 'sandwich' spacetime $M = M_- \cup M_0 \cup M_+$. Here sandwich spacetime means that in the regions M_- and M_+ there is no interaction present, but in the region M_0, gravitational interaction is considered to be turned on, briefly.

A scalar field solution to the Klein-Gordon equation can be expanded in M_- as

$$\Phi(x) = \sum_i [a_i u_i(x) + a_i^\dagger u_i^*(x)] \tag{2.26}$$

The functions $u_i(x)$ are unable to solve the K-G equation in M_0 unlike what they do in the M_- region. The eigenfunctions u_i change in M_0 from that of in M_-, then it changes again in M_+. This gives rise to some new function $\psi_i(x)$ in M_+ when it is continued through M_0. Thus, in M_+

$$\Phi(x) = \sum_i [a_i \psi_i(x) + a_i^\dagger \psi_i^*(x)] \tag{2.27}$$

The inner product matrix remains unchanged as the inner product on the Cauchy surface Σ. For some matrices A and B, this means that

$$\psi_i = \sum_j (A_{ij} u_j + B_{ij} u_j^*) \tag{2.28}$$

Therefore, in $M+$

$$\begin{aligned}\Phi(x) &= \sum_i (a_i\psi_i + a_i^\dagger \psi_i^*) \\ &= \sum_i \left[a_i \sum_j (A_{ij}u_j + B_{ij}u_j^*) + a_i^\dagger \sum_j (A_{ij}^* u_j^* + B_{ij}^* u_j) \right] \\ &= \sum_i [a_i' u_i(x) + a_i'^\dagger u_i^*(x)]\end{aligned} \tag{2.29}$$

Here,

$$\boxed{a_j' = \sum_i (a_i A_{ij} + a_i^\dagger B_{ij}^*)} \tag{2.30}$$

is the *Bogoliubov transformation.* A and B are the *Bogoliubov coefficients*, which follow the following properties:

$$\begin{aligned}[a_i', a_j'^\dagger] &= 0 \\ [a_i', a_j'^\dagger] &= \delta_{ij}\end{aligned} \tag{2.31}$$

If $B = 0$, then $A^\dagger A = AA^\dagger = 1$, which means that the definition of the vacuum remains unchanged when the basis u_i is changed to ψ_i. In this case, only the annihilation operators are permuted, hence it is a unitary transformation.

For the i^{th} mode of k, the particle number operator is

$$\begin{aligned}N_i &= a_i^\dagger a_i \qquad \textit{in } M_- \\ N_i' &= a_i'^\dagger a_i' \qquad \textit{in } M_+\end{aligned} \tag{2.32}$$

$|vac\rangle$ is the zero particle state in M_- such that $a_i |vac\rangle = 0 \ \forall i$. Thus the expectation value for the number of particles in the i^{th} mode in M_+ is

$$\begin{aligned}\langle N_i' \rangle \equiv \langle vac| N_i' |vac\rangle &= \langle vac| a_i'^{\dagger} a_i' |vac\rangle \\ &= \sum_{j,k} \langle vac| (a_k B_{ki})(a_j^{\dagger} B_{ji}^*) |vac\rangle \\ &= \sum_{k,j} \delta_{k,j} B_{ki} B_{ji}^* \\ &= \sum_{j} B_{ji}^* B_{ji} \\ &= \sum_{j} (B_{ij}^*)^T B_{ji} \\ &= (B^{\dagger} B)_{ii}\end{aligned} \tag{2.33}$$

Therefore, the expected value of the total number of particles is $\text{tr}(B^{\dagger}B)$, which is generally non-zero. This implies even though there were no particles in M_-, there will generally be particles in M_+. [2–4]

2.3 Hawking Radiation

The blackhole causes a collapse in spacetime and has a non-stationary metric. Thus it is expected that the non-static spacetime would cause particle production. However, particle production is necessarily a temporary event which depends on the properties of the collapse because the spacetime is stationary at late times. Moreover, in the region of the blackhole horizon, the particles experience an infinite time dilation, and hence might take unpredictably long durations to escape from the point of view of the outside observer. This implies that there might be particles coming out of the horizon at late times just because there exists an event horizon. It turns out that particles do come out of the horizon in the form of blackbody radiation. This is called *Hawking Radiation*. For a massless scalar field Φ in a Schwarzchild blackhole spacetime, near future null infinity, $\mathfrak{I}^+$, the positive outgoing modes have the property [5–7]

$$\Phi_\omega \sim e^{-i\omega u} \tag{2.34}$$

Since it is a wave it has e^{-ikx} form, and the convenient null coordinate is u, where $u = t - r^*$, and r^* is like r. Suppose, when the approximation is a geometric optics one, the particle with a null ray, γ as the world line, has its ray traced back in time from $\mathfrak{I}^+$. Here, u and v are related to U and V as follows:

$$U = -e^{-u/4M}, \qquad V = e^{v/4M} \tag{2.35}$$

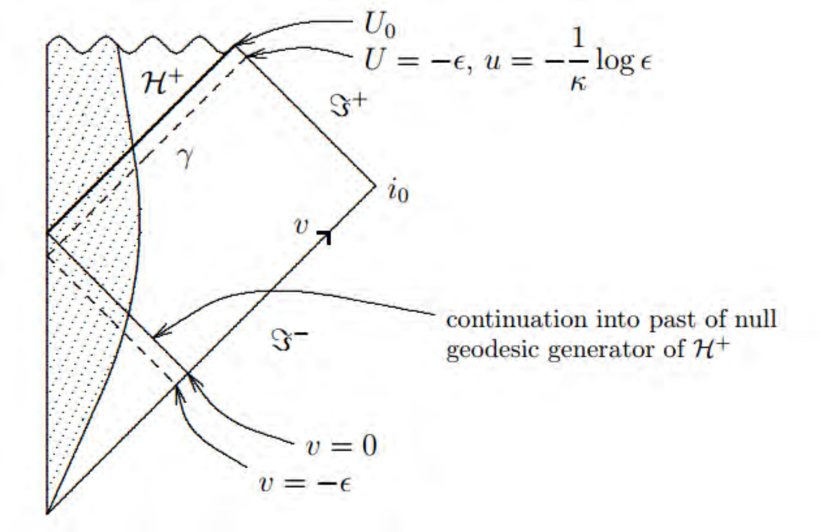


Figure 2.2: The geometric optics approximation

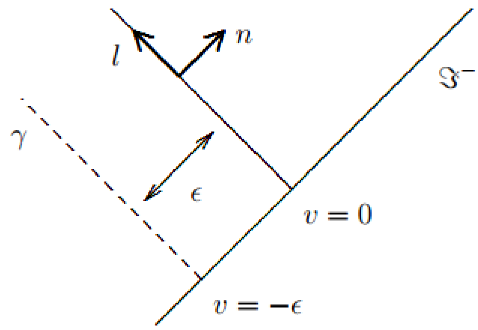


Figure 2.3: Parallel transport of n and l

As t approaches infinity, the ray γ becomes a free field. The ray γ can be parameterized by null coordinates u and v. Similarly, at future null infinity, $u \to \infty$, which means $U \to 0$. Now, let $U = -\epsilon$, so that u is

$$u = -\frac{1}{\kappa}\log\epsilon \qquad (\text{on } \gamma \text{ near } \mathcal{H}^+) \tag{2.36}$$

and the excitation becomes

$$\Phi_\omega \sim \exp\left(\frac{i\omega}{\kappa}\log\,\epsilon\right) \tag{2.37}$$

For small ϵ the frequency of oscillation increases quickly, and thus at late times the geometric optics approximation is reasonable.

The excitation parallel transported back to $\mathfrak{I}^-$ will have a similar form to the excitation at $\mathfrak{I}^+$. Therefore,

$$\Phi_\omega \sim \exp\left(\frac{i\omega}{\kappa}\log(-(-\epsilon))\right) \tag{2.38}$$

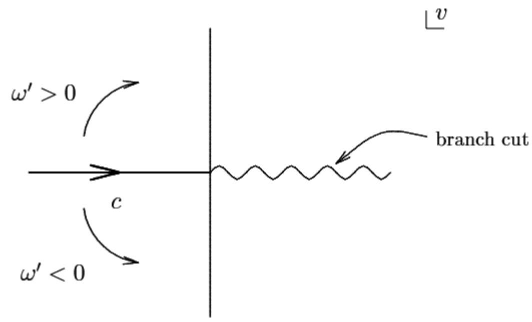


Figure 2.4: Branch cut along the complex plane v

Suppose, on $\mathfrak{I}^-$, $v = -\epsilon$ (for ϵ being positive), then

$$\Phi_\omega \sim \exp\left(\frac{i\omega}{\kappa}\log(-v)\right) \tag{2.39}$$

However, for outgoing excitations, $v > 0$. Thus an ingoing null ray from $\mathfrak{I}^-$ does not reach $\mathfrak{I}^+$ because it will never come out of the event horizon, $\mathcal{H}^+$. Thus the Fourier transform from $-\infty$ to $+\infty$ is actually a transform from $-\infty$ to 0:

$$\begin{aligned}\tilde{\Phi}_\omega &= \int_{-\infty}^{\infty} e^{i\omega' v}\Phi_\omega(v)dv \\ &= \int_{-\infty}^{0} \exp\Big[i\omega' v + \frac{i\omega}{\kappa}\log(-v)\Big]dv\end{aligned} \tag{2.40}$$

Fourier transform $\tilde{\Phi}_\omega(\omega)$ satisfies the following

Lemma

$$\boxed{\tilde{\Phi}_\omega(-\omega') = -\exp\left(-\frac{\pi\omega}{\kappa}\right)\tilde{\Phi}_\omega(\omega') \qquad \text{for } \omega' > 0} \tag{2.41}$$

Proof Suppose, a branch cut is chosen in a complex v-plane which lies on the real axis as follows

Firstly, the contour is rotated to the positive imaginary axis and then $v = ix$ is set for $\omega' > 0$. This yields

$$\begin{aligned}\tilde{\Phi}_\omega(\omega') &= -i\int_0^{\infty} \exp\left[-\omega' x + \frac{i\omega}{\kappa}\log\left(xe^{-i\pi/2}\right)\right]dx \\ &= -\exp\left(\frac{\pi\omega}{2\kappa}\right)\int_0^{\infty} \exp\left[-\omega' x + \frac{i\omega}{\kappa}\log(x)\right]dx\end{aligned} \tag{2.42}$$

Corollary *At late times*, a *positive* frequency mode on $\mathfrak{I}^+$ is equivalent to *mixed positive and negative* modes on $\mathfrak{I}^-$. The forward transmitted part is the positive frequency part of the wave in M_+. This is like the A_{ij} coefficient in equation 2.17. Thus

$$A_{\omega\omega'} = \tilde{\Phi}_\omega(\omega') \tag{2.43}$$

The reflected part of the ingoing wave is like the B_{ij} coefficients of the same equation 2.17. Thus B_{ij} correspond to $-\omega'$. Hence,

$$B_{\omega\omega'} = \tilde{\Phi}_\omega(-\omega') = -e^{\pi\omega/\kappa}\tilde{\Phi}_\omega(\omega') \tag{2.44}$$

are the Bogoliubov coefficients. Thus the Bogoliubov coefficients are related by

$$\boxed{B_{ij} = -e^{-\pi\omega_i/\kappa}A_{ij}} \tag{2.45}$$

Because of 2.16 the Bogoliubov coefficients satisfy

$$\begin{aligned}\delta_{ij} &= \left(AA^\dagger - BB^\dagger\right)_{ij} \\ &= \sum_k A_{ik}A^*_{jk} - B_{ik}B^*_{jk} \\ &= \left[e^{\pi(\omega_i+\omega_j)/\kappa} - 1\right]\sum_k B_{ik}B^*_{jk}\end{aligned} \tag{2.46}$$

Setting $i = j$ yields

$$\left(BB^\dagger\right)_{ii} = \frac{1}{e^{2\pi\omega_i/\kappa} - 1} \tag{2.47}$$

Now, the *inverse* Bogoliubov coefficients relative to a positive frequency mode on $\mathfrak{I}^-$ are required. They are identified with a mixed positive and negative frequency modes on $\mathfrak{I}^+$. It was shown before (2.18) that the inverse B coefficient is [3, 4]

$$B' = -B^T \tag{2.48}$$

If there is a vacuum on $\mathfrak{I}^-$, then the particle flux at late times through $\mathfrak{I}^+$ is given by

$$\langle N_i\rangle_{\mathfrak{I}^+} = \left(\left(B'\right)^\dagger B'\right)_{ii} = \left(B^*B^T\right)_{ii} = \left(BB^T\right)^*_{ii} \tag{2.49}$$

However, $(BB^T)_{ii}$ is real, therefore,

$$\boxed{\langle N_i\rangle_{\mathfrak{I}^+} = \frac{1}{e^{2\pi\omega_i/\kappa} - 1}} \tag{2.50}$$

This is the energy spectrum of an object radiating as a blackbody. The blackbody factor for a blackbody radiator is $\frac{1}{e^{h\nu/kT-1}}$. Thus a blackhole corresponds to a blackbody with temperature

$$\frac{h\nu}{k_B T_H} = \frac{2\pi\omega}{\kappa} \tag{2.51}$$

which implies that the Hawking Temperature is

$$k_B T_H = \frac{\hbar\kappa}{2\pi} \tag{2.52}$$

Using the Stephan-Boltzmann law the blackbody radiates energy as:

$$\frac{dE}{dt} \simeq -\sigma A T_H^4, \qquad \left(\sigma = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}\right) \tag{2.53}$$

Here, the area of the blackhole is represented by A. Since

$$E = Mc^2, \qquad A = \frac{MG^2}{c^2}, \qquad k_B \sim \frac{\hbar c^3}{GM} \tag{2.54}$$

this yields

$$\frac{dM}{dt} \sim \frac{\hbar c^4}{G^2 M^2} \tag{2.55}$$

from which the lifetime of a blackhole is found to be

$$\tau \sim \left(\frac{G^2}{\hbar c^4}\right) M^3 \tag{2.56}$$

It is to be noted that in deriving the Hawking radiation, M was taken to be fixed. This is good for large blackholes. This approximation does not hold in the final phases of evaporation, when $dM/dt \ll M$ does not hold.

2.4 Thermodynamics of Black Holes

Analogous to the first law of thermodynamics, $dU = TdS - PdV$, there exists a first law of black hole thermodynamics can be written as

$$dM = TdS_{BH} + \Omega_H dJ + \Phi_H dQ, \qquad (\Omega_H, \Phi_H \text{intensive}, J, Q \text{extensive}) \tag{2.57}$$

Where, Q is the charge, J is the angular momentum, Φ_H is the electric surface potential, Ω_H is the angular velocity and T is the Hawking Temperature.

$$T = \frac{\hbar\kappa}{2\pi} \tag{2.58}$$

Beckenstein guessed and Hawking gave persuasive arguments that the blackhole entropy is proportional to the area as

$$\boxed{S_{BH} = \frac{A}{4\hbar}}. \tag{2.59}$$

This is known as the Beckenstein-Hawking entropy or the entropy of the black hole.

The second law of black hole mechanics violated through the process of black hole evaporation since the S_{BH} decreases. However, the total entropy is

$$S = S_{BH} + S_{\text{ext}} \tag{2.60}$$

where the entropy of the matter in the exterior region of black hole spacetime is represented by S_{ext}. But, since black hole radiates itself away in the form of heat, the exterior entropy increases, thus causing an overall increase in the total entropy S as a function of time. From this, the second law of thermodynamics can be generalized as $S = S_{BH} + S_{\text{ext}}$ is always a non-decreasing function of time (in any process)

According to Beckenstein, if an encyclopedia is thrown into a black hole the entropy of the exterior spacetime could decrease. However, unless the black hole itself has an entropy, the second law of thermodynamics would be violated. Hence, the above equation was first put forward by him. [3, 4, 8, 9]

The Information Paradox

By the process of Hawking Radiation, a black hole will eventually evaporate away and the event horizon will disappear, as shown in the Carter-Penrose diagram 2.5. The diagram shows that the singularity disappears and the top part of the Penrose diagram looks like a Minkowki space.

For this spacetime, Σ_1 is a Cauchy surface. However, the black hole region is not included in the past domain of dependence $D^{-}(\Sigma_2)$, therefore, Σ_2 is not a Cauchy surface, from where information can not reach the black hole region, unlike that of Σ_1. Thus it seems like information has lost into the black hole. This puts the basic idea of quantum mechanics in conflict to the QFT of curved spacetime as it implies a *non-unitary* evolution from Σ_1 to Σ_2. On the contrary, the information might not be really lost, since a static observer does not 'see' anything passing through $\mathcal{H}^{+}$. One solution to this problem might be to do all the calculations regarding the back-reaction effects, while some argue that an understanding of Planck scale physics might be necessary. Moreover, the idea that when kT becomes the Planck energy $(\frac{\hbar c}{G})^{\frac{1}{2}}c^2$ then the prediction by QFT that at the horizon of the black hole the local temperature $T_{\text{loc}\to\infty}$ should be rendered invalid. This is because in this region the quantum gravity effects are significant and the temperature becomes of

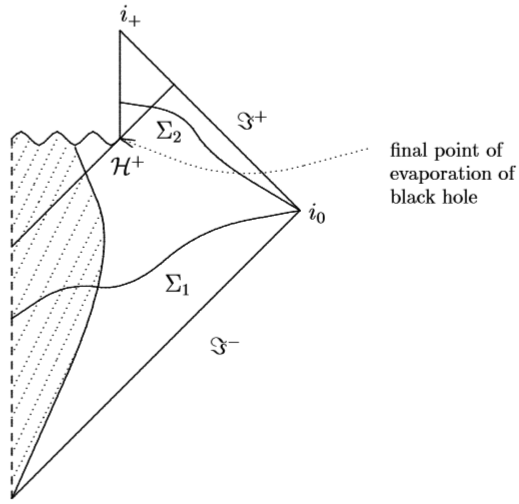


Figure 2.5: Evaporation of a black hole

the maximum order or the Hagedorn temperature in string theory.[4]

Chapter 3

Borel's Trick in Non-Perturbative QFT

3.1 First Hints of Non-perturbative effects

Perturbation theory in quantum mechanaics usually results in diverging series, with zero radius of convergence. This is manifest in even the elementary aspects such as that of harmonic oscillator with a quartic potential. The Hamiltonian is as follows [10–13]:

$$H = \frac{p^2}{2} + \frac{q^2}{2} + \frac{g}{4}q^4 \tag{3.1}$$

Here, g is the coupling constant. If we now calculate the ground state energy, which is a function of g, around $g = 0$, stationary perturbation theory gives us this divergent series:

$$E_0(g) \sim \sum_{n\geq 0} a_n g^n = \frac{1}{2} + \frac{3}{4}(\frac{g}{4}) - \frac{21}{8}(\frac{g}{4})^2 + \frac{333}{16}(\frac{g}{4})^3 + \mathcal{O}(g^4) \tag{3.2}$$

where $\hbar = 1$. It can be seen that the coefficients of g in the above expansion are factorially increasing as,

$$a_n \sim (\frac{3}{4})^n n!, \qquad n \gg 1 \tag{3.3}$$

Now, from the eigenvalue equation

$$H\left|\psi_n\right\rangle = E_n\left|\psi_n\right\rangle, \qquad n = 0, 1, ... \tag{3.4}$$

the ground state energy can be said to have a perturbative definition because it can be written as an asymptotic series, since any well defined function, $f(z)$ has a non-perturbative definition if $f(z)$ has $\phi(z)$ as its asymptotic series as such:

$$f(z) \sim \phi(z) = \sum_{n\geq 0} a_n z^n \tag{3.5}$$

However, as was shown above, the coefficients in the later terms increase and the series diverges as z increases. To counter this problem, one possible solution is to sum up only the first few terms up to which the series converges. This process is called *optimal truncation.* [14, 15]This is illustrated as follows.

Let us consider the series

$$a_n \sim A^{-n} n!, \qquad n \gg 1. \tag{3.6}$$

When N is minimum, the smallest term in the series, for $|z|$ constant is obtained from the following equation:

$$|a_N z^N| = cN!|\frac{z}{A}|^N. \tag{3.7}$$

This can be written using the Stirling approximation as

$$c\exp N(\log N - 1 - \log|\frac{A}{z})|. \tag{3.8}$$

When N is large, the saddle point of the above function is

$$N_* = |\frac{A}{z}|. \tag{3.9}$$

Now, at large N optimal truncation can be performed for small $|z|$. Howeve, as $|z|$ increases, lesser number of terms can be used. This results in an error, which can be estimated to be

$$\epsilon(z) = C_{N_*+1}|z|^{N_*+1} \sim e^{|A/z|} \tag{3.10}$$

This is the maximum resolution that can be obtained when a function $f(z)$ is reconstructed asymptotically. This problem appears exclusively in non-pertubartive field theory, hence it is called "non-perturbative ambiguity," and the absolute value of A determines the "strength" of the ambiguity. This is in fact the first clue of the existence of a non-perturbative effect.

3.2 Borel Summation to the Rescue

In order to resolve this ambiguity, Borel simply divided the equation

$$a_n \sim A^{-n} n! \tag{3.11}$$

by $n!$ and wrote it as a summation as

$$\hat{\phi}(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \zeta^n. \tag{3.12}$$

Due to the factorial term in the denominator, this series converges at large N with a radius of convergence $\rho = |A|$, and an analytic function in the circle $|\zeta| < |A|$ is defined by this. This is called the *Borel transformaiton*. Here, a singularity is present at $\zeta = A$. This is illustrated in the following examples:

Example 3.1 Suppose the series

$$\phi(z) = \sum_{n=0}^{\infty} (-1)^n n! z^n \tag{3.13}$$

can be identified as an asymptotic series of the form

$$a_n \sim A^{-n} n! \tag{3.14}$$

where $A = -1$. This can be Borel transformed as

$$\hat{\phi}(\zeta) = \sum_{n=0}^{\infty} (-1)^n \zeta^n \tag{3.15}$$

Here, the radius of convergence $\rho = 1$. Moreover, this series can be analytically continued to a meromorphic function shown below, with only one pole at $\zeta = -1$:

$$\hat{\phi}(\zeta) = \frac{1}{1+\zeta} \tag{3.16}$$

It can be seen off that there is a singularity of the Borel transformation, which is namely a pole at $\zeta = A = -1$.

Example 3.2 The Borel transform for the series

$$\phi(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+b)}{\Gamma(b)} A^{-k} z^k, \tag{3.17}$$

where b is not an integer, is

$$\hat{\phi}(\zeta) = \sum_{k=0}^{\infty} \frac{\Gamma(k+b)}{k!\Gamma(b)} A^{-k} \zeta^k = (1 - \zeta/A)^{-b}, \tag{3.18}$$

with a brach cut singularity at $\zeta = A$.

Now, if $b = 0$ in the above series, it turns out to be

$$\phi(z) = \sum_{k=1}^{\infty} \Gamma(k) A^{-k} z^k \tag{3.19}$$

with a Borel transform given by

$$\hat{\phi}(\zeta) = -\log(1 - \zeta/A) \tag{3.20}$$

where, a logarithmic singularity exists at $\zeta = A$.

Now, suppose that $\hat{\phi}(\zeta)$ can be analytically continued to a region near the positive real axis in the complex z-plane such that there exists the following Laplace transformation

$$s(\phi)(z) = \int_0^{\infty} e^{-\zeta} \hat{\phi}(z\zeta)\, \mathrm{d}\zeta = z^{-1} \int_0^{\infty} e^{-\zeta/z} \hat{\phi}(\zeta)\, \mathrm{d}\zeta. \tag{3.21}$$

This is when the series is said to be *Borel summable* and the *Borel sum* of $\phi(z)$ is given by $s(\phi)(z)$. It can be noted that $s(\phi)(z)$ can be asymptotically expanded around $z = 0$, and this expansion is identified with the original series $\hat{\phi}(\zeta)$, since

$$s(\phi)(z) = z^{-1} \sum_{n \geq 0} \frac{a_n}{n!} \int_0^{\infty} \mathrm{d}\zeta e^{-\zeta/z} \zeta^n = \sum_{n \geq 0} a_n z^n \tag{3.22}$$

For some values of z, the diverging series ϕz can be converted into a well defined function $s(\phi)(z)$ using this method. As was mentioned above for the case of the quartic harmonic oscillator, $\phi(z)$ is the asymptotic expansion of a well-defined function $f(z)$. If the Borel sum $s(\phi)(z)$ and the original function $f(z)$ matches, then Borel summation gives us the original non-perturbative solution.

Moreover, the generalized Borel resummation can be written as

$$s_\theta(\phi)(z) = \int_0^{e^{i\theta}\infty} e^{-\zeta} \hat{\phi}(z\zeta)\, \mathrm{d}\zeta. \tag{3.23}$$

along θ. This is done be doing the Laplace transform 3.21 along any direction θ, in the complex plane as such:

Example 3.3 The Borel transform shown in example 2.1 can be analytically extended on the complex plane $\mathcal{C}\backslash\{-1\}$ as

$$s(\phi)(z) = \int_0^{\infty} \frac{e^{-z}}{1 + z\zeta} d\mathrm{d}\zeta \tag{3.24}$$

which is valid for all $z \geq 0$.

Now even if we have a Borel summable series $\phi(z)$ and we expand it, we can only know a few coefficients in it. Thus, the analytic continuation of the Borel transformation around the positive real axis is very hard to do. A practical way to accurately estimate the resulting function is to incorporate the *Pade approximants.* Let us consider the series

$$\phi(z) = \sum_{k=0}^{\infty} a_k z^k. \tag{3.25}$$

For this, the Pade approximant is given by

$$[l/m]_\phi(z) = \frac{p_0 + p_1 z + \ldots + p_l z^l}{q_0 + q_1 z + \ldots + q_m z^m}, \tag{3.26}$$

where l, m, are positive integers and $q_0 = 1$ is fixed. Now, the coeffecients are set by

$$\phi(z) - [l/m]_\phi(z) = \mathcal{O}(z^{l+m+1}). \tag{3.27}$$

It is possible to reconstruct the analytic continuation of the Borel transform of a given series $\phi(z)$ using the Pade approximants. One of the many procedures is to use the Pade approximant given below:

$$\mathcal{P}_n^\phi(\zeta) = \big[[n/2]/[(n+1)/2]\big]\hat{\phi}(\zeta) \tag{3.28}$$

which works if the first $n+1$ coefficients of the original series is known. Now, an approximation to the Borel resummation is given by the integral

$$s(\phi)_n(z) = z^{-1} \int_0^\infty e^{-\zeta/z} \mathcal{P}_n^\phi(\zeta) \mathrm{d}\zeta \tag{3.29}$$

As n is increased, this gets better systematically. Since this integral connects the Pade approximant to the Borel resummation, this is often called the *Borel-Pade resumamtion.* This procedure is illustrated in the following example.

Example 3.4 The Borel transform of the series

$$\phi(g) = \sum_{k=0}^{\infty} a_k g^k, \qquad a_k = (-4)^{-k} \frac{(4k-1)!!}{k!}. \tag{3.30}$$

is given by

$$\hat{\phi}(\zeta) = \frac{2K(k)}{\pi(1+4\zeta)^{1/4}}, \qquad k^2 = \frac{1}{2} - \frac{1}{2\sqrt{1+4\zeta}}, \tag{3.31}$$

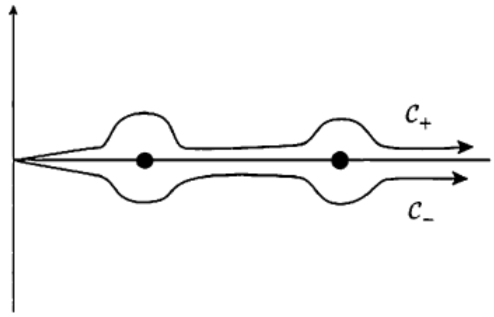


Figure 3.1: The integral path evading the poles on the x-axis

where $K(k)$ is an elliptic integral of the first kind. Now, we numerically show that

$$s(\phi)(g) = I(g) \tag{3.32}$$

for $g = 0.2$ and $g = 0.4$.

n	$s(\phi)_n(0.2)$	$s(\phi)_n(0.4)$
10	0.9079854376	0.8576207823
20	0.9079847776	0.8576086008
30	0.9079847774	0.8576085854
$I(g)$	0.9079847774	0.8576085853

In the above table, it can be seen that the values of the integral improves as the value of n increases. The numerical solution for $I(g)$ is written in the last line. The underlined digits are those which agree with the numerical results of 3.29.

However, there are instances when the the Borel transformation defined above does not exist. This is when there are poles on the positive real axis. The way to get across this problem is to deform the contours and carry out the integration on the paths $\mathcal{C}_\pm$ such that they are slightly above or below the positive real axis so that they dodge the singularities and branch cuts. This is illustrated in the figure 3.1.

Now, the *lateral Borel resummations* is defined as

$$s_\pm(\phi)(z) = z^{-1} \int_{\mathcal{C}_\pm} \mathrm{d}\zeta e^{-\zeta/z} \hat{\phi}(\zeta). \tag{3.33}$$

We get a complex number with an imaginary piece $\mathcal{O}(\exp(-A/z))$ in this situation.

However, if the lateral Borel resummation of the perturbative series does not give us the correct answer, some extra terms can be added to the series. A simple example is the formal power series in the form

$$\phi_l(z) = z^{b_l} e^{-lA/z} \sum_{n\geq 0} a_{n,l} z^n, \qquad l = 1, 2, ... \tag{3.34}$$

If we do perturbation theory around *non-trivial saddle points of the (Euclidean) path integral*, such as instantons, we get this series. The non-perturbative effects due to l-instantons is encoded in this series. A common example is the double-well potential in QM, which is illustrated as follows.

Example 3.5 The Hamiltonian of the double well potential is

$$H = \frac{p^2}{2} + W(x), \qquad W(q) = \frac{g}{2}\Big(q^2 - \frac{1}{4g}\Big)^2, \qquad g > 0. \tag{3.35}$$

Around the minima, there exists two degenerate ground states in perturbation theory, which are:

$$q_{\pm} = \pm\frac{1}{2\sqrt{g}}. \tag{3.36}$$

Now, stationary perturbation theory gives us the ground state energy as a formal power series as such:

$$\phi_0(g) = \frac{1}{2} - g - \frac{9}{2}g^2 - \frac{89}{2}g^3 - ... \tag{3.37}$$

When we do a path integral around the constant trajectory $q = q_{\pm}$ we get this series. Regardless, if a trajectory going from q_- to q_+ (or vice versa) is given, we can write the saddle-point of the Euclidean path integral as:

$$q_{\pm}^{t_0}(t) = \pm\frac{1}{2\sqrt{g}}\tanh\Big(\frac{t-t_0}{2}\Big). \tag{3.38}$$

From this we get the non-perturbative contribution to the ground state energy, which is

$$\phi_1(g) = -\Big(\frac{2}{g}\Big)^{1/2}\frac{e^{-1/6g}}{\sqrt{2\pi}}(1+\mathcal{O}(g)). \tag{3.39}$$

Now, we write down the *trans series* of the form

$$\Phi(z) = \phi_0(z) + \sum_{l=1}^{\infty} C^l\phi_l(z) \tag{3.40}$$

This series sometimes results in the function that we want, after we do the appropriate (lateral) Borel resummations and choose a value for the constant C in the form:

$$f(z) = s(\Phi)(z) + \sum_{l=1}^{\infty} C^l s(\phi_l)(z). \tag{3.41}$$

However, there are situations when a perturbative series is not Borel summable.

In such scenario, non-perturbative effects are inherently undefined, i.e, they are dependent on how we choose the lateral resummations for the perturbative series. The final answer and the underlying physical quantity has to remain unchanged, so the way the lateral resummation of the perturbative series is chosen accordingly affects the way the lateral resummation of the non-pertubartive effects are selected. [14, 16–20]

Chapter 4

Non-perturbative Extension to Yang-Mills

4.1 Elements of YM Theories

The first thing that is required to construct a Yang-Mills theory is to define a gauge group G, more specifically, a simple Lie group with $\mathbf{g}$ as its Lie algebra. This group will be of dimension $d(G)$. T_a are the generators of the Lie algebra, with $a = 1, ..., d(G)$, which are required to be Hermitian and follows the following commutation relations:

$$[T_a, T_b] = \mathrm{i} f_{abc} T_c, \qquad a, b = 1, ..., d(G), \tag{4.1}$$

here, f_{abc} are the structure constants of the algebra.

Example 3.1 If the gauge group is $SU(2)$, the generators of the Lie algebra are

$$T_a = \frac{1}{2}\sigma_a \qquad a = 1, 2, 3. \tag{4.2}$$

Here, σ_a are the Pauli matrices, and the structure constants are given by

$$f_{abc} = \epsilon_{abc}. \tag{4.3}$$

Moreover, the trace of the product of two generators of the Lie algebra can be found and defined in such a way that the generators are orthogonal to each other as such:

$$\mathrm{Tr}(T_a T_b) = \alpha \delta_{ab}. \tag{4.4}$$

Here, the coefficient α is the normalization constant. Note that , the Cartan inner product is defined as

$$(T_a, T_b) = \delta_{ab} \tag{4.5}$$

The most elementary field in YM theory is the gluon field, also known as the YM connection:

$$A_\mu = A_\mu^a T_a. \tag{4.6}$$

This is a field in the adjoint representation of the Lie algebra, or in other words, a vector field of Lie algebra values. Representations r of the Lie algebra are used to identify the fields in a YM theory. Each r, which have dimension $d(r)$, gives us a matrix representation of the generators $\left(T_a^r\right)^i_j$, where $i, j = 1, ..., d(r)$. Moreover, if r is the fundamental representation, we get back the basis T_a. Also, in this representation, the covariant derivative acting on a field ϕ is defined as

$$D_\mu \phi = \partial_\mu \phi - \mathrm{i} A_\mu^a T_a^r \phi. \tag{4.7}$$

Let us label the adjoint representation as $r = G$, such that the components of the matrices T_a^G are given by

$$\left(T_a^G\right)_{bc} = \mathrm{i} f_{bac}. \tag{4.8}$$

Similar to 4.6, if a field ϕ is in the adjoint representation, it can be written as

$$\phi = \phi_a T_a \tag{4.9}$$

with the covariant derivatives having the components

$$(D_\mu \phi)_a = \partial_\mu \phi_a + f_{abc} A_\mu^b \phi_c. \tag{4.10}$$

Side Note: In one-loop calculations, two important quantities of Lie algebra tend to appear always are the Casimir and the quadratic Casimir operators of the representation r, denoted by $C(r)$ and $C_2(r)$ respectively, and defined as

$$C(r)\delta_{ab} = \mathrm{Tr}(T_a^r T_b^r). \tag{4.11}$$

$$C_2(r)\delta^i_j = (T_a^r)^i_k (T_a^r)^k_j \tag{4.12}$$

We find the following relationship between these operators if we trace them, and hence, they are not independent,

$$C(r)d(G) = C_2(r)d(r), \tag{4.13}$$

Moreover, in case of the adjoint representation, i.e, when $r = G$, the two operators are equal and can be written in terms of the structure constants of the Lie algebra as

$$C_2(G)\delta_{ab} = f_{acd}f_{bcd} \tag{4.14}$$

Also,

$$f_{abc}f_{abc} = d(G)C_2(G). \tag{4.15}$$

For example, if $G = SU(N)$, we get

$$C(\text{fund}) = \frac{1}{2} \tag{4.16}$$

for the fundamental representation $r = \text{fund}$, and for the adjoint representation we get

$$C_2 = (SU(N)) = N. \tag{4.17}$$

Now, the YM field strength is defined as

$$F_{\mu\nu} = \mathrm{i}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu]. \tag{4.18}$$

The values of the curvature of the connection is found in the adjoint representation of the Lie algebra. In terms of the T_a basis its components can be written as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{abc}A^b_\mu A^c_\nu. \tag{4.19}$$

Furthermore, it is often useful to write the components with Lie algebraic values of the gauge connection A_μ and the field strength $F_{\mu\nu}$ collectively into a one-form and a two-form as

$$A = A_\mu \mathrm{d}x^\mu, \qquad F = \frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{4.20}$$

We define the exterior differential, with d as the exterior differential and $\wedge$ being the standard wedge product, as follows: if

$$\psi = \psi_{\mu_1\mu_2\ldots\mu_p}\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_p}. \tag{4.21}$$

is a p-form, then

$$\mathrm{d}\psi = \partial_\mu\psi_{\mu_1\mu_2\ldots\mu_p}\mathrm{d}x^\mu \wedge \mathrm{d}x^{\mu_1} \wedge \ldots \wedge \mathrm{d}x^{\mu_p} \tag{4.22}$$

From this, one gets

$$F = \mathrm{d}A - \mathrm{i}A \wedge A. \tag{4.23}$$

The gauge connection is acted upon by a gauge transformation as

$$A_\mu(x) \to A^U_\mu(x) = U(x)A_\mu(x)U^\dagger(x) + \mathrm{i}U(x)\partial_\mu U^\dagger(x). \tag{4.24}$$

Here, $U(x)$ has values in the Lie group G and is a function of spacetime. Now, let us suppose

$$U = e^{\mathrm{i}\phi}, \qquad \phi = \phi^a T_a. \tag{4.25}$$

infinitesimally, we get

$$\partial A_\mu = D_\mu \phi. \tag{4.26}$$

Now, the transformation of YM field is as follows

$$F_{\mu\nu}(x) \to F^U_{\mu\nu}(x) = U(x) F_{\mu\nu} U^\dagger(x), \tag{4.27}$$

and infinitesimally,

$$\delta F_{\mu\nu} = \mathrm{i}[\phi, F_{\mu\nu}]. \tag{4.28}$$

The Lagrangian of pure YM theory is

$$\mathcal{L}_{YM} = -\frac{1}{2g_0^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4g_0^2}F^a_{\mu\nu}F^{\mu\nu a}. \tag{4.29}$$

Here, the bare coupling constant is represented by g_0. Due to the cyclic property of the trace and the property 4.24 gauge connection, this Lagrangian is invariant under gauge transformation. The equation of motion that this Lagrangian gives is

$$D^\mu F_{\mu\nu} = 0. \tag{4.30}$$

Often it is suitable to rescale the fields such that the coupling constant is absorbed in the connection term and it is only seen in the interaction vertices, as

$$\hat{A}_\mu = \frac{1}{g_0} A_\mu \tag{4.31}$$

and the field strength is written as

$$\hat{F}^a_{\mu\nu} = \partial_\mu \hat{A}^a_\nu - \partial_\nu \hat{A}^a_\mu + g_0 f^{abc} \hat{A}^b_\mu \hat{A}^c_\nu \tag{4.32}$$

which gives the Lagrangian as

$$\mathcal{L}_{YM} = -\frac{1}{4}\hat{F}^a_{\mu\nu}\hat{F}^{\mu\nu a}. \tag{4.33}$$

Until now, we have shown all our calculations in Minkowski space. However, it is vital to work these out in the Euclidean space to study instantons. Therefore, we do the Wick rotation be redefining the time coordinate as follows

$$x_0 = -ix_4. \tag{4.34}$$

In our signature, the vector field in the Euclidean space will be written as

$$A_i^E = -A^i, \quad i = 1, 2, 3, \quad A_4^E = -iA_0. \tag{4.35}$$

Accordingly, the field strength becomes

$$F_{ij}^{aE} = F_{ij}^a, \qquad F_{0j}^{aE} = -\mathrm{i}F_{4j}^a. \tag{4.36}$$

Finally, the Euclidean Lagrangian turns out to be

$$\mathcal{L}_{YM}^E = \frac{1}{4g_0^2}(F_{\mu\nu}^{aE})^2 \tag{4.37}$$

and in the Euclidean path integral $e^{-S_{YM}^E}$ is the field configuration's weight, where the Euclidean action is given by

$$S_{YM}^E = \int \mathrm{d}^4x \mathcal{L}_{YM}^E. \tag{4.38}$$

We can renormalize YM theories at the quantum level, and it shows a *running coupling constant* and *asymptotic freedom*. In the $\overline{MS}$ scheme, after dimensional regularization, the connection between the regular coupling constant g_0^2 and the renormalized constant g^2 is as follows:

$$g_0^2 = \mu^{2\epsilon}\left\{g^2 + \sum_{k=1}^{\infty} a_k(g^2)\epsilon^{-k}\right\}, \tag{4.39}$$

in the dimensions $d = 4-2\epsilon$ and μ being the renormalization mass. Upto the quartic order g^4, one gets

$$a_1(g^2) = -\frac{g^4}{(4\pi)^2}\frac{11C_2(G)}{3}, \quad a_k = 0, \quad k \geq 2. \tag{4.40}$$

A one-loop calculation also yield this.

Now, we define the beta function as follows:

$$\beta(g) = \mu\frac{\partial g}{\partial \mu} = -\sum_{n=0}^{\infty} \beta_n g^{2n+3} = -\beta_0 g^3 - \beta_1 g^5 + ... \tag{4.41}$$

The regularization does not affect the coefficients β_0 and β_1. This beta function controls the conduct of the coupling constant as the renormalization scale μ is varied. From 4.39 and 4.40, the one-loop coefficient β_0 as $\epsilon \to 0$ can be found, which is

$$\beta_0 = \frac{1}{(4\pi)^2}\frac{11C_2(G)}{3}. \tag{4.42}$$

and

$$\beta_1 = \frac{34}{3} C_2^2(G). \tag{4.43}$$

Also, this is an asymptotically free theory because the beta function gives a negative coefficient in the very first term. When the coupling constant is run, it is found that the quantity

$$\Lambda = \mu(\beta_0 g^2)^{-\beta_1/(2\beta_0^2)} e^{-1/(2\beta_0 g^2)} \exp\left(- \int_0^g \left\{ \frac{1}{\beta(x)} + \frac{1}{\beta_0 x^3} - \frac{\beta_1}{\beta_0^2 x} \right\} \mathrm{d}x \right) \tag{4.44}$$

actually depends on μ and hence, determines a renormalization group (RG) invariant scale. The one-loop approximation of this quantity is

$$\Lambda \approx \mu \mathrm{e}^{-1/(2\beta_0 g^2)}. \tag{4.45}$$

The quantity λ is known as the *dynamically generated scale* of YM theory. This is dependent upon the regularization scheme that the beta function is computed with. Since, originally this theory had this coupling constant g which do not have any dimension, and from this a dimensional scale Λ is generated, this process is called *dimensional transmutation.* [1, 19]

4.2 Topological charge and θ vacua

In YM theory, another term called the *topological charge* can be added to the YM action apart from the standard action. This is given by

$$Q = \int q(x) \mathrm{d}^4 x. \tag{4.46}$$

Here,

$$q(x) = \frac{1}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{\mu\nu a} = \frac{1}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu a} F^{\rho\sigma a}. \tag{4.47}$$

This term is renormalizable and gauge invariant, therefore it can be essentially added to the action. This yields the Euclidean YM Lagrangian as follows:

$$\mathcal{L}^E_{YM,\theta} = \mathcal{L}^E_{YM} - \mathrm{i}\theta q(x)_E \tag{4.48}$$

where, a new parameter θ arises in the QCD Lagrangian. Moreover, the topological charge is quantized for any classical, continuous field configuration having a *finite* action.

Now, the observables of QCD has to be susceptible to the parameter θ. An example of such a quantity is the vacuum energy density $E(\theta)$, which is calculated in the large, finite spacetime volume V as

$$E(\theta) = -\lim_{V\to\infty}\frac{1}{V}\log Z(\theta), \tag{4.49}$$

where $Z(\theta)$ is the partition function in the presence of θ, which is in turn defined as

$$Z(\theta) = \int[\mathcal{D}A]\mathrm{e}^{-\int \mathrm{d}^4x\mathcal{L}^E_{YM,\theta}} \tag{4.50}$$

There are two important properties of this energy density. Firstly, the path integral that we calculate after the inclusion of $\mathrm{e}^{\mathrm{i}\theta Q}$ at $\theta \neq 0$, should be smaller than that of without this term, i.e, at $\theta = 0$. This is because, at $\theta \neq 0$ an oscillating function is calculated, which leads to a smaller value of $Z(\theta)$. Therefore,

$$E(0) \leq E(\theta), \qquad \theta \neq 0, \tag{4.51}$$

and the vacuum energy takes an absolute minimum value at $\theta = 0$. The second property is that Q is quantized in finite action smooth field configurations. Therefore, $E(\theta)$ should be harmonic, with period 2π:

$$E(\theta + 2\pi) = E(\theta). \tag{4.52}$$

When the function $E(\theta)$ is expanded around $\theta = 0$ we get the following power series

$$E(\theta) - E(0) = \frac{1}{2}\chi_t\theta^2 s(\theta) \tag{4.53}$$

where

$$s(\theta) = 1 + \sum_{n=1}^{\infty} b_{2n}\theta^{2n}. \tag{4.54}$$

The quantity χ_t^V is called *topological susceptibility*. It is an important quantity that gives us the leading dependence on of $E(\theta)$ on the angle θ around $\theta = 0$. It is defined as

$$\chi_t^V = \Big(\frac{\mathrm{d}^2E(\theta)}{\mathrm{d}\theta^2}\Big)_{\theta=0} = \frac{\langle Q^2\rangle}{V} = \int \langle q_{\mathrm{E}}(x)q_{\mathrm{E}}(0)\rangle \; \mathrm{d}^4(x). \tag{4.55}$$

Here, the last equation comes from

$$\langle Q^2\rangle = \int_V \mathrm{d}^4x \int_V \mathrm{d}^4y \, \langle 0|\, q_{\mathrm{E}}(x-y)q_{\mathrm{E}}(0)\,|0\rangle = V\chi_t^V \tag{4.56}$$

after incorporating the translation invariance of the vacuum. As $E(\theta)$ has a minimum at $\theta = 0$, $\chi_t^V \geq 0$. This quantity has an infinite volume limit, which is given by

$$\chi_t = \lim_{V\to\infty}\chi_t^V \tag{4.57}$$

The dependence of the YM observables on θ is very slight since the quantity $q(x)$ is fully divergent

$$q(x) = \partial_\mu K^\mu. \tag{4.58}$$

Here,

$$K_\mu = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}A^a_\nu(\partial_\rho A^a_\sigma + \frac{1}{3}f_{abc}A^a_\nu A^b_\rho A^c_\sigma). \tag{4.59}$$

The tensor above means that we use the three form, known as *Chern-Simons form*:

$$\omega_{CS}(A) = \frac{1}{16\pi^2}A^a_\nu\Big(\partial_\rho A^a_\sigma\frac{1}{3}f_{abc}A^a_\nu A^b_\rho A^c_\sigma\Big)\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma \tag{4.60}$$

$$= \frac{1}{8\pi^2}\mathrm{Tr}\Big(A\wedge\mathrm{d}A - \frac{2\mathrm{i}}{3}A\wedge A\wedge A\Big) \tag{4.61}$$

This results in the Fourier transform

$$\tilde{q}(p) = \int \mathrm{e}^{\mathrm{i}px}q(x)\mathrm{d}^4x \tag{4.62}$$

vanishing at zero momentum as it is of the form $p^\mu\tilde{K}_\mu(p)$. However, the topological susceptibily is written as

$$\chi_t = \lim_{k\to 0}U(k), \tag{4.63}$$

and

$$U(k) = \int \mathrm{d}^4x\mathrm{e}^{\mathrm{i}kx}\big\langle q(x)q(0)\big\rangle = \int\frac{\mathrm{d}^4p'}{(2\pi)^4}\big\langle\tilde{q}(k)\tilde{q}(p)\big\rangle. \tag{4.64}$$

This quantity vanishes order by order in perturbation theory since $\tilde{q}(0) = 0$. However, Witten noticed that this does not vanish in the whole theory. We may obtain a nonzero solution after adding infinitely many diagrams and taking the limit $k\to 0$. Infact, in $1/N$ expansion, we find nonzero value of topological susceptibility if we add an infinite number of planar diagrams.

The topological charge can be written using Stokes' theorem as

$$Q = \int \mathrm{d}\Sigma_\mu K^\mu \tag{4.65}$$

Let the surface of integration be two spatial planes at $t = \pm\infty$, such that

$$Q = \int \mathrm{d}^3\vec{x}K^0(t\to\infty,\vec{x}) - \int \mathrm{d}^3\vec{x}K^0(t\to-\infty,\vec{x}) \equiv K_+ - K_-. \tag{4.66}$$

These are Hermitian operators and are connected by time reversal, so their spectra are identified with each other. Let, their eigenstates are denoted by

$$|\nu_\pm\rangle$$

such that

$$K_{\pm} |\nu_{\pm}\rangle = \nu |\nu_{\pm}\rangle \tag{4.67}$$

The physical vacuum can now be expanded as

$$|\theta\rangle = \sum_{\nu} c_{\nu}(\theta) |\nu_{+}\rangle = \sum_{\nu} c_{\nu}(\theta) |\nu_{-}\rangle . \tag{4.68}$$

This is true because the vacuum is invariant if we apply time reversal operator and the first sum and the second one is interchanged. Moreover, we get the following identity:

$$\mathrm{i}\frac{\partial}{\partial\theta} \langle\theta| \mathcal{O} |\theta\rangle = \mathrm{i}\frac{\partial}{\partial\theta} \langle 0| \mathcal{O}\mathrm{e}^{-\int \mathrm{d}^x \mathcal{L}^E_{YM}|0\rangle} \tag{4.69}$$

$$= \int \mathrm{d}^4 x \langle 0| q(x)\mathcal{O}\mathrm{e}^{-\int \mathrm{d}^4 x \mathcal{L}^E_{YM}|0\rangle} \tag{4.70}$$

$$= \int \mathrm{d}^4 x \langle\theta| q(x)\mathcal{O} |\theta\rangle , \tag{4.71}$$

so the operator $\mathrm{i}\partial_\theta$ is equivalent to the insertion of Q. However, we get

$$\mathrm{i}\frac{\partial}{\partial\theta} \langle\theta| \mathcal{O} |\theta\rangle = \langle\theta| K_{+}\mathcal{O} |\theta\rangle - \langle\theta| \mathcal{O}K_{-} |\theta\rangle \tag{4.72}$$

In this case, the time ordering prescription is used, which dictates that K_+ and K_- should be inserted to the left and to the right respectively. Now, if we substitute the expansion of the physical vacuum in to this, we get

$$\mathrm{i}\frac{\partial}{\partial\theta} \sum_{\nu,k} c^*_{\nu}(\theta)c_k(\theta) = \sum_{\nu,k} (\nu - k)c^*_{\nu}(\theta)c_k(\theta), \tag{4.73}$$

which gives us

$$c_{\nu} = C\mathrm{e}^{\mathrm{i}\nu\theta} \tag{4.74}$$

Now, for simplicity we set the overall constant C to 1, and in terms of the eigenstates of $K_{\pm}$ we get, by forming a superposition:

$$|\theta\rangle = \sum_{\nu=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\nu\theta} |\nu\rangle . \tag{4.75}$$

This is the true vacuum of the theory, called the *theta vacuum* and ν is the winding number that describes the tunneling between the two vacua $|\nu_1\rangle$ and $|\nu_2\rangle$ where $\nu = \nu_2 - \nu_1$. [21–23]

4.3 YM theory instantons

The instantons in Yang Mills theory are defined as the field configurations which solve the equations of motion and have *finite action*. These configurations can begin some perturbative expansions, so these are crucial for semi-classical analysis.

The behavior of the fields at large distances is constrained by the condition of finite action. Their behavior at $r \to \infty$ is seen by writing the Euclidean action as

$$S_E \sim \int \mathrm{d}r \; r^3 F^2 \tag{4.76}$$

The integrand has to behave like $1/r^2$ at the least if we want a finite outcome. Let us say, if we have

$$F \sim \frac{1}{r^3} \tag{4.77}$$

as $r \to \infty$, and so $A(r)$ would behave as such:

$$A(r) \sim \frac{1}{r^2} \qquad r \to \infty \tag{4.78}$$

But, A is only defined upto a gauge transformation, which gives us the general behavior

$$A_\mu \to g\partial_\mu g^{-1} + \mathcal{O}(r^{-2}), \qquad r \to \infty. \tag{4.79}$$

This means that at infinity this gauge potential is pure gauge. As $r \to \infty$ the limiting behavior has to be well defined, and so the function g on the boundary at infinity is defined to be $S^3 \subset R^4$. If g only depends on the angular variables of R^4, then we get this. Thus, a map from S^3 to the gauge group is defined from the solution shown above, i.e.

$$g : S^3 \to G. \tag{4.80}$$

g will change under a gauge transformation. So, the *homotopy* type of mappings from S^3 to G are gauge invariant. For example, in the soliton theory, these homotopy types are labelled by

$$\pi_3(G). \tag{4.81}$$

Example 4.1 Instantons in Euclidean two dimensional space with $U(1)$ gauge group is a toy example, which has the homotopy group $\pi_1)(S^1) = Z$. The integer ν classifies the homotopy classes. If ν labels a map in the class, then this map is the covering

$$g^{(\nu)}(\theta) = \mathrm{e}^{\mathrm{i}\nu\theta} \tag{4.82}$$

Example 4.2 In the case of $G = SU(2)$, any element of $SU(2)$ can be written as

$$g = a + \mathrm{i}\mathbf{b}.\sigma, \quad a^2 + \mathbf{b}^2 = 1, \tag{4.83}$$

thus $SU(2)$ is homeomorphic to S^3. Now, let us think about the map

$$g : S^3 \to S^3 \tag{4.84}$$

The homotopy group associated to it is

$$\pi_3(S^3) = Z \tag{4.85}$$

which is true because

$$\pi_1(S^3) = 0. \tag{4.86}$$

Hence this theorem connects the homotopy groups to the homology groups. For this example,

$$\pi_2(S^3) = H_2(S^3), \qquad \pi_3(S^3) = Z. \tag{4.87}$$

Now, 4.85 suggests that a *winding number*, ν labels the homotopy classes that are related to the gauge group $SU(2)$. Explicitly, the map

$$g : S^3 \to SU(2) \tag{4.88}$$

with winding number ν is expressed as

$$g^{(\nu)}(x) = \Big(\frac{x_4 + \mathrm{i}\mathbf{x}.\sigma}{r}\Big)^{\nu} \tag{4.89}$$

Here, if $\nu = 0$, then this is the trivial map, and for $\nu = 1$ this gives us the identity. Furthermore, only angular variables can be used to express this.

To sum up, taking $SU(2)$ as our gauge group, we can characterize every field configuration of finite action by its winding number ν. Moreover, for a gauge field, its winding number is the topological charge itself. In order to elucidate this fact, we write

$$Q = \int \mathrm{d}\Sigma_\mu K^\mu \tag{4.90}$$

Now we integrate it over the boundary at infinity, namely the three sphere S^3 which gives us

$$Q = -\frac{1}{48\pi^2}\int \mathrm{d}\Sigma^\mu \epsilon_{\mu\nu\alpha\beta}(A^\nu, A^\alpha A^\beta) \tag{4.91}$$

where

$$\epsilon_{\mu\nu\alpha\beta}\partial^\alpha A^\beta = -\epsilon_{\mu\nu\alpha\beta}A^\alpha A^\beta \tag{4.92}$$

By taking note of how the gauge potentials behave at the boundary, we can express this quantity as

$$Q = \frac{1}{48\pi^2}\int \mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\theta_3\epsilon^{ijk}(g_{-1}\partial_i g, g^{-1}\partial_j g g^{-1}\partial_k g). \tag{4.93}$$

This is homotopy invariant and provides us with the winding number related to the homotopy class of g.

Example 4.3 We can use the integrals above to show that $g^{(1)}$ actually has $n = 1$. The inverse of the mapping is

$$g^{-1} = \frac{x_4 - \mathrm{i}\vec{x}.\vec{\sigma}}{r}. \tag{4.94}$$

We get

$$Q = -\frac{1}{24\pi^2}\int \mathrm{d}\Sigma_\mu\Big(-\frac{12x^\mu}{|x|^4}\Big). \tag{4.95}$$

By using the expression

$$\mathrm{d}\Sigma^\mu = x^\mu|x|^2\mathrm{d}\Omega_3, \tag{4.96}$$

we get

$$Q = \frac{1}{2\pi^2}\int \mathrm{d}\Omega_3 = 1. \tag{4.97}$$

We can tell from the above analysis that an integer winding number classifies the field configurations of finite action. Now, we will construct field configurations of the finite action that gives solutions to the equations of motion, and hence generate the various vacua of the YM theory. There is a configuration that minimizes the action in each topological sector, thereby solving the equations of motion, i.e, there exists an infinite set of classical vacua labelled by the integer n. In this way, we will find instantons as solutions to the first order differential equations. We will elucidate this below.

Firstly, there is a gauge configuration with a fixed topological charge $Q = \nu$. Now, we write the following identity

$$\int \mathrm{d}^4x\mathrm{Tr}\big\{(F \pm \tilde{F})_{\alpha\beta}(F \pm \tilde{F}^{\alpha\beta}\big\} \geq 0. \tag{4.98}$$

Since $\tilde{F}_{\alpha\beta}\tilde{F}^{\alpha\beta} = F_{\alpha\beta}F^{\alpha\beta}$, it gives us

$$\frac{1}{2g_0^2}\int \mathrm{d}^4x\mathrm{Tr}\big(F_{\alpha\beta}F^{\alpha\beta}\big) \pm \frac{1}{2g_0^2}\int \mathrm{d}^4\mathrm{Tr}\big(F_{\alpha\beta}\tilde{F}^{\alpha\beta}\big) \geq 0, \tag{4.99}$$

which is

$$S^E_{YM} \pm \frac{8\pi^2\nu}{g_0^2} \geq 0. \tag{4.100}$$

Finally,

$$S^E_{YM} \geq \frac{8\pi^2|\nu|}{g_0^2} \tag{4.101}$$

It can be noted that S^E_{YM} needs to be positive all the time in order for the inequality to be saturated. Thus, when $\nu > 0$ we get

$$F_{\alpha\beta} = \tilde{F}_{\alpha\beta}, \qquad S^E_{YM} = \frac{8\pi^2\nu}{g_0^2} \tag{4.102}$$

This means that we have a *self-dual* SD gauge field and a gauge theory instanton. Similarly, negative ν gives us

$$F_{\alpha\beta} = -\tilde{F}_{\alpha\beta}, \qquad S^E_{YM} = -\frac{8\pi^2\nu}{g_0^2}. \tag{4.103}$$

which represents an *anti-self-dual*(ASD) gauge field and we get a gauge theory anti-instanton. When any of these conditions are true, the action for a fixed topological class ν is minimized by the corresponding gauge field, and also solves the equations of motion. However, these are *first order equations*, unlike the standard YM theory EOMs.

Now, let us consider for example the gauge group $SU(2)$ and set $n = 1$ for the one-instanton solution. We will write the aymptotic expression fot the gauge field. Firstly, we set

$$A_\mu = \mathrm{i}U\partial_\mu U^\dagger, \tag{4.104}$$

where

$$U = \frac{x_4 + \mathrm{i}\mathbf{x}.\sigma}{r} \tag{4.105}$$

and $\mathbf{x}= (x_1, x_2, x_3)$, $\quad \sigma = (\sigma_1, \sigma_2, \sigma_3)$ with the σ_i being the Pauli matrices. Since

$$\partial_4 U = -\frac{x_4}{r^2}U + \frac{1}{r}, \qquad \partial_k U = -\frac{x_4}{r^2}U + \frac{\mathrm{i}\sigma_k}{r}, \quad k = 1, 2, 3, \tag{4.106}$$

From this we get

$$A_4 = -\frac{\mathbf{x}\cdot\sigma}{r^2} \tag{4.107}$$

and that

$$A_k = \frac{1}{r^2}(x_4\sigma_k + \epsilon_{klm}x_i\sigma_m). \qquad k = 1, 2, 3, \tag{4.108}$$

where the following identity is used

$$\sigma_j\mathbf{x}\cdot\sigma = x_k + \mathrm{i}\epsilon_{klm}x_l\sigma_m \tag{4.109}$$

Now, if we write the gauge connection in component form as

$$A_\mu = \frac{1}{2}\sigma_a A_\mu^a \tag{4.110}$$

and define the 't Hooft matrices $\eta_{\mu\nu}^a, \quad a = 1, 2, 3$ by

$$\eta_{ij}^a = \epsilon_{aij}, \eta_{i4}^a = \partial_{ai}, \quad i, j = 1, 2, 3, \tag{4.111}$$

we get

$$A_\mu^a = 2\eta_{\mu\nu}^a \frac{x^\nu}{r^2}. \tag{4.112}$$

From this asymptotic form, we have the following ansatz for the connection

$$A_\mu^a = 2\eta_{\mu\nu}^a \frac{x^\nu}{r^2} f(r^2). \tag{4.113}$$

when

$$f(r^2) \to 1, \qquad r \to \infty \tag{4.114}$$

Moreover, at the origin, regularity requires that

$$f(r^2) \sim r^2, \qquad r \to 0. \tag{4.115}$$

Substituting this ansatz in the gauge theory action gives us

$$S \propto \int_0^\infty \mathrm{d}r \left[\frac{r}{2}(f')^2 + \frac{2}{r} f^2 (1 - 2f)^2\right] \tag{4.116}$$

From the second order EOM of f we get

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right) + \frac{4}{r} f(1 - f)(1 - 2f) = 0 \tag{4.117}$$

The three constant solutions are: the trivial gauge connection for $f = 0$, a pure gauge transformation having a winding number 1 for $f = 1$ and finally, for $f = 1/2$ we get a half gauge transformation called a *meron.* Furthermore, this yields a space-dependent solution

$$f(r) = \frac{r^2}{r^2 + \rho^2} \tag{4.118}$$

Thus, we have the one-instanton solution of $SU(2)$ Yang-Mills theory which is expressed as

$$A_\mu = \frac{r^2}{r^2 + \rho^2} \mathrm{i} U \partial_\mu U^\dagger \tag{4.119}$$

It can be noted that this configuration is an interpolation between the trivial vacuum $f = 0$ at the origin and the homotopically non-trivial gauge transformation with

$n = 1$ as $r \to \infty$. Also, when r is large, it takes the form of 4.78.

Here, ρ is an integration constant that represents the *size* if the instanton. The interesting characteristic of the instanton in the YM theory is that it is a free parameter, unlike that of the instantons in the scalar field theory where the instantons were determined by the parameters of the potential. This is an example of a collective coordinate and it exists because of a symmetry of the theory, which is the *scale invariance* of the classical Yang-Mills action in this case. Moreover, the above solution is centered at the origin, but a more general solution can be written as

$$A^a_\mu = 2\eta^a_{\mu\nu}\frac{(x-x_0)^\nu}{((x-x_0)^2+\rho^2)^2} \tag{4.120}$$

Here, the centre of the instanton is at x_0. Due to translation invariance, we get four extra collective coordinates.

So far, we solved the second order YM EOM to find the space-dependednt action. similarly, we can solve the first order equation and substitute the ansatz and get

$$f(1-f) - r^2\frac{\mathrm{d}f}{\mathrm{d}r^2} = 0 \tag{4.121}$$

which also gives us the solutions $f = 0, 1$ and the one-instanton solution 4.118. However, the first order equation is not satisfied by the meron solution $f = 1/2$ and we get an infinite action. [19, 24, 25]

4.4 Instantons and theta vacua

As in Quantum Mechanics, the Yang-Mills instantons can be also thought of as the tunneling configurations between different vacua. For example, we can think of an instanton field in a sector with winding number n to be a field that goes from a vacuum in the infinite past $\tau = -\infty$ to a vacuum in the infinite future $\tau = +\infty$ in the Euclidean theory. We show this as follows.

Let there be a gauge field where $A_0 = 0$. An integral over an S^3 at infinity represents the winding number. Now, suppose that we get a cylinder by deforming this boundary parallel to the $x^0 = \tau$ axis. The curved surface of the cylinder does not contribute in the axial gauge $A_0 = 0$, so we get

$$n = n_+ - n_- \tag{4.122}$$

where

$$n_\pm = -\frac{1}{48\pi^2}\int \mathrm{d}^3x\epsilon_{ijk}(A_i, A_jA_k)\Big|_{\tau=\pm\infty} \tag{4.123}$$

At $\tau \to \pm\infty$ the field configurations match the with the various vacua whose homotopy numbers vary by the integer n, the instanton charge. We can choose the gauge such that $n_- = 0$. Thus, the semiclassical realization of all the vacua $|n_\pm\rangle$, labelled as integers, can be found. In fact, the amplitude of transition between two vacua is given by

$$\langle n|\, \mathrm{e}^{-HT}\, |m\rangle = \int \mathcal{D}A_{n-m} \exp\big[- \int \mathrm{d}^4x \mathcal{L}(A)\big] \tag{4.124}$$

where, the fact that the integration is done over all gauge fields, with the winding number $n - m$ being constant, is expressed by $\mathcal{D}A_{n-m}$. Here, we consider the limit $T \to \infty$. We write the sector's partition function having ν winding number as

$$Z_\nu = \int \mathcal{D}A_\nu \exp\big[- \int \mathrm{d}^4x \mathcal{L}(A)\big]. \tag{4.125}$$

From this, we get

$$\langle \theta'|\, \mathrm{e}^{-HT}\, |\theta\rangle = \sum_{n,m} \mathrm{e}^{\mathrm{i}n\theta - \mathrm{i}m\theta'} Z_{n-m} = \sum_{n,\nu} \mathrm{e}^{\mathrm{i}m(\theta-\theta')+\mathrm{i}\nu\theta} Z_\nu = \delta(\theta - \theta') \sum_\nu \mathrm{e}^{\mathrm{i}\nu\theta} Z_\nu \tag{4.126}$$

where the change of variables from $\nu = m = n$ is done in the second line. From this, we get the theta dependent partition function

$$Z(\theta) = \int [\mathcal{D}A] \mathrm{e}^{-\int \mathrm{d}^4x \mathcal{L}^E_{YM,\theta}}. \tag{4.127}$$

Here, the Lagrangian with a θ term has been introduced. The integration is done over all possible gauge fields, i.e, those belong to all possible homotopy classes. Now, we can write

$$\langle \theta'|\, \mathrm{e}^{-HT}\, |\theta\rangle = \delta(\theta - \theta') Z(\theta). \tag{4.128}$$

This shows that when we quantize the YM Lagrangian 4.48 we get the theta vacua.

In terms of this partition function, we get

$$VE(\theta) = -\log\Big\{ \sum_\nu \mathrm{e}^{\mathrm{i}\nu\theta} \int \mathcal{D}A_\nu \exp\big[- \int \mathrm{d}^4x \mathcal{L}(A)\big]\Big\} = -\log Z(\theta) \tag{4.129}$$

and also

$$\chi_t^V = \frac{1}{V} \sum_\nu \nu^2 P_\nu, \qquad P_\nu = \frac{Z_\nu}{Z(0)}. \tag{4.130}$$

It can be noted that the probability of finding a gauge field with charge k is denoted by P_ν. In these sums, we get the leading contributions from one instanton and one anti-instanton, since both have

$$S_c = \frac{8\pi^2}{g^2} \tag{4.131}$$

but opposite $\nu = \pm 1$. Thus we have

$$VE(\theta) = -\log Z_0 - \log\{1 + \mathrm{e}^{\mathrm{i}\theta - \frac{8\pi^2}{g^2}} K_1 + \mathrm{e}^{-\mathrm{i}\theta - \frac{8\pi^2}{g^2}} K_{-1} + ...\} \tag{4.132}$$

where we know that $K_{\pm 1} = KV$ at leading order in g, by the one-loop fluctuation around the instanton/anti-instanton solutions. The volume V is factored out, which we find when we integrate over the zero mode x_0 due to translation invariance. It is also noted that around the instanton and the anti-instanton, the one-loop fluctuations are equal. Now, we get

$$E(\theta) - E(0) =\approx 2(1 - \cos\theta) K \mathrm{e}^{-}\frac{8\pi^2}{g^2} \tag{4.133}$$

From this approximation, the topological susceptibility is given by

$$\chi_t \sim K \mathrm{e}^{-\frac{8\pi^2}{g^2}}. \tag{4.134}$$

When we calculate the path integral as a sum over instantons, we get the topological susceptibility as fully non-perturbative and we do not see this in perturbation theory.

Now, being careful about the collective coordinates, we calculate K. For instantons, there are eight of them in total, of which four of them represent the position of the instanton. Integrating over them yields the total volume space-time V which has been factored out above. One other collective coordiante is the instanton size ρ. The last three parameters are given by guage rotations. From these eight parameters, we get the factor

$$S_C^4 = (\frac{8\pi^2}{g^2})^4. \tag{4.135}$$

We have a constant factor when we integrate over gauge transformations. Therefore, the integral over ρ has to be of the form

$$\int_0^\infty \frac{\mathrm{d}\rho}{\rho^5} f(\rho\mu) \tag{4.136}$$

by dimensional analysis. The energy density is to be found out, hence this integral has dimensions of l^{-4} since ρ has length units. In order to renormalize a quantum gauge theory, we need μ, which $f(\rho\mu$ is a function of. The final solution must have renormalization group invariant quantities, and from this the form of f can be fixed. Therefore, we can say that a running coupling constant $g^2(\mu)$ must be used in the above computation as

$$\mathrm{e}^{-\frac{8\pi^2}{g^2(\mu)}} = \mathrm{e}^{-\frac{2\pi}{\alpha_s(\mu)}}. \tag{4.137}$$

Combining this with

$$\mu^{(-4\pi\beta_0)} \tag{4.138}$$

an RG-invariant integrand is produced, and the form of $f(\rho\mu)$ at leading order is determined to be

$$f(\rho\mu) = (\rho\mu)^{-4\pi\beta_0}. \tag{4.139}$$

Now,

$$\mathrm{e}^{-\frac{2\pi}{\alpha_s(\mu)}}\mu^{-4\pi\beta_0} = \mathrm{e}^{-\frac{2\pi}{\alpha_s(1/\rho)}}\rho^{4\pi\beta_0} = \Lambda^{-4\pi\beta_0} \tag{4.140}$$

due to invariance of RG. The integral becomes

$$\mathrm{e}^{-\frac{2\pi}{\alpha_s(\mu)}}\int_0^\infty \frac{\mathrm{d}\rho\rho^5}{\rho^5}(\rho\mu)^{-4\pi\beta_0} = \int_0^\infty \frac{\mathrm{d}\rho}{\rho^5}\mathrm{e}^{-\frac{2\pi}{\alpha_s 1/\rho}}. \tag{4.141}$$

This is the RG invariant way of writing the integral over instanton sizes. When ρ is smallm, the asymptotic freedom and the one-loop beta function can be used to write the integral for pure Yang-Mills theory as

$$\int_0^\infty \frac{\mathrm{d}\rho}{\rho^5}(\rho\Lambda)^{11N_c/3}. \tag{4.142}$$

In the UV $\rho \to 0$ region this integral is convergent but in the IR $\rho \to infty$ this is divergent, for all $N_c \geq 2$. For the large sized instantons, this is known as the famous IR embarrassment in instanton calculus.

In fact, in case of $\rho \to \infty$ the integral above does no tgive the right answer since we can not do reliable instanton computations. This is when the instanton size increases andthe running coupling constant $\alpha_s(1/p)$ enters the strong coupling region. To get rid of the problems of strong coupling we can have an IR cutoff in the instanton size. This is the only way to perform instanton calculus in gauge theory. Suppose we do the instanton calculation in a finite volume V space-time, such as a four sphere S^4. Here, the instanton size ρ cuts off naturally because it can not be greater than the characteristic scale of spacetime $V^1/4$. Now we can compute P_1 by instanton calculus. We know the natural scale in the problem is V, so from dimensional transmutation we get

$$P_1 \sim \exp\Big\{-\frac{8\pi^2}{\alpha_s(\Lambda V^{1/4})}\Big\} \sim (V\Lambda^4)^{\frac{11N_c}{12}}. \tag{4.143}$$

Natural cutoff can be obtained by one more way, that is if we have a Higgs-like field with a large VEV which sets the scale (for example in supersymmetric gauge theories), or by looking at the theory in finite temperature, in both cases we can perform instanton calculus successfully. In any other cases, instanton computations in QCD is ambiguous. In fact the dependence on θ is unaccepted in lattice calculations even though it appears to be an universal property of instanton calculations to the topological susceptibility. [20, 25, 26]

4.5 Renormalons

The large order behaviour in perturbation theory is dominated by instantons. The perturbation theory behaves in a way dictated by the number of diagrams growing factorially. Moreover, there is one more factor that affects the large order behavior of perturbation theory in *renormalizable* QFT, which is *renormalon* divergences. There is also a factorial growth due to this, which is not because of the rapid growth of the number of diagrams, but bue to the momentum integrals in the special cases of Feynman diagrams. As the loop order increases, these increase factorially.

Depending upon operator product expansion, analysis of diagrams or other indirect arguments, we can claim that renormalons exist. Now, we will show an hint of the existence of non-perturbative effect which is not of the instanton type on the basis of RG equations.

Suppose we have an RG invariant quantity in YM theory. In perturbation theory, we can calculate its asymptotic expansion around $g = 0$, and also take account of the non-perturbative effects, if any. Thus we write the general expression as

$$\phi(g) = \phi_{\mathrm{p}}(g) + \phi_{np}(g), \tag{4.144}$$

where

$$\phi_{\mathrm{p}}(g) = \sum_{n=0}^{\infty} a_n g^{2(n+1)} \tag{4.145}$$

is the term that comes from perturbation theory and $\phi_t extnp(g)$ is that of non-perturbative correction. The energy scale Q^2 and the normalization scale μ will also affect this quantity. Now. because $\phi(g)$ and $\phi_{\mathrm{p}}(g)$ are both RG invariant individually, we can say tha $\phi_{np}(g)$ should be too, and so the following equation must be satisfied

$$\Big(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma(g)\Big)\phi_{np}(g) = 0. \tag{4.146}$$

In this case, this quantity has an anomalous dimesion given by

$$\gamma(g) = \gamma_1 g^2 + ... \tag{4.147}$$

Due to this equation and supposing that $\phi_{np}(g)$ depends on a small coupling constant, we get

$$\phi_{np}(g) = C\Big(\frac{\mu^2}{Q^2}\Big)^d/2g^\delta \exp\Big(-\frac{d}{2\beta_0 g^2}\Big)(1+\mathcal{O}(g)). \tag{4.148}$$

Here d is the dimension of the ovservable. In fact, we cam check that the RG equation above is satisfied by this functional form because

$$\mu\frac{\partial}{\partial\mu}\phi_{np}(g) = d\phi_{np}(g), \tag{4.149}$$

and that

$$\beta(g)\frac{\partial}{\partial g}\phi_{np}(g) = -(\beta_0 g^3 + \beta_1 g^5 + ...)\Big(\frac{d}{\beta_0 g^3} + \frac{2\delta}{g} + ...\Big) \tag{4.150}$$

$$= -\Big\{d + \Big(2\delta\beta_0 + \frac{d\beta_1}{\beta_0}\Big)g^2 + ...\Big\}\phi_{np}(g). \tag{4.151}$$

From this we can say that if RG is to be invariant, we need

$$\delta = \frac{\gamma_1}{2\beta_0} - \frac{d}{2}\frac{\beta_1}{\beta_0^2}. \tag{4.152}$$

Now, using the Borel summation techniques outlined before, we will analyze the perturbative series 4.145 and also its non-perturbative counterpart. Let there be a Borel transformation of the form

$$\hat{\phi_{\rm p}}(\zeta) = \sum_{n=0}^{\infty}\frac{a_n}{n!}\zeta^n, \tag{4.153}$$

where we set g^2 as the coupling constant. We also know that a singularity exists at the Borel plane at

$$A = \frac{d}{2\beta_0}, \tag{4.154}$$

which is called the *IR renormalon singularity*. However, this causes a problem when we try to do the Borel summation since the singularity lies on the positive real axis, as $\beta_0 > 0$. Therefore, we need to do the lateral Borel resummations with a discontinuity, multiplied by the extra term g^2. Thus an imaginary ambiguity arises in the Borel summation of the perturbative series, which is of order

$$(g^2)^{1-b}{\rm e}^{-A/g^2}. \tag{4.155}$$

For the full observable to be real, we need the imaginary part in the above equation to be cancelled out by the imaginary part that arises from the non-perturbative contribution, which must have been absorbed into the coefficient C. Due to this cancellation and by matching the exponents in the leading terms in G^2 we conclude that the power b in the above expression must be expressed as

$$1 - b = \delta \tag{4.156}$$

Thus, we interrelate the perturbative and non-perturbative behaviors using the RG arguments. Moreover, the coefficient of the one-loop beta function affects the position of the IR renormalon and the value of d is determined by the quantity that is being analyzed.

Now, similar to the fact that an instanton–anti-instanton pair is responsible for the behavior of the double-well potential at large order, the leading singularity in the Borel plane is due to the instanton configuration that has zero net topological charge. Its action is double that of YM instanton action

$$S = \frac{16\pi^2}{g^2} \tag{4.157}$$

From this the singularity of the Borel plane will be at

$$\zeta = 16\pi^2. \tag{4.158}$$

Yet, when d is low, renormalon effects are more significant than instantons because they generate singularities that ar nearer to the origin. In fact, for pure $SU(N)$ YM, the position of the IR renormalon singularity is found to be at [27–29]

$$\zeta = 16\pi^2 \frac{3d}{22N}. \tag{4.159}$$

Chapter 5

Sigma Models at Large N

5.1 The O(N) non-linear sigma model

In Quantum Field Theory there are many toy models, one of the best is the $O(N)$ non-linear sigma model in two dimensions. Two of its properties matches with that of non-abelian YM theory. First of all, it is asymptotically free, and secondly it has a mass gap. It is a theory which is defined on a sphere of unit size, and it has N fields, $\sigma^a, a = 1, ..., N$. The unit sphere is such that

$$\sigma^a \sigma^a = 1. \tag{5.1}$$

Since σ_a transform in the vector representation of $O(N)$, an $O(N)$ global symmetry exists. We get the action to be

$$S = \frac{1}{2g^2} \int \mathrm{d}^2 x \partial_\mu \sigma^a \partial^\mu \sigma^a = \frac{N}{2t} \int \mathrm{d}^2 x \partial_\mu \sigma^a \partial^\mu \sigma^a \tag{5.2}$$

and

$$t = Ng^2 \tag{5.3}$$

is known as the *'t Hooft parameter*. The limit is chosen such that t is fixed, and for that we need g^2 to be small and N to be large. If we solve the constraint 5.1, we can see that the action above describes a theory of $N-1$ independent fields. Perturbation theory in two dimensions gives us a theory of $N-1$ massless bosons, namely the Goldstone bosons, which has an $SO(N)$ symmetry. However, Coleman-Mermin-Wagner showed that Goldstone bosons can not exist in two dimensions, so the perturbative picture is not true in this case. In fact, in a vector representation of $O(N)$ we get N *massive* particles when we look at the non-perturbative spectrum. Moreover, the mass gap of the theory is accounted for by their masses. Therefore, large N solution of the model gives us those particles which we do not find in perturbation theory. [25, 30, 31]

The way large N calculations are done is as follows. Firstly, a canonically normalizes kinetic term is obtained by renormalizing the fields as,

$$\sigma^a \to \sqrt{\frac{t}{N}}\sigma^a \tag{5.4}$$

Now the constraint 5.1 is applied through an extra field α, which gives us the action

$$S = \frac{1}{2}\int \mathrm{d}^d x\Big\{\partial_\mu\sigma^a\partial^\mu\sigma^a - \mathrm{i}\alpha\Big(\sigma^a\sigma^a - \frac{N}{t}\Big)\Big\}. \tag{5.5}$$

The generating functional of the correlation function is calculated as

$$Z[J] = \int \mathcal{D}\sigma\mathcal{D}\alpha\mathrm{exp}\Big\{-S + \int \mathrm{d}^2 x J^a(x)\sigma^a(x)\Big\} \tag{5.6}$$

We get an effective action for α by integrating σ^a

$$Z[J] = \int \mathcal{D}\alpha\mathrm{exp}\Big\{-S_{\text{eff}} + \int \mathrm{d}^2x\mathrm{d}^2y J^a(x)(-\partial^2 - \mathrm{i}\alpha)^{-1}(x,y)J^a(y)\Big\} \tag{5.7}$$

where

$$S_{\text{eff}}(\alpha) = \frac{N}{2}\mathrm{Trlog}(-\partial^2 - \mathrm{i}\alpha(x)) + \frac{\mathrm{i}N}{2t}\int \mathrm{d}^2 x\alpha(x) \tag{5.8}$$

and

$$(-\partial^2 - \mathrm{i}\alpha)^{-1}(x,y) \tag{5.9}$$

is the Green function of the operator of $-\partial^2 - \mathrm{i}\alpha(x)$. It can be noted that N equals $1/\hbar$ in this effective action. We can see the stationary point α and calculate the path integral for large N. We also get this from Lorentz invariance. The equations of motion for α is give by

$$\frac{\delta}{\delta\alpha}\Big[\frac{\mathrm{i}N}{g}\int \mathrm{d}^2 x\alpha(x) + N\mathrm{Tr}log\Big(-\partial^2 - \mathrm{i}\alpha\Big) = 0 \tag{5.10}$$

or

$$\frac{1}{t} = \mathrm{Tr}\frac{1}{-\partial_\mu^2 - \mathrm{i}\alpha} = 0. \tag{5.11}$$

If we calculate the trace in momentum space we get

$$\frac{1}{t} - \int \frac{\mathrm{d}^2 k}{(2\pi)^2}\frac{1}{k^2 - \mathrm{i}\alpha} = 0. \tag{5.12}$$

This integral is divergent, so a cutoff Λ needs to be used for $|k|$. Now, we do a coordinate transformation to polar coordinates and find

$$\int \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{1}{k^2 - \mathrm{i}\alpha} = \int \frac{\mathrm{d}k}{2\pi} \frac{k}{k^2 - \mathrm{i}\alpha} \tag{5.13}$$

$$= \frac{1}{4\pi} \log\Big(k^2 - \mathrm{i}\alpha\Big)\Big|_0^\Lambda \tag{5.14}$$

$$= \frac{1}{4\pi} \log\Big(\frac{\mathrm{i}\Lambda^2}{\alpha} + 1\Big) \approx \frac{1}{4\pi} \log \frac{\Lambda^2}{m^2}, \qquad \Lambda \gg 1. \tag{5.15}$$

where it is assumed that 5.12 has the form

$$\alpha = \mathrm{i}m^2, \qquad m^2 > 0. \tag{5.16}$$

Now, we get

$$\frac{1}{t} - \frac{1}{4\pi} \log \frac{\Lambda^2}{m^2} = 0. \tag{5.17}$$

The cutoff tells us that the coupling constant must be renormalized. We set the renormalization scale to be μ, and write the running coupling constant as

$$\frac{1}{t(\mu)} = \frac{1}{t} + \frac{1}{4\pi} \log \frac{\mu^2}{\Lambda^2}. \tag{5.18}$$

and we get

$$\frac{1}{t(\mu)} - \frac{1}{4\pi} \log \frac{\mu^2}{m^2} = 0. \tag{5.19}$$

that is satisfied by

$$m^2 = \mu^2 \mathrm{e}^{-4\pi/t(\mu)}. \tag{5.20}$$

This is a crucial outcome. Firstly, it is identified as the phenomenon of dimensional transmutation in an asymptotically free theory. Similar to the model in Λ^2_{QCD} we have a dimensionful parameter m^2 which is generated dynamically. Moreover, at large N we have

$$\beta_0 = -N \tag{5.21}$$

Secondly, we can find that the mass of the fields σ^a is in fact given by m^2. This dynamical effect is obtained when we go to large N.

We can expand around the vacuum now that we have extracted the nonperturbative effects by analyzing the constant field configuration. We call the fluctuation around the α field as α too:

$$\alpha \to \mathrm{i}m^2 + \alpha \tag{5.22}$$

We find that the fluctuation α has a natural normalization as

$$\alpha \to \alpha/\sqrt{N}. \tag{5.23}$$

Now, we can write the effective action as

$$\frac{N}{2}\mathrm{Tr}\ \log\Big(-\partial^2+m^2-\mathrm{i}\frac{\alpha(x)}{\sqrt{N}}\Big), \tag{5.24}$$

or,

$$\frac{N}{2}\log(-\partial^2+m^2)+\frac{N}{2}\log\Big[1+\Delta\big(-\mathrm{i}\alpha(x)/\sqrt{N}\big)\Big]. \tag{5.25}$$

Here,

$$\Delta=(-\partial^2+m^2)^{-1}. \tag{5.26}$$

α is a two point operator in this case, such that

$$\alpha(x,y)=\alpha(x)\partial(x-y) \tag{5.27}$$

and,

$$(\Delta\alpha)(x,y)=\int \mathrm{d}^2z\Delta(x,z)\alpha(z,y)=\Delta(x,y)\alpha(y). \tag{5.28}$$

Then it is expanded in powers of N^{-1}. It can be noted that there is no linear term in α by the definition of saddle point, thus we get

$$-\frac{1}{4}\mathrm{Tr}(\Delta\alpha)^2=-\frac{1}{4}\int \mathrm{d}^2x\mathrm{d}^2y(\Delta\alpha)(x,y)(\Delta\alpha(y,x) \tag{5.29}$$

$$=-\frac{1}{4}\int \mathrm{d}^2x\Delta(x,y)\alpha(y)\int \mathrm{d}^2y\Delta(y,x)\alpha(x). \tag{5.30}$$

Here,

$$\Delta(x,y)=\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{e}^{\mathrm{i}p(x-y)}}{p^2+m^2}. \tag{5.31}$$

When we take the Fourier transforms of the fields, we find this to be

$$-\frac{1}{4}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\tilde{\alpha}(p)\tilde{\gamma^s}(p)\tilde{\alpha}(-p). \tag{5.32}$$

Here,

$$\tilde{\Gamma^s}(p)=\int\frac{\mathrm{d}^2q}{(2\pi)^2}\frac{1}{(q^2+m^2)((p+q)^2+m^2)}. \tag{5.33}$$

When we calculate the above integral we get

$$\tilde{\Gamma^\alpha}(p)=f(p)\equiv\frac{1}{2\pi\sqrt{p^2(p^2+4m^2}}\log\frac{\sqrt{p^2+4m^2}+\sqrt{p^2}}{\sqrt{p^2+4m^2}-\sqrt{p^2}}. \tag{5.34}$$

Now, we can write the calculation of correlation function in terms of $1/N$ expansion

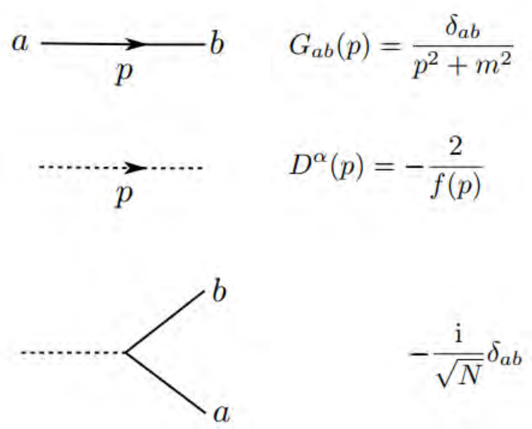


Figure 5.1: Feynman rules for the $1/N$ expansion of the $O(N)$ sigma model

as follows. Given that there are N massive particles obeying the Green function

$$G_{ab}(p) = \frac{\delta_{ab}}{p^2 + m^2} \tag{5.35}$$

and and α particle with propagator

$$D^\alpha(p) = -\frac{2}{f(p)}. \tag{5.36}$$

The interaction of these particles occur through a trivalent vertex having

$$-\frac{i\partial_{ab}}{\sqrt{N}}. \tag{5.37}$$

Thus, we can see that if the coupling constant is $1/N$, the interactions are suppressed, while $1/\sqrt{N}$ acts as the effective coupling constant of the theory. The $1/N$ expansion for the $O(N)$ sigma model theory, the Feynman rules are illustrated in the diagram below. In the diagram, the α particles are represented as the dashed lines.

5.2 The $\mathbf{P}^{N-1}$ sigma model

Let us consider another toy model in nonperturbative QFT that involves resummation of an infinite number of diagrams. Here, it will be shown that we can get a two dimensional analogue of the topological susceptibility, which is a purely nonperturbative outcome.

5.2.1 The model and its instantons

An N-component vector with unit norm forms the basic field of the $\mathbf{P}^{N-1}$ sigma model. If the spacetime is two dimensional, we express this as

$$z_1(x), ..., z_N(x), \qquad \sum_{i=1}^{N} |z_i|^2 = 1. \tag{5.38}$$

We also have an $U(1)$ gauge symmetry

$$z_i \to \mathrm{e}^{\mathrm{i}\alpha(x)} z_i. \tag{5.39}$$

From the z_i we get a gauge field since the real composite field

$$A_\mu = \frac{i}{2}\big(\overline{z}_i \partial_\mu z_i - (\partial_\mu z_i)\overline{z}_i\big), \tag{5.40}$$

has a transformation property such as

$$A_\mu \to A_\mu - \partial_\mu \alpha(x). \tag{5.41}$$

We can check this as follows:

$$\partial_\mu z_i \to \overline{z}_i \partial_\mu z_i + \mathrm{i}\partial_\mu \alpha z_i z_i = \overline{z}_i \partial_\mu z_i + \mathrm{i}\partial_\mu \alpha. \tag{5.42}$$

The $\mathbf{P}^{N-1}$ sigma model is defined by the gauge invariant action below, which tells us how this field behaves.

$$S = \frac{1}{g^2} \int \mathrm{d}^2 x \overline{D_\mu z} D^\mu z, \qquad D_\mu = \partial_\mu + \mathrm{i}A_\mu. \tag{5.43}$$

It can be noted that since a kinetic term is absent here, it is an auxiliary field. The Lagrangian of this action is

$$\mathcal{L} = \overline{D_\mu z} D^\mu z \tag{5.44}$$

and when we expand it, we get

$$\mathcal{L} = \partial^\mu \overline{z}_i \partial_\mu z_i - \mathrm{i}A_\mu \overline{z}_i \partial^\mu z_i + \mathrm{i}A_\mu \partial^\mu \overline{z}_i z_i + A_\mu A^\mu \overline{z}_i z_i \tag{5.45}$$

or,

$$\mathcal{L} = \partial^\mu \overline{z}_i + A_\mu^2 - A_\mu \mathrm{i}\big(\overline{z}_i \partial^\mu z_i - (\partial^\mu \overline{z}_i) z_i\big). \tag{5.46}$$

We know that

$$z_i \overline{z}_i = 1 \tag{5.47}$$

and that

$$(\partial^\mu \overline{z}_i)z_i + \overline{z}_i \partial^\mu z_i = 0. \tag{5.48}$$

Therefore,

$$A^\mu = \mathrm{i}\overline{z}_i \partial^\mu z_i = -\mathrm{i}(\partial^\mu \overline{z}_i)z_i, \tag{5.49}$$

and

$$\mathcal{L} = \partial^\mu \overline{z}_i \partial_\mu z_i - A_\mu^2 = \partial^\mu \overline{z}_i \partial_\mu z_i - (\overline{z}_i \partial^\mu z_i)(z_j \partial_\mu z_j). \tag{5.50}$$

Or

$$\mathcal{L} = \partial^\mu \overline{z}_i \partial_\mu z_i + (\overline{z}_i \partial^\mu z_i)(\overline{z}_j \partial_\mu z_j). \tag{5.51}$$

Interestingly, there are many similarities of the instanton solutiuons of the $\mathbf{P}^{N-1}$ model to that of the Yang-Mills theory. These instantons have finite action and are topologically nontrivial configurations. Also, it is implied by the finite action that

$$D_\mu z_i = 0, \quad \mathrm{at}|x| \to \infty, \quad i = 1, ..., n. \tag{5.52}$$

thus, z_i becomes covariantly constant when it goes to infinity. This means, upto a phase, it has to be a constant vector. Now,

$$z_i = n_i \mathrm{e}^{\mathrm{i}\sigma(x)}, \quad |x| \to \infty, \quad n_i \overline{n}^i = 1. \tag{5.53}$$

This implies that

$$-\mathrm{i}A_\mu = \frac{\partial_\mu z_i}{z_i} = \frac{\partial |z_i|}{z_i} + \mathrm{i}\partial_\mu \phi_i. \tag{5.54}$$

Here the phase of z_i is represented by ϕ_i. Because $\mathrm{i}A_\mu$ does not depend on the index i, we can say that, at infinity

$$\partial_\mu |z_i| = 0, \quad \phi_i = \sigma(\theta), \quad i = 1, ..., N. \tag{5.55}$$

Instantons are classified by the topological charge

$$Q = \frac{1}{2\pi} \int \mathrm{d}^2 x \epsilon_{\mu\nu} \partial_\mu A_\nu. \tag{5.56}$$

Since we know

$$\epsilon_{\mu\nu} \partial_\mu A_\nu = \mathrm{i}\epsilon_{\mu\nu} \partial_\mu \overline{z}_i \partial_\nu z_i \tag{5.57}$$

the topological charge can be written as

$$Q = \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}^2 x \epsilon_{\mu\nu} \partial_\nu (\overline{z}_i \partial_\mu z_i). \tag{5.58}$$

Now, the topological charge must be quantized before we deal with instantons. Since our boundary is at infinity, we apply Stokes' theorem to write down the integral as

$$Q = \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}x^\mu \overline{z}_i \partial_m u z_i. \tag{5.59}$$

Substituting the boundary condition from 5.53, we get

$$Q = \frac{1}{2\pi} \oint \mathrm{d}x^\mu \frac{\partial \sigma}{\partial x^\mu} = \frac{1}{2\pi} \Delta\sigma. \tag{5.60}$$

Here, σ varies as we go around the integral loop, the change is given by $\Delta\sigma$. We can tell that $\Delta\sigma$ is quantized since a phase is only defined as multiples of 2π.

Furthermore, the action in the topological sector is minimized by instantons. To show that this is also true in this toy model, we first express the topological density as follows

$$q(x) = \frac{1}{2\pi} \epsilon_{\mu\nu} \partial_\mu A_\nu = \frac{\mathrm{i}}{2\pi} \epsilon_{\mu\nu} \overline{D}_\mu z D_\nu z. \tag{5.61}$$

Now, we note that the last term is equal to

$$\frac{\mathrm{i}}{2\pi} \epsilon_{\mu\nu} (\partial_\mu \overline{z}_i - \mathrm{i} A_\mu \overline{z})(\partial_\nu z + \mathrm{i} A_\nu z_i) \tag{5.62}$$

and since $\epsilon_{\mu\nu}$ is antisymmetric, we just confirm that the following term becomes zero

$$-\mathrm{i}\epsilon_{\mu\nu} (A_\mu \overline{z}_i \partial_\nu z_i + A_\nu z_i \partial_\mu \overline{z}_i). \tag{5.63}$$

From 5.48, we write the above expression as

$$-\mathrm{i}\epsilon_{\mu\nu} (A_\mu \overline{z}_i \partial_\nu z_i + A_\nu \overline{z}_i \partial_m u z_i) = 0. \tag{5.64}$$

Thus, we now write the topological charge as

$$Q = \frac{\mathrm{i}}{2\pi} \int \mathrm{d}^2 x \epsilon_{\mu\nu} \overline{D}_\mu z \cdot D_\nu z. \tag{5.65}$$

The inequatlity

$$\left| D_\mu z \mp \mathrm{i}\epsilon_{\mu\nu} \overline{D_\nu z} \right|^2 \geq 0 \tag{5.66}$$

gives us

$$\overline{D_\mu z} \cdot D_\mu z + \epsilon_{\mu\rho} \epsilon_{\mu\sigma} \overline{D_\mu z} \cdot D\sigma_z \mp 2\mathrm{i}\epsilon_{\mu\nu} \overline{D}_\mu z \cdot D_\nu z \geq 0, \tag{5.67}$$

and hence $\epsilon_{\mu\rho}\epsilon_{\mu\sigma} = \delta_{\rho\sigma}$ we finally find

$$\overline{D_\mu z} \cdot D_\mu z \geq \mathrm{i}\epsilon_{\mu\nu} \overline{D_\mu z} \cdot D_\nu z, \tag{5.68}$$

and after integration we then have

$$\frac{1}{g^2}\int \mathrm{d}^2x\overline{D_\mu z}\cdot D_\mu z \geq \frac{\mathrm{i}}{g^2}\int \mathrm{d}^2x\epsilon_{\mu\nu}\overline{D_\mu z}\cdot D_\nu z, \tag{5.69}$$

that is,

$$S \geq \frac{2\pi}{g^2}|Q|. \tag{5.70}$$

This is known as the BPS bound. Only when the bound is saturated this equality is true, and from this we get the equation that tells us about the model's instanton configuration, which is

$$D_\mu z \mp \mathrm{i}\epsilon_{\mu\nu}\overline{D}_\nu z = 0. \tag{5.71}$$

From this equation, we get the instanton and anti-instanton solutions when we consider the $\pm$ signs respectively. These are analogous to the (anti) self-duality conditions for QCD instantons. [26, 32]

5.2.2 The effective action at large N

We write the Euclidean action for the $\mathbf{P}^{N-1}$ model as

$$S = \int \mathrm{d}^2x\Big[\frac{1}{g^2}\overline{D_\mu z}\cdot D_\mu z - \frac{\mathrm{i}\lambda}{g^2}(z_i\overline{z}_i - 1) + \frac{\mathrm{i}\theta}{2\pi}\epsilon_{\mu\nu}\partial_\mu A_\nu\Big] \tag{5.72}$$

where a Lagrange multiplier λ and a theta term analogue is used. Now, once again we define the 't Hooft parameter such that t is constant, and for that we go to the limit where N is large and g^2 is small. A_μ and λ are auxiliary fields in the above equation. After performing the integration, we get the action for the fields z_i along with the constraint 5.38. However, the action takes a quadratic form over z_i as follows

$$\int \mathrm{d}^2x\overline{z}_i\Delta z_i. \tag{5.73}$$

Here,

$$\Delta = -\frac{N}{t}D_\mu D^\mu - \frac{N\mathrm{i}\lambda}{t}, \tag{5.74}$$

and the N bosonic, complex variables z_i can be integrated out. We get a factor of

$$\frac{1}{\det\Delta} \tag{5.75}$$

from each of them. We get N such factors, and from that we write the determinant as the exponential of a trace of a log as

$$\exp\Big[-n\mathrm{Tr}\,\log\big(-(\partial_\mu+\mathrm{i}A_\mu)^2-\mathrm{i}\lambda\big)\Big]. \tag{5.76}$$

From this we get the following effective action

$$S\text{eff} = N\text{Tr}\,\log\big(-(\partial_\mu + \mathrm{i}A_\mu)^2 - -\mathrm{i}\lambda\big) + \frac{\mathrm{i}n\lambda}{t} - \frac{\mathrm{i}\theta}{2\pi}\epsilon_{\mu\nu}\partial_\mu A_\nu \tag{5.77}$$

that is dependent upon the fields A_μ and λ. The Fourier transform of the field is

$$\tilde{\lambda}(p) = \int \mathrm{d}^2x \mathrm{e}^{-\mathrm{i}px}\lambda(x). \tag{5.78}$$

N represents $1/\hbar$ in this effective action too. We evaluate the path integral from the stationary points for large N

$$A_\mu = 0, \qquad \lambda = \text{constant} \tag{5.79}$$

This is infact given by Lorentz invariance. We get the EOM of λ from

$$\frac{\delta}{\delta\lambda}\Big[\frac{\mathrm{i}N}{t}\int \mathrm{d}^2x\lambda + N\text{Tr}\,\log\big(-(\partial_\mu + \mathrm{i}A_\mu)^2 - \mathrm{i}\lambda\big)\Big] = 0 \tag{5.80}$$

which is the same as that of the α in the $O(N)$ sigma model, and of the form

$$\lambda = \mathrm{i}m^2, \qquad m^2 > 0. \tag{5.81}$$

Here, m^2 is the scale of the theory, which is dynamically generated. A running coupling constant $t(\mu)$ is also present that satisfies the $O(N)$ sigma model RG equation

$$m^2 = \mu^2 \mathrm{e}^{-4\pi/t(\mu}. \tag{5.82}$$

Moreover, the $\mathbf{P}^{N-1}$ theory is an asymptotically free one. The mass for the z_i fields is given by expectation value for λ in 5.72.

Now, let us express the fluctuation around the vev 5.81 of the field λ by λ also. We get the natural normalization for A_μ and the fluctuation λ to be

$$A_\mu \to \frac{1}{\sqrt{N}}A_\mu, \qquad \lambda \to \lambda/\sqrt{N}. \tag{5.83}$$

Now,

$$N\text{Tr}\,\log\big(-(\partial_\mu + \mathrm{i}A_\mu/\sqrt{N})^2 + m^2 - \mathrm{i}\frac{\lambda}{\sqrt{N}}\big) \tag{5.84}$$

can be written as

$$N\text{Tr}\,\log(-\partial^2 + m^2) + N\text{Tr}\,\log\Big[1 + \Delta\big(A^2/N - \mathrm{i}\lambda/\sqrt{N} - \mathrm{i}\{A, \partial\}/\sqrt{N}\big)\Big] \tag{5.85}$$

where Δ is defined in 5.26. Expanding this in inverse powers of N we get at leading order

$$\Delta A^2 + \frac{1}{2}\Delta^2(\partial A + 2A\partial)^2 + \frac{1}{2}(\Delta\lambda)^2. \tag{5.86}$$

In case of the $O(N)$ sigma model, the last term is

$$\frac{1}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\tilde{\lambda}(p)\tilde{\Gamma}^s(p)\tilde{\lambda}(-p), \tag{5.87}$$

where $\tilde{(\gamma)}^s$ is given by 5.33. Now, for the quadratic terms of the A_μ fields, we get

$$\frac{1}{2}\int \frac{\mathrm{d}^2 p}{(2\pi)^2}\tilde{A}^\mu(p)\tilde{\Gamma}^A_{\mu\nu}(p)\tilde{A}^\nu(-p), \tag{5.88}$$

in which

$$\tilde{\Gamma}^A_{\mu\nu}(p) = 2\delta_{\mu\nu}\int \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{1}{(q^2+m^2} - \int \frac{\mathrm{d}^2 q}{(2\pi)^2}\frac{(p_\mu + 2q_\mu)(p_\nu + 2q_\nu)}{(q^2+m^2)((p+q)^2+m^2)}. \tag{5.89}$$

If we calculate this we get

$$\tilde{\Gamma}^A_{\mu\nu}(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\{(p^2+4m^2)f(p) - \frac{1}{\pi}\}, \tag{5.90}$$

where $f(p)$ was defined in 5.34. Because of the fact that

$$f(p) = \frac{1}{4\pi m^2} - \frac{p^2}{24\pi m^4} + \mathcal{O}(p^4) \tag{5.91}$$

around $p^2 = 0$, we have the quadratic term in $\tilde{A}$ to be of the form

$$(\delta_{\mu\nu}p^2 - p_\mu p_\nu)(c + \mathcal{O}(p^2)) \tag{5.92}$$

where

$$c = \frac{1}{12\pi m^2}. \tag{5.93}$$

This results from gauge invariance and gives us the kinetic energy of the standard gauge field

$$(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \tag{5.94}$$

expressed in momentum space. This means that a kinetic energy has been generated for A_μ by the quantum corrections. This is another dynamical effect that is manifest at large N. Also, if we add up an infinite number of large N dominating diagrams, we see this effect in the original z variables.

Furthermore, in this model the quarks and anti-quarks are given by z and $\overline{z}$ respectively, which interact through the gauge field A_μ. However, in two dimensions, Coulomb's law gives us a linear potential, so the gauge field $U(1)$ in two dimensions is confining. Thus, the appearance of a dynamical gauge field in this theory confines the charges, that only forms singlets or triplets. [33–35]

5.2.3 Topological susceptibility at large N

One unique aspect of large N is that in it the value of topological susceptibility is nonzero. Now, we know from YM theory that the topological susceptibility is written as

$$\chi_t = \lim_{k\to 0} U(k) \tag{5.95}$$

We now calculate

$$U(p) = \int \mathrm{d}^2x \mathrm{e}^{\mathrm{i}px} \langle q(x)q(0)\rangle = \int \frac{\mathrm{d}^2p'}{(2\pi)^2} \langle \tilde{q}(-p)\tilde{q}(p')\rangle \tag{5.96}$$

with $q(x)$ being the topological density. When we normalize A_μ we get the factor $1/N$ in this quantity. We write the Fourier transform of $q(p)$ as

$$\tilde{q}(p) = -\frac{\mathrm{i}}{2\pi\sqrt{N}} \epsilon_{\mu\nu} p_\mu \tilde{A}_\nu. \tag{5.97}$$

Finally,

$$\langle \tilde{q}(-p)\tilde{q}(p')\rangle = \frac{1}{4\pi^2 N} \epsilon_{\mu\nu}\epsilon_{\rho\sigma} p_\mu p'_\rho \langle \tilde{A}_\nu(-p)\tilde{A}_\sigma(p')\rangle. \tag{5.98}$$

We want to calculate the two point function of the gauge field, and for that firstly the Lorentz gauge is chosen

$$\partial_\mu A_\mu = 0. \tag{5.99}$$

We get the two-point function from 5.90 which is

$$\langle \tilde{A}_\nu(p)\tilde{A}_\sigma(-p')\rangle = (2\pi)^2\delta(p-p')\Big(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big) D_A(p), \tag{5.100}$$

where

$$D_A(p) = \big\{(p^2+4m^2)f(p) - \frac{1}{\pi}\big\}^{-1}. \tag{5.101}$$

The kinetic term 5.88 in momentum space causes the factor $(2\pi)^2$ to arise in the above expression. Because of the fact that

$$\epsilon_{\mu\nu}\epsilon_{\rho\sigma} p_\mu p_\rho \big(\delta_{\nu\sigma} - \frac{p_\nu p_\sigma}{p^2}\big) = (\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}) p_\mu p_\rho \big(\partial_{\nu\sigma} - \frac{p_\nu p_\sigma}{p^2}\big) = p^2 \tag{5.102}$$

we get

$$\langle \tilde{q}(-p)\tilde{q}(p')\rangle = \frac{p^2}{4\pi^2 N}(2\pi)^2 D_A(p)\delta(p-p'). \tag{5.103}$$

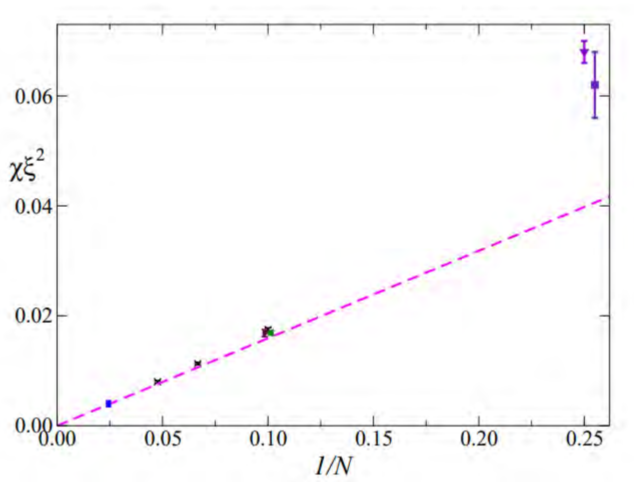


Figure 5.2: Lattice calculations of $\chi_t\zeta^2$ in the CP^{N-1} model

Thus,

$$\int \frac{\mathrm{d}p'}{(2\pi)^2}\langle\tilde{q}(p)\tilde{q}(p')\rangle = \frac{p^2}{N}D_A(p) = \frac{3m^2}{\pi N} + \mathcal{O}(p^2), \tag{5.104}$$

and so, finally we have the topological susceptibility to be

$$\chi_t^{\text{large } N} = \frac{3m^2}{\pi N} \tag{5.105}$$

at leading order when $1/N$ is expanded. In perturbation theory this quantity goes to zero order by order, which makes this an important outcome of nonperturbative theory. What we have done here is that we did the resummation procedure over an infinite number of large N dominating diagrams, before we considered taking the limit $p \to 0$. This ensured that the quantity does not vanish. Another interesting factor is that the above result was experimentally verified in lattice calculations of the topological susceptibility. The results are given in the graph in figure 5.2.

In this graph, the results for the lattice calculations of $\chi_t\zeta^2$ in the $\mathbf{CP}^{N-1}$ model for different N values are shown. These are represented as a function of $1/N$. The length scale is determined by the quantity $\zeta^2 = (6m^2)^{-1}$. From this it can be clearly seen that for $N \geq 10$ the calculations highly agree with theory.

To sum up the nonperturbative effects at large N, we can say mention the few points: 1. In the coupling constant and the 't Hooft parameter of the perturbation theory the masses of the quarks z_i and $\overline{z}_i$ are absent, in contrast, which is manifest in nonperturbative analysis at large N. 2. The A_μ field appears to be a dynamic gauge field in this theory. On the other hand, this was just a spare field in perturbation theory. 3. Finally, the topological susceptibility is found to be of order $\mathcal{O}(1/N)$ instead of zero. [20, 21, 36]

Chapter 6

The 1/N expansion in QCD

6.1 Fatgraphs

Just as a small recap, we write the QCD Lagrangian as follows

$$\mathcal{L} = \frac{N}{t}\Big[\frac{1}{4}(F_{\mu\nu}, F^{\mu\nu}) + \sum_{f=1}^{N_f} \overline{q}_f(i\not{D} - m_f)q_f\Big] \tag{6.1}$$

where the *t' Hooft parameter* is set to be

$$t = g^2 N. \tag{6.2}$$

Also, the large N limit is written as

$$N \to \infty, \quad g^2 \to 0. \qquad t \text{ fixed.} \tag{6.3}$$

Thus the theory remains nontrivial. One hint that this is true is the one-loop β function of QCD which we write as

$$\mu\frac{\mathrm{d}g}{\mathrm{d}\mu} = -\Big(\frac{11}{3}N - \frac{2}{3}N_f\Big)\frac{g^2}{16\pi^2}. \tag{6.4}$$

After we multiply by $N^{\frac{1}{2}}$, we have

$$\mu\frac{\mathrm{d}t}{\mathrm{d}\mu} = -\Big(\frac{11}{3}N^{\frac{3}{2}} - \frac{2}{3}N^{\frac{1}{2}}N_f\Big)\frac{t^3/N^{\frac{3}{2}}}{16\pi^2} = -\Big(\frac{11}{3} - \frac{2}{3}\frac{N_f}{N}\Big)\frac{t^3}{16\pi^2}. \tag{6.5}$$

Therefore, we can see that in the large N limit, it is well defined. It can also be seen that the effect of quark loops do not appear. Moreover, different important quantities in QCD can be expressed in terms of the expansion in terms of $1/N$, with the leading term preserved my the large N limit.

Now, the rescaled fields below

$$A_\mu = g\hat{A}_\mu, \qquad q = g\hat{q}, \tag{6.6}$$

that were mentioned earlier, can be written in terms of the 't Hooft parameter as follows

$$A_\mu = \frac{t}{\sqrt{N}}\hat{A}_\mu, \qquad q = \frac{t}{\sqrt{N}}\hat{q}. \tag{6.7}$$

The main factor that determines the behavior that we observe in the $1/N$ expansion is the presence of a latent variable N, which is the rank of the gauge group $SU(N)$, along with the other quantities such as g, etc. In fact, a polynomial in N with various powers of N arise for each Feynman diagram that we calculate. Thus, the *group factors* related to the Feynman diagrams causes the N dependence in the perturbative expansion to appear. So we can use Feynman diagrams to account for the powers of the coupling constants, but we can not say what powers of N do the diagrams give us. Therefore, in order to know how our diagrams depend on N, we break them up in various parts, each of which represent a particular power of N. This was first shown by 't Hooft. He drew the Feynman diagrams as double line graphs, which we now call "fatgraphs." This is illustrated as follows.

We write the quark propagator as

$$\langle \psi^i(x)\overline{\psi}^j(y)\rangle = \frac{t}{N}\delta^{ij}S(x-y), \qquad i,j = 1,...,N. \tag{6.8}$$

We denote this by a single line as shown below.

Due to the Kronecker delta in the above equation, we have the color at the beginning and at the ending of the line to be equal. Now, the gluon propagator, with a and b being the indices in the adjoint representation, is written as

$$\langle A^a_\mu(x)A^b_\nu(y)\rangle = \frac{t}{N}\delta^{ab}D_{\mu\nu}(x-y). \tag{6.9}$$

Now, it is better that we treat a gluon as a $N\times\overline{N}$ matrix having two indices in the N and $\overline{N}$ representations, rather than considering it as a field with a single adjoint index. Mathematically,

$$(A_\mu)^i_j = A^a_\mu(T_a)^i_j, \tag{6.10}$$

where $(T_a)^i_j$ is a Lie Algebra basis satisfying the normalization condition below

$$(Tr)(T_aT_b) = \delta_{ab}, \qquad a,b = 1,...,N^2. \tag{6.11}$$

Another condition that, for $U(N)$ these bases fulfill

$$\sum_a (T_a)^i_j (T_a)^k_l = \delta^i_l \delta^k_j \tag{6.12}$$

and for $SU(N)$ they fulfill

$$\sum_a (T_a)^i_j (T_a)^k_l = \delta^i_l \delta^k_j - \frac{1}{N} \delta^i_j \delta^k_l. \tag{6.13}$$

Thus, the $U(N)$ gluon propagator can be written as

$$\langle A^i_{\mu j}(x) A^k_{\nu l}(y) \rangle = \frac{t}{N} D_{\mu\nu}(x-y) \delta^i_l \delta^k_j. \tag{6.14}$$

The figure below shows the group structure of this propagator, which is represented by the double line notation.

The interaction vertices are expressed in the double line notation too. There are structure constants f_{abc} present in the three-gluon vertices, defined as

$$[T_a, T_b] = f_{abc}. \tag{6.15}$$

When it is multiplied by T_d and the trace is taken, we get

$$f_{abc} = \mathrm{Tr}(T_a T_b T_c) - \mathrm{Tr}(T_b T_a T_c). \tag{6.16}$$

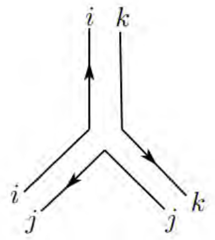
We can construe the trace of the three generators of Lie algebra as a cubic vertex. In fact, it yield from

$$\mathrm{Tr}(A_\mu A_\nu A_\rho) = A^a_\mu A^b_\nu A^c_\rho \mathrm{Tr}(T_a T_b T_c). \tag{6.17}$$

However, when we consider this in terms of the double line notation, we have

$$\sum_{i,j,k} (A_\mu)^i_j (A_\nu)^j_k (A_\rho)^k_i \tag{6.18}$$

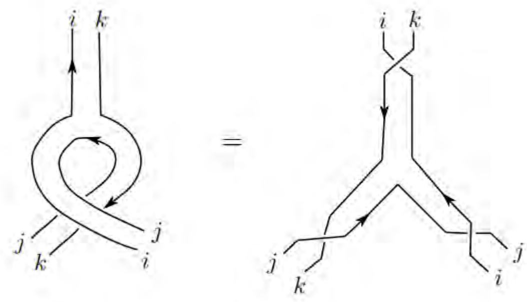
which is illustrated diagrammatically as follows:



Moreover, there is also a commutator, so we get an extra term

$$-\sum_{i,j,k}(A_\nu)^i_j(A_\mu)^j_k(A_\rho)^k_i, \tag{6.19}$$

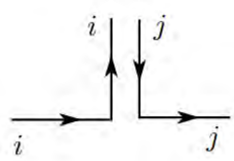
which is drawn as a "twisted" double-line vertex such as



The last rule that we need to know is that of the interaction between a quark bilinear and a gluon. This vertex is represented by the group structure

$$\psi^i(x)(A_\mu)^j_i(x)\overline{\psi}^j(x), \tag{6.20}$$

thus we denote this using the double line notation as



Here, the vertical line represents the quark and the horizontal line represents the gluon.

The topological characterization of the fatgraphs are usually done by three variables E, V, and h, which represents the number of propagators or edges, vertices, and closed loops respectively. Moreover, the propagators and the interaction vertices give the factors and the powers of g respectively. Lastly, the factor of h results from the closed loops having sums over color indices. Thus we get the overall factor

$$N^h g^{2(E-V)}, \tag{6.21}$$

which, when written in terms of the 't Hooft parameter, looks like

$$N^{V-E+h}t^{V-E}. \tag{6.22}$$

Now, the fatgraphs can be thought of as Riemann surfaces. If no quark loops are present, we call them *closed*. To illustrate this, let us consider each closed loop to be the perimeter of a polygon. Then, a double line tells us to attach the polygons by identifying one edge of the polygon to that of the other, provided that they both lie on the same double line. Lastly, the boundary of the surface is represented by the closed quark loop (single-line).Now, the Euler's relation can be used to write

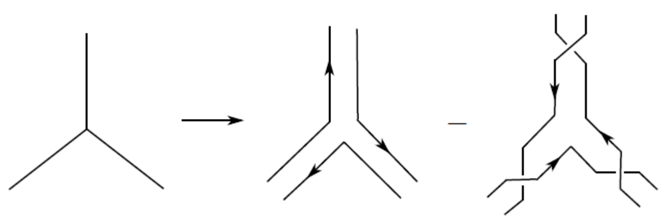
$$h + V - E = \chi = 2 - 2g - b. \tag{6.23}$$

Here, b is the number of boundaries and g denotes the genus of the Riemann surface. Thus it turns out that N has a factor of

$$N^{2-2g-b} = N^{\chi}. \tag{6.24}$$

Planar diagrams are those which have $g = 0$ and the nonplanar are those which have $g > 0$. Different Feynman diagrams give us different fatgraphs with varying genera.

Now, we show how to find the group factor of any diagram in QCD. Considering the group theory structure we can decompose the Yang-Mills quartic vertex into two cubic vertices connected by an extra edge, thereby converting each diagram to trivalent diagrams. Therefore, if we have a trivalent diagram G, consisting of V vertices, we do a summation over 2^V possible "resolutions" of the vertices by incorporating 6.16, as shown in the diagram below.

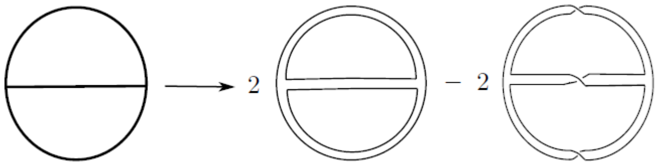


Each of these diagrams is a fatgraph Σ, the weight of which will be given by the number of closed loops, namely N to the power $h(\Sigma)$. Mathematically,

$$r(G) = \sum_{\Sigma} N^{h(\Sigma)}. \tag{6.25}$$

Also, a factor of $g^{2(E-V)}$ will be present in the contribution of the diagram to the large N expansion. Here, $E - V$ remains unchanged.

Example 5.1 Suppose we have the theta diagram, which is formed as two-loops connected by cubic vertices. When we resolve the vertices using the 6.16, we get two separate diagrams. One of them is twisted and the other is untwisted, both having a multiplicity of 2, as illustrated in the figure below.



Thus, we get the group factor

$$2N^3 - 2N. \tag{6.26}$$

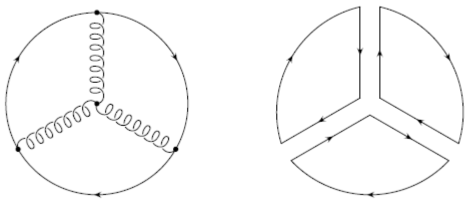
The first fatgraph has $g = 0$, and so its weight in the $1/N$ expansion is

$$N^3 g^2 = N^2 t^2. \tag{6.27}$$

On the other hand, the second one in nonplanar, so it has $g = 1$ and weight

$$N g^2 = t. \tag{6.28}$$

Example 5.2 Below is a diagram with a closed quark line and its respective fatgraph.



We can see that the number of closed cycles, vertices and propagators are three, four and six, respectively and hence, we get the diagram's weight to be [19, 37–40]

$$N^3 g^4 = N t^2. \tag{6.29}$$

6.2 Large N rules for correlation functions

Now we illustrate the counting rules for large N correlation functions by using the fatgraphs. In fact, we can also conduct these calculations for any quantum theory that had an $U(N)$ symmetry with the fields being in the adjoint and fundamental representations.

We have mentioned before that the Feynman diagrams can be considered to be two dimensional surfaces when we denote the perturbative expansion in terms of fatgraphs. These surfaces are characterized by two topological quantities, namely the genus g and the number of boundaries with a weight given by 6.24. When the surface is a closed one, we have $g = 0$ and so the value of $\chi = 2$, that is the largest. However, if the surface has boundaries, the value of χ becomes equal to 1 because then we have $g = 0$ and $b = 1$. Therefore, we can say

1. The order of the leading vacuum-to-vacuum connected graphs is N^2. They are composed of gluons and are planar in nature.
2. On the other hand, the order of the leading connected vacuum-to-vacuum diagrams in N if their boundary is made up of only one quark loop. They are also planar graphs.

Finally, the free energy of the pure $U(N)$ gauge theory (that is, all connected vacuum-to-vacuum diagrams summed over) is written as

$$F(N,t) = \sum_{g=0}^{\infty} F_g(t) N^{2-2g}, \tag{6.30}$$

Here,

$$F_g(t) = \sum_{h \geq 0} a_{g,h} t^{2g-2+h} \tag{6.31}$$

in which we have summed over all the fatgraphs with a fixed topology. When N becomes large, only the planar diagrams with $g = 0$ remains.

Now we discuss about correlation functions. We consider the gauge-invariant operator G_i formed solely from gluons, such as

$$\mathrm{Tr} F_{\mu\nu} F^{\mu\nu}, \quad \mathrm{Tr}_R U_\gamma, \tag{6.32}$$

where

$$U_\gamma = \mathrm{P} \;\; \exp \oint_\gamma A \tag{6.33}$$

γ is a closed loop and U_γ is a Wilson line operator. Now with the action we add

$$S \to S + N \sum_i J_i G_i \tag{6.34}$$

where the sources are given by J_i. The counting rules for the Lagrangian do not change due to the presence of the overall factor N. Now, the *connected* correlation functions are generated by a functional which is the sum of connected vacuum-to-vacuum diagrams along with these sources. Therefore we write

$$\langle G_1...G_r\rangle^{(c)} = \frac{1}{N^r}\frac{\partial^r\Gamma(J)}{\partial J_1...\partial J_r}\Big|_{J=0}. \tag{6.35}$$

Now that the generating functional has its leading contribution of order N^2, we can say that at leading order in N we get

$$\langle G_1...G_r\rangle^{(c)} \sim N^{2-r} \tag{6.36}$$

Moreover, if this correlation function is expanded fully in $1/N$, we have

$$W^{(r)}(N,t) = \langle G_1...G_r\rangle^{(c)} = \sum_{g=0}^{\infty} W_g^{(r)}(t)N^{2-2g-r} \tag{6.37}$$

in which

$$W_g^{(r)}(t) = \sum_{n\geq 0} W_{n,g}^{(r)} t^n \tag{6.38}$$

is all the fatgraphs contributing to the correlation function with a fixed topology being summed over.

We can also think of gauge-invariant operators M_i that involve quark bilinears, such as

$$\overline{\psi}\psi, \qquad \overline{\psi}(y)\mathrm{P}e^{\int_x^y A}\psi(x) \tag{6.39}$$

and so on. Now, the action is perturbed as follows

$$S \to S + N\sum_i J_i M_i \tag{6.40}$$

where the sources are given by b_i, and

$$\langle M_1...M_r\rangle^{(c)} = \frac{1}{N^r}\frac{\partial^r\Gamma(b)}{\partial J_1...\partial J_r}\Big|_{J=0}. \tag{6.41}$$

The order of the leading contribution to the generating functional is N. It also includes a quark loop at the boundary where the bilinears are put in. This is illustrated in the figure 6.1.

Here, $g = 0$ and $b = 1$. Therefore we can say that

$$\langle M_1...M_r\rangle^{(c)} \sim N^{1-r}. \tag{6.42}$$

From these we get the counting rules for the amplitudes of meson and glueball

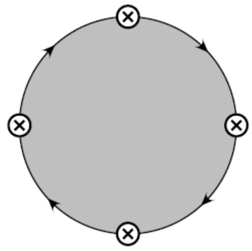


Figure 6.1: Quark bilinears

scattering. The glueball states are made by the gluon operators G_i, and the meson states are made by the quark bilinears B_i as follows

$$G_i \left|0\right\rangle \sim \left|G_i\right\rangle, \qquad M_i \left|0\right\rangle \sim \left|M_i\right\rangle. \tag{6.43}$$

When we try to appropriately normalize the states, we get

$$\langle G_1|G_2\rangle \sim \langle G_1G_2\rangle^{(c)} \sim \mathcal{O}(N^0). \tag{6.44}$$

Thus the amplitude of the glueball states created by G_i is unitary. Also,

$$\langle M_1|M_2\rangle \sim \langle M_1M_2\rangle^{(c)} \sim \mathcal{O}(1/N). \tag{6.45}$$

Thus, the appropriately normalized meson state becomes

$$\sqrt{N}M_i \left|0\right\rangle. \tag{6.46}$$

Therefore we say that the interactions by the mesons and glueballs are suppressed by factors of N. The order of suppression of an $r-$glueball vertex is N^{2-r} and subsequent glueballs have additional $1/N$ orders of suppression. Furthermore, the suppression of a normalized r meson vertex will be as

$$\langle\sqrt{N}M_1...\sqrt{N}M_r\rangle^{(c)} \sim N^{1-r/2} \tag{6.47}$$

and suppressions of order $1/\sqrt{N}$ are added by each extra meson. Lastly, the suppression of the mixed glueball-meson correlators will be as follows

$$\langle G_1...G_s\sqrt{N}M_1...\sqrt{N}M_r\rangle^{(c)} \sim N^{1-s-r/2}. \tag{6.48}$$

This means that if we consider $1/N$ to be a coupling constant, from QCD we derive a theory of glueballs and mesons weakly interacting with each other. Also the rescaling 6.6 gives us the counting rules for the original fields of the Lagrangian. [30, 41, 42]

Example 5.3 Suppose we have the following

$$\langle 0|\,\mathrm{Tr}(FF)\,|M\rangle\,, \qquad \langle 0|\,\mathrm{Tr}(FF)\,|G\rangle\,. \tag{6.49}$$

From the rules given above we get

$$\langle 0|\,\mathrm{Tr}(FF) \sim \frac{1}{\sqrt{N}}, \qquad \langle 0|\,\mathrm{Tr}(FF)\,|G\rangle \sim \mathcal{O}(1). \tag{6.50}$$

When we write them in terms of the rescaled fields we get $\mathrm{Tr}(\tilde{F}\tilde{F}) \sim \sqrt{N}\mathrm{Tr}(FF)$, and so

$$\langle 0|\,\mathrm{Tr}(\tilde{F}\tilde{F}\,|M\rangle \sim \sqrt{N}, \qquad \langle 0|\,\mathrm{Tr}(\tilde{F}\tilde{F})\,|G\rangle \sim N. \tag{6.51}$$

Example 5.4 *Large N scaling of F_π.* The constant for the pion decay is given by

$$\langle 0|\,A^{\mu}_{ud}(x)\,|\pi(p)\rangle = \mathrm{i}p^{\mu}C_{\pi}\mathrm{e}^{-\mathrm{i}p\cdot x}, \tag{6.52}$$

where C_π is a constant which is parameterized by

$$\frac{\sqrt{2}F_\pi}{(2\pi)^3/2\sqrt{2E_p}} \tag{6.53}$$

and we get

$$F_\pi \sim 93\mathrm{MeV}. \tag{6.54}$$

Now the decay constant has the structure

$$\langle 0|\,M_1\,|M_2\rangle \sim 1/\sqrt{N}. \tag{6.55}$$

Because $\hat{q} \sim \sqrt{N}q$, we have $\hat{A}_{ud} \sim NA_{ud}$, and finally we get

$$F_\pi \sim \sqrt{N}. \tag{6.56}$$

6.3 QCD spectroscopy at Large N: mesons and glueballs

The analysis of mesons and glueballs and the above mentioned properties of QCD tell us the following:

1. Both mesons and glueballs are stable and do not interact at large N. They are infinite in number and their masses have a smooth large N limit.

2. The order for the meson decay amplitudes is $1/\sqrt{N}$. The tree diagrams of an

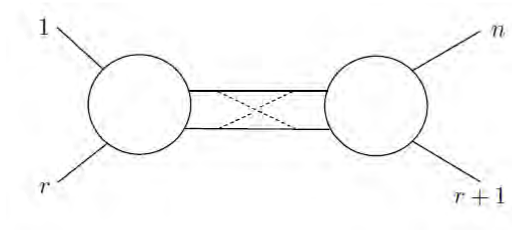


Figure 6.2: The appearance of one quark-antiquark pair in QCD spectroscopy

effective local Lagrangian that involves meson fields can illustrate the large N limit.

3. When we decouple mesons, we get the glueball states in the lowest order in $1/N$. When the mesons and the glueballs are mixed, the order in $1/\sqrt{N}$, and the order of mixing solely the glueballs is $1/N$.

Now, in order to show that the first point above holds, we suppose that we have a two-point function of a current J which are formed from quark bilinears and can also create a meson. Just like any other two-point functions, We can express this two-point function, by its spectral representation, in terms of a sum over poles and a more involved term given by the multiparticle states. Now, the punch line is that we only get the contribution from the poles summed over, at large N. Mathematically,

$$\langle J(k)J(-k)\rangle = \sum_n \frac{a_n^2}{k^2 - m_n^2} \tag{6.57}$$

where the one particle meson states $|n\rangle$, which have masses m_n are summed over, and up to a kinetic factor, we have

$$a_n = \langle 0| J |n\rangle \,. \tag{6.58}$$

This is true because, when N is large, only the Feynman diagrams that have only one quark loop at the boundary contribute to this correlator. Thus, we get exactly one $q\bar{q}$ pair as shown in the figure below, when we try to detect the intermediate states by cutting this diagram (figure 6.2). Now, this must be a single meson state given that the confinement is assumed to be true.

The equation 6.57 also tells us that at large N we have a spectrum of mesons with well defined masses since the r.h.s of this equation is well-defined in this limit. Suppose that we normalize the current as we have done in 6.47, both the r.h.s of the equation and the masses of the mesons m_n^2 with a smooth limit do not depend on N. There are infinite number of states because as k^2 becomes large, the two-point function becomes logarithmic in k^2. We would have a k^{-2} dependence instead of the k^2 dependence if the number of terms in the summation was not infinite.

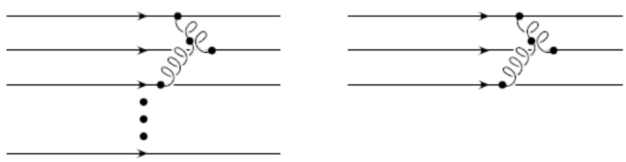


Figure 6.3: N counting rules for the Baryon graphs

6.4 Baryons at large N

Quarks form the color singlet hadrons called the Baryons. This is an N-quark state because there are N indices in the $SU(N)$ invariant ϵ-symbol. It is expressed as

$$\epsilon_{i_1 ... i_N} q^{i_1} ... q^{i_N} . \tag{6.59}$$

Since all the indices in the ϵ symbol must be different for the quantity to be nonzero, we can consider the baryons to be made up of N quarks, each one having a differnt color. Quarks are fermions and the baryons are antisymmetric in color since the ϵ itself is antisymmetric. Therefore, in other quantum numbers such as spin and flavor, the baryons must be completely symmetric.

The number of quarks in a baryon increases as N increases, so we may mistakenly think that large N baryons do not have much in common with the baryons with $N = 3$. However, if we expand in $1/N$ in a systematic way, we can retrieve the characteristics of baryons. The outcome that we get are also backed by experimental data, and also give us information regarding the spin-flavor structure of baryons.

Furthermore, we can use the results from the meson graphs to deduce the N-counting rules for that of the baryon graphs. For this, firstly the incoming baryons are drawn as N-quarks with color arrangement being in the order $1...N$. We also need to find the N-counting rules for the connected diagrams. To get this, we consider the incoming and outgoing quark lines to be ending up on independent vertices. This is shown in the figure 6.3. In this figure we can see that the connected component of the diagram on the left is given by the diagram on the right side.

An n-body interaction is a connected piece that has n quark lines. When we permute the colors on the incoming quarks, we get the distinct colors of the outgoing quarks in an n-body interaction. We uniquely identify each outgoing line with an incoming line if they have the same color. Now, the planar diagrams with a single quark loop can be compared to the connected diagrams for the baryon interactions. [20, 39, 43]

Chapter 7

Painleve Equation I

We know that *resurgence* is the theory which tells us information about non-perturbative aspects of a theory from its perturbative information. Now, we attempt to show whether the predictions about the large-order relations given by the resurgence theories are actually valid by using the PI equation. We choose this equation since this enables us to use both analytical and numerical approaches, whereas in other cases only numerical solutions are obtainable. We start of with the power series method, and then apply Borel summation in order to set a value for the resulting divergent sum. After that, We will analytically figure out a full solution to the instanton action and instanton sectors. Finally, we will use numerical methods to determine the instanton action and the instanton sector by incorporating the large-order relations on the perturbative coefficients. The Painleve-I equation is given as

$$\phi(z)^2 - \frac{1}{6}\phi''(z) = z \tag{7.1}$$

This is a second order, non-linear differential equation. One of its use is in two dimensional quantum gravity, in which it gives the all-genus solution.

7.1 Power Series Method

By the method of dominant balance, we get the following ansatz, which we substitute into the PI equation to find the power series solution

$$\phi(z) = z^{1/2} \sum_{n=0}^{\infty} \phi_n z^{-5n/2} \tag{7.2}$$

Then we find the derivatives and put them into the PI equation. Finally,

$$z\Big(\sum_{n=0}^{\infty}\phi_n z^{-5n/2}\Big)\Big(\sum_{n=0}^{\infty}\phi_n z^{-5m/2}\Big)-\frac{1}{6}\sum_{n=0}^{\infty}\phi_n\Big(\frac{1-5n}{2}\Big)\Big(\frac{-1-5n}{2}\Big)z^{(-3-5n)/2}=z \tag{7.3}$$

or,

$$z\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{n}\phi_m\phi_{n-m}\Big)z^{-5n/2}-\frac{1}{6}\sum_{n=0}^{\infty}\phi_n\Big(\frac{1-5n}{2}\Big)\Big(\frac{-1-5n}{2}\Big)z^{(-3-5n)/2}=z. \tag{7.4}$$

Now, for the z^1 in both the sides of the above equation to be equal, it is required that ϕ_0^2 be equal to 1. Therefore, $\phi_0 = \pm 1$, from which we take the positive part. Since our ansatz is a formal solution, we let the factor $z^{1/2}$ to absorb the sign, and also all the powers of z gets cancelled. Now, shifting the second term in the above equation gives us

$$\sum_{n=0}^{\infty}\phi_m\phi_{n-m}=\frac{1}{6}\phi_{n-1}\Big(\frac{1-5(n-1)}{2}\Big)\Big(\frac{-1-5(n-1)}{2}\Big) \tag{7.5}$$

$$\sum_{m=1}^{n-1}\phi_m\phi_{n-m}+2\phi_0\phi_n=\frac{\phi_{n-1}}{24}(25(n-1)^2-1) \tag{7.6}$$

and,

$$\phi_n=\frac{\phi_{n-1}}{48}(25(n-1)^2-1)-\frac{1}{2}\sum_{m=1}^{n-1}\phi_m\phi_{n-m} \tag{7.7}$$

We have solved this recursion relation for the perturbative coefficients numerically in Mathematica using the code:

```
f[n_] := f[n]

f[0] = 1;
n = 1;
While[ n < 21,
  f[n] = f[n - 1]/48 (25 (n - 1)^2 - 1) - 1/2 ( Sum[f[m] * f[n - m], {m, n - 1}]);
  Print[f[n]];
  n++];
```

We have also listed the first twenty one terms in the table below.

ϕ_0	1
ϕ_1	$-\frac{1}{48}$
ϕ_2	$-\frac{49}{4608}$
ϕ_3	$-\frac{1225}{55296}$
ϕ_4	$-\frac{4412401}{42467328}$
ϕ_5	$-\frac{73560025}{84934656}$
ϕ_6	$-\frac{245229441961}{21743271936}$
ϕ_7	$-\frac{7759635184525}{36691771392}$
ϕ_8	$-\frac{21630993344469560445}{400771988324352}$
ϕ_9	$-\frac{2433521765777765537625}{1352605460594688}$
ϕ_{10}	$-\frac{1261548258446836126698067 43}{16620815899787526144}$
ϕ_{11}	$-\frac{307996788703417873806157775}{779100745302540288}$
ϕ_{12}	$-\frac{381621650814403922234841017 5181221}{1531774393324418409431 04}$
ϕ_{13}	$-\frac{4472139245793702477426700875742975}{239339748956940376473 6}$
ϕ_{14}	$-\frac{3869659109987312485704943494193912966 1339}{2352805468146306676886 07744}$
ϕ_{15}	$-\frac{96442427363337689886991695129573972897 5}{574415397496656903536 64}$
ϕ_{16}	$-\frac{5119569911358056679589066513337449330170941 9998829}{2602014623332363480101850762 4448}$
ϕ_{17}	$-\frac{1234465053761258456327734513118479122195306 44725}{4705610936292613353772154 88}$
ϕ_{18}	$-\frac{5260649001580551386721455751207683243044235876590074 90115}{13322314871461701018121475903717376}$
ϕ_{19}	$-\frac{2925884059020607616500208022636411653250934134613430 87375}{43908996768733633726718731616256}$
ϕ_{20}	$-\frac{1538267551991111681653771480335893218249571536284824096420896 27159}{1227784538553910365830075219286593 37216}$

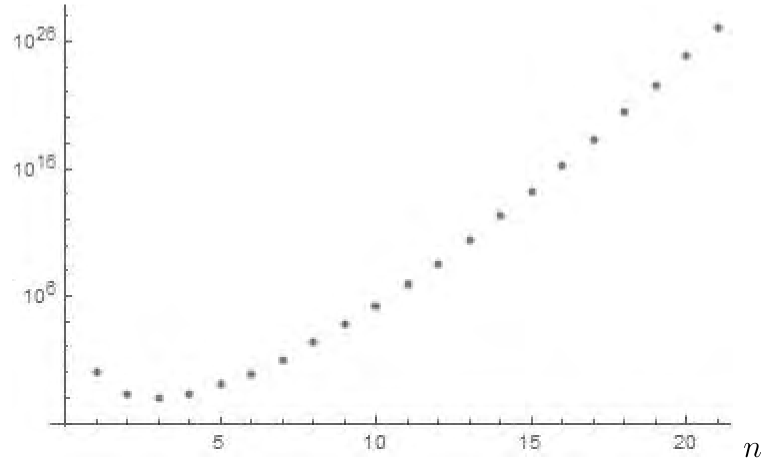


Figure 7.1: Perturbative coefficients, ϕ_n (y-axis) diverges when n increases

When the absolute values of these coefficients are plotted in the y-axis against the values of n in the x-axis, we get a graph as follows:

As we can see above, the series is divergent, and we would not be able to do anything with it if we did not know how to do Borel summation. Now, we will apply this technique to get finite values for z. [17, 18, 44, 45]

7.2 Power series solution to the PI equation Borel summed

We have at hand the series 7.2, the coefficients of which are given by the 7.7. Now, we take the Borel transformation of this series and we get

$$\mathcal{B}[\phi](\zeta) = \zeta^{1/2} \sum_{n=0}^{\infty} \frac{a_n}{n!} \zeta^{-5n/2}. \tag{7.8}$$

Then, the original sum for a z is given a value by taking the inverse Borel transform of the series. For that, we need to use the technique of Pade approximant in order to approximate the Borel transformation, which would otherwise be required to be tediously done term by term. However, now we get convergent values for large-order N in n, using Pade approximant. If we have a polynomial of even degree N, this function is expanded by the Pade approximant as a ratio of two power series in degrees $N/2$ such that the first $N+1$ derivatives are unchanged.

$$\mathcal{B}^{[N]}[\phi](\zeta) \approx \frac{P^{[N/2]}[\phi](\zeta)}{Q^{[N/2]}[\phi](\zeta)}. \tag{7.9}$$

From this, we can see off that the value of the polynomial on the *l.h.s* grows quickly as $\zeta \to \infty$ but that of the ratio of the polynomials in the *r.h.s* remains fixed since both of them are of the same degree. The integration is done along a given path, after a convenient value of the order N in n, and the Pade approximant is taken. We use the built-in Mathematica function called the PadeApproximant for taking the Pade approximants of polynomials. Now, a value for z is chosen in the original series, which we want to sum. After that, we put everything into the mathematica function NIntegrate, and try to evade the singularities on the positive real axis of the PI equation. We do this by first integrating from 0 to $\varepsilon.i$ followed by the integration from $\varepsilon.i$ to ∞. In this case, we have taken the values $z = 10$, $N = 100$ and $\varepsilon = 1$, which yielded:

$$\phi(10) \approx \int_{\mathcal{C}} \mathcal{B}^{[100]}[\phi](10) \approx \int_{\mathcal{C}} \frac{P^{[50]}[\phi](10)}{Q^{[50]}[\phi](10)} = 0.316207 - 4.16334 \times 10^{-16} i \quad (7.10)$$

We can show that there lies a non-perturbative ambiguity around the singularity, by computing the same integration again, except now with the value of ε taken to be -1, in which case the result would be $0.316207 + 4.16334 \times 10^{-16} i$.

7.3 Transseries Solution

In this section, we will show how to find an exact solution for the instanton action. We will also determine the coefficients of the one-instanton sector of the transseries.

Now, although the power series expansion of the perturbative expansion was of the order $z^{-5/2}$, but string theory requires that that the powers should be of the order $z^{-5/4}$ for the PI equation to be physically plausible. Thus, we now have our transseries ansatz to be

$$\phi(z) = z^{1/2} \sum_{n=0}^{\infty} \sigma_1^n e^{-nAz^{5/4}} z^{-\frac{5n\beta}{4}} \sum_{g=0}^{\infty} \phi_g^{(n)} z^{-5g/4} \quad (7.11)$$

After differentiating it twice with respect to z, we substitute it into the PI equation and set the scaling to be $x = z^{-5/4}$ so that the expression is clean. Now, we have

$$x^{-4/5} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \sigma_1^n e^{-nA/x} x^{n\beta} \sum_{g=0}^{\infty} \sum_{k=0}^{g} \phi_k^{(m)} \phi_{g-k}^{(n-m)} x^g$$
$$- \frac{1}{96} \sum_{n,g}^{\infty} \sigma_1^n \phi_g^(n) e^{-nA/x} x^{n\beta+g-2/5} \big[25(nA)^2 x^{-2/5} - 5nAx^{3/5}$$
$$- 10nA(-5n\beta - 5g + 2)x^{3/5} + (-5n\beta - 5g + 2)(-5n\beta - 5g - 2)x^{8/5} \big] = x^{-4/5}$$

Perturbative Sector

We now test whether we get our perturbative coefficients back to check our transseries ansatz. When we choose $n = 0$ and $g = 0$, we get $\phi_0^(0) = 1$, and

$$x^{-4/5} \sum_{g=0}^{\infty} \sum_{k=0}^{\infty} \phi_k^(0) \phi_{g-k}^{(0)} x^g - \frac{1}{96} \sum_{g=0} \phi_g^{(0)} x^{g+6/5} (-5g+2)(-5g-2) = x^{-4/5} \quad (7.12)$$

Let, $g \to g-2$ and we have,

$$\sum_{k=0}^{g} \phi_k^{(0)} \phi_{g-k}^{(0)} = \frac{1}{96} \phi_{g-2}^{(0)} (-5(g-2)+2)(-5(g-2)-2) \quad (7.13)$$

$$\phi_g^{(0)} = \frac{1}{192} \phi_{g-2}^{(0)} (25(g-2)^2 - 4) - \frac{1}{2} \sum_{k=1}^{g-1} \phi_k^{(0)} \phi_{g-k}^{(0)} \quad (7.14)$$

From this we can see off the recurrence formula that gives us the perturbative coefficients, which yields us

$$\phi_g^{(0)} = \begin{cases} \phi_{g/2}, & \text{even g} \\ 0, & \text{odd g} \end{cases} \quad (7.15)$$

which is just because of the rescaling of the perturbation parameter to $z^{-5/4}$ from $z^{-5/2}$.

One-instanton Sector

Now we will find an exact solution of the instanton action and a recurrence formula for the one-instanton coefficients. We choose n to be equal to 1 to get

$$\begin{aligned} &2\sigma_1 e^{-A/x} x^{\beta-4/5} \sum_{g=0}^{\infty} \sum_{k=0}^{g} \phi_k^{(0)} \phi_{g-k}^{(1)} x^g \\ &= \frac{1}{96} \sum_{g=0}^{\infty} \sigma_1 \phi_g^{(1)} e^{-A/x} x^{\beta+g-2/5} \big[25A^2 x^{-2/5} - 5Ax^{3/5} - 10A(-5\beta-5g+2)x^{3/5} \\ &+ (-5\beta-5g+2)(-5\beta-5g-2)x^{8/5}\big] \end{aligned}$$

Setting the powers of $x^{\beta-4/5}$, we can see that

$$2 = \frac{25}{96} A^2 \implies A = \pm \frac{8\sqrt{3}}{5} \quad (7.16)$$

which is the instanton action's exact solution. Now, we cancel the negative part since in the transseries, the exponential part is suppressed. However, had we tried to determine the two-parameters transseries, we would also take the negative value of A. Now, we do the same thing with the powers of $x^{\beta+1/\beta}$ to find the values of β. We get

$$2\phi_1^{(1)} - \frac{25}{96}A^2\phi_1^{(1)} = \frac{25A}{96}\big(1-2\beta\big)\phi_0^{(1)}, \tag{7.17}$$

and we find that $\beta = 1/2$.

With A and β at hand, we now proceed to reduce the expression into a recurrence formula for the one-instanton sector coefficients.

$$2\sigma_1 e^{-A/x}x^{-3/10}\sum_{g=0}^{\infty}\sum_{k=0}^{g}\phi_k^{(0)}\phi_{g-k}^{(1)}x^g$$
$$= \frac{1}{96}\sum_{g=0}^{\infty}\sigma_1\phi_g^{(0)}e^{-A/x}x^{1/10}x^g\Big[192x^{-4/10} + 80\sqrt{3}gx^{6/10} + (25g^2 + 25g + \frac{g}{4})x^{16/10}\Big]$$

Using similar calculations as to derive the recurrence relation for the perturbative sector, we obtain

$$\phi_{2g+1}^{(1)} = \frac{-1}{320\sqrt{3}(g+1)}\Big((100g^2+100g)\phi_g^{(1)} - 768\sum_{k=2}^{g+2}\phi_k^{(0)}\phi_{g+2-k}^{(1)}\Big) \tag{7.18}$$

Finally, the first two sectors of the transseries solution of the PI equation are as follows:

$$\phi^{(0)} = 1 - \frac{1}{48}x^2 - \frac{49}{4608}x^4 - \frac{1225}{55296}x^6 - ... \tag{7.19}$$
$$\phi^{(1)} = 1 - \frac{5}{64\sqrt{3}}x + \frac{75}{8192}x^2 - \frac{341329}{23592960\sqrt{3}}x^3 + ... \tag{7.20}$$

So far, we have used the transseries ansatz to find the instanton action and the different instanton sectors of the result. Now, we will take our results to show how resurgence theory gives us large-order relations to a high precision. [15, 18]

7.4 Large-order relations

In the cases where it is not possible to derive the exact solutions for the instanton action analytically, we resort to the large-order relations given by resurgence theory. Here, we will test its outcome by comparing it to the results given by the analytic solutions.

To begin, we present the large order relation

$$\phi_g^{(0)} \simeq \sum_{k=1}^{+\infty}\frac{S_1^k}{2\pi i}\frac{\Gamma(g-k\beta)}{(kA)^{g-k\beta}}\sum_{h=1}^{+\infty}\frac{\Gamma(g-k\beta-h+1)}{\Gamma(g-k\beta)}\phi_h^{(k)}(kA)^{h-1} \tag{7.21}$$

When we write out the first few terms of the above equation, we have

$$\phi_g^{(0)} \approx \frac{S_1}{2\pi i}\frac{\Gamma(g-\beta)}{A^{g-\beta}}\Big(\phi_1^{(1)} + \frac{A}{g-\beta-1}\phi_2^{(1)} + ...\Big) + \\ + \frac{S_1^2}{2\pi i}\frac{\Gamma(g-2\beta)}{(2A)^{g-2\beta}}\Big(\phi_1^{(2)} + \frac{2A}{g-2\beta-1}\phi_2^{(2)} + ...\Big) + \\ \frac{S_1^3}{2\pi i}\frac{\Gamma(g-3\beta)}{(3A)^{g-3\beta}}\Big(\phi_1^{(3)} + \frac{3A}{g-3\beta-1}\phi_2^{(3)} + ...\Big) + ...$$

After modifying the above equation to the PI equation, we get

$$\phi_{2g}^{(0)} \approx \frac{S_1}{2\pi i}\frac{\Gamma(2g-1/2)}{A^{2g-1/2}}\Big(\phi_0^{(1)} + \frac{A}{2g-3/2}\phi_1^{(1)} + ...\Big) \\ \frac{S_1^2}{2\pi i}\frac{\Gamma(2g-1)}{(2A)^{2g-1}}\Big(\phi_0^{(2)} + \frac{2A}{2g-2}\phi_1^{(2)} + ...\Big) + \mathcal{O}(3^{-g})$$

Now, we can find the instanton action and the different instanton sector coefficients from this expression. For that, we take the known sequence of the perturbative coefficients and set up a new sequence such as

$$\chi_g = \frac{\phi_{2(g+1)}^{(0)}}{4g^2\phi_{2g}^{(0)}} \tag{7.22}$$

We know,

$$\chi_g = \frac{1}{A^2}\Big(1 + \frac{1}{g} + \mathcal{O}\Big(\frac{1}{g^2}\Big)\Big) \tag{7.23}$$

This means that we need to see how $\chi_g^{-1/2}$ converges as $g \to \infty$. Since this convergence is very slow, the technique of Richardson transform is incorporated to make it faster. In case of the twenty fifth Richardson transform $\Big(\chi_1^{[25]}\Big)^{-1/2}$ we have a huge improvement in accuracy from the first coefficient of the original series, with an error of only 1.646×10^{-7} from the exact value of $\frac{8\sqrt{3}}{5}$, whereas previously the error was as large as 11.085. We can improve the precision further by taking the fifth Richardson transform's hundredth element $\Big(\chi_{100}^{[25]}\Big)^{-1/2}$, where the uncertainty is only 1.553×10^{-31}. Due to the fact that the previous errors are carried forward in every subsequent calculations, it is important that these uncertainties be as low as possible.

Now, let us find the first coefficient of the one-instanton

$$\frac{2\pi i A^{2g-1/2}}{\Gamma(2g-1/2)}\phi_{2g}^{(0)} \sim S_1\phi_0^{(1)}. \tag{7.24}$$

However, because we have the freedom to take $\phi_0^{(1)} = 1$, adapting the trasseries parameter with respect to it, and since we know that the Stokes factor $S_1 = -i\frac{3^{1/4}}{2\sqrt{\pi}}$, we immediately find the coefficient after that. We thus have

$$\frac{2g}{A}\left(\frac{2\pi i A^{2g-1/2}}{\Gamma(2g-1/2)}\phi_{2g}^{(0)} - \phi_0^{(1)}\right) \sim \phi_1^{(1)}. \tag{7.25}$$

When we plot the above equation, we can see that it converges as we approach $\phi_1^{(1)} = \frac{5}{64\sqrt{3}} = 0.045105....$ We repeat the same procedures to find all the other coefficients. First, a term is approximated and the its difference with that of the left had side is calculated, after which the next term is approximated and so on.

So far it was shown how we can find out the instanton action and the coefficients of the instanton sectors by using the large-order relation, which is itself a verified outcome of resurgence theory. This technique can be used to perform calculations in physical problems like that of the quantum anharmonic oscillator, etc. [15, 17, 45–47]

Chapter 8

Conclusion

Nonperturbative phenomena are very important elements of modern physics. This is apparent in many modern physical phenomena such as the blackhole evaporation., instantons in field theories, topological defects, spharelons, string theory, etc.

The perturbative expansion for all but the simplest systems are divergent. This is mainly because the expansion might be done about the wrong point. It might also be due to the fact that the correct spectrum of excitations is unknown. Moreover, non-perturbative effects are generic.

In this thesis, we explained the Borel summation technique along with the Borel-Pade approximation, and applied them to Painleve-I equation to solve it numerically. Apart from that, these techniques are being applied to all types of non-perturbative problems such as the blackhole information paradox, calculating amplitudes, topological field theories, etc. The key breakthrough in the twenty-first century physics will likely be in understanding how to calculate the nonperturbative terms in a perturbative expansion.

Appendix A

Richardson Transform

When we wish to converge a sequence quickly, we can use the technique of Richardson transform. Suppose we have the following series:

$$\chi_g = a_0 + \frac{a_1}{g} + \frac{a_2}{g^2} + \frac{a_3}{g^3} + \mathcal{O}\Big(\frac{1}{g^4}\Big). \tag{A.1}$$

First we generalize the problem that we are dealing with into a general structure like this. We have already shown this in 7.23. After that, we compute the N-th Richardson transformation of the sequence χ_g. We get:

$$\chi_g^{[N]} \equiv \sum_{k=0}^{N} (-1)^{N-k} \frac{(g+k)^N}{k!(N-k)!} \chi_{g+k} \tag{A.2}$$

Then Richardson transformation converges this series to the limit by simply throwing away the terms before the second last term, which raises the negative exponent of g in A.1. This results in the following:

$$\chi_g^{[N]} = a_0 + \frac{a_{N+1}}{g^{N+1}} + \mathcal{O}\Big(\frac{1}{g^{N+2}}\Big). \tag{A.3}$$

Now, since the number of terms in the series χ_g is limited, we can take only a limited numbers of Richardson transforms. For example, if the number of elements is K, we can only take $K - N$ elements for the N-th Richardson transform. [17, 48]

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