In Quest of an Improved Algorithm for Transforming Plane Triangulations by Simultaneous Flips

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Declaration

It is hereby declared that

1. The thesis submitted is our own original work while completing degree at Brac University.
2. The thesis does not contain material previously published or written by a third party, except where this is appropriately cited through full and accurate referencing.
3. The thesis does not contain material which has been accepted, or submitted, for any other degree or diploma at a university or other institution.
4. We have acknowledged all main sources of help.

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Abstract

In this thesis we have studied the problem of transforming one plane triangulation into another by simultaneous flips of diagonals. A triangulation is a simple planar graph consisting of only 3-cycle (triangles) faces including outerface. In a triangulation, every edge lies on two faces that form a quadrilateral. An edge flipping is an operation that replaces this edge which is a diagonal of its corresponding quadrilateral with the other diagonal of the quadrilateral. A simultaneous flip set is an edge set of a triangulation that when flipped, the resulting graph is still a triangulation. Initially, it was proved that any two triangulations of equal order (number of vertices of a graph) can be transformed from one to another using a finite sequence of edge flip operations. Later on, it was observed that to complete this transformation, $\mathcal{O}(n \log n)$ individual flips are enough. In the continuation of the research, an algorithm was established which states that the transformation can be done in $327.1 \log(n)$ simultaneous flips. Lately, two algorithms were introduced to improve the leading coefficient of this bound for transforming any plane triangulation into another. These two algorithms lower this bound down to $85.8 \log(n)$ and $45.6 \log(n)$ respectively. In this thesis, we have developed an algorithm to introduce two dominant vertices simultaneously. Using our algorithm, any pair of vertices of the triangulation can be made dominant. The process requires at most $60.8 \log(n)$ simultaneous flips.

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Chapter 1

Introduction

Computational Geometry is a well-studied field of Computer Science where different algorithms are studied to solve various geometry-based problems. In the late 1970s, it emerged from the field of algorithm design and analysis. Computational geometry has two branches - combinatorial computational geometry and numerical computational geometry. Graph theory, another popular field, is intimately related to combinatorial computational geometry. The study of graphs and their properties is known as graph theory. It is one of the most visually appealing fields in mathematics, with a wide range of applications including the internet, web search engines, social networks like Facebook, Whatsapp, Twitter, etc., electronic circuits, data mining, image segmentation, clustering, image capturing, computer network security, GSM mobile phone networks, map coloring and so on. Our topic is a well-studied problem related to graph theory which is the transformation of one triangulation to another in an efficient way.

1.1 Preliminaries

A *simple graph* is a graph that does not have loops or parallel edges. A *simple planar graph* is a graph that has an embedding on a plane such that no two edges of it will intersect each other excluding their endpoints. A *triangulation* is a simple planar graph consisting of only 3-cycle faces (triangles) including the outer face. *Edge flipping* is an operation that replaces one edge with another without losing the planarity of the graph while keeping the resulting graph in the same class as the original graph. The subject of our research is triangulation. It is known that any triangulation can be transformed into any other of the same order by simultaneous flips of diagonals. In a plane graph $G = (V, E)$, every cycle defines two separate regions called - *internal region* (in the interior of the cycle) and *external region* (in the exterior of the cycle). If a region does not contain any vertex inside it, then it is called a *face*. In a triangulation, each edge is on two distinct triangular faces. These two faces form a quadrilateral and the common edge is the diagonal. After flipping this, we will add the other diagonal removing the previous one.

1.2 Literature Review

Transforming one triangulation to another by simultaneous flip is a well-studied problem in computational geometry. Wagner [3] studied the problem of transforming one triangulation to another by diagonal flips. He was the first to introduce the idea of canonical triangulation as an intermediate stage to make the transformation accomplished. Next, Negami [5] proved the fact that transformation of two triangulations into each other on a sphere can be done by diagonal flip operations if both the triangulations contain an adequately large and equal number of vertices. Similarly, according to Brunet, Nakamoto, and Negami [6], the transformation of any two triangulations into each other can be done up to homomorphism via diagonal flips on the condition that the two triangulations have the same number of vertices and belong to the same class. Then, Komuro [7] showed that to transform an n-vertex triangulation into another one on the sphere, it needs a maximum of $8n-54$ diagonal flips where $n \geq 13$ and $8n-48$ diagonal flips where $n \geq 7$. Later on, Hurtado, Noy, Urrutia [8] demonstrated that any triangulation carries no less than $\frac{n-4}{2}$ flippable edges. Mori, Nakamoto, Ota [10] proved that transformation of two hamiltonian triangulations on the sphere having n vertices where $n \geq 5$ can be accomplished with no more than $4n-20$ diagonal flips conserving the hamiltonian cycle of the triangulation. Using this result they came up with the result that no more than $6n-30$ diagonal flips are required to transform one triangulation to another having n vertices. In addition, they also proved that given a maximal outerplane graph a dominant vertex can be constructed in a linear number of flips. However, according to Bose et al. [12], every single triangulation with order ≥ 5 has a simultaneous flip which results into a hamiltonian triangulation and this operation requires $\mathcal{O}(n)$ time to be executed. Using this result, they demonstrated that there exists a sequence of $\mathcal{O}(\log n)$ simultaneous flips to accomplish the transformation of triangulations into each other. The number of total flipped edges is $\mathcal{O}(n)$ in this sequence. Furthermore, it was proved by Bose et al. [12] that each triangulation consists of a simultaneous flip of no less than $\frac{n-2}{3}$ edges. On the other hand they proved that each simultaneous flip has no more than $n-2$ edges and there can be triangulations where simultaneous flip of at most $\frac{6n-2}{7}$ edges can exist. Along with this, Bose et al. [13] stated that any combinatorial triangulation having n vertices can be converted into a hamiltonian triangulation by using maximum $\frac{3n-9}{5}$ edge flips. They also improved the upper bound of the number of flips required to transform a 4-connected triangulation into a canonical triangulation. Furthermore, they showed that the transformation of one triangulation to another can be done in approximately $327.1 \log n$ simultaneous flips. De Carufel, and Kaykobad [15] devised an algorithm leading to an improvement of the coefficient of the upper bound previously given by Bose et al. [12] The upper bound was lowered to approximately $85.8 \log n$ from $327.1 \log n$. Kaykobad further proved that $45.6 \log n$ simultaneous flips are adequate to execute the transformation of one maximal outerplane graph to another.

1.3 Problem Statement

Summary of Necessary Known Results		
Author Name	**Findings**	**Improvement**
Wagner [3]	Transforming one triangulation to another by diagonal flips	$2n^2 - 14n + 24$ flips are needed
Mori, Nakamoto and Ota [10]	A maximal outerplane graph a dominant vertex can be constructed in linear time	$\mathcal{O}(n)$ number of flips are enough for that
Bose et al. [12]	If the number of simultaneous flips are equal to the diameter, then it is enough to make a vertex dominant in a maximal outerplane graphs	$4 \times (\frac{2}{\log \frac{54}{53}} + \frac{2}{\log \frac{6}{5}}) \log n + 2$
Kaykobad[15]	Improved the coefficient of the logarithm function	$4 \times (\frac{2}{\log \frac{12}{11}} + \frac{2}{\log \frac{9}{7}}) \log n + 2$

Table 1.1: Result Summary

In Table 1.1, a summary of known results on the problem of transforming one triangulation to another is shown. For decades, researchers have been trying to improve the bound and the process of transformation. In 1936, Wagner [3] at first showed that the transformation can be done in $\mathcal{O}(n^2)$ flips. In 2007, Bose et al. [12] developed an algorithm that could transform any triangulation to any other using no more than $327.1\log(n)$ flips. After that, more improvements have been done in the bound and it decreased the bound to $85.8\log(n)$. Since the main aim is to decrease the number of flips as it defines the efficiency of the algorithm, so we are exploring a way to decrease the number of flips so that the transforming process and algorithm become more efficient. Note that, for all of the calculations of this paper, we consider the base of logarithm to be 2.

1.4 Objective of the Thesis

In this thesis, we have studied simple and plane triangulation, simultaneous diagonal flip, flippable set and many other terms and theorems related to our topic. Also, we are analysing the procedure of the transformation of a plane triangulation into another plane triangulation via simultaneous diagonal flips. Previously, a lot of research has been done in this field and these have been discussed in literature review.

Bose et al. [12] came up with an algorithm to transform any triangulation with n vertices into any other triangulation with same number of vertices requiring approximately $327.1\log n$ simultaneous flips. In continuation, De Carufel and Kaykobad [15] have introduced another algorithm to improve the algorithm given by Bose et al. [12] with approximately $85.8\log n$ simultaneous flips. In this paper, we are trying to improve further the interlaced algorithm devised by Kaykobad [14] requiring approximately $45.6\log n$ simultaneous flip. The Kaykobad algorithm has four steps and to reach our goal, we are following the first, second and fourth steps same as before. We are working on the third step for constructing two dominant vertices parallelly in inner and outer subgraphs to get the canonical triangulation and trying to make the process more efficient. For this we are making a pair of randomly pre-selected vertices dominant in inner and outer subgraph parallelly so that the process of the transformation of triangulation becomes more efficient.

Chapter 2

Literature Review

In this chapter, we have discussed some basic definitions with figures and explanations related to our problem which will help us understand the topic in a better way. In addition, some important results are also highlighted with necessary explanations.

Definition 2.1 (Graph)**.** *A graph $G = (V, E)$ consists of a non-empty set of vertices (or nodes), V and a set of edges, E. Every edge $e = (u, v)$ connects two vertices u and v. If $u = v$ then the edge is said to be self-loop. On the other hand if there are two edges $e_1 = ((u, v)$ and $e_2 = ((u, v)$ they are called parallel edges. Graphs without loops and parallel edges are called simple graphs. The number of vertices of G is considered as the order of G. Also, the number of edges of G is considered as the size of G.*

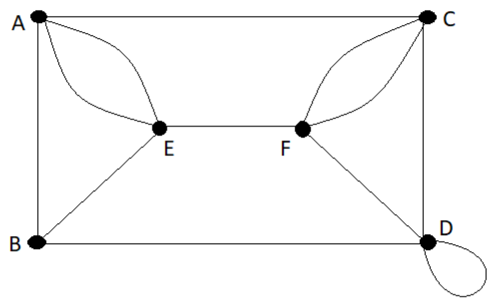


Figure 2.1: Graph

A graph is illustrated in Figure 2.1. A graph has two components, edge and vertex. Every edge connects one or two vertices. A graph can be connected or disconnected. A graph $G = (V, E)$ is called connected if for any pair of its vertices u and v there is a sequence of edges $e_i = (w_i, w_{i+1}),\ i = 1, ..., k$ such that $w_1 = u,\ w_k = v$ for some k, and $e_i \in E, \forall i = 1, ..., k - 1$. Otherwise the graph is disconnected.

Definition 2.2 (Planar Graph)**.** *Any graph is called planar graph if it can be embedded on a plane where no pair of edges intersect each other other than at their end points.*

Planar graph is basically a blueprint of a plane graph where plane graph is a graph embedded on a plane with no intersection among the edges of the graph except their endpoints. Note that a planar graph can have a non-planar embedding, that is an embedding where some edges may intersect other than at the endpoints.

Definition 2.3 (Plane Graph)**.** *If an embedding in the plane has been provided for a planar graph such that no two edges in the given embedding intersect except at their endpoints, then the embedded graph is called a plane graph.*

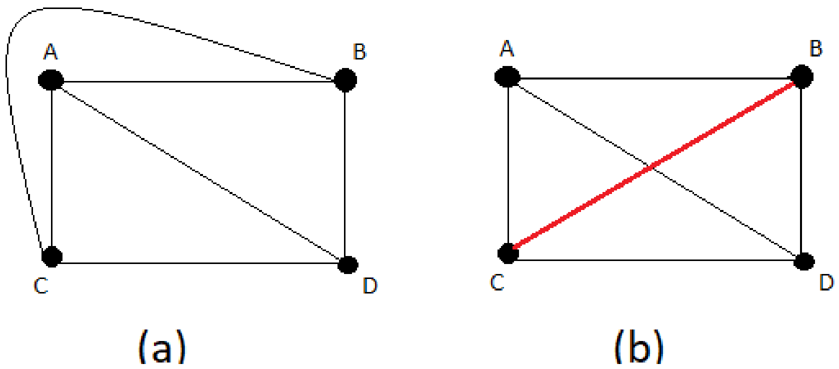


Figure 2.2: fig-(a): Plane graph, fig-(b): Non plane graph

Figure 2.2(a) is a plane graph since it has an embedding on a plane so that no pair of edges intersect each other excluding their endpoints. On the other hand, figure 2.2(b) is a non plane graph since there is an intersection between the edges AD and BC.

Definition 2.4 (Simple Planar Graph)**.** *A graph $G = (V, E)$ that has an embedding on a plane such that it has no parallel edges, no self-loops and, no two edges will intersect each other excluding at their endpoints.*

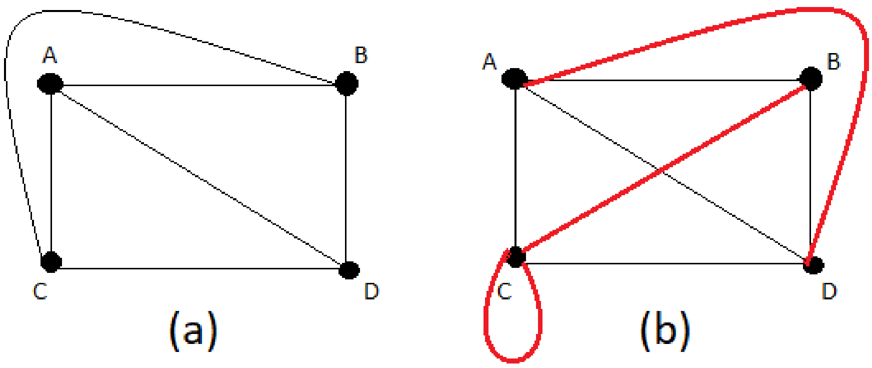


Figure 2.3: fig-(a): Simple planar graph, fig-(b): Non simple planar graph

Figure 2.3(a) is a simple planar graph as it does not have any parallel edges or self loops. Figure 2.3(a) is a simple graph as well as a planar graph since it is fulfilling the conditions of planar graph. Contrastingly, figure 2.3(b) is a non simple planar graph since it has parallel edge, self loop and also intersection between the edges AD and BC.

Definition 2.5 (Triangulation)**.** *Any simple planar graph $G = (V, E)$ consisting of only 3-cycle faces (triangles) including outer face that has an embedding on a plane such that no pair of edges intersect each other except at their endpoints is called a triangulation.*

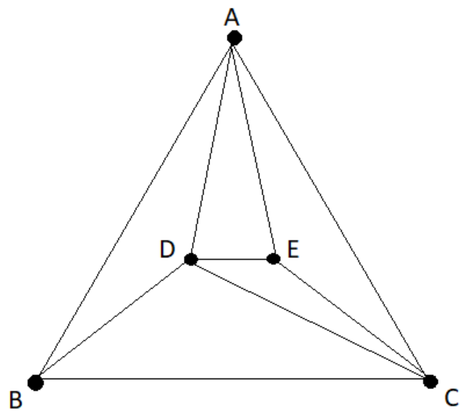


Figure 2.4: Triangulation

Figure 2.4 is a triangulation since it is a simple planar graph each face of which is a triangle incorporating the outer face.

Definition 2.6 (Region, Face). *For a plane graph $G = (V, E)$, every cycle creates two separate regions called - internal region (inside the cycle) and external region (outside the cycle). If a region does not consist any vertex inside it or in other words does not consist anymore regions inside it, it is called a face.*

In Figure 2.4, ABC is a cycle, so it creates two regions such as - internal region and external region. Here, ADE is also a cycle so it similarly creates two regions.Since ADE does not contain any vertex in it, it is called a face.

Definition 2.7 (Seeing). *For a planar graph $G = (V, E)$ any vertex of a 3-cycle face is called seeing its opposite edge.*

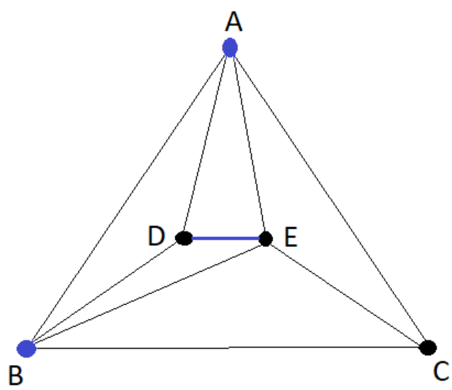


Figure 2.5: Vertex A and B are two seeings of the edge DE

An illustration of seeing is shown in figure 2.5. In the figure 2.5, ADE is a 3-cycle face. As A sees edge DE. So, A is a seeing of DE. Again, vertex B is also another seeing of DE since DE is the diagonal of the quadrilateral $ADBE$, edge DE has two seeing (A and B).

Definition 2.8 (Flip). *Flip is an operation where one edge gets replaced with another by adding two seeing vertices of that edge. Also, flip is a reversible operation.*

Definition 2.9 (Diagonal Flip)**.** *In a triangulation each edge is on two distinct triangular faces. These two faces form a quadrilateral of which the common edge is the diagonal. To flip this edge, we delete this diagonal, and add the opposite diagonal of the same quadrilateral, provided that:*

1. *The new diagonal is also in the same region as the previous one*
2. *The resulting graph is also a triangulation after the flip.*

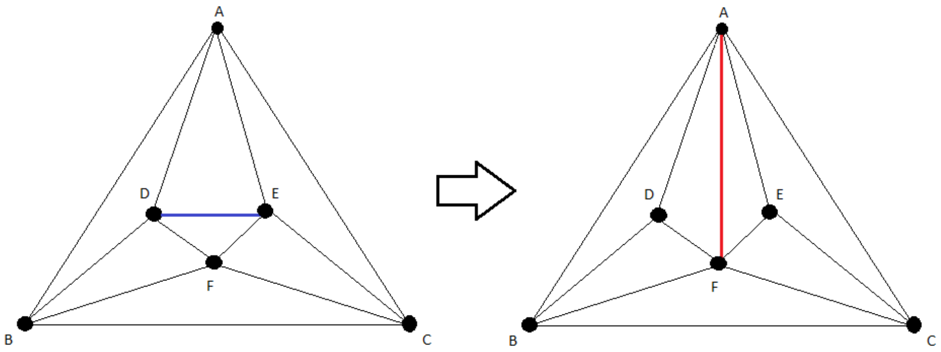


Figure 2.6: Edge DE is the diagonal of quadrilateral $ADFE$, DE has been flipped; the new edge is AF

A diagonal flip has been shown in Figure 2.6 where $ADFE$ is a quadrilateral and DE is one of its diagonals. So the diagonal DE is flipped and connect AF which is the new diagonal of the quadrilateral.

Definition 2.10 (Consecutive Edges)**.** *Two edges of a triangulation that are incident to a common face are called consecutive edges.*

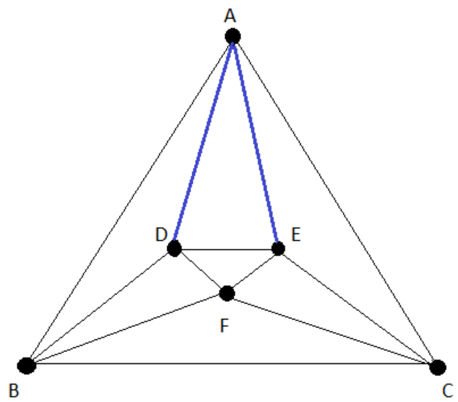


Figure 2.7: AD and AE is sharing the same face ADE

Figure 2.7 shows two consecutive edges. In the figure, AD and AE are consecutive edges as these two edge are sharing the same face ADE.

Definition 2.11 (Adjacent Faces)**.** *Two faces of a triangulation that have a common edge are called adjacent faces.*

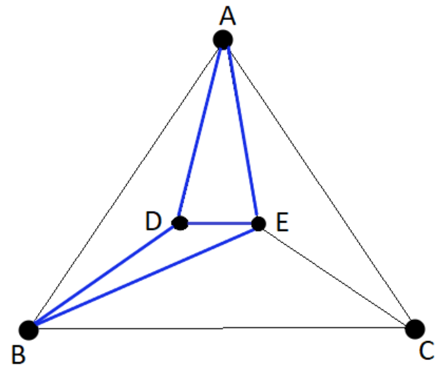


Figure 2.8: ADE & BDE are adjacent faces (highlighted in blue); DE their common edge

An illustration of adjacent face is presented in figure: 2.8. Here, ADE and BDE are two faces of triangulation sharing a common edge DE. Therefore, ADE and BDE are adjacent faces.

Definition 2.12 (Bad Pair)**.** *A pair of edges form a bad pair if both of them are seen by the same two vertices.*

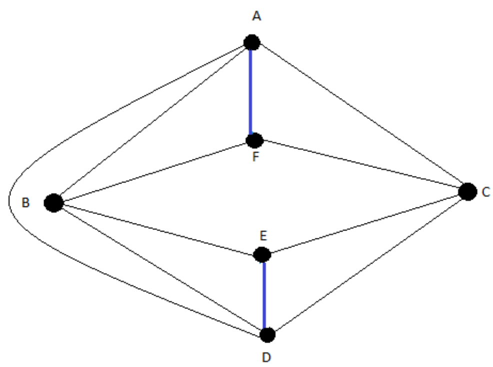


Figure 2.9: AF and ED form a bad pair as both are seen by B and C

An illustration of a bad pair is showed in the Figure 2.9. The two edge AF(blue) and edge ED(blue) form a bad pair since both of the edges can be seen by the vertices B and C.

Definition 2.13 (Blocked edge, Blocking edge)**.** *Given that,* $G = (V, E)$ *is a triangulation and two edges* $uv, xy \in E$*. If* uv *is seen by both* x *and* y *then* uv *is a blocked edge and* xy *is its blocking edge.*

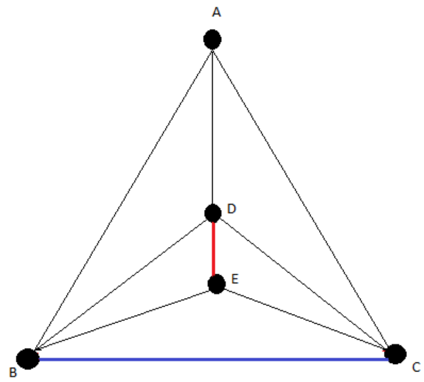


Figure 2.10: DE is blocked edge and BC is blocking edge as DE is seen by both B and C

Figure 2.10 is represented blocked edge and blocking edge. As the edge DE (red) is seen by both of the vertex B and vertex C then DE is the blocked edge and BC (blue) is the blocking edge.

Lemma 2.14 (Lemma 2.1 of Bose et. al [12]). *An arbitrary set of edges, S is called a flippable set where*

1. *No pair of edges are consecutive*
2. *No pair of edges form a bad pair*
3. *If there is any blocked edge present then the blocking edge must also be present in S*

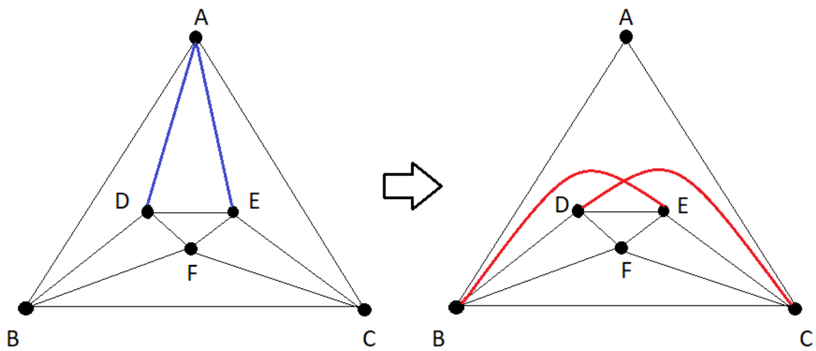


Figure 2.11: Flipping consecutive edges AD (blue) and AE (blue) results into edge intersection (red)

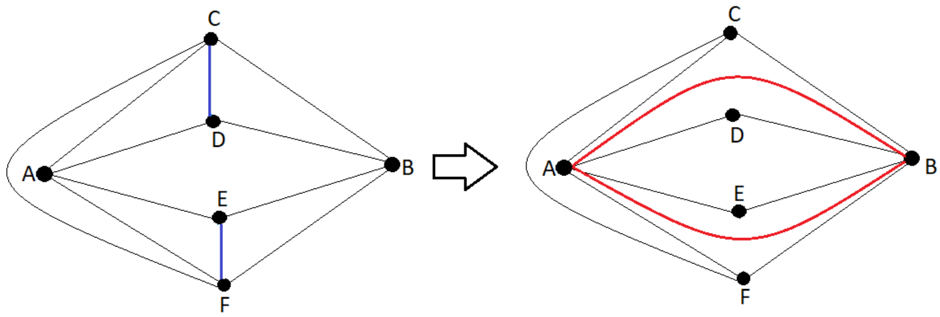


Figure 2.12: Flipping bad pair CD (blue) and EF (blue) results into parallel edge AB (red)

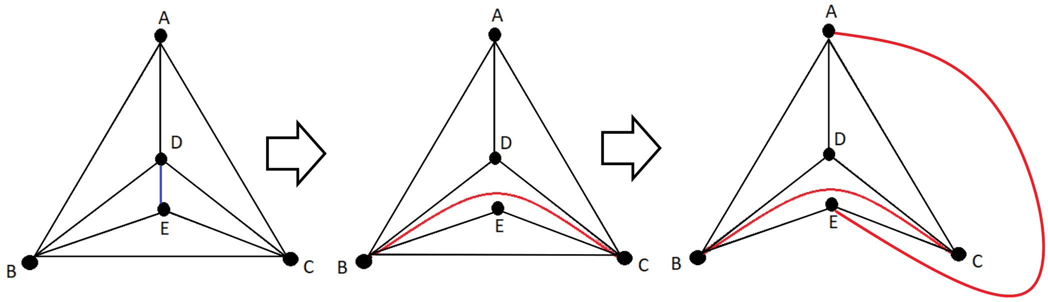


Figure 2.13: In case of flipping blocked edge DE, the blocking edge BC must be flipped

A flippable set is defined in figure 2.11, 2.12, 2.13. In figure 2.11, by flipping two consecutive edges AD (blue) and edge AE (blue), we get the edge BE (red) and edge CD (red) and they intersect each other so it could not fulfill the condition of a flippable set. Again, in figure 2.12, by flipping a bad pair CD (blue) and EF (blue) results into parallel edge AB (red) and it also violates the condition of a flippable set. In figure 2.13, to flip edge DE (blue), blocking edge BC also has to be flipped as only flipping DE, it creates a parallel edge BC. Note that, $G^{i'} = G^i\langle S_1^i \rangle$ notation denotes that S_1^i is a flippable set and this flippable set is being executed on G^i and the resulting graph is $G^{i'}$.

Definition 2.15 (Ordered Set of Simultaneous Flip)**.** *An ordered set of simultaneous flip is a set that contains multiple edges which are flippable in that order. It is denoted by S, therefore the resulting graph after executing S sequentially on a graph, G is denoted by $G\langle S \rangle$.*

Definition 2.16 (Dominant Vertex)**.** *Dominant vertex is a vertex of any graph that is incident to all the vertices of the same graph. It is also known as universal vertex.*

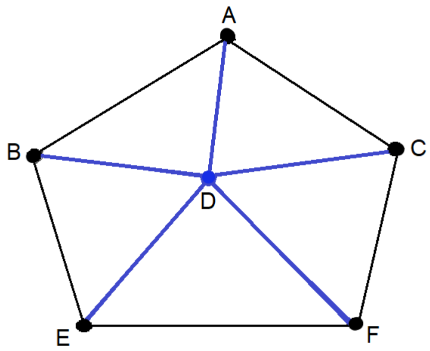


Figure 2.14: All the vertices are connected to F (blue), F (blue) is the dominant vertex

An illustration of Dominant Vertex is presented in figure 2.14. In the figure the vertex F (blue) is the dominant vertex as all the vertices are the incidents of F.

Definition 2.17 (Canonical Triangulation)**.** *A triangulation that contains two dominant vertices is called a canonical triangulation where* $n \geq 3$.

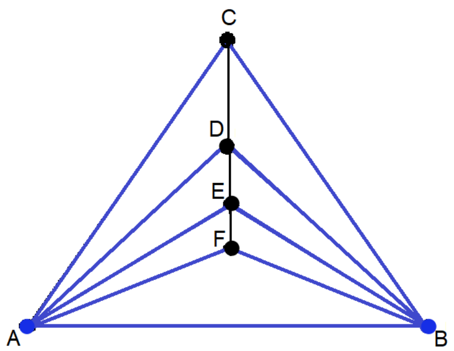


Figure 2.15: ABC triangulation contains two dominant vertices B (blue), C (blue) so it is canonical triangulation

Canonical Triangulation is presented in Figure 2.15. In the figure, the triangulation ABC has more than one dominant vertex. B (blue) and C (blue) are two dominant vertices. So, ABC is a canonical triangulation.

Definition 2.18 (Isomorphism)**.** *Two graphs* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *are isomorphic if there are bijections* $\theta : V_1 \to V_2$, *and* $\psi : E_1 \to E_2$ *such that* $uv \in E_1 \leftrightarrow \theta(u)\theta(v) = \psi\ (u\ v)$ $\in E_2$.

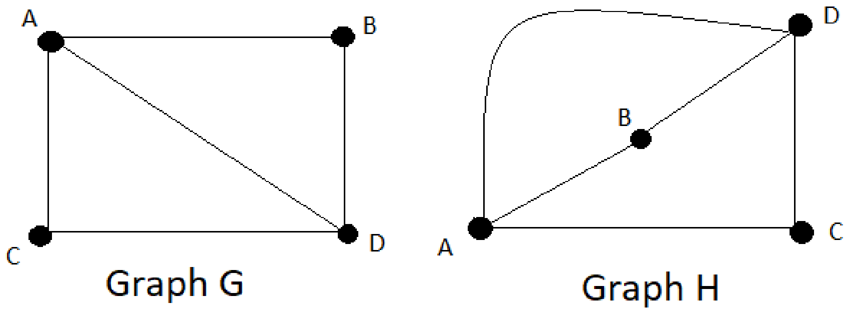


Figure 2.16: Graph G and graph H are isomorphic

Figure 2.16 is a demonstration of isomorphism.The vertices of graph G can be mapped to the vertices of graph H. Similarly, the edges of Graph G can be mapped to the edges of Graph H. The mapping is bijective. Their structure preserves the one to one correspondence between the vertices and edges, so their basic structure is same.

2.1 Wagner's Findings

Wagner [3] was the first to prove that with diagonal flips any two triangulations can be transformed into each other. To do that he introduced the idea of canonical triangulation. He also showed that diagonal flips can be standardized by transforming each of them into a canonical triangulation which works as an intermediate triangulation in the transforming process.

Definition 2.19 (Maximal Outerplane Graph)**.** *Maximal Outerplane Graph is a simple plane graph where all its vertices are on outer cycle and if any other pair of vertices are connected in the inner cycle, it will lose its planarity.*

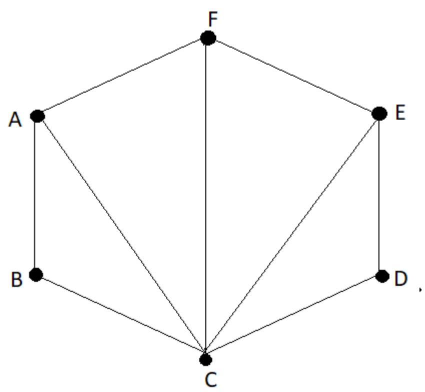


Figure 2.17: All six vertices are on the maximal cycle

A representation of maximal outer plane graph is shown in figure 2.17 where all the vertices of graph are on the maximal cycle of the graph.

Definition 2.20 (Chord)**.** *An edge of a graph G which connects any two non adjacent vertices of the cycle C is referred as the chord of the cycle. A chord is called internal chord when it passes through the interior of the cycle and it is considered as external chord when it belongs to the exterior of the cycle.*

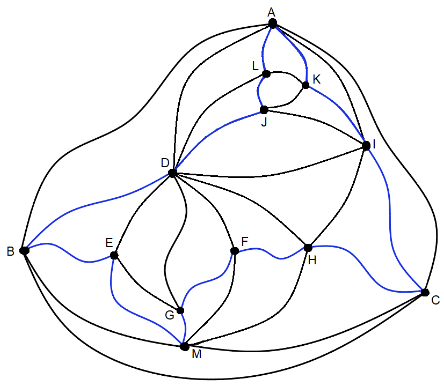


Figure 2.18: Chord

An illustration of chord is represented in figure 2.18. Here, the cycle C denoted by blue color is the maximal cycle of the graph and the edges belongs to the interior region of the cycle $\{IJ, DG, DI, ...\}$ are the internal chords of the cycle. However, the edges that are associated with the external region of the cycle $\{AB, HM, BM, ...\}$ are called external chords.

Definition 2.21 (Dual Tree)**.** *A dual tree T^* has all its vertices in each face of its primal graph G (except the outer face) and each edge of T^* connects a pair of vertices (green) if the corresponding two faces of primal graph are adjacent.*

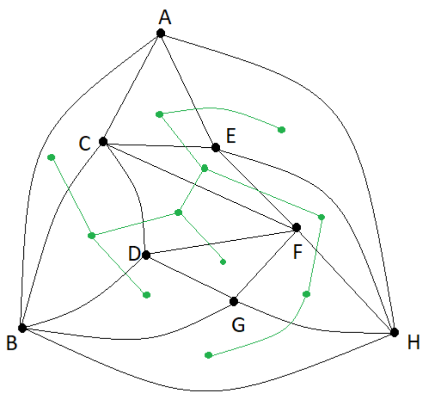


Figure 2.19: Dual tree (green) has vertices (green) in every faces of its primal graph and every edge connects its neighbouring pair of vertices

figure 2.19 corresponds to a dual tree. ABH is a triangulation and it contains a dual tree(green) which has a vertex in every face of the triangulation and each of the edge of dual tree connects a pair of vertices. For example: There are vertices in dual tree corresponding to faces ABC and BCD and both the vertices are adjacent, that is connected with an edge.

Definition 2.22 (Subgraph)**.** *A graph $G' = (V', E')$ is called a subgraph of a graph $G = (V, E)$ if $(V') \subseteq V$, $E' \subseteq E \cap (V' \times V')$.*

A subgraph is a part of main graph having similar vertices and edges. When some of the edges and vertices are removed from the main graph except the endpoints of any remaining edges, we get a smaller graph. This smaller graph is called a subgraph of the main graph.

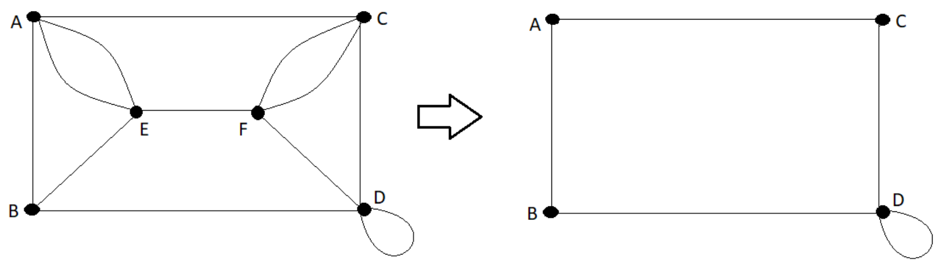


Figure 2.20: fig: The left one is Main Graph and the right one is subgraph

A demonstration of subgraph is shown in figure 2.20. Figure 2.20(b) contains the endpoints of its primal graph which is in figure 2.20(a) and two vertices are removed from the primal graph to make it subgraph.

2.2 Paper by Mori, Nakamoto and Ota

Mori, Nakamoto and Ota [6] showed that a dominant vertex can be introduced in a maximal outerplane in linear time.
In figure 2.21(a) a maximal outerplane with its dual tree is illustrated. In figure 2.21(b) edge BE (dotted blue) is flipped and after flipping vertex A and H is connected through a new edge(red). In figure 2.21(c), 2.21(d), 2.21(e) and 2.21(f) edge EH, EF, FD and GD are flipped respectively.
Finally, A is the dominant vertex as all the vertices are incident to A. To make A dominant, 5 flips were needed which is equal to the diameter of its dual tree. It is the worst case where diameter number of flips were needed. If any other vertex other than A or C was chose then it would have taken less number of flips.

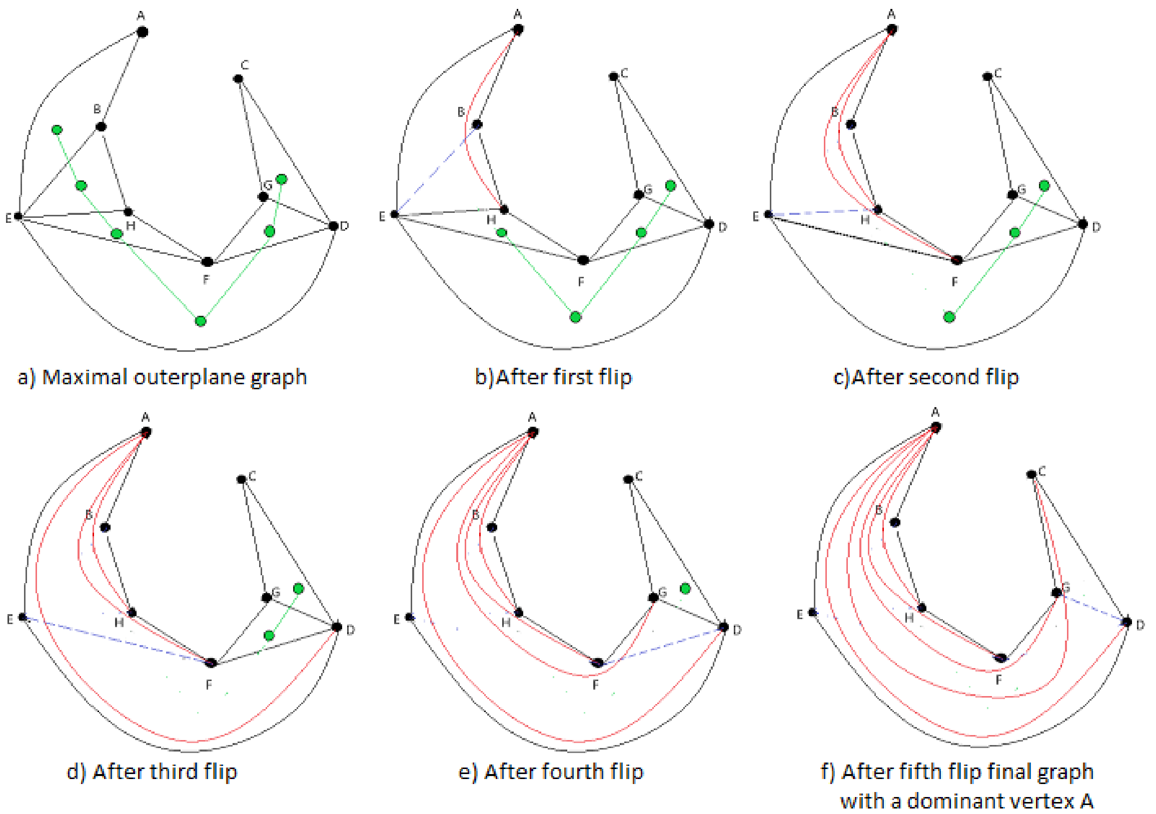


Figure 2.21: A vertex(A) is made dominant in a maximal outerplane graph with diameter number of flips

Definition 2.23 (K-vertex Connected). *A graph $G = (V, E)$, is called k-vertex-connected if at least k vertices need to be deleted to make G disconnected.*

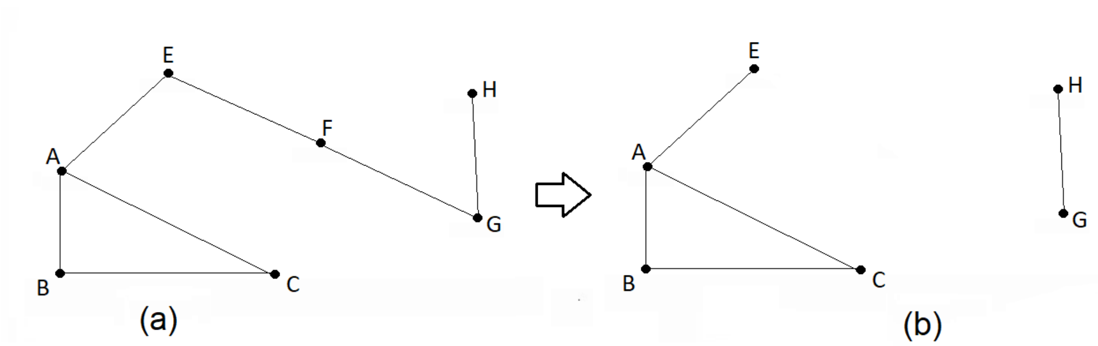


Figure 2.22: fig:-(a): Main graph, fig:-(b): Disconnected graph

figure 2.22 shows a 1-vertex connected graph where figure 2.22(a) is the main graph and after deleting one vertex F, the graph becomes disconnected which is shown in figure 2.22(b).

Definition 2.24 (Hamiltonian Cycle). *Let $G = (V, E)$ be a graph with a cycle C that contains every vertex of G once then C is called the hamiltonian cycle of the graph.*

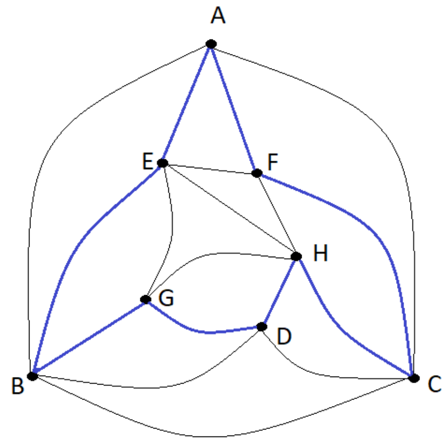


Figure 2.23: ABC triangulation is containing a hamiltonian cycle (A, E, B, G, D, H, C) (blue)

Figure 2.23, illustrates a hamiltonian cycle of the graph. Here, the cycle starts from vertex A, it goes through every vertex of the cycle and returns to the same vertex by visiting each vertiex once.

Definition 2.25 (Hamiltonian Triangulation)**.** *A triangulation that contains a hamiltonian cycle is called a Hamiltonian triangulation.*

A demonstration of hamiltonian triangulation is presented in figure 2.23 where ABC triangulation is a hamiltonian triangulation as it contains a hamiltonian cycle (A, E, B, G, D, H, C) (blue) in it.

Definition 2.26 (Matching, Perfect Matching)**.** *In a graph $G = (V, E)$, matching is a set of edges M where two edges are non-consecutive to each other and perfect matching is when every vertex of G is adjacent to an edge of the M.*

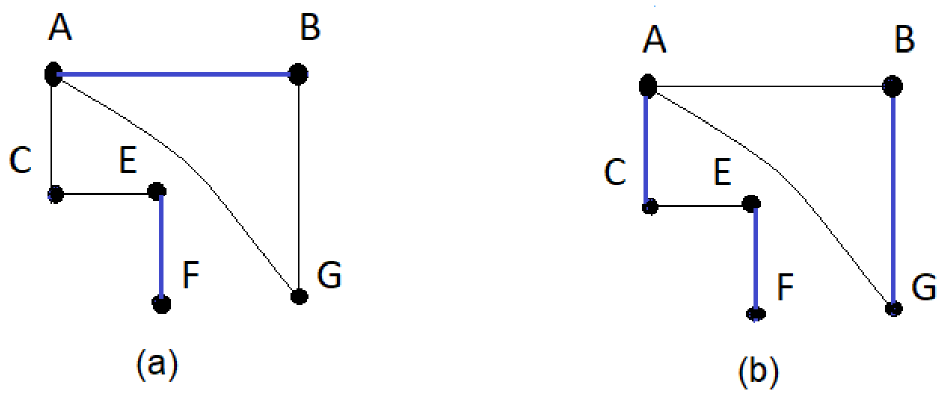


Figure 2.24: fig:-(a): Matching Set $M = AB, EF$, fig:-(b): Perfect Matchings are $\{AC$, BG, $EF\}$

An illustration of Perfect Matching is shown in the Figure 2.24. Here, the figure 2.24 (a) is an example of matching where AB and EF are matching to each other as both are non-consecutive edges and the figure 2.24 (b) represents perfect matching. In this figure 2.24 (b), the edges AC, BG and EF are not adjacent to each other. Thus every vertex is connected to the edge of the matching set.

Definition 2.27 (Separating Triangle). *If deletion of all vertices of a triangle T of a connected graph $G = (V, E)$ transforms G into a disconnected graph, then T is called a separating triangle.*

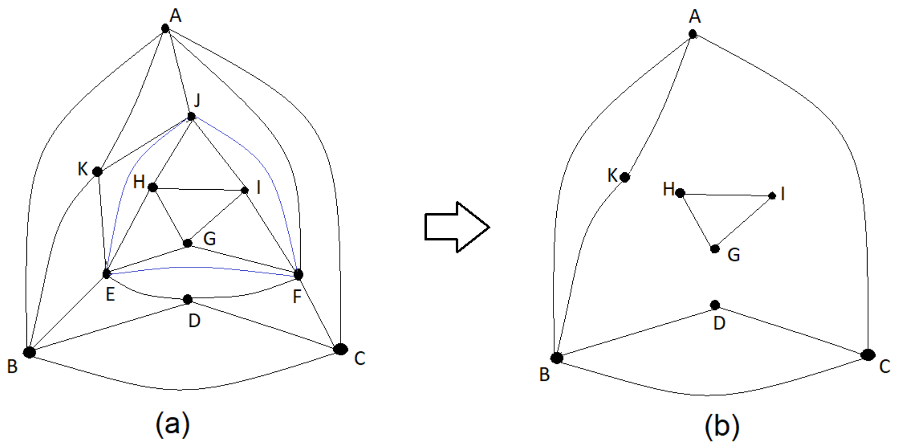


Figure 2.25: fig:-(a): Main graph, fig:-(b): Separated graph

Figure 2.25 is an illustration of separating triangle. Figure 2.25(a) is the primal graph and after deleting one of its face JEF (blue), the graph becomes separated as in figure 2.25(b). So the triangle JEF (blue) is the separating triangle of the primal graph.

Definition 2.28 (Edge Sub-division). *The edge subdivision operation for an edge $uv \in E$ of a graph $G = (V, E)$, is the deletion of the edge uv and adding two edges uw and vw such that w is a new vertex in the graph. The obtained new graph after the operation is $G' := (V' := V \cup w \mid w \notin V,\ E' := (E \setminus uv) \cup \{uw, vw\})$.*

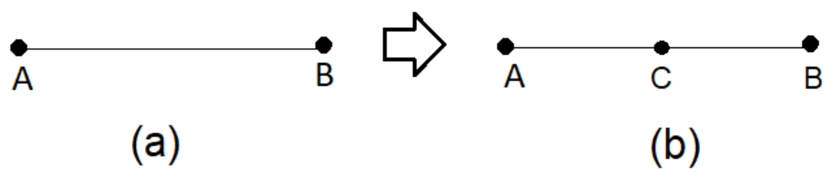


Figure 2.26: fig:-(a): Main graph, fig:-(b): Edge-Sub divided graph

Figure 2.23 represents an edge sub-division. In figure 2.26(a), AB is an edge and in figure 2.26(b), the vertex C divided the edge AB and created two new edges AC and CB.

Definition 2.29 (Graph Sub-division). *A graph $G' = (V', E')$ is called a subdivision of G, if it can be generated by a sequence of edge subdivision operations on G.*

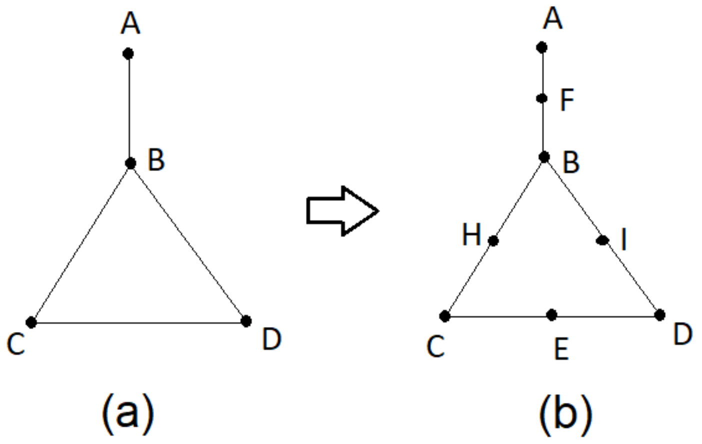


Figure 2.27: fig:-(a): Main graph, fig:-(b): Sub-divided graph

A demonstration of graph sub-division is shown in figure 2.20. Here, figure 2.20(a) is the main graph and by operating a sequence of edge sub-division on the main graph, it becomes sub-divided in figure 2.20(b). In the figure 2.20(b), the vertex F divided the edge AB and introduced two new edges AF & FB. In the same process, the vertex H splits the edge BC and established two new edges BH & CH. Also, CD edge is separated by vertex E and two new edges CE & DE are introduced. Similarly, BD edge is separated by vertex I and this vertex established BI & DI two new edges.

Definition 2.30 (Complete Graph)**.** *Given a simple graph $G = (V, E)$, if all the vertices of G are adjacent to each other, then G is called a $CompleteGraph$. It is denoted by K_n.*

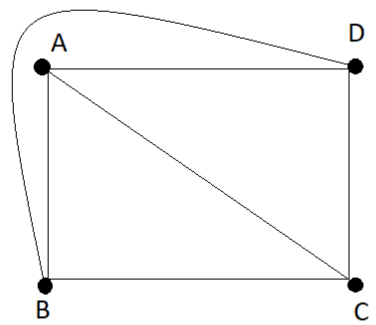


Figure 2.28: $ABCD$ quadrilateral is a complete graph

Figure 2.28 is an illustration of a complete graph where all the vertices of G are adjacent to each other. Vertices A & B, B & C, C & D, D & A, A & C all are adjacent to each other.

Definition 2.31 (Dual Graph)**.** *Given a planar graph $G = (V, E)$ with its set of faces F, the dual graph $G^* = (V^*, E^*)$ of G is also a graph with a bijection between F and V^*, and another bijection between E and E^*, such that for every edge $e \in E$ on two faces $f_i, f_j \in F$ there is an edge $e^* = (v_i^*, v_j^*) \in E^*$, and v_i^*, v_j^* correspond to the faces f_i and f_j respectively.*

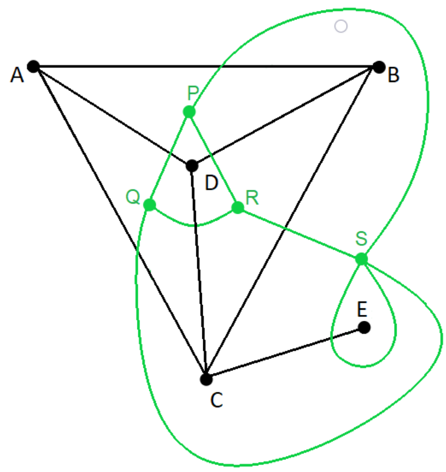


Figure 2.29: $PQRS$ (Green) is a Dual Graph of $ABCE$

A representation of dual graph is presented in figure 2.29. Here, $ABCE$ is the main graph and $PQRS$(green) is the dual graph of it. Each face of the main graph contains a vertex of the dual graph. The 3-cycle face ABD of the main graph contains a vertex P of the dual graph. Similarly, vertex Q is on the ACD 3-cycle face and vertex R is on the BCD 3-cycle face. Also, there is a vertex S on the outer face of the main graph. Besides, the total number of edges $\{AB, AC, BC, AD, BD, CD, CE\}$ of the main graph and the edges of the dual graph $\{PQ, PR, QR, PS, QS, RS, SS\}$ are same. Every face of the dual graph contains a vertex of the main graph as well. For instance, PQR triangle contains the vertex D, also, PRS 3-cycle face contains the vertex R, vertex C is on the QRS triangle, the self-loop face S contains the vertex E and the vertex A is situated on the outer face of the dual graph.

Definition 2.32 (Ear)**.** *If two edges of a 3-cycle of a maximal outerplanar $G = (V, E)$ is on the outerface, and the remaining edge is a chord then the 3-cycle is called an ear.*

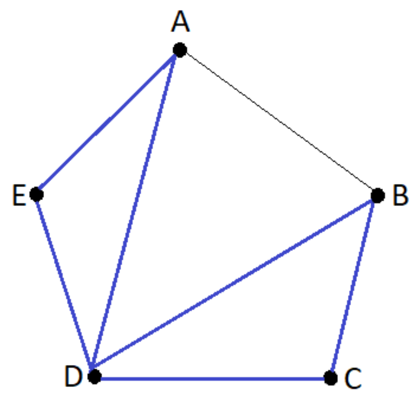


Figure 2.30: Triangles AED and BCD both are Ears (blue)

An illustration of ear is presented in figure 2.30. In the figure, both AED and BCD (blue) 3-cycle faces are ears where $\{AE, ED\}$ & $\{BC, CD\}$ edges are on the maximal cycle of the graph. In addition, AD & BD both edges are chords.

Theorem 2.33 (Appel and Haken[4])**.** *Every planar graph is four-colorable.*

The four color theorem above states that if the graph is planar, then its vertices can be colored with 4 colours in such a way that no two adjacent vertices are coloured the same. Using this theorem, Bose et al. [12] showed that using 3-colors any planar graph edges can be colored, so that no two edges sharing the same face have the same color.

Definition 2.34 (Independent Set). *A subset of the vertex set of a simple plane graph $G = (V, E)$ is independent if and only if it contains no pair of adjacent vertices.*

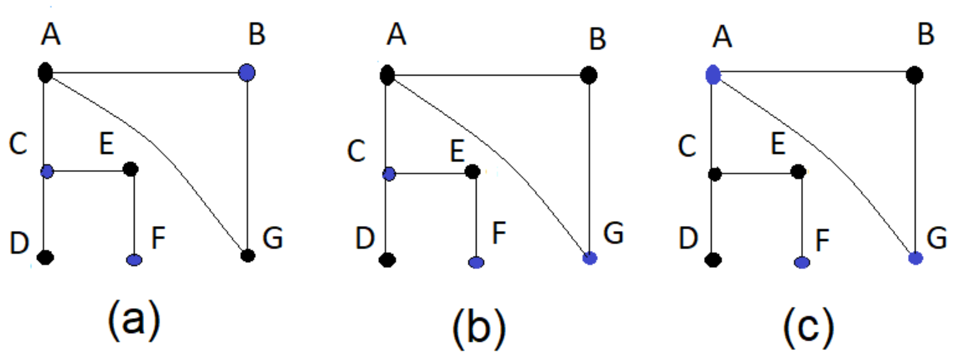


Figure 2.31: fig:-(a): $\{B, C, F\}$, fig:-(b):$\{C, F, G\}$ and fig:-(c):$\{A, F, G\}$ contains Independent Set as these vertices don't share any edges

A representation of independent set is presented in the figure 2.31. In 2.31, the vertices A, C, F (blue) are independent as these vertices are not adjacent to each other and in 2.31, the vertices C, F, G (blue) don't share any faces, so they are independent as well. Similarly, A, F, G (blue) vertices are independent for not sharing any face in 2.31.

Theorem 2.35 (Pigeon Hole Theorem). *If m items are placed among n containers, then there must be a container with at least $\lceil m/n \rceil$ items.*

2.3 Explanation of Bose et al. [12] Algorithm

Bose et al. [12] introduced an $\mathcal{O}\log(n)$ algorithm to transform any triangulation to any other triangulation having the same number of vertices by simultaneous flips. Bose et al. [12] also proved that every combinatorial triangulation of at least six vertices has a simultaneous flip into a 4-connected triangulation. They Bose et al. [12] further proved in this paper that it needs approximately $327.1 \log n$ simultaneous flips to transform from one triangulation to another. To transform one triangulation to another Bose et al. [12] showed the following steps.

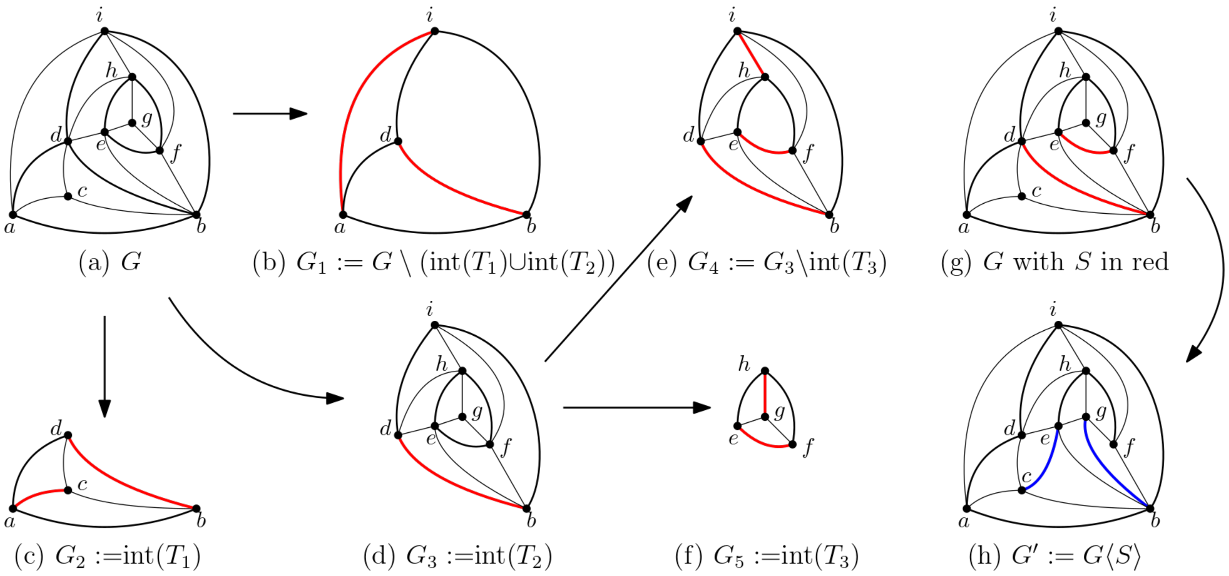
1. Transforming into Hamiltonian triangulation
2. Introduce the second dominant vertex
3. Introduce the second dominant vertex
4. Do the reverse to transform to get target triangulation

2.3.1 Transforming into Hamiltonian triangulation

To accomplish the triangulation, the first step is to transform any n-vertex triangulation into Hamiltonian triangulation. Bose et al. [12] proved that with only one simultaneous flip any triangulation with vertex number $n \geq 6$ can be converted into a hamiltonian triangulation. Let, S be a set of edges in a triangulation G with $n \geq 6$ vertices and no two edges in S occur in a common triangle as well as every edge of S belongs to a separating triangle. Also, in every separating triangle T, there must be an edge which exists in S. Then in the triangulation, S is simultaneously flippable and the new triangulation is 4-connected.

Whitney and Hassler [2] proved that every 4-connected graph is a hamiltonian triangulation. A triangulation G can be converted into a hamiltonian triangulation with only one simultaneous flip. For this, at first need to identify the separating triangles from the triangulation, then flipping one edge from each separating triangle converts the triangulation into hamiltonian triangulation. This whole process can be computed within $\mathcal{O}\log(n)$ time.

Every edge in set S is individually flippable and no two edges are incident in a common separating triangle. Therefore neither can they be blocked nor can they be bad pairs. Because if one edge is from a separating triangle T_1 and another one is from a different separating triangle T_2, to form a bad pair both edges from S must be seen from two distinct vertices and without loss of generality, one vertex should be inside and the other one is outside So it means both the edges are in the same separating triangles, that is, $T_1 = T_2$. But $T1$ and $T2$ both are different separating triangles. So there cannot be any bad pairs and blocking edges. Therefore, S must be flippable.



(a) G (b) $G_1 := G \setminus (\text{int}(T_1) \cup \text{int}(T_2))$ (e) $G_4 := G_3 \setminus \text{int}(T_3)$ (g) G with S in red

(c) $G_2 := \text{int}(T_1)$ (d) $G_3 := \text{int}(T_2)$ (f) $G_5 := \text{int}(T_3)$ (h) $G' := G\langle S \rangle$

Figure 2.32: Hamiltonian Triangulation Transformation Source: [14]

To obtain such a set of S where no edges are incident in a same separating triangle and at least one edge belongs to a separating triangle, perfect matching is convenient. According to Petersen [1], any graph of 3-regular or cubic which is bridgeless contains a perfect matching. In triangulation, all the faces are triangles, so every face of G is also a triangle, which means the dual graph of G is cubic. Again, the dual graph $G*$ is 2-edge-connected or can be said bridgeless as there is no loop in G and it is also 3-connected. Thus, dual graph $G*$ has a perfect matching that means from every face of G there is at least one edge which is the property of S, so none of the edges are consecutives.

Here, there are 3 separating triangles $T_1 = (a, b, d)$, $T_2 = (b, d, i)$ and $T_3 = (e, h, f)$ in G and the outermost triangles of G are T_1 and T_2. Next we are computing $G_2 := int(T_1)$, $G_3 := int(T_2)$ & $G_5 := int(T_3)$, also the outermost triangles of$G_1 := G\ (int(T_1)Uint(t_2))$. According to Bied et al. [9], by doing perfect matching on separating triangles have selected the non-consecutive edges. G_2 and G_3 both contain bd. Again, as T_3 belongs to G_3, so after computing we get $G_4 := G_3\ int(T_3)$ and $G_5 := int(T_3)$. From G_4 and G_5, we get the perfect matches ih and ef. Taking all the matched edges, established them on the graph G. After getting all the perfect matching of G, we define S as the set of perfect matching edges that are on a separating triangle. After flipping the edges bd & ef, we get the new edges ce & bg. Finally, we obtained

$G' := G(S)$ our 4-connected or hamiltonian triangulation.

2.3.2 Introducing the First Dominant Vertex

Hamiltonian cycle separates a hamiltonian triangulation into two subgraphs. One is called inner subgraph and the other one is called outer subgraph. Now, the target is to make one dominant vertex in each of the subgraphs. Thus, it will be become a canonical triangulation. Both inner and outer subgraphs are maximal outerplane graphs.

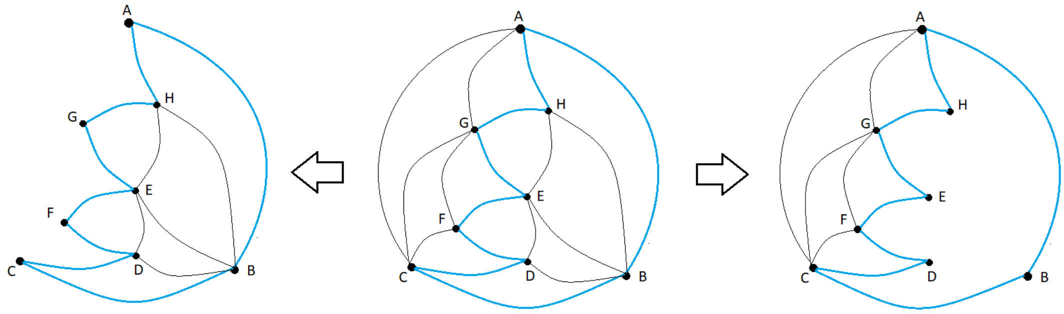


Figure 2.33: Main graph division into two subgraphs

If the number of flips required is equal to the diameter of the dual tree, then $\mathcal{O}\log(n)$ number of flips will be needed to make a vertex dominant. Later on, Bose et al. [12] increased the efficiency by reducing the number of flips. If the diameter of the dual tree can be made logarithmic, it will require much less number of flips. Also, we could spend $\mathcal{O}\log(n)$ simultaneous flips for transforming that outerplane graph into an outerplane graph with diameter $\mathcal{O}\log(n)$. So, now the target is to make the diameter logarithmic in order of the graph.
To transform the maximal outerplane graph into one whose dual tree has a diameter of $\mathcal{O}\log(n)$, Bose et al. [12] used a strategy where in each iteration they transformed a fraction of the vertices into degree 2 vertices. In a maximal outerplane graph each of these degree two vertices will correspond to an ear. By removing the ears from the graph, the graph can be made smaller. To reduce the degree of the vertices the following method will be followed.
At first, we need to find an independent set, I consisting of vertices with degree no more than 4. Now, we will create a flippable set, S by inserting one corresponding arbitrary chord of each vertex from set I where these chords have at least degree 3. Here, all the vertices in set S are non-consecutive since I is an independent set. Still, these edges can be blocked by external chords. To solve this problem, according to the properties of flippable set both blocking and blocked edge should be inserted in S. Again, external chords can be consecutive amongst them. To remove this connectivity Bose et al. [12] used 3-coloring rule. The highest number of external chords with same color will be inserted in S. At this point, S can have some blocking external without having its corresponding blocked chords in S. Hence, we will add those blocked chords in S. Again, there are are still some edges that are neither blocked nor blocking. We can insert these edges in S as well so that the number of flips in one simultaneous flip becomes highest.

Theorem 2.36 (Bose et al. [11] Theorem 8)**.** *According to Bose, Dujmović, and Wood [11],*

$$\alpha_d(OP_n) \geq \tfrac{d-3}{3d-6}n + \tfrac{2}{d-2} \text{ ; for all } d \geq 4 \text{ and } n \geq 5$$

Here, $\alpha_d(OP_n)$ is the largest cardinality of an independent set of any outerplane graph (OP_n); where n is the number of vertices

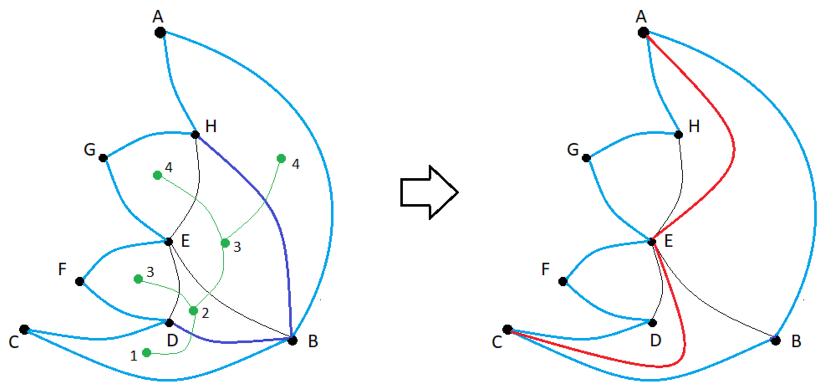


Figure 2.34: Dominant Vertex in Inner-subgraph

According to the theorem 2.36, the cardinality of I will be at least $\frac{n}{6}$. By using the pigeon hole principle 2.35 , Bose et al. [12] proved that at least half the blocking chords will be in set S, it can be assured that $\frac{1}{4}$ of the blocking chords will be in set S. In worst case, after 1st simultaneous flip $\frac{n}{18}$ ($\frac{1}{3}$ of $\frac{n}{6}$ vertices) vertices will have degree at most 3. Similarly, after second simultaneous flip, repeating the process $\frac{n}{54}$ ($\frac{1}{3}$ of $\frac{n}{18}$ vertices) vertices will have degree 2. This process ensures at least $\frac{n}{54}$ ears corresponding to distinct leaf nodes of the dual tree since every ear has one vertex of degree two. This process leads to a maximal outer-plane graph of at most $\frac{53n}{54}$ $(n - \frac{n}{54})$vertices.Here, after certain number of iterations the maximal outer-plane graph is smaller than 6 (since all graphs with order less than 6 are isomorphic) Now from recurrence relation we get,

$$
\begin{aligned}
2 + c_2 \log \tfrac{53n}{54} &= c_2 \log n \\
\rightarrow c_2 &= \tfrac{2}{\log \frac{53}{54}}
\end{aligned}
$$

So, the diameter of the dual tree is $c_2 \log n$. In the smaller graph the hamiltonian cycle is preserved and we can make any one of its vertex dominant with at most $c_2 \log n$ simultaneous flips.Then a vertex v is chosen by using following rules:

1. v is not incident to any external chord

2. v should have degree two in outer sub-graph

Now, the target is to make a dominant vertex in inner subgraph. A vertex E is chosen from inner subgraph which has degree two in outer subgraph so that it ensures there will be no chords of E in the outer graph. After choosing the vertex, with at most $c_2 \log n$ simultaneous flips, E will be the first dominant vertex.

2.3.3 Introducing the Second Dominant Vertex

As, outerplane subgraph is a maximal outerplane graph so it does not have any external chords. Like before, we will create an independent set I with the cardinality $\frac{n}{6}$.To create a flippable set S, one arbitrary chord from each vertex of I is included.But, none of them are blocked by external chords nor it'll be blocking any internal chords. No bad pair can be formed since its a maximal outerplane graph which does not contain any subdivision of K_4. So, the cardinality of S is $\frac{n}{6}$ in worst case. After two iterations, again all the vertices of I have degree no more than two which ensures $\frac{n}{6}$ corresponded ears. After completing the process, the maximal cycle becomes a 3-cycle Now we get,

$$2 + c_1 \log \frac{5n}{6} = c_1 \log n \text{ (From recurrence relation)}$$
$$\rightarrow c_1 = \frac{2}{\log(\frac{5}{6})}$$

So, the diameter of the dual tree= $c_1 \log n$ To make the second dominant vertex in the other outer plane graph, we remove the first dominant vertex with all of its incidents. So that the first dominant vertex, E is not part of the outer sub-graph anymore as a result v cannot have any internals chords in the outer plane graph.Again,at most $c_1 \log n$ simultaneous flips will require to make the any of its vertex dominant.

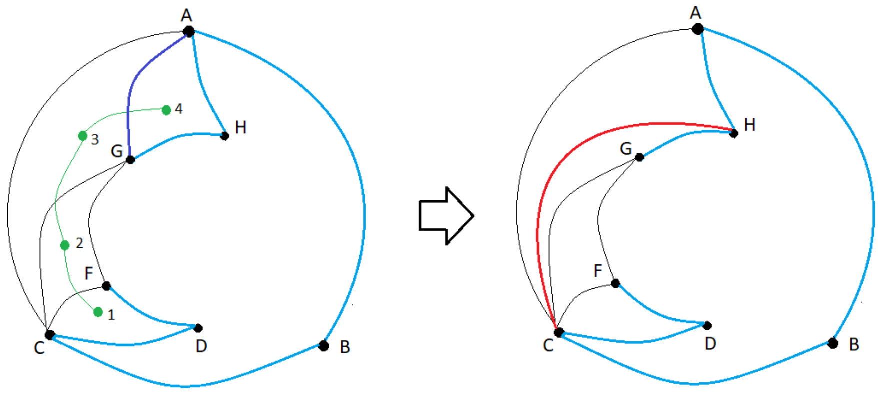


Figure 2.35: Dominant Vertex in Outer-subgraph

2.3.4 Reverse Transformation to Get to Target Triangulation

At this point,we have two dominant vertices in two sub-graph.By merging two sub-graphs we will get our canonical triangulation with two dominant vertices. Finally, to get our target triangulation, we can do the reverse as simultaneous flips is an reversible operation.

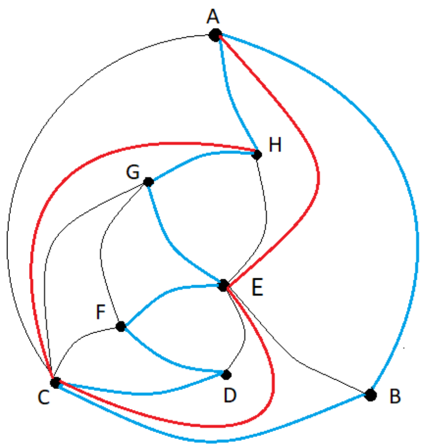


Figure 2.36: Target Triangulation

2.4 Improvement of Bose et al.'s [12] algorithm by Carufel and Kaykobad [14]

Carufel and Kaykobad [14] improved the algorithm of Bose et al. [12] based on the theorem 2.14. They improved the value of the leading coefficients c_1 and c_2 and lower the upper bound to $85.8 \log(n)$.

Definition 2.37 (Empty Cycle)**.** *Given graph G with a cycle C, if any of the regions created by the cycle is empty (contains no vertex), then we call such cycle as an empty cycle. The empty region is considered as local region and the other region is considered as foreign region.*

A representation of empty cycle is presented in figure 2.37. Here, $ADEA$ is an empty cycle since its internal region does not hold any vertex inside it and is considered as local region. The external region is the foreign region in this case. Again, $ADBCEA$ is a Hamiltonian cycle as well as an empty cycle since neither of its empty regions contain any vertices. Note that, every Hamiltonian cycle is an empty cycle. Besides, every cycle of the graph which fulfills the mentioned condition is considered as an empty cycle.

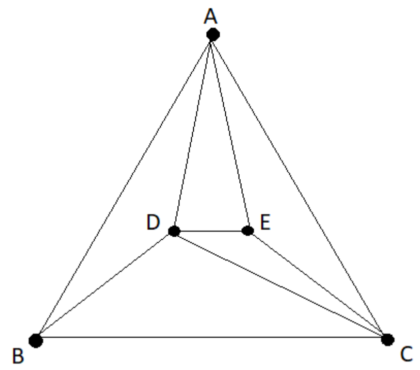


Figure 2.37: Empty Cycle

Definition 2.38 (Corner Chord)**.** *Given a graph $G = (V, E)$, a vertex $u \in V$ has three internal chords uv, uw and ux arranged in clockwise form whereas uv and ux are considered as corner chords.*

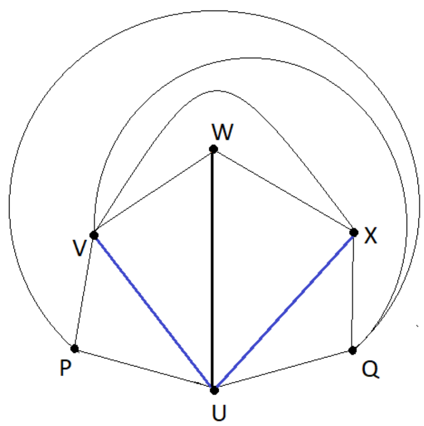


Figure 2.38: UV and UX are corner chords

An illustration of corner chord is given in figure 2.38. Here, the vertex U has three internal chords UV, UW and UX which are arranged in clockwise form. Therefore, UV and UX are corner chords (shown in blue).

2.4.1 Summary of Improvement

To create flippable set S, By using the theorem of Bose, Dujmović, and Wood [11],we choose an independent set I such that the cardinality of I is at least $\frac{2}{9}$, where the degree $d \geq 5$. Now, for each degree five vertices, two corner chords are inserted in set S and for each three and four vertices, one of its incident internal chords are inserted in the set S. The authors then show that S is pairwise non-consecutive and there are no blocked chords or bad pairs. So due to Theorem 2.14 the set S is flippable and after the first flip of S, the degree of all vertices with three and four are reduced by one and the vertices with degree five are reduced by two. We follow the process until we get a three cycle face which is a maximal outerplane graph. This process ensures at least $\frac{2}{9}n$ ears which corresponding to distinct leaf nodes of the dual tree since every ear has one vertex of degree two and the diameter of the dual tree is precisely two less than the previous one. Thus, to make the diameter of the resulting graph's dual tree smaller, it requires $\frac{2\log n}{\log\frac{1}{(1-\frac{2}{9})}}$ simultaneous flips and the diameter of the dual tree is $c_1 = \frac{2}{\log\frac{9}{7}}$.

Again, to create a flippable set S for inner subgraph at first we create an independent set I. Using the Theorem 8 of [11], we get the cardinality of I is $\frac{n}{6}$ where the degree $d \leq 4$. In set I_4, the degree of all vertices is 4 and one unblocked incident internal chord of each of these vertices will be inserted in flippable set S according to the Lemma 2 by Carufel and Kaykobad [15]. Using Lemma 3, 2, 1 of [15] all the conditions of Theorem 2.14 gets satisfied and thus S is a flippable set.
Every three and four degree vertices of I has one remaining internal chord in the resulting graph after performing S. All these chords are inserted to flippable set C' and all the blocked chords of C' are inserted to A'. According to Lemma 7 of [15], we get a flippable set N' subset of A' so that $|N'| \geq \frac{1}{2}|A'|$. Now, all the blocked and unblocked chords of C' those have blocking chords in N' is added in C'' and its cardinality is at least $\geq \frac{1}{2}|C'|$. As the edges of C'' and N' is in different regions separated by hamiltonian cycle so they cannot contain any pairwise edges which are consecutive and also $C'' \cup N'$ pairwise non-consecutive. According to the Lemma 3 of [15],there are no bad pairs in C'' and N' and also due to the Lemma 2.5 of Bose et al. [12], there are no bad pairs in C'' so that no bad pair can be formed in $C'' \cup N'$. Moreover, due to the Lemma 2.5 of Bose et a. [12],$C'' \cup N'$ are non-blocked as each blocked chords of C'' has its corresponding blocking chords in N', all the blocking chords has been its corresponding blocked of C''. Thus, $C'' \cup N'$ become a flippable set according to the Lemma 1 by Carufel and Kaykobad [15].Again, consecutive pair of vertices of any vertex of I gets connected to create a new chord. So, the degree of the vertices of I will remain same. Now, we have $\frac{n}{12}$ ears that corresponds to the vertices of I with degree two. We will follow the same process as mentioned in Lemma 9 by Carufel and Kaykobad [15] on a new cycle excluding $\frac{n}{12}$ earing vertices. It is ensured that the ears that has already been introduced will remain the same as we will flip the internal chords of the empty cycle. With at most $\frac{2\log n}{\log\frac{12}{11}}$, we can get the maximal outer plane graph and the diameter of its the dual tree is $c_2 = \frac{2}{\log\frac{12}{11}}$.
Therefore, the value of leading coefficient has been decreased using the Lemma 9 and 10 by Carufel and Kaykobad [15] in their paper. According to their improvement, any triangulation can be transformed to another with at most with $2 + 4(c_1 + c_2)\log n$ simultaneous flips where $c_1 = \frac{2}{\log\frac{9}{7}}$ and $c_2 = \frac{2}{\log\frac{12}{11}}$, using the similar process mentioned in the paper by Bose et al. [12] along with stated modifications.

2.5 Interlaced Algorithm Developed by Kaykobad [14]

In this section, the interlaced algorithm proved by Kaykobad [14] has been discussed briefly. The interlaced algorithm has four steps which are as follow -

1. Transform the triangulation into Hamiltonian triangulation
2. Make the dual tree of both the maximal outerplane graphs defined by the Hamiltonian cycle logarithmic in graph order
3. Create two dominant vertices to get canonical triangulation
4. Follow the steps in reverse to get the canonical triangulation to target triangulation

The first and last step of the interlaced algorithm are same as the algorithm given by Bose et al. [12] and the second as well as the third steps were improved by Kaykobad [14] by implementing preprocess (preparation method) and process (making ears) to the inner and outer subgraphs defined Hamiltonian cycle and making dominant vertices parallelly in the both maximal outerplane graphs respectively. Some of the relevant definitions and the procedure are discussed below.

Definition 2.39 (Earing Vertex)**.** *In a graph G^i, the vertices whose degree are decreased by 2 to convert them into ears is called earing vertices.*

Definition 2.40 (Avoiding Vertex, 2-Chord)**.** *In a given graph $G = (V, E)$ where the vertices $\{m, n, o\}$ creates an empty cycle L such that mn shares a triangulation with the vertex o and creates the edges mo and no where o is the avoiding vertex corresponding to mn and mn is the 2-chord of the vertex o as well as the empty cycle.*

Definition 2.41 (Avoiding Vertex Set)**.** *In a triangulation, the avoiding vertex set (I^i_{avoid}) is the set of all the avoiding vertices of the graph G^{i+1}_{ear}.*

2.5.1 The Process of Making the Diameter of Dual Tree Logarithmic to Order

The objective of making the diameter of the dual tree of both the inner and outer subgraphs logarithmic to order $O(\log n)$ is reducing the number of flips needed to make a vertex dominant. The process begins with a Hamiltonian triangulation G_H which consists a Hamiltonian cycle H. In every iteration, two simultaneous flips take place in the mentioned graph. At the beginning of the i-th iteration, the graph is symbolized as G^i. The first iteration is starting with G_H, therefore, $G^1 = G_H$. There are two empty cycles present in G_H which are C^1_{ear} and C^1_{prep} defined by H. The subgraph which consists C^i_{ear} is considered as G^i_{ear} and the other subgraph G^i_{prep} contains C^i_{prep}. In a single iteration, two different processes are being conducted parallelly:

1. Make ears in the graph G^i_{ear} and
2. Prepare G^i_{prep} for the next iteration where $G^i_{prep} = G^{i+1}_{ear}$.

An independent vertex set I^i_{ear} containing the earing vertices is defined in G^i_{ear} in every iteration. The target is to reduce the degree of these vertices to 2 and make them ears. At the same moment, in G^i_{prep} an avoiding vertex set I^i_{prep} is identified through preparation method to avoid the vertices (without compromising the cardinality). The motive is to avoid these vertices from making ears so that they can not create blocking chords during making ears in the next iteration. In a single iteration two simultaneous flips S^i_1 and S^i_2 take place. Mainly, this S^i_1

consists $S^i_{ear_1} \cup S^i_{prep_1}$. Similarly, S^i_2 consists $S^i_{ear_2} \cup S^i_{prep_2}$. The ears are created through $S^i_{ear_1}$ and $S^i_{ear_2}$, at the same time preparation method takes place through $S^i_{prep_1}$ and $S^i_{prep_2}$. This clearly defines while creating ears in a subgraph G^i_{ear}, the other subgraph G^i_{prep} gets prepared to become G^{i+1}_{ear} in the next iteration. It is important to mention that since the first iteration doesn't have any preparation method, the first iteration takes place in a different manner. As we already know, maximal outerplane graph on n vertices are isomorphic where $n \leq 5$, so the flipping process needs to stop when G^i_{ear} has maximum 5 vertices. Hence, S^i_{ear} is made an empty set. After having maximum 5 vertices in both G^i_{ear} and G^i_{prep}, the process is completed and we get the resulting graph G'_H. During this procedure, the hamiltonian cycle H is preserved in G'_H and the dual tree of the subgraphs $G'_{H_{interior}}$ & $G'_{H_{exterior}}$ are $O(\log n)$. In $i-th$ iteration, we get $G^{i'}$ after the first simultaneous flip, $G^{i'} = G^i\langle S^i_1\rangle$ and we get $G^{i''} = G^{i'}\langle S^i_2\rangle$ after completing the second simultaneous flip. $G^{i''}$ is the initial state of the following iteration, G^{i+1}. However, $G^{i''}_{prep}$ becomes the G^{i+1}_{ear} in $(i+1)$-th iteration. When we get the expected ears from $G^{i''}_{ear}$, we exclude the earing vertices $(degree - 2)$ from the $G^{i''}_{ear}$. Thus it works as the G^{i+1}_{prep} in upcoming iteration. Therefore, the cycle $C^{i''}_{ear}$ after exclusion of the earing vertices is the same cycle as C^{i+1}_{prep} in $(i+1)$-th iteration. In other words, the processing and prepossessing regions are interchanging their regions between them in every iteration.

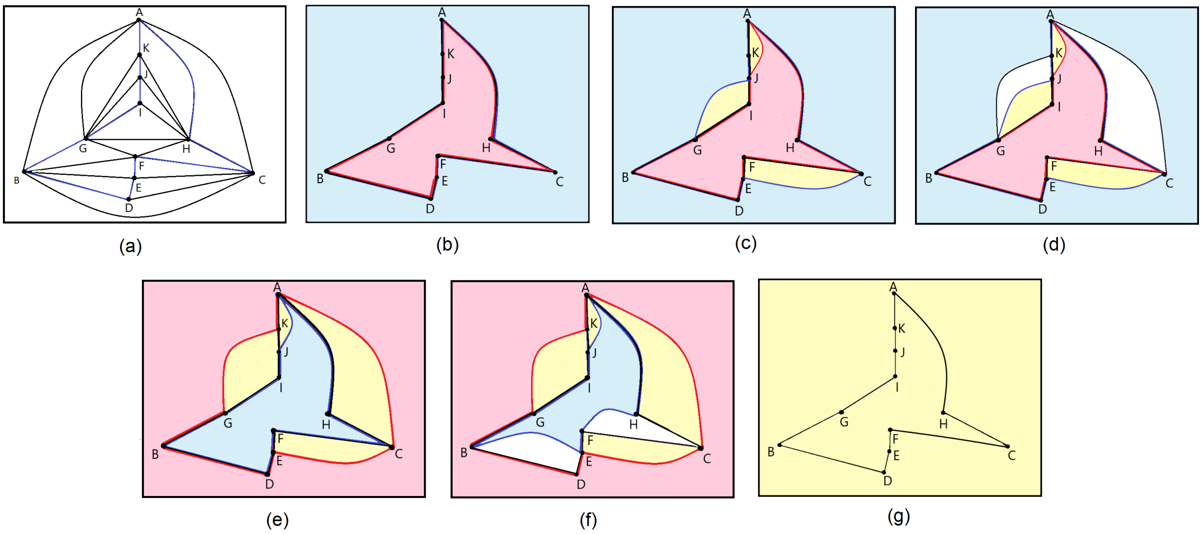


Figure 2.39: Transforming the diameter to logarithmic

Figure 2.39 is an illustration of the process of transforming the diameter to Logarithmic. Here, figure 2.39 (a) is a Hamiltonian triangulation G_H which is our main graph. The Hamiltonian cycle H creates two subgraphs G_{ear} (denoted by blue region) and G_{prep} (denoted by pink region) in figure 2.39 (b). There is no processed region (denoted by yellow) in this figure. The Hamiltonian cycle H along with other cycles C^x_{ear} and C^x_{prep} where $1 \leq x \leq i$ is a subgraph of G^{i+1}. The figure 2.39 (c) is the initial figure of i-th iteration of G^i along with some processed regions and with each iteration this region covers more graph. In the figure 2.39 (d), $G^{i''}$ is the final phase of the i-th iteration as well as the preliminary state of the $(i+1)$-th iteration where after the both S^i_1 and S^i_2 simultaneous flips, ears (denoted by white region) has been produced. The figure 2.39 (e) denotes as G^{i+1} whereas both regions in blue and pink swaps between them and as a result, the blue region where before ears created has become now the G^{i+1}_{prep} and on the contrary, the pink region has converted into G^{i+1}_{ear}. Further, the figure 2.39 (e) is exact same as the figure 2.39 (d) except here the ears has been processed and transformed into yellow region. The next figure 2.39 (f) represent the graph $G^{(i+1)''}$ of the $(i+1)$-th iteration where two simultaneous flips S^{i+1}_1 and S^{i+1}_2 has occurred, also ears BDE and CFH has been

generated in interior of the G_H which symbolizes the blue region of the graph. Following the final iteration G^x, the graph has processed in the figure 2.39 (g) and the diameter of the dual tree of both interior and exterior of $G^x\{H\}$ has become $O(\log n)$.
In every i-th iteration without the first iteration, the diameter reduces by exactly 2 from the diameter of G^i_{ear} to G^{i+1}_{prep} and the order reduces by at most $\frac{5}{6}$ times. Finally, the diameter becomes $\frac{2\log n}{\log \frac{6}{5}}$ after reducing the diameters of the dual tree in both regions.

2.5.2 Introducing Dominant Vertices

In this paper, the diameter of the dual tree of both inner and outer subgraphs are same. Thus we need same number of simultaneous flips to make distinct dominant vertex in both of the graphs. In the paper written by Bose et al. [12] we have seen that we nee $c_1 \log n$ and $c_2 \log n$ simultaneous flips. However, Kaykobad [14] in his paper improved the leading coefficient to make c_1 and c_2 equal. The researcher worked on the graphs with more than 5 vertices since the lower order graphs already has 2 dominant vertices. After dividing the main graph into two subgraphs according to the hamiltonian cycle we get one distinct corresponding empty cycle in both of the the sub-graphs. Inner subgraph has inner empty cycle and outer subgraph has outer empty cycle. Both of the cycles have minimum 2 foreign 2-chords, with one avoiding vertex for each of them because both of the dual trees have minimum two leaf nodes. We will chose two distinct vertices from avoiding vertices to make them dominant in inner and outer subgraph respectively. We will remove the first candidate from the outer subgraph. Now we need to make the vertices dominant by using simultaneous flip operation from the ordered sets of simultaneous flip as done in the paper by Bose et al. [12].

- Inner and outer maximal outerplane subgraphs are non-overlapping and thus the edges in the flippable are not consecutive.
- Since the the vertex chose to be dominant in inner subgraph is removed from outer subgraph, the vertex is not a seeing for the edges of the outer subgraph. Only for the chords inserted in the flippable set of inner subgraph, this vertex is the common seeing.
- For the same reason mentioned in point 2, two edges of the flippable sets can not block each other.

So, the three conditions of Theorem 2.14 are satisfied. So the set is flippable and the triangulation can be made canonical triangulation.

2.5.3 Complexity of the Interlaced Algorithm

Bose et al. [12] claimed in his paper that only one simultaneous flip is required to transform one triangulation to Hamiltonian triangulation. Therefore, Kaykobad [14] make the diameter logarithmic with $\frac{4\log n}{\log \frac{6}{5}}$ simultaneous flips and the diameter of the dual tree become $\frac{2\log n}{\log \frac{6}{5}}$ for the both of the inner and outer subgraph. To make two dominant vertex parallelly in both inner and outer subgraph, at most $\frac{2\log n}{\log \frac{6}{5}}$ simultaneous flips are required and same number of flips are enough to convert the canonical triangulation to the target triangulation. Therefore, with at most $2 \times (\frac{4\log n}{\log \frac{6}{5}} + \frac{2\log n}{\log \frac{6}{5}}) \approx 45.6 \log n$ simultaneous flips any triangulation can be transformed to other.

Chapter 3

Algorithm

Before presenting our algorithm we first start with some definitions and list out the existing lemmas and theorems we use for the proof of correctness of our algorithm.

3.1 Definitions and Necessary Lemmas

In previous chapters, we have gained some insights to improve the existing algorithms and are in the process of establishing our algorithm. In this chapter, the process of making pre-selected vertices dominant parallelly in inner and outer subgraphs has been described. Let us present some of the relevant definitions which are explained below.

Definition 3.1 (Prov Chord)**.** *Given a Triangulation T with a Hamiltonian cycle H which defines two maximal outerplane graph G_1 and G_2 where D_1 and D_2 are selected to be the dominant vertices in G_1 and G_2 respectively. If any chord e except the edge of H is flipped and the resulting chord increases the degree of D_1 in G_1 or D_2 in G_2, then that mentioned chord is called $prov_{in}$ or $prov_{out}$ chord respectively.*

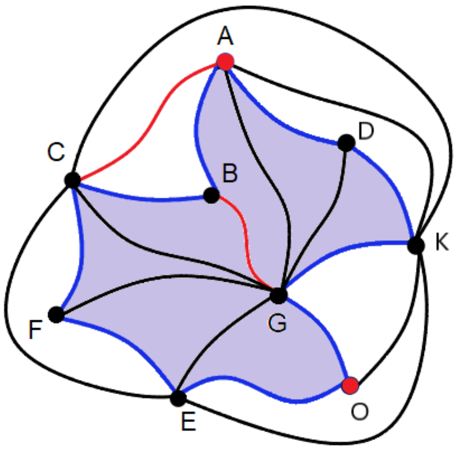


Figure 3.1: Here, the shaded region is the inner subgraph G_1 and the rest is the outer subgraph G_2. A and O are to be dominant internally and externally respectively. For A, BG is a $prov_{in}$ chord and AC is its corresponding $obst_{out}$ chord (highlighted in red)

A demonstration of *prov* chord is illustrated in figure 3.1. In this figure, A is to be dominant in inner subgraph. On t-th iteration, BG is $prov_{in}$ chord because flipping BG increases the degree of A in inner subgraph.

Definition 3.2 (Obst Chord)**.** *Obst chord is a blocking chord of a prov chord. The obst chord is denominated as* $obst_{in}$ *and* $obst_{out}$ *chord when it exists in the internal subgraph and the external subgraph respectively.*

An illustration of *obst* chord is presented in figure 3.1. Here, A is to be dominant in inner subgraph and BG is $prov_{in}$ chord on t-th iteration where AC is a blocking chord of BG. Therefore, AC is its corresponding $obst_{out}$ chord. Since, the *obst* chord AC exists in outer subgraph, it's called $obst_{out}$ chord.

Definition 3.3 (Eccentricity)**.** *Given a graph* $G = (V, E)$ *where eccentricity* e *of a vertex* $v \in V$ *is the maximum distance to any other vertex* $u \in V$ *from* v*. Eccentricity* $e(v) = max\{distance(u, v)\,|u \in V(G)\}$*.*

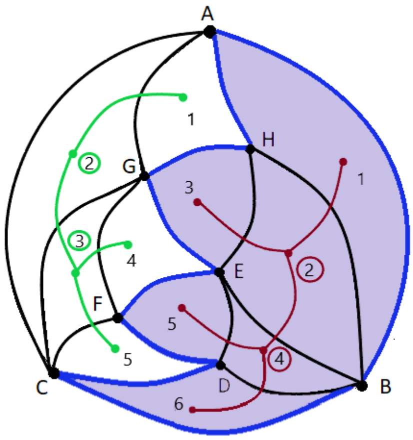


Figure 3.2: Here, the shaded region is the inner subgraph G_1 and the rest is the outer subgraph G_2. Considering vertex 1 of G_1^*, the eccentricity is 3. The dual radius of G_1^* is 2 due to the vertex 2 and 3. Therefore, the center of G_1 is $f = \{BEH, BDE\}$. Considering B as the target dominant vertex, dual eccentricity for B is 2

A representation of eccentricity can be seen in figure 3.2. Here, graph G has been divided into inner subgraph G_1 and outer subgraph G_2 by the Hamiltonian cycle H (highlighted in blue). G_1^* and G_2^* are the dual trees of G_1 and G_2 respectively. To demonstrate the eccentricity, vertex 1 of G_1^* is being considered. The longest distance to any other vertices of G_1^* from vertex 1 is 3. Therefore, 3 is the eccentricity of vertex 1. Note that, eccentricity of any vertex v of any graph G can be found in similar way.

Definition 3.4 (Dual Radius)**.** *Given a maximal outerplane graph* G *and its dual tree* G^**, the* dual radius *of* G *is the length of the shortest longest path in amongst all vertex pairs in* G^**.*

An illustration of dual radius is presented in figure 3.2. Here, graph G has two maximal outerplane graph G_1 and G_2 defined by Hamiltonian cycle H. G_1^* and G_2^* are the dual trees of G_1 and G_2 respectively. Considering G_1^*, the dual radius is 2. It is the lowest eccentricity among all the vertices of G_1^*. Similarly, the dual radius of G_2^* is 2.

Definition 3.5 (Center)**.** *Given a maximal outerplane graph* G*, let* f *be a set of faces in* G*, whose corresponding vertices have minimum eccentricity in the dual tree* G^* *of* G*. Then we call the set of faces* f *as the center of* G*.*

An illustration of center is presented in figure 3.2. Here, G_1^* and G_2^* are the dual trees of G_1 and G_2 respectively. Since dual radius of G_1^* is 2 due to the vertex 2 and 3. Therefore, the set of faces $f = \{BEH, BDE\}$ is the center of G_1. In similar manner, $f = \{AGC, GFC\}$ is the center of G_2.

Definition 3.6 (Dual Eccentricity of a vertex). *Given a maximal outerplane graph G and a vertex v in G, let f be the set of faces that v is on, excepting the outerface. Then the dual eccentricity of v is the maximum of all the shortest paths of the corresponding vertices of f in the dual tree G^*.*

A representation of dual eccentricity is illustrated in figure 3.2. Here, the center of G_1 is $f = \{BEH, BDE\}$. Selecting B as the target dominant vertex for G_1. Therefore, dual eccentricity for B is 2.

Definition 3.7 (Interrupted Chord). *Given an ordered set of simultaneous flip set S, let e be a chord in its t-th flipset S_t. Then e is an* interrupted *chord in $S_t \in S$ if it is removed from S_t and is inserted in S_{t+1}. When a chord is interrupted in S_t, we also interrupt any chord blocked by it in S_t at the same time as well. An interrupted chord is called an* interrupted prov chord *or an* interrupted obst chord, *depending on whether it is a prov chord or an obst chord.*

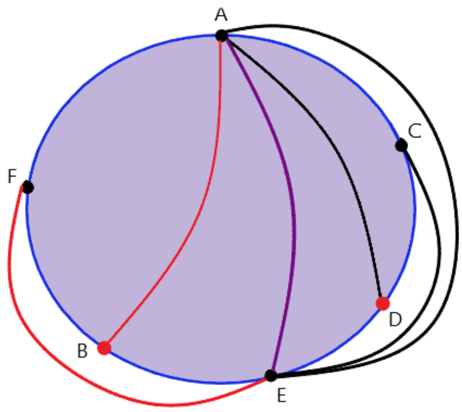


Figure 3.3: Here, the shaded region is the inner subgraph G_1 and the rest is the outer subgraph G_2. AE is an interrupted chord (shown in violet)

A demonstration of interrupted chord is represented in figure 3.3. Here, D and B are selected to be dominant in internal and external subgraphs respectively. On t-th iteration, AE is $prov_{in}$ chord and FE is $prov_{out}$ where AB is its corresponding $obst_{in}$ chord. AB and AE are consecutive edges. Therefore, AE is skipped from flipping on t-th iteration and inserted in the flip set of $(t+1)$-th iteration. Here, AE is an interrupted chord (highlighted in violet).

Definition 3.8 (Safe Chord). *Let G be a triangulation with a Hamiltonian cycle H that separates the graph into two maximal outerplane graphs G_1 and G_2. Furthermore, let D_1 and D_2 be the two vertices to be made dominant in G_1 and G_2 respectively and S be the set of ordered flipsets for G where we are currently flipping the set $S_t \in S$. Let e be a chord in S_t that WLOG is present in G_1. Then a corresponding safe chord e' of e in S_t is a consecutive chord of e such that e' is not blocked in G and flipping e' does not decrease the degree of D_1 (the vertex to be made dominant in its respective maximal outerplane graph).*

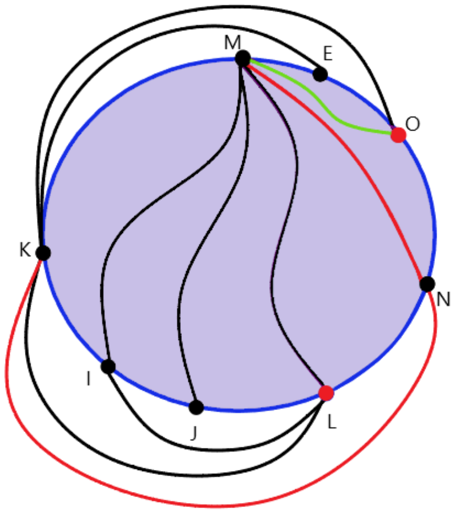


Figure 3.4: Here, the shaded region is the inner subgraph G_1 and the rest is the outer subgraph G_2. MO is a safe chord (shown in green) which is consecutive to MN

A demonstration of safe chord is showed in figure 3.4. Here, O and L are selected to be dominant in G_1 and G_2 respectively. In the $t-th$ iteration, MN and KN are $Prov_{in}$ and $Prov_out$ chords respectively. Moreover, MO is a safe chord of MN which is consecutive to MN and flipping MO does not decrease the degree of L.

Now, the important lemmas given by other researchers, those have been used in our algorithm are mentioned below.

Lemma 3.9 (Lemma 2.5 of Bose et. al [12]). *Suppose that vw and xy are a bad pair in a triangulation G, both seen by vertices p and q. Suppose that vw blocks some edge ab. Then xy and ab are consecutive, and vw and xy are in a common triangle (amongst other properties).*

Lemma 3.10 (Lemma 4.2 of Bose et. al [12]). *A set S of internal edges in a maximal outerplane graph G is flippable if and only if the corresponding dual edges S^* form a matching in G^*.*

Lemma 3.11 (Lemma 7 of Carufel and Kaykobad [15]). *Let G be a triangulation with an empty cycle C, with a set S of blocking external chords of C. Then at least half of the chords in S can be simultaneously flipped.*

3.2 Proposed Algorithm

Suppose we want to transform a triangulation $G_{initial}$ to G_{target}, we consider both maximal outerplane graphs of a triangulation $G_{initial}$ defined by the Hamiltonian cycle H. After finishing the process of making the diameter logarithmic proposed by Kaykobad [14], the diameter of dual trees of both the maximal outerplane graphs are at most $2\frac{\log n}{\log \frac{6}{5}}$. We make an improvement by introducing a process of choosing a pair of vertices randomly to be dominant simultaneously in two subgraphs with another $4\frac{\log n}{\log \frac{6}{5}}$ simultaneous flips where $\frac{\log n}{\log \frac{6}{5}}$ is the radius of the dual tree. The proposed algorithm consists of the following steps:

1. Transform initial triangulation $G_{initial}$ into a Hamiltonian triangulation T where its Hamiltonian cycle H separates T into two maximal outerplane graphs G_1 and G_2.

2. Transform T into $T\prime$ preserving Hamiltonian cycle H such that the diameters of the dual tree of both G_1 and G_2 are $O(\log n)$.

3. Transform $T\prime$ into a canonical triangulation Δn by simultaneously introducing two dominant vertices, one in each of the maximal outerplane subgraphs.

4. Follow the previous three steps in reverse order to obtain the target triangulation G_{target}.

Step 1 of the proposed algorithm is done exactly as described by Bose et al. [12]. Step 2 is done exactly as described in the paper written by Kaykobad [14]. Step 3 is explained in Section 3.3 and 3.4. Simultaneous flips are inversible operations. Thus to complete the transformation of $G_{initial}$ to G_{target}, in step 4 we execute the inverse process of producing $G_{initial} \rightarrow \Delta n$ on the canonical triangulation that is generated in Step 3.

3.3 Overview of the Algorithm

Given a triangulation T with a Hamiltonian cycle H that defines two maximal outerplane graphs G_1 and G_2. We choose two arbitrary vertices D_1 and D_2 to be dominant in the internal subgraph G_1 and external subgraph G_2 respectively from the center of the triangulation. Our algorithm can be useful to make any arbitrary pair of vertices of the triangulation to be dominant. However, for better efficiency we choose vertices from the center. Let, i_1 and i_2 denote the dual eccentricity of D_1 and D_2 in G_1 and G_2 respectively. Since it does not require any flips to nominate any existing face to be the outer face, for convenience of demonstrating the algorithm, we pick the outerface of the graph such that the external region's dual eccentricity i_2 is the larger value between i_1 and i_2. Thus, $i_1 \leq i_2$. In the following algorithm, $S_{in} = \{prov_{in}, obst_{in}\}$ and $S_{out} = \{prov_{out}, obst_{out}\}$ have been considered as the ordered set of simultaneous flip sets of G_1 and G_2 respectively. The total flip set S is the union of S_{in} and S_{out}. Then, every element of the ordered set of simultaneous flip set S consists of $\{prov_{in}, prov_{out}, obst_{in}, obst_{out}\}$. In this algorithm, we will prove that S is flippable and $2 \times (i_1 + i_2)$ flips are sufficient where $i = max(i_1, i_2)$ to introduce two dominant vertices in the resulting graph.

Lemma 3.12. *Given a triangulation T with a Hamiltonian cycle H that defines the two maximal outerplane graphs G_1 and G_2 where D_1 and D_2 are selected to be the dominant vertices in G_1 and G_2 respectively. The blocking chords of the corresponding prov chords in G_1 and G_2 are defined as obst chords. Hence, obst chords are always flippable.*

Proof. In G_1 and G_2, for any $prov_{in}$ or $prov_{out}$ chord on the t-th iteration, if there is any corresponding blocking chords, then those blocking chords are inserted in the flippable set S_t as $obst_{in}$ or $obst_{out}$ chord. According to the lemma 2.4 stated by Bose et. al [12], *obst* chords are always flippable.

□

3.4 Algorithm

We first present the pseudocode of the algorithm, followed by an explanation of how it works.

Algorithm 1 Process of Making Dominant Vertices

procedure CREATE-FLIPPABLE-SET
  Let i_1 and i_2 be dual eccentricity from the vertex D_1 and D_2 respectively of G_1 and G_2 such that $i_2 \geq i_1$
  Let iterator, $t = 1$
  while $i_2 > 0$ **do** $\triangleright$ When i_2 becomes 0, i_1 must also be 0 and there must be two dominant vertices in G
    Let $S_{in_t} = prov_{in} \cup obst_{in}$ on the t-th iteration
    Let $S_{out_t} = prov_{out} \cup obst_{out}$ on the t-th iteration
    Let $S_t = S_{in_t} \cup S_{out_t}$
    if $i_1 > i_2$ **then**
      swap G_1 & G_2
      swap i_1 & i_2
      swap S_{in} & S_{out}
    end if
    $success$ = Flip (S_t)
    if success **then**
      Decrease i_1 and i_2 by 1
    else
      Decrease i_2 by 1
    end if
    Increase t by 1
  end while
end procedure

procedure FLIP(S_t)
  $success := True$
  if There exists two chords $e_1 \in prov$ chords and $e_2 \in obst$ chords such that e_1 and e_2 are consecutive **then**
    Remove interrupted chords from S_t
    $success := False$
    Insert the interrupted chords in S_{t+1}
  end if
  if There exists two *prov* chords $e_1 \in S_{in}$ and $e_2 \in S_{out}$ such that e_1 and e_2 form a bad pair **then**
    if Safe chord e_3 exist **then**
      Replace any *prov* chord $e \in \{e_1, e_2\}$ with e_3 in S_t
    else
      Remove any *prov* chord $e \in \{e_1, e_2\}$ from S_t
    end if
    Insert e in S_{t+1}
  end if
  if S_t contains *obst* chords that are consecutive **then**
    Separate the edges in S_t into S_{t_1} and S_{t_2} such that for each pair of *obst* chords that are consecutive, only one of them and their corresponding *prov* chord are in S_{t_1} and the other and its corresponding *prov* chord are in S_{t_2}.
    Flip S_{t_1}
    Flip S_{t_2}
  else
    Flip S_t
  end if
  Return $success$
end procedure

Procedure $Create - Flippable - Set$ takes two maximal outerplane subgraphs G_1 and G_2 defined by Hamiltonian cycle H in a triangulation T. WLOG $i_2 \geq i_1$ where i_1 and i_2 are the dual eccentricity from the target vertices D_1 and D_2 respectively of G_1 and G_2. Next, iterator t is initialized which is for keeping the track of the iteration number. The following procedure keeps continuing itself until $i_2 = 0$ because i_2 becomes 0 when two dominant vertices are already established in G. Thus, the procedure should be stopped. S_{in_t} and S_{out_t} consist of $prov_{in}$, $obst_{in}$ and $prov_{out}$, $obst_{out}$ chords respectively of the t-th iteration. Finally, $S_t = S_{in_t} \cup S_{out_t}$. After initializing all of the stated factors, the first thing is to check that whether $i_1 > i_2$. If so, then G_1 and G_2 are to be swapped along with their dual eccentricity i_1 and i_2. Also, flippable sets S_{in} and S_{out} are to be swapped with each other. Afterwards the flag *success* is initialized, the procedure calls another procedure $Flip$ which is going to be explained further and the return result is stored in *success*. The next thing is to check if success is $True$. If so, i_1 and i_2 are decreased by 1, otherwise only i_2 is decreased by 1. The final step is to increase t by 1 which means one successful iteration has been completed.
Procedure $Flip$ takes S_t where S_t is the flippable set for t-th iteration. Basically, the flips are executed in this part according to S_t after being ensured that S_t is flippable. At first *success* is made $True$. This flag is used to keep the track of flip penalization of G_1. However, the task is to flip S_t, S_t needs to be freed from blocked edges, consecutive edges and bad pairs. S_t already contains the *obst* chords of the corresponding *prov* chords which solves the issue of blocked edges. So, next the procedure checks if there exists any consecutivity case, then it removes the interrupted chords from S_t, *success* is made $False$ and the interrupted chords are inserted in S_{t+1}. The following thing is to inspect that whether e_1 and e_2 form a bad pair such that e_1 and e_2 are *prov* chords of G_1 and G_2 respectively. Bad Pair formation only takes place when D_1 and D_2 try to get connected in the same iteration. To handle this scenario, any of the Prov chords $e \in \{e_1, e_2\}$, is replaced with a safe chord e_3 if and only if e_3 exists in that orientation. If e_3 does not exist, simply e is removed from S_t and inserted in S_{t+1}. Finally, the procedure checks if S_t contains *obst* chords those are consecutive then, S_t is split into S_{t_1} and S_{t_2} such that each set contains only one of the *obst* chords and their corresponding *prov* chords from the consecutive *obst* chord pairs. Afterwards, S_{t_1} and S_{t_2} are flipped. Otherwise, S_t is flipped. Finally, the $Flip$ procedure ends with returning the value of success. Therefore, all of the problems that could arise in the whole process are handled through the necessary condition and solution. Consequently, S_t is ensured as a flippable set.

3.5 Proof of the Algorithm

Lemma 3.13. *By running Algorithm 1, when executing a simultaneous flip, the flipset does not contain any consecutive chords.*

Proof. Let us assume that there exist a pair of consecutive chords in S_t. Since the chords in G_{in} and G_{out} are separated by the Hamiltonian cycle H, no chord in G_{in} can be consecutive to any chords in G_{out}. Hence, WLOG, let us assume that the pair of consecutive chords (e_1 and e_2) are in G_{out}.
Both e_1 and e_2 cannot be $prov_{out}$ chords due to the Lemma 3.10 of Bose et al. [12]. In Lemma 3.10, Bose et al. proved that the chords that increase the degree of the dominant vertices in the flipset, does not contain consecutivity among them. By definition 3.1, the *prov* chords increase the degree of the dominant vertices, therefore $prov_{out}$ chords cannot be consecutive. Moreover, e_1 and e_2 can be $obst_{out}$ chords. To handle the situation, it is enough to follow the method presented in the Lemma 3.11 of Carufel and Kaykobad. Due to Lemma 3.11, we sacrifice at most half of those *obst* chords and their corresponding *prov* chords. According to Algorithm1 line 38, S_t is separated into S_{t_1} and S_{t_2} where e_1 and its corresponding $prov_{out}$

chord are in S_{t_1} and e_2 and its corresponding $prov_{out}$ chord in S_{t_2}.

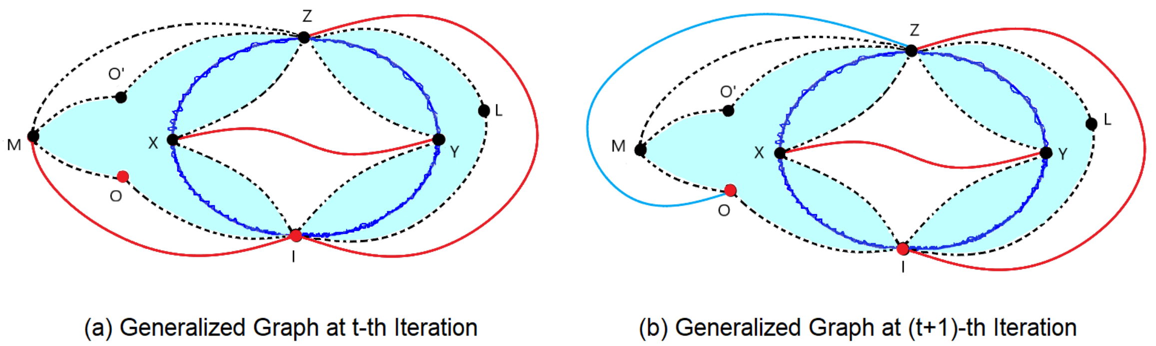


Figure 3.5: Consecutivity Analysis Case-1: At t-th iteration, XY is a $prov_{in}$ chord (red) in G_1 and its corresponding $obst_{out}$ chord is IZ(red) where IM is a $prov_{out}$ chord (red) in G_2. IM and IZ are consecutive to each other. To eradicate this situation, XY is removed with IZ from the flip set S_t and inserted both chords to S_{t+1}. Flipping IM results OZ (sky blue). The zigzagged curve (blue) is a Hamiltonian cycle and shaded regions (paste) are further triangulated.

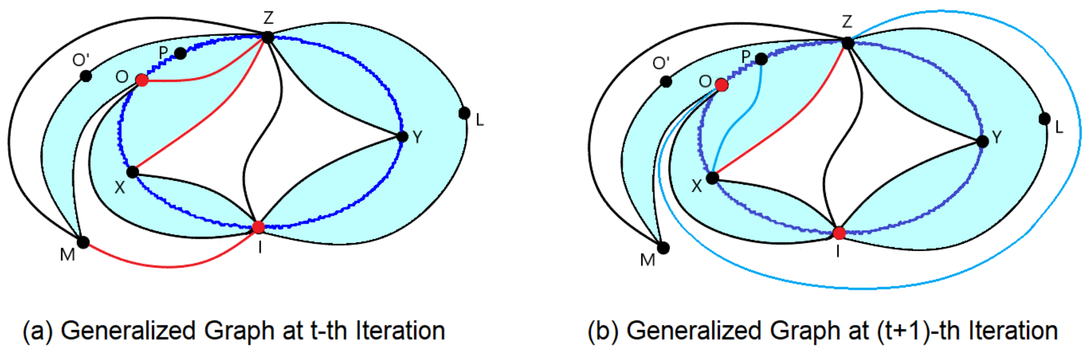


Figure 3.6: Consecutivity Analysis Case-2: IM is a $prov_{out}$ chord (red) in G_2 and XY is a $prov_{in}$ chord (red) and its corresponding $obst_{in}$ chord OZ (red) in G_1 at t-th iteration. To handle this situation, xz is eliminated from the flip set S_t and is inserted in S_{t+1}. Flipping IM and OZ, results in OZ and PX (sky blue). Shaded regions are further triangulated and the Hamiltonian cycle is drawn in blue (the zigzagged curve).

WLOG, let e_1 be an $obst_{out}$ and e_2 be a $prov_{out}$ chord in Algorithm 1. Since we only flip $prov$ chords and $obst$ chords, now we prove the final case by contradiction. Let us assume that at iteration t, a pair of $prov_{out}$ chord and $obst_{out}$ chord are consecutive in the flip set S_t due to Algorithm 1 line 30. Similarly, $prov_{in}$ chord and $obst_{in}$ chord can also be consecutive in S_t. WLOG, in figure 3.5 let I and O be the two vertices that are selected to be dominant in G_1 and G_2 respectively. On the t-th iteration, the flippable set S_t contains $prov_{in}$ chord XY, its corresponding $obst_{out}$ chord IZ and $prov_{out}$ chord IM. The $prov_{out}$ chord IM and $obst_{out}$ chord IZ are consecutive in G_2. Thus, we cannot flip both IM and IZ at the same iteration. To avoid this situation, $prov_{in}$ chord xy with its corresponding $obst_{out}$ chord IZ is removed from S_t in Algorithm1 line 26 and is inserted in S_{t+1} in Algorithm1 line 28. On the other hand, in figure: 3.6, on the t-th iteration, the flippable set S_t contains $prov_{in}$ chord XZ, $prov_{out}$ chord IM and its corresponding $obst_{in}$ chord OZ where XZ and OZ are consecutive to each other in G_1. Here, to handle the situation, G_1 is being penalized by removing the $prov_{in}$ chord from S_t and added to S_{t+1}. In both cases, the removed chords are called as interrupted chords.

To eliminate consecutivity from S_t, the interrupted chords are delayed and G_1 is always being penalized. The consecutivity case can occur more than once in T and whenever this case occurs, it is always handled by penalizing G_1. Therefore, there is no consecutivity in the flip set S_t. Since, $\{S_{t_1}, S_{t_2}\} \subset S_t$ hence S_{t_1} and S_{t_2} do not contain consecutivity as well.

□

Lemma 3.14. *Given a Triangulation T with a Hamiltonian cycle H which defines two maximal outerplane graphs G_1 and G_2 respectively where we have a pair of prov chords $e_1 \in G_1$ and $e_2 \in G_2$ form a bad pair. Then each of e_1 and e_2 have a corresponding safe chord in T and safe chord is flippable.*

Proof. According to the lemma 2.14 given by Bose et. al [12] safe chord can only be flippable if and only if it satisfies three conditions of being flippable.

A safe chord does not have any blocking chord: To Prove by contradiction, assuming in figure 3.7, KL is a safe chord in G_2 which is blocked by IN. Let, L and O are the two vertices selected to be dominant in G_1 and G_2 respectively. For having a blocking chord of KL, there must be a connection between the two vertices of the both sides of the edge LM in G_1 which is IN. Here, KN and NM form a bad pair that means both KN and NM see the vertices O and L. As a result, NLM must be a face in G_1, hence, LM must be an edge and I and N cannot be adjacent to each other because if IN exists, it intersects LM, then T loses its planarity. Let, KL be seen by I and N in G_2. Notice that the edge LM separates I from N in G_1 and LM can not be flipped as it is connected with the dominant vertex L in G_1. Therefore, KL cannot be blocked. Again, KL cannot be a blocking chord as blocking chords are always flippable due to the lemma 2.4 given by Bose et. al [12].

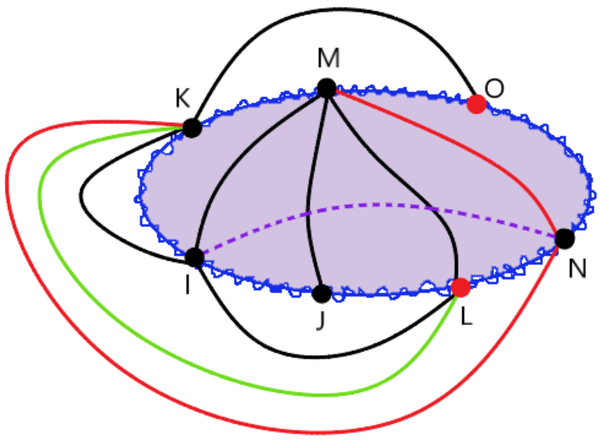


Figure 3.7: Safe Chord Analysis: MN ia a $prov_{in}$ chord and KN is a $prov_{out}$ chord where KL is a safe Chord. NLM cannot be a face, therefore, IN cannot be adjacent to each other. Thus, a safe chord does not have any blocking chords and cannot form a bad pair. The Hamiltonian Cycle (zigzagged curve) is shown in blue which is further triangulated.

A safe chord cannot form a bad pair: Let, L and O be the dominant vertices of G_1 and G_2 respectively where KL is a safe chord in G_2 and by flipping KL, I and N becomes adjacent in G_2 in figure 3.7. In G_1, every *prov* chord increases the degree of the dominant vertex L. As a result, there is no *prov* chord which connects two non-dominant vertices N and I in G_1. Therefore, a safe chord cannot form a bad pair with a *prov* chord. Again, KL can only form a bad pair with any *obst* chord in G_1, only if LM itself is an *obst* chord. However, LM cannot be an *obst* chord as it increases the degree of L in G_1. As, L is supposed to be dominant in G_1, thus the degree of L will not be increased in G_2. The degree of M in G_2 will be increased only if M were supposed to be dominant in G_2. But since O is supposed to be dominant in G_2, the degree of M will not be increased in G_2. Moreover, if any *obst* chord exist in G_1 which may form a bad pair with KL, then by flipping KL, the resulting chord must intersect LM, thus, T does not remain planar anymore. Therefore, a safe chord cannot form a bad pair with any *obst* chord either.

A safe chord does not have consecutivity with *obst* chord: At the t-th iteration, a safe chord and an *obst* chord can be consecutive in the flippable set S_t. Due to Lemma 3.13, this situation cannot happen. Thus, the safe chords in S_t do not have any consecutive chords in S_t. Therefore, a safe chord is a flippable chord in S as it satisfies the three constraints that mentioned in the lemma 2.14 of being flippable given by Bose et. al [12].

□

Lemma 3.15. *By running Algorithm 1, when flipping S_{t_1}, S_{t_2} or S_t, the flip sets cannot have two prov chords forming a bad pair.*

Proof. In the figure: 3.8 let L and O be the two vertices that are selected to be dominant in the G_1 and G_2 respectively. On the $t-th$ iteration, two *prov* chords $NM \in S_{prov_{in}}$ and $KN \in S_{prov_{out}}$ form a bad pair due to Algorithm 1 line 30 while connecting two dominant vertices internally and externally in the same iteration. To handle this situation, between both of the *prov* chords only one of them NM will be flipped and KN will be removed from the flippable set of current iteration t due to Algorithm 1 line 34. Vice versa KN can also be flipped and NM will be removed from S_t in this case. In addition, a safe chord KL which is consecutive to KN replaces KN from the current flippable set S_t due to Algorithm 1 line 33. Therefore, the inserted safe chord KL is a flippable chord due to lemma 3.14. After changing the order of KL in S, the flippable set S is recomputed along with the *prov* and *obst* chords based on the resulting graph. Suppose, there is no safe chord in G_1 or G_2 in the $t-th$ iteration, the *prov* chord KN is removed from S_t. Thus, some of the next flips get delayed too which requires the flippable set S to be recomputed accordingly. Whenever there is an insertion of any safe chord or reduction of any *prov* chord occurs, it still preserves the flippability of S. Further, it is mentioned that while flipping $prov_{in}$ and $prov_{out}$ chords and the resulting edges connect the dominant vertices L & O in both G_1 and G_2 at the same iteration, then $prov_{in}$ and $prov_{out}$ chords form a bad pair. This situation can not occur more than once as to become dominant vertices, L has the tendency to increase the degree in G_1 whereas O has the tendency to increase its degree in G_2. Therefore, L and O must not get connected more than once in the triangulation. Thus, the only chance of arising bad pair is when L and O gets connected. Hence, there cannot be *prov* chords forming bad pairs the flip set S_t. Thus, $\{S_{t_1}, S_{t_2}\} \subset S_t$ hence S_{t_1} and S_{t_2} do not have any *prov* chords that can form bad pair.

□

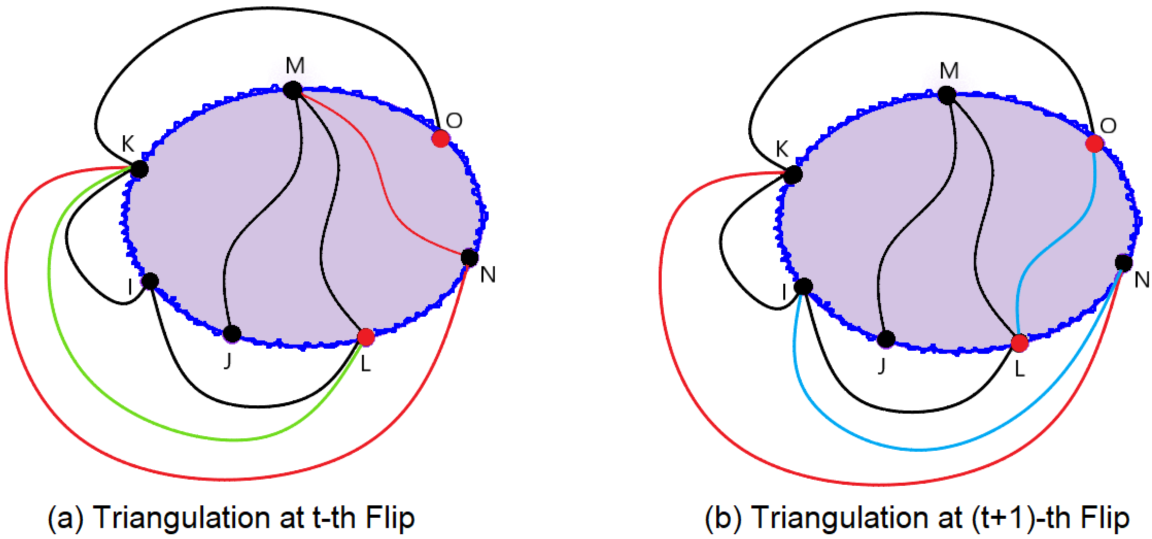


Figure 3.8: Bad Pair Analysis: At t-th iteration MN is a $prov_{in}$ chord (red) in G_1 and KN is a $prov_{out}$ chord (red) in G_2. After flipping both MN and KN, the vertices L and O gets connected. To avoid this situation, instead of flipping KN, a safe chord KL (shown in green) is selected in G_2 and KL is inserted in S_{t+1}. The zigzagged curve (blue) is a Hamiltonian cycle which is further triangulated.

Lemma 3.16. *By running Algorithm 1, when flipping S_{t_1}, S_{t_2} or S_t, the flip sets cannot have two obst chords forming a bad pair.*

Proof. . In Algorithm 1 line 25 and line 38, we already deal with consecutive chords, so by line 25 and 38, S_{t_1}, S_{t_2} and S_t, no longer contains any consecutive chords. Due to Lemma 2.5 by Bose et. al [12] for bad pairs to be blocked consecutivity is necessary. Thus there cannot be *obst* chords forming bad pairs in S_t. Therefore, $S_{t_1}, S_{t_2} \subset S_t$ hence S_{t_1} and S_{t_2} do not contain any *obst* chords that can form bad pairs.

□

Lemma 3.17. *By running Algorithm 1, when flipping S_{t_1}, S_{t_2} or S_t, the flip sets cannot have a prov chord and an obst chord forming a bad pair.*

Proof. WLOG let a $prov_{in}$ chord ab and an $obst_{out}$ chord vw form a bad pair. Since ab is a *prov* chord, it increases the degree of a dominant vertex $D_1 \in G_1$. On the other hand, vw is an *obst* chord which connects two non-dominant vertices in G_2 and thus flipping vw decreases the degree of D_1. Therefore, ab and vw cannot form bad pair as they cannot create parallel edge in the same iteration t. □

Lemma 3.18. *When flipping S_t, S_{t_1} or S_{t_2}, they form a valid simultaneously flippable set.*

Proof. As a consequence of Lemmas 3.15, 3.16 and 3.17, S_{t_1}, S_{t_2} and S_t do not contain any bad pair. Due to Lemma 3.12, S_{t_1}, S_{t_2} and S_t contain the corresponding blocking chords of all blocked chords in S_{t_1}, S_{t_2} and S_t. Finally, due to Lemma 3.13, S_{t_1}, S_{t_2} and S_t do not contain any consecutive edges either. Thus due to Lemma 2.14 by Bose et. al [12], S_{t_1}, S_{t_2} and S_t is simultaneously flippable.

□

Theorem 3.19. *When running Algorithm 1, let the dual eccentricity of the two vertices D_1 and D_2 to be made dominant be i_1 and i_2 respectively. Then with no more than $2(i_1 + i_2)$ simultaneous flips, the triangulation is transformed into a canonical triangulation.*

Proof. Notice that the while loop in Algorithm 1 Procedure Create-Flippable-Set iterates until both i_1 and i_2 become 0 where i_1 and i_2 denote the dual eccentricity of the two vertices to be made dominant in their respective maximal outerplane graphs. On each of these iterations, one of the following occurs:

1. The value of both i_1 and i_2 reduce by one.

2. Only one of i_1 and i_2 is reduced by one.

Hence the algorithm must terminate after at most i_1+i_2 iterations. On each iteration, Procedure Flip can make at most two simultaneous flips. Thus the algorithm terminates after at most $2(i_1+i_2)$ simultaneous flips.
Notice that, the only way the degree of D_1 or D_2 can be increased in their respective maximal outerplane subgraphs is to flip their *prov* chords. Therefore, we do not flip any chords that decrease the degree of D_1 or D_2 in their respective maximal outerplane subgraphs. Furthermore, when the algorithm terminates, we have flipped all the *prov* chords for both the dominant vertices. Thus, when the algorithm terminates, these vertices no longer have any *prov* chords, which means that they have become dominant vertices in their respective maximal outerplane subgraphs. □

3.6 Complexity Analysis

We calculate the complexity of our algorithm by determining the number of flips required in every step mentioned in the Section 3.2.

1. According to Bose et al. [12], to transform $G_{initial}$ into a Hamiltonian triangulation T, only one simultaneous flip is enough.

2. Kaykobad [14] showed that $4\frac{\log n}{\log \frac{6}{5}} + 2$ simultaneous flips are sufficient to make $T\prime$ with logarithmic diameter in both of the maximal outerplane graphs and the diameter becomes $2\frac{\log n}{\log \frac{6}{5}}$.

3. Due to the Theorem 3.19, at most $4\frac{\log n}{\log \frac{6}{5}} + 4$ simultaneous flips are required to introduce the dominant vertices in both of the maximal outerplane graphs where dual eccentricity i_1 and i_2 is not more than $\frac{\log n}{\log \frac{6}{5}} + 1$. Thus, $T\prime$ is turned into a canonical triangulation Δn.

4. An equal number of simultaneous flips are required to convert Δn to G_{target} as the previous three steps combined.

Theorem 3.20. *Let $G_{initial}$ and G_{target} be two triangulations with n vertices. There is a sequence of no more than $\frac{16\log n}{\log \frac{6}{5}} + 14 \approx 60.83\log n + 14$ simultaneous flips to transform $G_{initial}$ into G_{target}.*

Proof. To transform $G_{initial}$ to G_{target} atmost $2 \times (1 + 4\frac{\log n}{\log \frac{6}{5}} + 2 + 4\frac{\log n}{\log \frac{6}{5}} + 4)$ simultaneous flips are required. Here, in the step 4 of complexity analysis, all the previous steps are repeated again in reverse order thus we are multiplying the cost of previous steps with 2. □

Chapter 4

Conclusion

In this thesis, our goal was to introduce a new algorithm to transform one to triangulation to another via simultaneous diagonal flips. The notability of this algorithm is to pick any two vertices arbitrarily from a Hamiltonian triangulation and transform them into dominant vertices parallelly in the internal and external subgraphs in order to make the triangulation a canonical triangulation. Hence, there is no special conditions to select the target vertices. To accomplish the whole procedure, $\frac{16 \log n}{\log \frac{6}{5}} + 14 \approx 60.83 \log n + 14$ simultaneous flips are required. While this does not yet reduce the upper bound for transforming triangulations using simultaneous flips, we are hopeful that with further research we may make an improvement on the bound using a similar approach. Some of our future prospects are:

1. We hope to find an effective solution to handle the problem of consecutivity between prov chord and obst chord. On the t-th iteration, we flip S_t where S_t is a flip set and this set contains prov chords and their corresponding obst chords for both of the subgraphs. Although flippable sets cannot have consecutive edges, S_t can have prov chords and obst chords which are consecutive shown in Figure 3.5. At present we handle this scenario by penalizing each time the flips of inner subgraph and continue the flips of outer subgraph. This requires $(i_1 + i_2)$ simultaneous flips.

2. To handle the consecutivity between two obst chords with more efficiency. In t-th iteration, S_t can have Obst chords of different Prov chords which are consecutive shown in figure : 4.1. To handle this scenario, S_t is divided into two flip sets and the Obst chords are equally distributed in them. So, there are two simultaneous flips S_{t_1} and S_{t_2}. In worst case, this case can appear in every iteration and so it requires $2 \times (i_1 + i_2)$ simultaneous flips.

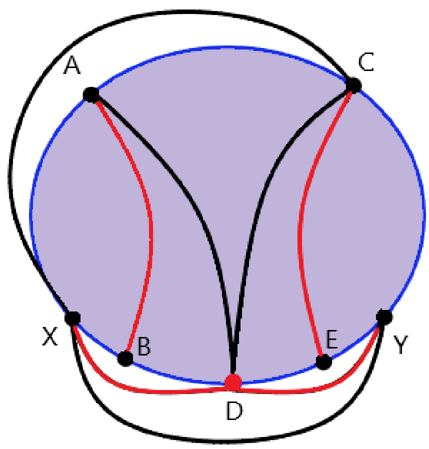


Figure 4.1: Here, D is selected to be dominant. On t-th iteration, AB and CE are $prov_{in}$ chords and their corresponding $obst_{out}$ chords are XD and YD. XD and YD are consecutive to each other.

However, we have introduced a promising new approach in this thesis for introducing dominant vertices which is essential for transforming triangulations using simultaneous flips.

Bibliography

[1] J. Petersen, "Die theorie der regulären graphs," *Acta Mathematica*, vol. 15, no. 1, pp. 193–220, 1891. DOI: https://doi.org/10.1007/BF02392606.

[2] H. Whitney, "A theorem on graphs," *Annals of Mathematics*, pp. 378–390, 1931. DOI: https://doi.org/10.2307/1968197.

[3] K. Wagner, "Bemerkungen zum vierfarbenproblem.," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 46, pp. 26–32, 1936. [Online]. Available: http://eudml.org/doc/146109.

[4] K. Appel and W. Haken, "Every planar map is four colorable," *Bulletin of the American Mathematical Society*, vol. 82, no. 5, pp. 711–712, 1976. DOI: bams/1183538218.

[5] S. Negami, "Diagonal flips in triangulations of surfaces," *Discrete Mathematics*, vol. 135, no. 1-3, pp. 225–232, 1994. DOI: https://doi.org/10.1016/0012-365X(93)E0101-9.

[6] R. Brunet, A. Nakamoto, and S. Negami, "Diagonal flips of triangulations on closed surfaces preserving specified properties," *journal of combinatorial theory, Series B*, vol. 68, no. 2, pp. 295–309, 1996. DOI: https://doi.org/10.1006/jctb.1996.0070.

[7] H. Komuro, "The diagonal flips of triangulations on the sphere," *Yokohama Mathematical Journal*, vol. 44, no. 2, pp. 115–122, 1997. [Online]. Available: http://hdl.handle.net/10131/5673.

[8] F. Hurtado, M. Noy, and J. Urrutia, "Flipping edges in triangulations," *Discrete & Computational Geometry*, vol. 22, no. 3, pp. 333–346, 1999. DOI: https://doi.org/10.1007/PL00009464.

[9] T. C. Biedl, P. Bose, E. D. Demaine, and A. Lubiw, "Efficient algorithms for petersen's matching theorem," *Journal of Algorithms*, vol. 38, no. 1, pp. 110–134, 2001. DOI: https://doi.org/10.1006/jagm.2000.1132.

[10] R. Mori, A. Nakamoto, and K. Ota, "Diagonal flips in hamiltonian triangulations on the sphere," *Graphs and Combinatorics*, vol. 19, no. 3, pp. 413–418, 2003. DOI: https://doi.org/10.1007/s00373-002-0508-6.

[11] P. Bose, V. Dujmović, and D. R. Wood, "Induced subgraphs of bounded degree and bounded treewidth," in *International Workshop on Graph-Theoretic Concepts in Computer Science*, Springer, 2005, pp. 175–186. DOI: https://doi.org/10.1007/11604686_16.

[12] P. Bose, J. Czyzowicz, Z. Gao, P. Morin, and D. R. Wood, "Simultaneous diagonal flips in plane triangulations," *Journal of Graph Theory*, vol. 54, no. 4, pp. 307–330, 2007. DOI: https://doi.org/10.1002/jgt.20214.

[13] P. Bose, D. Jansens, A. van Renssen, M. Saumell, and S. Verdonschot, "Making triangulations 4-connected using flips," *Computational Geometry*, vol. 47, no. 2, Part A, pp. 187–197, 2014, ISSN: 0925-7721. DOI: https://doi.org/10.1016/j.comgeo.2012.10.012.

[14] M. T. Kaykobad, "Transforming plane triangulations by simultaneous diagonal flips," M.S. thesis, Université d'Ottawa/University of Ottawa, 2020. DOI: http://dx.doi.org/10.20381/ruor-24732.

[15] J.-L. De Carufel and T. Kaykobad, "Transforming plane triangulations by simultaneous diagonal flips," *Information Processing Letters*, vol. 170, p. 106 120, 2021. DOI: https://doi.org/10.1016/j.ipl.2021.106120.