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MULTIPARAMETER BIFURCATION AND STABILITY OF SOLUTIONS AT A DOUBLE EIGENVALUE

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ABSTRACT

This paper deals with some problems of bifurcation theory for general non-linear eigenvalue problem for 2-dimensional parameter space. An explicit analysis of the bifurcation for 2-dimensional parameter space is done and the structure of the non-trivial solution branches of the bifurcation equation near origin is given. Since the study of the bifurcation problem is closely related to change in the qualitative behaviour of the systems, and to exchange of stability, analysis of the stability of the bifurcating solutions is done here. It is proved that the stability of the bifurcating solutions is determined, to the lowest non-vanishing order, by the eigenvalues of the Fréchet derivative of the reduced bifurcation equation.

Key words: Non-linear eigenvalue problem, Bifurcating solutions, Linearised operator, Lyapunov-Schmidt method, Stability.

I. INTRODUCTION

In this paper we will establish the structure of the bifurcating solutions and discuss the stability of the solutions of the equation

$$F(\lambda, x) := L(\lambda)x + N(\lambda, x) = 0 \qquad (1)$$

near the origin, where F is a C^m, $m \geq 2$ mapping, $L(\lambda)$ is a linear operator on a Banach space for $\lambda \in \nabla^2$ and $N(\lambda, x)$ is a non-linear operator with

$$N(\lambda, 0) = 0, \quad D_x N(0,0) = 0,$$

where $D_x N(0,0)$ is the Fréchet derivative of N with respect to *x* at $(0,0)$. We assume that the principle of linearised stability holds i.e.,

> "*a solution x of (1) is stable if all the eigenvalues of the derived operator* $D_x F(\lambda, x)$ *have negative real parts and x is unstable if some of the eigenvalues have positive real parts".*

Thus the stability of a solution of Eq. (1) is determined by the eigenvalues of the linearised operator $D_x F(\lambda, x)$ We will prove that the stability of the bifurcating solutions is determined, to the lowest non-vanishing order, by the eigenvalues of the Fréchet derivative of the reduced bifurcation equation.

The known results are quite complete for a 1-dimensional parameter space i.e., for real λ, see [2,3,4]. In [2], McLeod and Sattinger have shown that, at a double eigenvalue, the stability of the bifurcating solutions is determined by the eigenvalues of the Fréchet derivative of the bifurcation equation. For a 1-dimensional parametric space ∇, this result has been extended by Sattinger in [4] to multiple eigenvalues i.e., when the dimension of the null space of the linearised operator $D_x F(0,0)$ is $n \geq 1$.

We will extend the results given by McLeod and Sattinger in [1] to the case of a 2-dimensional parameter space at a double eigenvalue.

Some preliminaries are given in Section 2. The structure of the bifurcating solutions is given in Sections 3. The Fréchet derivative of the reduced bifurcation equation is evaluated in Section 4. In Section 5, we have discussed two examples to give the precise information about the stability of the bifurcating solution using CAS (Mathematica).

II. PRELIMINARIES

Let $X \subseteq Y$ be Banach spaces and X^*, Y^* be the dual spaces of X, Y respectively. Let us suppose that for 7λ7 and 7x7 sufficiently small, N(λ,x) has the form

$$N(\lambda, x) = B(x, x) + h(\lambda, x),$$

where $B(x, z)$ is a bilinear operator from $X \times X \to Y$, independent of λ and $h(\lambda, x)$ is a C^m, $m \geq 2$ mapping from $\nabla^2 \times X$ to Y with the property that

$$h(\lambda, \varsigma x) = \varsigma^2 h_1(\lambda, \varsigma, x),$$

where h_1 and $\partial h_1/\partial x$ tend to zero as $\varsigma, \lambda \to 0$. Also suppose that,

$$L(\lambda) = L_0 + \lambda_1 L_1 + \lambda_2 L_2,$$

where L_0 is a Fredholm operator of index zero with 2-dimensional null-space spanned by φ_1, φ_2 and L_1, L_2 are linear operators from X to Y. Since $L_0 : X \to Y$, the adjoint operator L_0^* maps from Y^* to X^*. Note that since $X \subseteq Y$ we have $Y^* \subseteq X^*$. Let us suppose that the null space $N(L_0)$ and range $R(L_0)$ of L_0 have zero intersection. Then as a consequence of Hann-Banach Theorem, there exists a set of linear functionals $\varphi_1^*, \varphi_2^* \in Y^*$ such that

$$\langle \varphi_\iota, \varphi_j^* \rangle = \partial_{\iota j},$$

$$\langle \psi, \varphi_j^* \rangle = 0, \; y \in R(L_0), \quad i, j = 1,2,$$

where $\langle .,. \rangle$ is a duality pairing between a Banach space and its dual space [5, Theorem 2.7 on page 129]. Thus we have, $\langle L_0 \psi, \varphi_j^* \rangle = 0$, for all $\psi \in X$, $j = 1,2$. Also there exist two closed subspaces X_0, Y_0 of X, Y respectively such that

$$X = X_0 \oplus N(L_0),$$

$$Y = Y_0 \oplus R(L_0).$$

Then L_0 is an isomorphism from X_0 to $R(L_0)$. Let Q be the projection of Y onto $R(L_0)$ given by,

$$Qy = y - \langle y, \varphi_1^* \rangle \varphi_1 - \langle y, \varphi_2^* \rangle \varphi_2, \quad y \in Y.$$

Then the projection (I–Q) from Y onto Y_0 is given by

$$(I - Q)y = \langle y, \varphi_1^* \rangle \varphi_1 + \langle y, \varphi_2^* \rangle \varphi_2, \quad y \in Y.$$

III. ANALYSIS OF THE BIFURCATION EQUATION

In this section we will do an explicit analysis of the bifurcation equation. We will also give the structure of the non-trivial solution branches of Eq. (1) near the bifurcation point $(0,0) \in \nabla^2 \times X$. By the Lyapunov–Schmidt method, we can reduce the infinite-dimensional problem Eq. (1) to a 2-dimensional equation

$$(I - Q)F(\lambda, v + x_0(\lambda, v)) = 0, \tag{2}$$

where $v \in N(L_0)$ and $x_0(\lambda, v) \in X_0$ is the unique solution of the equation

$$QF(\lambda, v + x_0(\lambda, v)) = 0,$$

for (λ, v) in a sufficiently small neighbourhood of (0,0) with $x_0(\lambda, 0) = 0$, $D_v x_0(0,0) = 0$. So by Taylor's theorem we have

$$x_0(\lambda, v) = O(\|v\|(\|v\| + \|\lambda\|)), \quad \|v\|, \|\lambda\| \to 0.$$

Putting $v = u_1\varphi_1 + u_2\varphi_2$, $u = (u_1, u_2) \in \nabla^2$, we find that Eq. (2) is equivalent to the system of equations,

$$f_i(\lambda, u) := u_1 \langle (\lambda_1 L_1 + \lambda_2 L_2)\varphi_1, \varphi_i^* \rangle + u_2 \langle (\lambda_1 L_1 + \lambda_2 L_2)\varphi_2, \varphi_i^* \rangle + q_i(u) + G_i(\lambda, u) = 0,$$

for i=1,2, where q_i: $\nabla^2 \to \nabla$ is a quadratic function given by

$$q_i(u) = \langle B(u_1\varphi_1 + u_2\varphi_2, u_1\varphi_1 + u_2\varphi_2), \varphi_i^* \rangle$$

and G_i: $\nabla^2 \times \nabla^2 \to \nabla$ is defined by

$$|G_i(\lambda, u)| = o(\|u\|^2 + \|u\| \, \|\lambda\|).$$

Hence, we get a map $f_i : \nabla^2 \times \nabla^2 \to \nabla$ defined by

$$f(\lambda, u) := \lambda_1 M_1 u + \lambda_2 M_2 u + q(u) + G(\lambda, u),$$

where,

$$M_j = \begin{pmatrix} \langle L_j\varphi_1, \varphi_1^* \rangle & \langle L_j\varphi_2, \varphi_1^* \rangle \\ \langle L_j\varphi_1, \varphi_2^* \rangle & \langle L_j\varphi_2, \varphi_2^* \rangle \end{pmatrix}, \; j = 1,2,$$

are 2□2 real matrices, $q(u) = \begin{pmatrix} q_1(u) \\ q_2(u) \end{pmatrix}$ is a homogeneous quadratic mapping from ∇^2 to ∇^2 and $G(\lambda, u)$ is an operator from $\nabla^2 \times \nabla^2$ to ∇^2 satisfying $\|G(\lambda, u)\| = o(\|u\|^2 + \|u\|\|\lambda\|)$ Thus our original problem Eq. (1) is reduced to the 2-dimensional problem

$$f(\lambda, u) = \lambda_1 M_1 u + \lambda_2 M_2 u + q(u) + G(\lambda, u) = 0 \tag{3}$$

with $f(\lambda, 0) = 0$ for all λ in a neighbourhood of $(0,0) \in \nabla^2 \times \nabla^2$.

Now suppose that the following spanning condition holds,

(C) $\forall \varphi \in N(L_0)$ with $\|\varphi\| = 1$, $\text{span}[M_1\varphi, M_2\varphi, q(\varphi)] = \nabla^2$.

This condition holds generically in the sense that the set of matrices (L_1, L_2, q) satisfying (C) is open and dense in the set of all such collections.

We are now interested in finding the non-trivial solutions of the equation Eq. (3), using the spanning condition (C). Let us denote S^1 as the unit circle in ∇^2. Then for each $\varphi \in S^1$, let us define,

$$N(\varphi) = \{(\nu, t) \in \nabla^2 \times \nabla : \nu_1 M_1 \varphi + \nu_2 M_2 \varphi + tq(\varphi) = 0\},$$
$$M(\varphi) = N(\varphi)^{\perp} \subset \nabla^2 \times \nabla.$$

Hence $N(\varphi)$ and $M(\varphi)$ are 1 and 2 dimensional subspaces of $\nabla^2 \times \nabla$ respectively. Define,

$$E_N = \{(\nu, t, \varphi) \in \nabla^2 \times \nabla \times S^1 : (\nu, t) \in N(\varphi)\},$$
$$E_{NM} = \{(\nu, t, \varphi, \mu, r) \in \nabla^2 \times \nabla \times S^1 \times \nabla^2 \times \nabla : (\nu, t) \in N(\varphi), (\mu, r) \in M(\varphi)\}.$$

E_N and E_{NM} are manifolds. With these notations we get the following result:

Theorem 1: Let the assumption on L_0 and condition (C) hold, then there exist unique C^{m-1} mappings $r : E_N \to \nabla$, $\mu : E_N \to \nabla^2$ such that the point

$$(\lambda(\nu, t, \varphi), u(\nu, t, \varphi)) := (\nu + \mu(\nu, t, \varphi), (t + r(\nu, t, \varphi))\varphi)$$

solves the Eq. (3). In addition, all non-trivial solutions of Eq. (3) are of this form. Also,

$$D_{(\nu,t)} r(0,0,\varphi) = 0, \quad D_{(\nu,t)} \mu(0,0,\varphi) = 0.$$

Proof. Let $u = \gamma\phi$, $\gamma \in \nabla$. Then substituting this value in Eq. (3) we get (after dividing by γ),

$$g_1(\lambda, \gamma, \varphi) := \lambda_1 M_1 \varphi + \lambda_2 M_2 \varphi + \gamma q(\varphi) + G_1(\lambda, \gamma),$$

where $\|G_1(\lambda, \gamma)\| = o(\gamma + \|\lambda\|)$. Now for each $\varphi \in S^1$, decompose $(\lambda,\gamma) \in \nabla^2 \times \nabla$ as,

$$(\lambda, \gamma) = (\nu + \mu, t + r), \quad (\nu, t) \in N(\varphi), (\mu, r) \in M(\varphi).$$

Then consider the mapping g: $E_{NM} \to \nabla^2$ defined by,

$$g(\nu, t, \varphi, \mu, r) = g_1(\nu + \mu, t + r, \varphi)$$
$$= (\nu_1 + \mu_1) M_1 \varphi + (\nu_2 + \mu_2) M_2 \varphi + (t + r) q(\varphi) + G_1(\nu + \mu, t + r).$$

Note that, $g(0,0,\phi,0,0) = 0$, and

$$D_{(\mu,r)} g(0,0,\varphi,0,0)(\mu', r') = \mu'_1 M_1 \varphi + \mu'_2 M_2 \varphi + r' q(\varphi), \quad (\mu', r') \in M(\varphi).$$

The operator $D_{(\mu,r)} g(0,0,\varphi,0,0) : M(\varphi) \to \nabla^2$ is an isomorphism for each $\varphi \in S^1$. Hence by the Implicit Function Theorem and by the compactness of S^1 there exists a neighbourhood U of $E_0 = \{0\} \times \{0\} \times S^1$ in E_N and unique C^{m-1} functions $r : U \to \nabla$, $\mu : U \to \nabla^2$ such that

$$r(0,0,\varphi) = 0, \quad \mu(0,0,\varphi) = 0 \text{ and}$$
$$g(\nu, t, \varphi, \mu(\nu, t, \varphi), r(\nu, t, \varphi)) = 0, \quad \forall\, (\nu, t, \varphi) \in U,$$

i.e.,

$$\sum_{i=1}^{2} (\nu_i + \mu_i(\nu, t, \varphi)) M_i \varphi + (t + r(\nu, t, \varphi)) q(\varphi) + G_1(\nu + \mu(\nu, t, \varphi), t + r(\nu, t, \varphi)) = 0 \tag{4}$$

This completes the proof of the first part of Theorem 1.

Now differentiating Eq. (4) with respect to (ν,t) at $(0,0,\phi)$ we get, for all $(\nu',t') \in N(\phi)$,

$$\sum_{i=1}^{2} D_{(\nu,t)} \mu_i(0,0,\varphi)(\nu', t') M_i \varphi + D_{(\nu,t)} r(0,0,\varphi)(\nu', t) q(\varphi) = 0$$

($(\nu',t') \in N(\phi)$, so $\nu'_1 M_1 \varphi + \nu'_2 M_2 \varphi + t' q(\varphi) = 0$).Bu t$(\mu,r) \in M(\phi)$and hence $(D_{(\nu,t)} \mu(0,0,\varphi), D_{(\nu,t)} r(0,0,\varphi)) \in M(\varphi)$. Also, since $M(\varphi) = N(\varphi)^{\perp}$, the above equation can only hold if

$$D_{(\nu,t)} \mu(0,0,\varphi)(\nu', t') = 0, \quad \forall\, (\nu', t') \in N(\varphi)$$
$$D_{(\nu,t)} r(0,0,\varphi)(\nu', t') = 0, \quad \forall\, (\nu', t') \in N(\varphi)$$
$$\Rightarrow \quad D_{(\nu,t)} \mu(0,0,\varphi) = 0, \quad D_{(\nu,t)} r(0,0,\varphi) = 0.$$

This completes the proof.

Remark 1

These results show that if $(\nu, t) \in N(\varphi)$ with $t \neq 0$ then there is a curve of non-trivial solutions of the form

$$\{(\lambda(\alpha\nu, \alpha t, \varphi), u(\alpha\nu, \alpha t, \varphi)) : |\alpha| < \varepsilon\},$$

and this curve is tangential (at (0,0)) to the ray $\{\alpha(\nu, t) : \alpha \in \nabla\}$ of zeros of Q where

$$Q(\lambda, u) = \lambda_1 M_1 u + \lambda_2 M_2 u + q(u).$$

In this sense the solution set of the general problem is similar to the solution set of $Q(\lambda, u) = 0$. If t=0, we may get $N(\varphi) = \{0\}$and (C) does not hold for roughly half of the pairs (M_1,M_2) (for details, see [1]).

Remark 2

It can happen that all the solutions found by Theorem 1 are trivial. For example, consider the system

$$\lambda x + \mu y = 0$$
$$x^2 + y^2 = 0.$$

Here the only solutions are trivial ones.

Now by Taylor's expansion, for each $\varphi \in S^1$, the non-trivial solutions of the equation (3) can be written as:

$$\lambda(\nu, t) = \nu + O\left(t^2 + \|\nu\|^2\right),$$
$$u(\nu, t, \varphi) = \varphi\left(t + O\left(t^2 + \|\nu\|^2\right)\right), \tag{5}$$

in a neighbourhood of $\{0\}\times\{0\}\times S^1$ in $\nabla^2\times \nabla\times S^1$. Now for fixed $\varphi\in S^1$, let us take a fixed normalised vector $\left(\hat{v}_1, \hat{v}_2, \hat{t}\right) \in N(\varphi)$ with $\hat{t} \neq 0$. Then the solutions (5) have the form

$$\lambda(s) = s\hat{v} + O\left(s^2\right),$$
$$u(s, \varphi) = s\hat{t}\varphi + O\left(s^2\right), \tag{6}$$

in a neighbourhood of $s = 0 \in \nabla$ for each $\varphi\in S^1$ (i.e., we look at a curve of solutions tangential to a ray in $\nabla^2\times\nabla$ space). Thus substituting in (3) and dividing by s^2 and $\hat{t}$ we obtain the system of equations of the form

$$\hat{v}_1 M_1\varphi + \hat{v}_2 M_2\varphi + \hat{t}q(\varphi) + O(s) = 0\,. \tag{7}$$

Letting $s\to 0$ we get the *reduced bifurcation equation*

$$\hat{v}_1 M_1\varphi + \hat{v}_2 M_2\varphi + \hat{t}q(\varphi) = 0\,. \tag{8}$$

If the Fréchet derivative of this equation with respect to φ at a solution φ^0 is nonsingular, then the Implicit Function Theorem implies that there exists a unique C^{m-1} solution φ (s) for |s| sufficiently small, with $\varphi(0)=\varphi^0$, of (8).

Now by direct calculation, we get that the Fréchet derivative of the reduced bifurcation equation (8) with respect to φ, evaluated at a solution φ, is 2×2 matrix given by,

$$\hat{v}_1 M_1 + \hat{v}_2 M_2 + 2\hat{t}\left(\left\langle B(\phi, \phi_i), \phi_j^*\right\rangle\right)_{ij}, \quad i, j = 1,2.$$

In the next section we will prove that the stability of the bifurcating solutions is determined, to the first order in s, by the eigenvalues of the Fréchet derivative of equation (8), i.e., the stability is determined by the reduced bifurcation equation, so we do not have to look at the eigenvalues of the linearised operator of the original equation (1).

IV. STABILITY OF THE BIFURCATING SOLUTIONS

Let (λ,x) be a solution of Eq. (1), i.e., $L(\lambda)x + N(\lambda, x) = 0$ and the derived operator associated with this solution is,

$$A(\lambda, x) = L(\lambda) + D_x N(\lambda, x), \tag{9}$$

where $D_x N(\lambda, x)$ is a linear operator from X to Y given by,

$$D_x N(\lambda, x)\tilde{x} = 2B(x, \tilde{x}) + D_x h(\lambda, x)\tilde{x}, \quad \tilde{x} \in X.$$

Substituting the solution Eq. (6) into Eq. (9) we get the operator $A_1 : \nabla\times S^1 \to L(X, Y)$ defined by

$$A_1(s, \varphi) = A\left(s\hat{v} + O\left(s^2\right), s\hat{t}\varphi + O\left(s^2\right)\right)$$
$$= L_0 + s\tilde{L}(\varphi) + R(s, \varphi), \tag{10}$$

where,

$$\tilde{L}(\varphi) = \hat{v}_1 L_1 + \hat{v}_2 L_2 + 2\hat{t}B(\varphi,.),$$

and $R(s, \varphi)$ is an operator satisfying $\|R(s, \varphi)\| = O\left(s^2\right)$ as |s| tends to zero. So we will study the eigenvalues of the operator $A_1(s, \varphi)$ in a neighbourhood of $s = 0 \in \nabla$ for all $\varphi \in S^1$. Note that for s = 0, the operator $A_1(0, \varphi)$ is a Fredholm operator of index zero with 2-dimensional null space. In the following Theorem we will prove that the stability of the bifurcating solution can be reduced to the study of the eigenvalues of a 2×2 matrix (this theorem is proved by McLeod and Sattinger [2] for real parameter λ. We will prove it for two dimensional parameter space i.e, for $\lambda\in \nabla^2$).

Theorem 2: For |s| sufficiently small and for each $\varphi \in S^1$ the operator Eq. (10) has a uniquely defined two-dimensional invariant subspace spanned by the vectors of the form

$$\tilde{\varphi}_i(s, \varphi) = \varphi_i + O(s) \quad (i = 1,2) \tag{11}$$

and there exists a 2×2 matrix $C(s, \varphi)=(c_{ij}(s, \varphi))$ with

$$c_{ij}(0, \varphi) = \left\langle \tilde{L}(\varphi)\varphi_i, \varphi_j^*\right\rangle, \quad (i, j = 1,2) \tag{12}$$

such that

$$A_1(s, \varphi)\tilde{\varphi}_i(s, \varphi) = s\sum_{j=1}^{2} c_{ij}\tilde{\varphi}_j \tag{13}$$

where $c_{ij} = c_{ij}(s, \varphi)$.

Remark 3

Theorem 2 shows that, for sufficiently small |s|, the eigenvalues of $sC(s, \varphi)$ are eigenvalues of $A_1(s, \varphi)$. So, if we suppose that all the eigenvalues of $A_1(s, \varphi)$ (other than those two of $sC(s,\varphi)$) have negative real parts, for |s| sufficiently small, then by the principle of linearised stability, the solution $x(s,\varphi)=s\hat{t}\varphi+O(s^2)$ for fixed $\varphi\in S^1$, of Eq. (1) is *stable*, if both eigenvalues of sC(s, φ) have negative real parts. If, for |s| sufficiently small and for fixed $\varphi\in S^1$, at least one of the eigenvalues of sC(s, φ) is positive, the solution is *unstable*.

Proof. (of Theorem 2) We will use the Implicit Function Theorem to show that for |s| sufficiently small and for all $\varphi\in S^1$, the equation (13) has a solution $\widetilde{\varphi}_i(s,\varphi), c_{ij}(s,\varphi), i,j=1,2$ satisfying (11) and (12). We look for basis vectors $\widetilde{\varphi}_i$ in the form

$$\widetilde{\varphi}_i=\varphi_i+\psi_i \tag{14}$$

where $\left\langle\psi_i,\varphi_j^*\right\rangle=0, i,j=1,2.$ Substituting (14) in (13) we get

$$L_0\psi_i+s\widetilde{L}(\varphi)\widetilde{\varphi}_i+R(s,\varphi)\widetilde{\varphi}_i-s\sum_{j=1}^{2}c_{ij}\widetilde{\varphi}_j=0,\ (i=1,2)\,. \tag{15}$$

This equation is equivalent to the equations (for i =1,2)

$$Q\left[L_0\psi_i+s\widetilde{L}(\varphi)\widetilde{\varphi}_i+R(s,\varphi)\widetilde{\varphi}_i-s\sum_{j=1}^{2}c_{ij}\widetilde{\varphi}_j\right]=0 \tag{16}$$

$$(I-Q)\left[L_0\psi_i+s\widetilde{L}(\varphi)\widetilde{\varphi}_i+R(s,\varphi)\widetilde{\varphi}_i-s\sum_{j=1}^{2}c_{ij}\widetilde{\varphi}_j\right]=0 \tag{17}$$

From (16) we get, for i =1,2,

$$L_0\psi_i+sQ\widetilde{L}(\varphi)\widetilde{\varphi}_i+QR(s,\varphi)\widetilde{\varphi}_i-s\sum_{j=1}^{2}c_{ij}\psi_j=0$$

where we have used the fact that

$$Qc_{ij}\widetilde{\varphi}_j=c_{ij}Q\widetilde{\varphi}_j=c_{ij}\psi_j.$$

For each i = 1, 2, let us define a map Φ_i: $X_0\times\nabla^4\times\nabla\times S^1\to R(L_0)$ by,

$$☞_i(\psi_i,C,s,\varphi)=L_0\psi_i+sQ\widetilde{L}(\varphi)\widetilde{\varphi}_i+\ QR(s,\varphi)\widetilde{\varphi}_i-s\sum_{j=1}^{2}c_{ij}\psi_j$$

where C denotes the matrix (c_{ij}) which we regard as an element of ∇^4. Now from (16) we obtain

$$s\left\langle\widetilde{L}(\varphi)\widetilde{\varphi}_i,\varphi_j^*\right\rangle+\left\langle R(s,\varphi)\widetilde{\varphi}_i,\varphi_j^*\right\rangle-sc_{ij}=0,\quad i,j=1,2.$$

Dividing by s we obtain the equations

$$\begin{aligned}☞_{ij}(\psi_i,C,s,\varphi)&:=\left\langle\widetilde{L}(\varphi)(\varphi_i+\psi_i),\varphi_j^*\right\rangle+\\&s^{-1}\left\langle R(s,\varphi)\widetilde{\varphi}_i,\varphi_j^*\right\rangle-c_{ij}=0,\end{aligned} \tag{18}$$

where Φ_{ij}: $X_0\times\nabla^4\times\nabla\times S^1\to\nabla$, for i,j = 1,2. Also for s = 0, we have

$$☞_{ij}(\psi_i,C,0,\varphi)=\left\langle\widetilde{L}(\varphi)(\varphi_i+\psi_i),\varphi_j^*\right\rangle-c_{ij},$$

(since $\|R(s,\varphi)\|=O(s^2)$).

Now consider the map Φ: $X_0\times X_0\times\nabla^4\times\nabla\times S^1\to R(L_0)\times R(L_0)\times\nabla^4$ defined by,

$$☞(\psi_1,\psi_2,C,s,\varphi)=\begin{pmatrix}☞_1(\psi_1,C,s,\varphi)\\☞_2(\psi_2,C,s,\varphi)\\☞_{11}(\psi_1,C,s,\varphi)\\☞_{12}(\psi_1,C,s,\varphi)\\☞_{21}(\psi_2,C,s,\varphi)\\☞_{22}(\psi_2,C,s,\varphi)\end{pmatrix}.$$

Thus we get equivalent system of equations

$$☞(\psi_1,\psi_2,C,s,\varphi)=0 \tag{19}$$

to the equation (15), of six unknowns ψ_i, (i = 1,2), and C = (c_{ij}), depending on s and φ. We will apply the Implicit Function Theorem to solve the equation (19). Now at $s=\psi_i=0$ and $c_{ij}=\left\langle\widetilde{L}(\varphi)\varphi_i,\varphi_j^*\right\rangle$, i, j = 1, 2 we have

$$☞\left(0,0,\left(\left\langle\widetilde{L}(\varphi)\varphi_i,\varphi_j^*\right\rangle\right),0,\varphi\right)=0\text{, for all }\varphi\in S^1.$$

Also the Fréchet derivative of Φ with respect to ψ_i and c_{ij} at $s=\psi_i=0$ and $c_{ij}=\left\langle\widetilde{L}(\varphi)\varphi_i,\varphi_j^*\right\rangle$ is,

$$\begin{pmatrix}L_0&&&&&\\&L_0&&&0&\\&&-1&&&\\&&&-1&&\\&0&&&-1&\\&&&&&-1\end{pmatrix}$$

which from $X_0\times X_0\times\nabla^4$ to $R(L_0)\times R(L_0)\times\nabla^4$ is non-singular since $L_0: X_0\to R(L_0)$ is bijective. So by Implicit Function Theorem there exists a neighbourhood of $s=0\in\nabla$ for all $\varphi\in S^1$ and unique C^{m-1} functions $\psi_i(s,\varphi)$, $c_{ij}(s,\varphi)$ such that $\psi_i(0,\varphi)=0$, $c_{ij}(0,\varphi)=\left\langle\widetilde{L}(\varphi)\varphi_i,\varphi_j^*\right\rangle$, for each i,j = 1, 2,

and

$$☞(\psi_1(s,\varphi),\psi_2(s,\varphi),C(s,\varphi),s,\varphi)=0,$$

i.e,

$$A_1(s,\varphi)(\varphi_i+\psi_i(s,\varphi))=s\sum_{j=1}^{2}c_{ij}(s,\varphi)(\varphi_j+\psi_j(s,\varphi)),$$

with $\psi_j(s,\varphi) = O(s)$, $c_{ij}(0, \varphi) = \left\langle \tilde{L}(\varphi)\varphi_i, \varphi_j^* \right\rangle$. Hence the proof of the lemma is complete.

Thus the stability of the bifurcating solutions in the infinite dimensional space is reduced, to the first order in s, to the study of the eigenvalues of the 2×2 matrix

$$
\begin{aligned}
C(0,\varphi) &= \left(\left\langle \tilde{L}(\varphi)\varphi_i, \varphi_j^* \right\rangle\right)_{ij} \\
&= \left(\hat{v}_1\left\langle L_1\varphi_i, \varphi_j^*\right\rangle + \hat{v}_2\left\langle L_2\varphi_i, \varphi_j^*\right\rangle + 2\hat{t}\left\langle B(\varphi(\varphi_i), \varphi_j^*\right\rangle\right)_{ij} \\
&= \hat{v}_1 M_1 + \hat{v}_2 M_2 + 2\hat{t}\left(\left\langle B(\varphi, \varphi_i), \varphi_j^*\right\rangle\right)_{ij}, \quad i, j = 1,2,
\end{aligned}
$$

which is the Fréchet derivative of the reduced bifurcation equation (10) at a solution $\varphi \in S^1$.

Remark 4

Thus to the first order in s, if both eigenvalues of C(0, φ), for fixed $\varphi \in S^1$, have positive real parts, then the corresponding bifurcating solution of equation (1) is stable for small negative s and unstable for small positive s. The situation is reversed if both eigenvalues have negative real parts, while the bifurcating solutions are never stable if the eigenvalues of C(0, φ) have real parts of opposite signs. If neither of these holds, the stability is said to be indeterminate and an analysis of higher order terms may be carried out.

V. EXAMPLES

In this section we will consider two examples. We will discuss the stability situations of the bifurcating solution near origin finding the eigenvalues of C(0, φ), for fixed $\varphi \in S^1$. These examples were studied by Bari [1], and similar stability situations were found there by counting the indices of solutions.

Example 1: Let us consider

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ M_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ q(u) = \begin{pmatrix} u_1 u_2 \\ u_1^2 - u_2^2 \end{pmatrix}.$$

Then for all $\varphi \in S^1$ with $\|\varphi\| = 1$, the spanning condition (C) holds. Now we will use the following steps in MATHEMATICA to solve the equation

$$\lambda_1 M_1 u + \lambda_2 M_2 u + q(u) = 0$$

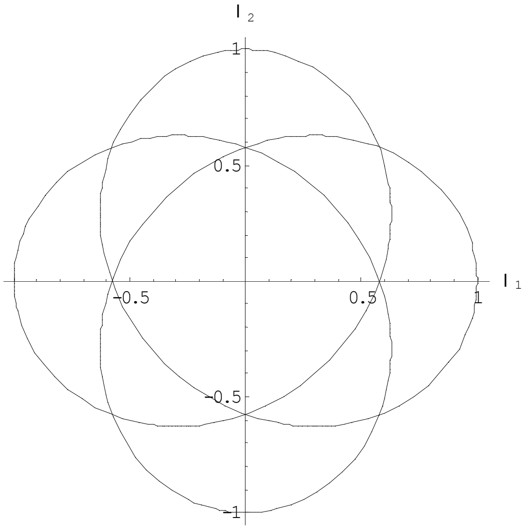
and to find the eigenvalues of the 2×2 matrix C(0, φ).

```
Solve[ ... u1 u2 ... ,
  ... u2 - ... - u2^2 ... l1, l2 ...
  u1 -> Cos ... , u2 -> Sin ... ]
```



```
l1 ® - 2 Cos ... / Cos ...
l2 ® - - Cos ... / Cos ...
Cos ...
..., 2 p , AxesLabel -> ..., "l2" ,
AspectRatio -> Automatic
```

Fig. 1: Parametric Plot of λ(φ) for φ∈S^1




```
l1 = - 2 Cos ... / Cos ...
l2 = - - Cos ... / Cos ...
l ... , l2 ;

= MatrixForm ... Sin ... Cos ...
  - l2 + 2 Cos ... 2 Sin ...
B = Eigenvalues ...
  ... + Sin ... Cos ...
  - l2 + 2 Cos ... 2 Sin ...

= ...
  ... 2 p , p  4
  ... TableHeadings ...,
  ..., "l ..., f ..., Eigenvalues" ,
TableAlignments -> Center
```

Table 1

φ	λ(φ)	C(0,φ)	Eigen-values
0	0 1.	0 2. 1. 0	-1.4 1.4
$\frac{\pi}{4}$	-0.35 -0.35	0.[illegible]5 0.[illegible] 1.[illegible]8 -1.[illegible]8	-2. 0.62
$\frac{\pi}{2}$	1. 0	2. 0 0 -1.	-1. 2.
$\frac{3\pi}{4}$	-0.35 0.35	0.[illegible]5 -0.[illegible] -1.[illegible]8 -1.[illegible]	-2. 0.62
π	0 -1.	0 -2. -1 0	-1.4 1.4
$\frac{5\pi}{4}$	0.35 0.35	-[illegible].35 -[illegible]35 -[illegible].8 [illegible]	-0.6 2.
$\frac{3\pi}{2}$	-1. 0	-2[illegible] 0 0 1.	-2. 1.
$\frac{7\pi}{4}$	0.35 -0.35	-[illegible].35 [illegible]35 1[illegible]8 1.[illegible]	-0.6 2.
2π	0 1.	0 2. 1. 0	-1.4 1.4

In Fig 1, we get three non-trivial solution curves λ(φ) in every direction for all $\varphi\in S^1$ in the (λ_1,λ_2)-space and the Table 1 shows that, for different values of φ, the real parts of the eigenvalues of C(0, φ) are of opposite signs. Hence all the solutions are unstable.

Example 2 : Consider M_1, M_2 as before and take

$$q(u)=\begin{pmatrix} u_1^2+2u_2^2 \\ 2u_1u_2 \end{pmatrix}.$$

Then for all $\varphi\in S^1$ with $\|\varphi\| = 1$, the spanning condition (C) holds. Now we will use the following steps in MATHEMATICA to solve the equation

$$\lambda_1 M_1 u + \lambda_2 M_2 u + q(u) = 0$$

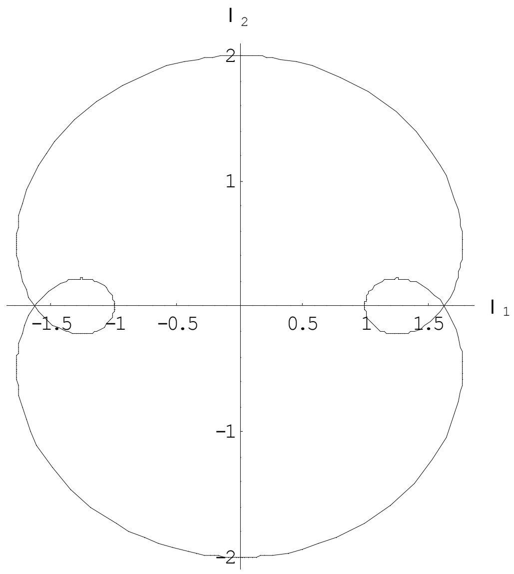
and to find the eigenvalues of the 2×2 matrix C(0, φ).

```
So[illegible] + u1^2 + 2[illegible]
[illegible]u2 - l[illegible] u[illegible] + [illegible]u2 == 0[illegible] l 2
[illegible] > Cos[illegible] > Sin[illegible]
```

```
l 1 ® - Cos[illegible]
        Cos[illegible]
l 2 ® - - Cos[illegible]
        Cos[illegible]
```

```
ParametricPlot[illegible]
8 [illegible]Cos[illegible]
   Cos[illegible]
[illegible], 2 p , AxesLabel -> [illegible], "l 2" ,
AspectRatio -> Automatic
```

Fig. 2: Parametric Plot of λ(φ) for $\varphi\in S^1$




```
l 1 = - Cos[illegible]
        Cos[illegible]
l 2 = - - Cos[illegible]
[illegible]
l [illegible] l 2
```

```
A = MatrixForm[illegible]
  [illegible] + 2 Cos[illegible] Sin[illegible]
  - l 2 + 2 Sin[illegible] 2 Cos[illegible]
```

```
B =
Eigenvalues[illegible] 2 Cos[illegible] Sin[illegible]
  - l 2 + 2 Sin[illegible] 2 Cos[illegible]
```

```
t = Table[illegible]
  N[B, [illegible] 2 p , p [illegible] 4 [illegible]
TableForm[illegible]
```

Table 2

φ	λ(φ)	C(0,φ)	Eigen- values
0	-1. 0	1. 0 0 1.	-1 1.
π/4	-1.8 -0.35	-0.35 2.5 1.8 -0.35	-2 0.
π/2	0 -2.	0 2. 4. 0	-1 2.
3π/4	1.8 -0.35	0.35 2.5 1.8 0.35	-2 0.
π	1. 0	-1. 0 0 -1.	-1 1.
5π/4	1.8 0.35	0.35 -2. -1.8 0.35	-0 2.
3π/2	0 2.	0 -2. -4. 0	-2 1.
7π/4	-1.8 0.35	-0.35 2.5 -1.8 -0.35	-0 2.
2π	-1. 0	1. 0 0 1.	-1 1.

Fig 2 shows that in certain directions in the (λ_1,λ_2)-space there exists only one non-trivial solution curve $\lambda(\varphi)$, while in some other directions we get three non-trivial solutions. Table 2 shows that, when we get only one non-trivial solution, the real parts of the eigenvalues of $C(0, \varphi)$ have opposite signs. Hence the solution is unstable for fixed $\varphi \in S^1$. But for some values of $\varphi \in S^1$, such as $\varphi = 0$, π, 2π, the real parts of the eigenvalues of $C(0, \varphi)$ have same signs. Thus by Remark 4, for $\varphi = 0$, the corresponding non-trivial solution is stable for small negative *s* and unstable for small positive *s*, that is the solution changes its stability as it passes through the origin. We get the similar stability situations for $\varphi = 2\pi$. But the stibility situations is reversed for $\varphi = \pi$.

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