BRAC University Journal, Vol. II, No. 1, 2005, pp. 99-103

FINITE VOLUME METHODS FOR SOLVING HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS ON CURVED MANIFOLDS

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ABSTRACT

The natural mathematical arena to formulate conservation laws on curve manifolds is that of differential geometry. Ricci developed this branch of mathematics from 1887 to 1896. Subsequent work in differential geometry has made it an indespensible tool for solving in mathematical physics. The idea from differential geometry is to formulate hyperbolic conservation laws of scalar field equation on curved manifolds. The finite volume method is formulated such that scalar variables are numerically conserved and vector variables have a geometric source term that is naturally incorporated into a modified Riemann solver. The orthonormalization allows one to solve Cartesian Riemann problems that are devoid of geometric terms. The new method is tested via application to the linear wave equation on a curved manifold.

Key words: finite volume methods, curved manifolds, conservation law, wave propagation.

I. INTRODUCTION

In many physical applications, geometric considerations require the use of numerical grids that are not Cartesian. If the solution domain is a flat manifold but contains complicated internal or external boundaries, it is often possible to introduce a curvilinear grid that conforms to the boundaries. A fundamentally different situation arises when the solution domain is a curved manifold, such as the surface of a sphere of radius r embedded in $\Re^3$. In this situation, the curvature of the manifold modifies the underlying dynamics.

In this paper we are specifically interested in the solution of hyperbolic partial differential equations on curved manifolds. For curvilinear grids, a standard numerical approach is to update the hyperbolic system in Cartesian form, or sometimes referred to as strong conservation form, in order to avoid the introduction of source terms. However, this strategy does not work for general curved manifolds. Philosophically speaking, there are two approaches one can take in order to solve PDEs on a curved manifold $M \in \Re^n$. The first approach is not to directly solve the equations on the manifold, but instead to solve the equations in Cartesian form in $\Re^n$ with the help of a Lagrange multiplier to force the solution to remain on $M \in \Re^n$. The advantage is that all of the geometry is hidden in a relatively simple source term; the disadvantage is that one must solve on a higher-dimensional domain. The alternative to this strategy is to solve directly on the manifold. This removes the extra space dimension, but introduces geometric source terms and flux functions that explicitly vary in space. In particular, we present in this work a numerical scheme that is to solve hyperbolic partial differential equations on curved manifolds and the basic methods apply to equations in Cartesian coordinates. For simplicity, we focus specifically on manifolds that can be described by two independent coordinates.

II. CURVED MANIFOLDS

Differential geometry describes the geometric structure of a curved differentiable manifold, *M*. For example, a manifold *M* may represent the nearly spherical surface of a planet, a curved spacetime in relativity theory. A manifold is a set of points that looks locally Euclidean in that this set can be entirely covered by a collection of local coordinate mappings. Consider a two-dimensional curved manifold that is embedded in $\Re^3$. Let the coordinates $\left(x^1, x^2\right)$ be the coordinates on the manifold *M*. This coordinate system can be related to the standard Cartesian coordinate system, (*x, y, z*), through the transformations

$$\left.\begin{aligned} x &= \bar{x}\left(x^1, x^2\right) \\ y &= \bar{y}\left(x^1, x^2\right) \\ z &= \bar{z}\left(x^1, x^2\right) \end{aligned}\right\} \qquad (1)$$

A vector, ($\left(\mu^1, \mu^2\right)$, in contravariant form on the manifold *M* can be transformed to a vector, $\left(\mu^x, \mu^y, \mu^z\right)$, in Cartesian space through the Jacobian *J* in the following way:

$$\begin{bmatrix} \mu^x \\ \mu^y \\ \mu^z \end{bmatrix} = J\begin{bmatrix} \mu^1 \\ \mu^2 \end{bmatrix} = \begin{bmatrix} \frac{\delta \bar{x}}{\delta x^1} & \frac{\delta \bar{x}}{\delta x^2} \\ \frac{\delta \bar{y}}{\delta x^1} & \frac{\delta \bar{y}}{\delta x^2} \\ \frac{\delta \bar{z}}{\delta x^1} & \frac{\delta \bar{z}}{\delta x^2} \end{bmatrix}\begin{bmatrix} \mu^1 \\ \mu^2 \end{bmatrix} \qquad (2)$$

Therefore, the coordinate transformations directly give us a natural basis in which to represent vectors on *M*. We will refer to such a basis as a coordinate basis.

III. THE METRIC TENSOR

The metric tensor, $\overleftrightarrow{g}$, is a symmetric tensor that provides a measure of length on *M*. The metric relates true distances as measured in $\Re^3$ to the coordinate distances measured in the coordinate system of the manifold. In particular, the line element $ds^2 = (dx)^2 + (dy)^2 + (dz)^2$ in $\Re^3$ is related to dx^1 and dx^2 through

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \qquad (3)$$

The distance along a curve $C(\lambda)$ parameterized by λ from *C(a)* to *C(b)* is given by

$$L = \int_a^b \left| g_{\alpha\beta}\left(\frac{dx^\alpha}{d\lambda}\right)\left(\frac{dx^\beta}{d\lambda}\right) \right|^{1/2} d\lambda \qquad (4)$$

The surface area of $\Omega \subseteq M$ can be evaluated by computing the following integral:
Surface Area of

$$\Omega = \int_{\Omega\left(x^1, x^2\right)} \sqrt{g}\, dx^1 dx^2 \qquad (5)$$

where $\sqrt{g}$ is the square root of the determinant of the metric tensor. The components of the tensor $\Gamma^k_{\alpha\beta}$ are referred to as the Christoffel symbols or as connection coefficients. They involve spatial derivatives of the metric tensor $\overleftrightarrow{g}$. In particular, in a coordinate basis they can be written as follows

$$\Gamma^k_{\alpha\beta} = \frac{1}{2} g^{hk}\left(\frac{\delta}{\delta x^\beta} g_{h\alpha} + \frac{\delta}{\delta x^\alpha} g_{h\beta} - \frac{\delta}{\delta x^h} g_{\alpha\beta}\right) \qquad (6)$$

The Christoffel symbols play an important role in wave propagation on curved manifolds.

IV. CONSERVATION LAWS ON CURVED MANIFOLDS

Consider the flow of a substance with *M* state variables, $q\left(\vec{x}, t\right) \in \Re^M$. In the absence of any sources or sinks, the time rate of change of the integral of each state variable over the volume *V* is only dependent on the flux of that variable through the boundary, δV. Mathematically, this is expressed with the following integral conservation law

$$\frac{\delta}{\delta t}\int_V q\left(t, \vec{x}\right) dV + \int_{dV} \vec{f}(q).\vec{n} d\, s = 0 \qquad (7)$$

where $\vec{f}(q) \in \Re^{M\times 2}$ is the flux function, $\vec{n}$ is the outward pointing unit normal vector to δV, and *s* is the arclength parameterization of δV.The differential form of (7) is written

$$\frac{\delta}{\delta t} q + \vec{\nabla}.\vec{f}(q) = 0 \qquad (8)$$

In order to solve (8), it is necessary to express the conservation law in some basis. In flat space, equation (8) is often written in a Cartesian basis:

$$\frac{\delta}{\delta t} q + \frac{\delta}{\delta x} f^x(q) + \frac{\delta}{\delta y} f^y(q) = 0 \qquad (9)$$

In many interesting applications, we need to solve (8) on a smooth manifold *M* covered by a set of non-Cartesian basis vectors. In general, the coordinate basis representation of (8) takes the form of a balance law:

$$\frac{\delta}{\delta t} q + \frac{\delta}{\delta x^1} f^1\left(q, \vec{x}\right) + \frac{\delta}{\delta x^2} f^2\left(q, \vec{x}\right) = \psi_c\left(q, \vec{x}\right) \tag{10}$$

where the flux has now gained an explicit dependence on the spatial coordinates and a geometrically induced source term has appeared.

Let us consider a conservation law in which q is comprised of a scalar quantity density $\rho\left(\vec{x},t\right)$ and a vector quantity momentum, $\overline{\mu}\left(\vec{x},t\right)$:

$$q\left(\vec{x},t\right) = \begin{bmatrix} \rho \\ \vec{\mu} \end{bmatrix}$$

Let the corresponding flux function be

$$\vec{f}\left(q, \vec{x}\right) = \begin{bmatrix} \vec{U} \\ \overleftrightarrow{T} \end{bmatrix}$$

We can rewrite equation (8) in tensor form:

$$\frac{\delta}{\delta t}\rho + \frac{1}{\sqrt{g}}\frac{\delta}{\delta x^k}\left(\sqrt{g}\, U^k\right) = 0 \tag{11}$$

$$\frac{\delta}{\delta t}\mu^m + \frac{1}{\sqrt{g}}\frac{\delta}{\delta x^k}\left(\sqrt{g}\, T^{km}\right) = -\Gamma^m_{nk} T^{kn} \tag{12}$$

The focus of this paper is to develop an accurate numerical method for the solution of equations (11) and (12).

V. CARTESIAN FINITE VOLUME METHODS

Let us fist consider Cartesian conservation laws of the form (9). We construct a Cartesian grid with grid spacing Δx and Δy and let

$$x_i = x_m + \left(i - \frac{1}{2}\right)\Delta x$$

$$y_j = y_k + \left(j - \frac{1}{2}\right)\Delta y$$

where $\left(x_m, y_k\right)$ is the lower-left corner of the rectangular computational domain.

In each grid cell centered at $\left(x_i, y_j\right)$ and at each time t^n, a finite volume method will produce an approximation to the average of $q\left(\vec{x},t\right)$:

$$Q^n_{i\,j} \approx \frac{1}{\Delta x \Delta y}\int_{y_{j-1/2}}^{y_{j+1/2}}\int_{x_{i-1/2}}^{x_{i+1/2}} q\left(\xi,\eta,t^n\right) d\xi\, d\eta \tag{13}$$

If $\Delta t \equiv t^{n+1} - t^n$, then the time averaged flux of q from $t = t^n$ to $t = t^{n+1}$ across the cell interface located at $x_{i-1/2}$ and the cell interface located at $y_{j-1/2}$ can be written as

$$F^1_{i-1/2,j} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f^x\left(q\left(\overrightarrow{x}_{i-1/2,j}, \tau\right)\right) d\tau \tag{14}$$

$$F^2_{i,j-1/2} \approx \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f^y\left(q\left(\overrightarrow{x}_{i,j-1/2}, \tau\right)\right) d\tau \tag{15}$$

respectively. Conservation now tells us that $Q^{n+1}_{i\,j}$ must be equal to $Q^n_{i\,j}$ minus the flux of Q out of the grid cell centered at $\left(x_i, y_j\right)$:

$$Q^{n+1}_{i\,j} = Q^n_{i\,j} - \frac{\Delta t}{\Delta x}\left(F^1_{i+1/2,j} - F^1_{i-1/2,j}\right) - \frac{\Delta t}{\Delta y}\left(F^2_{i,j+1/2} - F^2_{i,j-1/2}\right) \tag{16}$$

All Cartesian finite volume methods can be written in the form (16). A full numerical scheme is obtained by choosing a specific strategy for constructing the numerical fluxes F^1 and F^2.

VI. THE SCALAR FIELD EQUATION

We apply the methods for solving hyperbolic equations to the scalar field equation on a curved manifold *M*. This equation models the propagation of acoustic waves in a thin membrane whose shape is given by the manifold *M*. The scalar field equation can be written as

$$\frac{\delta^2\varphi}{\delta t^2} - \vec{\nabla}.\left(\vec{\nabla}\varphi\right) = 0. \tag{17}$$

The pressure, $p\left(\vec{x},t\right)$, and the fluid velocity, $\vec{u}\left(\vec{x},t\right)$, can be obtained by taking appropriate temporal and spatial gradients of the scalar field:

$$p\left(\vec{x},t\right) = -\frac{\delta\varphi\left(\vec{x},t\right)}{\delta t} \tag{18}$$

$$\vec{u}\left(\vec{x},t\right) = \vec{\nabla}\varphi\left(\vec{x},t\right)$$

Replacing $\vec{\varphi}\left(\vec{x},t\right)$ in equation (17) by the above definitions and imposing that

$$\vec{\nabla}\left(\frac{\delta\varphi}{\delta t}\right) = \frac{\delta}{\delta t}\left(\vec{\nabla}\varphi\right) \tag{19}$$

results in the following system of balance laws for the pressure and the components of the fluid velocity:

$$\frac{\delta q}{\delta t} + \frac{1}{\sqrt{g}}\frac{\delta f^k}{\delta x^k} = \psi_c \tag{20}$$

one can easily show that the orthonormal equations form a strictly hyperbolic system of conservation laws.

Consider the propagation of sound waves on a surface given by $z = b(x,y)$, where *x*, *y*, and *z* are the standard Cartesian coordinates. In this case, the transformation to surface coordinates is quite simple since x and y parameterizes the surface:

$$\begin{aligned} x &= \bar{x}\left(x^1, x^2\right) = x^1 \\ y &= \bar{y}\left(x^1, x^2\right) = x^2 \\ z &= \bar{z}\left(x^1, x^2\right) = b\left(x^1, x^2\right). \end{aligned} \tag{21}$$

The surface that defines *M* for a Gaussian dip with the functional form

$$b(x,y) = \alpha \exp\left(\frac{-\left((x-x_0)^2 + (y-y_0)^2\right)^{1/2}}{w^2}\right) \tag{22}$$

The point $\left(x_0, y_0\right)$ define the location in the coordinate plane of the minimum value surface. Figures 1 and 2 show this surface embedded in a three-dimensional Euclidean space with different values of the amplitude α . The Euclidean space in which the surface is embedded has the Cartesian metric $\left\lfloor g_{\alpha\beta} \right\rfloor = dia(1,1,1)$.

The Jacobian matrix that transforms a vector on the surface of *M* into a Cartesian vector is

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \dfrac{\delta b}{\delta x^1} & \dfrac{\delta b}{\delta x^2} \end{bmatrix}$$

The metric and the square root of the determinant of the metric for this coordinate system are

$$\overleftrightarrow{g} = \begin{bmatrix} 1 + \left(\dfrac{\delta b}{\delta x^1}\right)^2 & \dfrac{\delta b}{\delta x^1}\dfrac{\delta b}{\delta x^2} \\ \dfrac{\delta b}{\delta x^1}\dfrac{\delta b}{\delta x^2} & 1 + \left(\dfrac{\delta b}{\delta x^2}\right)^2 \end{bmatrix}$$

$$\sqrt{g} = \sqrt{1 + \left(\frac{\delta b}{\delta x^1}\right)^2 + \left(\frac{\delta b}{\delta x^2}\right)^2}$$

respectively.

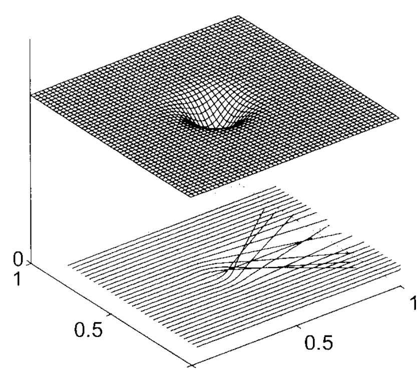

Figure-1: The surface that defines *M* for geodesics initialized as right-going curves along the left boundary is projected down onto a coordinate plane. The depression in the surface causes the geodesics to converge.

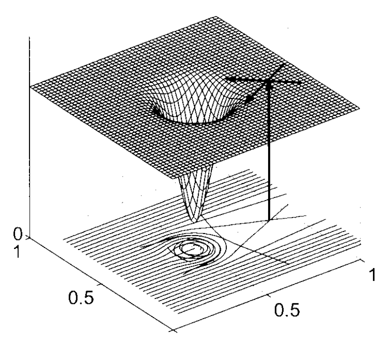

Figure-2: The dip gets larger in amplitude, the geodesics converge more and eventually wrap around the center of the surface.

The fact that the off-diagonal terms of this metric are nonzero implies that the coordinate lines on the surface are skewed. Figure-2 shows the skewing of coordinating lines that occurs in the region near the dip.

The metric $g_{\alpha\beta}$ derived above defines the geometry of the two-dimensional manifold on which we solve equation (17). One way to get a feel for the geometry defined by $g_{\alpha\beta}$ is to compute geodesics on the surface. Geodesics are curves that represent the "straightest possible lines" in *M*. In figgur-1 shows geodesics that are initialized along the left boundary to be right-going. These geodesics are projected down to the coordinate plane for visualization purposes. The figure shows that the geodesics that avoid the dip begin to converge and their paths are altered. If the amplitude of the surface is increased as in Figure-2. the direction of geodesic curves in *M* can be changed entirely.

The geodesics shown in Figure-1 and 2 resemble the paths of photons traveling past large clusters of galaxies. Due to Einstein's general theory of relativity, a cluster of galaxies will warp the fabric of spacetime similar to the surface shown in the figures. Clusters with more mass will have larger dips. Light from distant objects will arrive at the cluster traveling in straight and parallel paths much like the geodesics shown in the figures. However, when the light encounters the curvature of the dip produced by the cluster, light rays converge as they do in the figures. Figure-2 shows the intersection of two light rays that have passed by the dip. An observer at this intersection would see the same object coming from two different directions. This phenomena is called gravitational lensing, and is observed by astronomers.

7. CONCLUSIONS

We have presented in this paper a finite volume method for hyperbolic partial differential equations on curved manifolds. The equation is solved in a coordinate basis resulting from the choice of coordinates on the manifold. The claim is verified by using the algorithm to compute the solution to the acoustic equations on a curved manifold. The Fortran code that is used to obtain the solution by using the standard clawpack software package.

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