An Introduction to Cosmic Inflation Theory

by

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Declaration

It is hereby declared that

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Abstract

This thesis provides a detailed derivation of the Friedmann Equations in the framework of general relativity and for an FLRW universe. An introduction to the theory of cosmic inflation, the motivations for it, and, the scalar field dynamics associated with inflation presently follows. It further attempts to discuss models of inflation and endeavors to make a classical introduction to cosmological perturbations generated in the course of inflation, which are thought to be the originator of structure in the universe. Our metric signature is $(+,-,-,-)$ and we will use the Einstein summation convention.

Keywords: FLRW metric; Hot Big Bang; Inflation; Inflaton; Cosmological perturbations

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I express my earnest gratitude to my Thesis supervisor Dr. Mahbub Majumdar, who has always been extraordinarily kind and considerate in every respect.

I have turned the mode of being stupid into an actual art form, and despite that, I thank my family here by my side and the other half eleven-thousand kilometers away, for not yet deigning to disown me. To be at the receiving end of their overwhelming, unconditional love is infinitely gratifying.

I thank Chinku Chowdhury, who shares my bleak outlook in life. I profess, you shall forever be the most endearing cat this side of the Milky Way.

The universe has decreed a myriad of wonderful, benevolent human beings all around me, and for their presence in my life right now or at some point in the past, I am ever grateful.

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The Homogeneous Universe

Aleksandr Alexandrovich Friedmann[1] was a Soviet mathematician and physicist most renowned for first coming forth with a set of equations that act as the theoretical basis for an expanding universe. After the earliest general relativistic models suggested that a universe which was dynamical and contained ordinary gravitational matter would collapse upon itself, Einstein had went onto introduce the ad hoc cosmological constant Λ, which would ensure a static, spherical, spatially closed universe. Einstein's first veritable encounter with a dynamic universe was in fact Friedman's 1922 article "Über die Krümmung des Raumes" (About the curvature of space) [1].
The Friedmann equation incorporates the cosmological constant Λ, and is based on the Einstein field equations. It models the expanding homogeneous and isotropic 4-dimensional universe(that is positively curved)[2] for a perfect fluid with a given mass density[3] ρ and pressure p. Besides, the equation contains the gravitation constant G, the Hubble parameter H, a scaling parameter R, and a curvature factor k which tells us about the geometry of the universe.

1.1 Friedmann–Lemaître–Robertson–Walker (FLRW) metric

The FLRW metric is built upon the assumption of homogeneity and isotropy of space. It also assumes that the spatial component of the metric can be time-dependent. Although one may lack any prior understanding of general relativity and still be sufficiently able to derive Friedmann equation from Newton's theory of gravity, the conventional derivation of the Friedmann equations begins by inserting the FLRW metric into the Einstein field equations.
The square of two infinitesimally separated simultaneous[4] events with coordinates

[1] His name in his own language is Friedman and not Friedmann, the more popular variation in the spelling was in fact Einstein's German rendition of the name!

[2] A flat surface is said to have zero curvature, a spherical surface is said to have positive curvature, and a saddle-shaped surface is said to have negative curvature. General Relativity pronounces that space itself can be curved. The space of General Relativity has 3 space-like dimensions and one time dimension.

[3] The natural unit system has been adopted in this text and since the energy density ϵ is related to the mass density ρ by $\epsilon = \rho c^2$ and c is set equal to 1, thus ρ and ϵ may be used interchangeably.

[4] For an expanding universe, the positions of the two points must be noted at the exact same

(t, x, y, z) and $(t + dt, x + dx, y + dy, z + dz)$ may be given by the metric for flat space-time:

$$(ds)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{1.1.1}$$

$$\Delta s = \int_{p_1}^{p_2} \sqrt{(ds)^2} \tag{1.1.2}$$

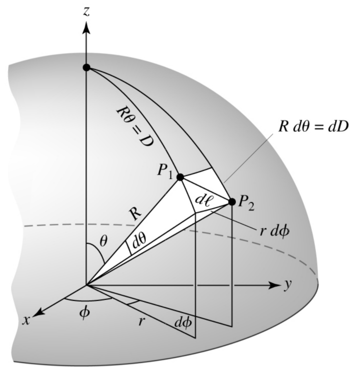


Figure 1.1: Infinitesimally separated points on the surface of a sphere

The distance between two points on the surface of a sphere, in terms of the 2-D plane polar coordinates r and ϕ may be given by:

$$(dl)^2 = (dD)^2 + (rd\phi)^2 = (Rd\theta)^2 + (rd\phi)^2 \tag{1.1.3}$$

Since $r = R \sin\theta$, we have $dr = R\cos\theta = d\theta$ and hence

$$Rd\theta = \frac{dr}{\cos\theta} = \frac{Rdr}{\sqrt{R^2 - r^2}} = \frac{dr}{\sqrt{1 - r^2/R^2}} \tag{1.1.4}$$

thus:

$$(dl)^2 = \left(\frac{dr}{\sqrt{1 - r^2/R^2}}\right)^2 + (rd\phi)^2 \tag{1.1.5}$$

The Gaussian curvature of a sphere of radius R is defined to be $K \equiv 1/R^2$ everywhere. Thus (1.1.5) is reduced to:

$$(dl)^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (rd\phi)^2 \tag{1.1.6}$$

A three dimensional interpretation of the effect of curvature on the spatial distances can then be said to be:

$$(dl)^2 = \left(\frac{dr}{\sqrt{1 - Kr^2}}\right)^2 + (rd\theta)^2 + (r \sin\theta \, d\phi)^2 \tag{1.1.7}$$

instant for the measure of their separation to have any quantifiable meaning and not be complete hogwash.

where the radial coordinate is now considered to be r.

Adding in the temporal term:

$$(ds)^2 = (cdt)^2 - \left(\frac{dr}{\sqrt{1-Kr^2}}\right)^2 - (rd\theta)^2 - (r\sin\theta\, d\phi)^2 \tag{1.1.8}$$

The real (proper) distance[5] is $\sqrt{-(\Delta s)^2}$ where $dt = 0$.
Now, changing the radial coordinate to a comoving[6] coordinate,

$$r(t) = a(t) \cdot x \tag{1.1.9}$$

Since the expansion of the universe verily affects all of its geometric properties it may be prudent to further express the curvature in terms of the scale factor[7] and a time independent constant k.

$$K(t) = \frac{k}{a^2(t)} \tag{1.1.10}$$

Now, Substituting (1.1.9) and (1.1.10) into (1.1.8) we obtain the final form of the FLRW metric.

$$(ds)^2 = (cdt)^2 - a^2(t)\left[\left(\frac{dx}{\sqrt{1-kx^2}}\right)^2 + (xd\theta)^2 + (x\sin\theta\, d\phi)^2\right] \tag{1.1.11}$$

where r is now the comoving distance in an infuriatingly confusing turn of notations.
The components of the FLRW metric may be written as a matrix[8]

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a^2r^2 & 0 \\ 0 & 0 & 0 & -a^2r^2\sin^2\theta \end{pmatrix} \tag{1.1.12}$$

1.2 Einstein field equations

The Einstein equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.2.1}$$

[5]Proper distances may vary over time unlike comoving distances which are by definition fixed.

[6]Comoving coordinates are carried along with the expansion.

[7]$a(t)$ is known as the scale factor. It characterizes the rate of expansion of the universe by describing how physical separations grow with time.

[8]N.B. $g^{\mu\nu} = \frac{1}{g_{\mu\nu}}$

where,

$R_{\mu\nu}$ - Ricci curvature tensor
$g_{\mu\nu}$ - The metric of the manifold where the equations apply
R - Ricci scalar curvature
Λ - Cosmological constant
G - Universal gravitational constant
c - Speed of light
$T_{\mu\nu}$ - Energy-momentum tensor

(1.2.1) in essence demonstrates that it is the mass that inherently impels the curvature of spacetime, and it is the curved spacetime that in turn dictates how the mass must move.

1.3 Evaluating the Ricci Tensor and the Ricci Scalar

From §**1.1** it is pretty much obvious that the FLRW metric is diagonal and has a connection which is torsion-free. This vastly reduces the number of Christoffel symbols one must compute, as the majority of them are either null or symmetric.

$$\Gamma^l_{ji} = \frac{1}{2} g^{lm} (\partial_j g_{mi} + \partial_i g_{mj} - \partial_m g_{ij}) \tag{1.3.1}$$

Since $g^{lm} \equiv \delta^{lm}$ and $\delta^{lm} = 0$ for every $l \neq m$, and 1 for $l = m$, Γ^l_{ji} would also therefore vanish whenever $l \neq m$. Now, using (1.3.1), the Christoffel symbols that remain are as follows:

$$\begin{aligned}
\Gamma^t_{rr} &= \frac{a\dot{a}}{1-kr^2} \\
\Gamma^t_{\theta\theta} &= r^2 a\dot{a} \\
\Gamma^t_{\phi\phi} &= r^2 a\dot{a}\sin^2\theta \\
\Gamma^r_{tr} = \Gamma^r_{rt} = \Gamma^\theta_{t\theta} = \Gamma^\theta_{\theta t} = \Gamma^\phi_{t\phi} = \Gamma^\phi_{\phi t} &= \frac{\dot{a}}{a} \\
\Gamma^r_{rr} &= \frac{kr}{1-kr^2} \\
\Gamma^r_{\theta\theta} &= -r(1-kr^2) \\
\Gamma^r_{\phi\phi} &= -r(1-kr^2)\sin^2\theta \\
\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} &= \frac{1}{r} \\
\Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} &= \frac{1}{\tan\theta}
\end{aligned} \tag{1.3.2}$$

We may now proceed to calculate the Riemann tensor.

$$R^l_{kji} = -\partial_i \Gamma^l_{kj} + \partial_j \Gamma^l_{ki} - \Gamma^m_{kj}\Gamma^l_{mi} + \Gamma^m_{ki}\Gamma^l_{mj} \tag{1.3.3}$$

For 4D spacetime there are 4×4×4×4=256 components and we must now compute all of them. I jest. Fortunately for us, the symmetries of the Riemann tensor imply that its *only* non-zero contraction is in fact the Ricci tensor.[9]

$$R_{ij} = R^m_{imj} \tag{1.3.4}$$

Hence, it follows that:

- $$\begin{aligned} R_{tt} &= R^t_{ttt} + R^r_{trt} + R^\theta_{t\theta t} + R^\phi_{t\phi t} \\ &= 0 - 3\frac{\partial}{\partial t}\left(\frac{\dot{a}}{a}\right) - 3\left(\frac{\dot{a}}{a}\right)^2 \\ &= -3\left(\frac{\ddot{a}}{a}\right) \end{aligned} \tag{1.3.5}$$

- $$\begin{aligned} R_{rr} &= R^t_{rtr} + R^r_{rrr} + R^\theta_{r\theta r} + R^\phi_{r\phi r} \\ &= \frac{\partial}{\partial t}\left(\frac{a\dot{a}}{1-kr^2}\right) - 2\frac{\partial}{\partial r}\frac{1}{r} \\ &+ \frac{\dot{a}^2}{1-kr^2} + \frac{2k}{1-kr^2} - 2\left(\frac{1}{r}\right)^2 \\ &= \frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2} \end{aligned} \tag{1.3.6}$$

- $$\begin{aligned} R_{\theta\theta} &= R^t_{\theta t\theta} + R^r_{\theta r\theta} + R^\theta_{\theta\theta\theta} + R^\phi_{\theta\phi\theta} \\ &= r^2(a\ddot{a}+2\dot{a}^2+2k) \end{aligned} \tag{1.3.7}$$

- $$\begin{aligned} R_{\phi\phi} &= R^t_{\phi t\phi} + R^r_{\phi r\phi} + R^\theta_{\phi\theta\phi} + R^\phi_{\phi\phi\phi} \\ &= r^2\sin^2(\theta)(a\ddot{a}+2\dot{a}^2+2k) \end{aligned} \tag{1.3.8}$$

$$R_{\mu\nu} = \begin{pmatrix} -3\left(\frac{\ddot{a}}{a}\right) & 0 & 0 & 0 \\ 0 & \frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2} & 0 & 0 \\ 0 & 0 & r^2(a\ddot{a}+2\dot{a}^2+2k) & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta)(a\ddot{a}+2\dot{a}^2+2k) \end{pmatrix}$$

i.e. $R_{ii} = \frac{-g_{ii}}{a^2}(a\ddot{a}+2\dot{a}^2+2k)$ spatially. Finally,[10] we attain the Ricci scalar:

$$\begin{aligned} R &= R_{\mu\nu}g^{\mu\nu} \\ &= R_{tt}g^{tt} + R_{rr}g^{rr} + R_{\theta\theta}g^{\theta\theta} + R_{\phi\phi}g^{\phi\phi} \\ R &= -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) \end{aligned} \tag{1.3.9}$$

[9] In principle, $R^a_{acd} \neq R^a_{bad} \neq R^a_{bca}$. Contracting over the first index,

$$R^a_{acd} = g^{ae}R_{eacd} = -g^{ae}R_{aecd} = -g^{ea}R_{aecd} = -R^e_{ecd} = -R^a_{acd}$$

The only way that can possibly be true is if $-R^a_{acd} = 0$. There exists exactly one non-zero contraction of the Riemann curvature tensor, which we call the Ricci tensor. [2]

[10] After an excruciatingly painful amount of work. Period.

1.4 Energy momentum tensor for a perfect fluid

The perfect fluid form of the energy momentum tensor is a perfect description of a homogeneous and isotropic universe.

$$
\begin{aligned}
T_{\mu\nu} &= (\rho + p)u_\mu u_\nu - pg_{\mu\nu} && (1.4.1)\\
T^\mu{}_\nu &= g^{\mu\alpha}T_{\alpha\nu} = (\rho + p)g^{\mu\alpha}u_\alpha u_\nu - pg^{\mu\alpha}g_{\alpha\nu} && (1.4.2)\\
&= (\rho + p)u^\mu u_\nu - p\delta^\mu{}_\nu && (1.4.3)
\end{aligned}
$$

Here, u_α is the macroscopic speed of the fluid, and its isotropy ensures that the perfect fluid must look exactly same in all directions, and hence it has only the time component,

$$
u_\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

Therefore,

$$
T^\mu{}_\nu = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix} \tag{1.4.4}
$$

and

$$
\begin{aligned}
\mathrm{Tr}(T^\mu{}_\nu) &= \rho - 3p && (1.4.5)\\
T_{tt} &= \rho g_{tt} = \rho && (1.4.6)
\end{aligned}
$$

For the spatial parts:

$$
T_{\mu\mu} = (\rho + p)\cdot 0 - pg_{\mu\mu} = -pg_{\mu\mu} \tag{1.4.7}
$$

1.5 Friedmann equations

Now we may simply slip in all of the elements we have so far derived into the Einstein equations.

$$
G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{1.5.1}
$$

We begin by deriving the first Friedmann equation with the temporal part of the Einstein tensor:

$$
\begin{aligned}
G_{tt} &= R_{tt} - \frac{1}{2}Rg_{tt} && (1.5.2)\\
&= -3\frac{\ddot{a}}{a} + 3\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2} + 3\frac{k}{a^2} && (1.5.3)\\
&= 3\left(\frac{\dot{a}^2 + k}{a^2}\right) && (1.5.4)
\end{aligned}
$$

Plugging in:

$$G_{tt} = 8\pi G T_{00} + \Lambda g_{00} \tag{1.5.5}$$

$$3\left(\frac{\dot{a}^2 + k}{a^2}\right) = 8\pi G\rho + \Lambda \tag{1.5.6}$$

$$\frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G\rho + \Lambda}{3} \tag{1.5.7}$$

$$\boxed{\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho(t) + \frac{\Lambda}{3} - \frac{k}{a^2(t)}} \tag{1.5.8}$$

and the Hubble parameter H,

$$H \equiv \frac{\dot{a}}{a} \quad \text{usually parametrized as } \ 100h\ \text{kms}^{-1}\text{Mpc}^{-1}$$

We now proceed to examine the effects of incorporating the spatial part into the EFE:

$$\Longrightarrow \qquad \frac{-g_{ii}}{a^2}(a\ddot{a} + 2\dot{a}^2 + 2k) - \frac{1}{2}Rg_{ii} - \Lambda g_{ii} = 8\pi G(-p)g_{ii} \tag{1.5.9}$$

$$\Longrightarrow \qquad g_{ii}\left[\frac{-1}{a^2}(a\ddot{a} + 2\dot{a}^2 + 2k) - \frac{1}{2}R - \Lambda\right] = 8\pi G(-p)g_{ii} \tag{1.5.10}$$

$$\Longrightarrow \qquad -\frac{\ddot{a}}{a} - 2\left(\frac{\dot{a}}{a}\right)^2 - \frac{2k}{a^2} + 3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3k}{a^2} - \Lambda = -8\pi G p \tag{1.5.11}$$

$$\Longrightarrow \qquad \frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -4\pi G p + \frac{\Lambda}{2} - \frac{1}{2}\frac{k}{a^2} \tag{1.5.12}$$

It may be observed that 2×(1.5.12)−(1.5.8) eliminates $\left(\frac{\dot{a}}{a}\right)^2$ and we then obtain:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3p(t) + \rho(t)) + \frac{\Lambda}{3}} \tag{1.5.13}$$

Otherwise known as the **acceleration equation**. The Friedmann equations are amongst the most consequential equations required to intuit the fate of the universe.

1.6 Simple cosmological models

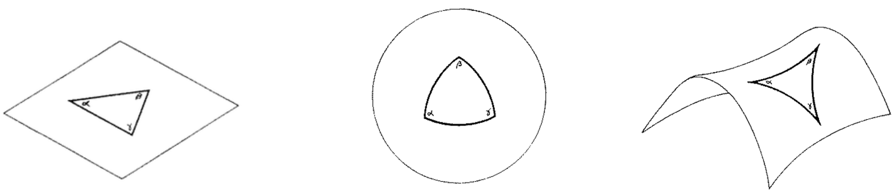


Figure 1.2: $k = 0$ (flat) ; $k > 0$ (spherical) ; $k < 0$ (hyperbolic)

The simplest type of geometry that preserves the homogeneity and isotropy of our universe is the flat geometry with k=0. However other geometries that correspond to non-zero values of k are still altogether feasible.

1.6.1 Solutions for the scale factor

To verily assess how the flat universe may evolve, it is imperative to first discuss what is contained within it, and we may further proceed by first describing the relationship between the mass density ρ and the pressure p. For an expanding volume V with physical radius a, the energy is given by

$$E = mc^2 \tag{1.6.1}$$
$$= \frac{4\pi a^3}{3}\rho c^2 \tag{1.6.2}$$

The change in volume in a time dt,

$$\frac{dV}{dt} = 4\pi a^2 \dot{a} \tag{1.6.3}$$

Thus the change in energy,

$$\frac{dE}{dt} = 4\pi a^2 \rho c^2 \dot{a} + \frac{4\pi a^3}{3}\dot{\rho} c^2 \tag{1.6.4}$$

Considering the first law of thermodynamics and assuming a reversible expansion $dS = 0$,

$$dE + pdV = TdS = 0 \tag{1.6.5}$$

we obtain,

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{1.6.6}$$

This is the **fluid equation**.

Matter

In a cosmological context, matter may be anything in the universe that is non-relativistic and exerts a negligible pressure, $p = 0$.
Having set $p = 0$ and $\rho = \rho_{mat}$ in the fluid equation, it may be alternatively written as,

$$\dot{\rho}_{mat} + 3\frac{\dot{a}}{a}\rho_{mat} = 0 \implies \frac{1}{a^3}\frac{d}{dt}(\rho_{mat}a^3) = 0 \implies \frac{d}{dt}(\rho_{mat}a^3) = 0 \tag{1.6.7}$$

i.e.

$$\rho_{mat} \propto \frac{1}{a^3} \tag{1.6.8}$$
$$\rho_{mat} = \frac{\rho_0}{a^3} \tag{1.6.9}$$

Now substituting ρ_{mat} into Friedmann equation, for a flat universe[11], we have:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho_0}{3a^3} \tag{1.6.10}$$

$$\dot{a} = \sqrt{\frac{8\pi G\rho_0}{3}}\frac{1}{\sqrt{a}} \tag{1.6.11}$$

$$= \frac{\alpha}{\sqrt{a}} \tag{1.6.12}$$

$$\sqrt{a}da = \alpha dt \tag{1.6.13}$$

Integrating both sides:

$$\frac{2}{3}a^{2/3} = \alpha t \quad \Longrightarrow \quad a = \left(\frac{3}{2}\alpha\right)^{2/3} t^{2/3} \quad \Longrightarrow \quad a(t) \propto t^{2/3} \tag{1.6.14}$$

Therefore,

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \tag{1.6.15}$$

In spite of the gravitational pull of matter, the universe does not collapse and instead expands for all time - for the scale factor solution of a matter dominated ρ. However, as the universe becomes infinitely old, the rate of expansion $H(t)$ becomes infinitely slow, as $H(t)$ declines with increasing time:

$$H = \frac{\dot{a}}{a} = \frac{2}{3t} \tag{1.6.16}$$

This is also known as the Einstein-de Sitter model.

Radiation

All particles travelling at relativistic speeds have the equation of state,

$$p = \frac{\rho_{rad}c^2}{3} \tag{1.6.17}$$

By substitution of p into the **fluid equation** (1.6.6) we get,

$$\dot{\rho}_{rad} + 4\frac{\dot{a}}{a}\rho_{rad} = 0 \tag{1.6.18}$$

likewise this implies,

$$\rho_{rad} \propto \frac{1}{a^4} \tag{1.6.19}$$

$$\rho_{rad} = \frac{\rho_0}{a^4} \tag{1.6.20}$$

and hence,

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2} \tag{1.6.21}$$

[11]The flat universe has one very useful symmetry, $a(t)$ may be rescaled anyhow since it only ever appears in the friedmann equation as the combination $\frac{\dot{a}}{a}$. Conventionally a is set to be equal to 1 at the present time, and consequently physical and comoving coordinate systems cooincide.

It is further observed that a radiation dominated universe expands at a slower rate in comparison to a matter dominated universe,

$$H = \frac{\dot{a}}{a} = \frac{1}{2t} \tag{1.6.22}$$

Mixtures

If one were to contemplate the cases of mixed matter and radiation, the density will then be,

$$\rho = \rho_{mat} + \rho_{rad} \tag{1.6.23}$$

Now, considering the unambiguous and less complex setting where either of the two components is predominant, we can safely assume that the Friedmann equation may be solely governed by that dominant component.

Case 1: Radiation dominating over matter

$$a(t) \propto t^{1/2} \quad ; \quad \rho_{rad} \propto \frac{1}{t^2} \quad ; \quad \rho_{mat} \propto \frac{1}{a^3} \propto \frac{1}{t^{3/2}} \tag{1.6.24}$$

Case 2: Matter dominating over radiation

$$a(t) \propto t^{2/3} \quad ; \quad \rho_{mat} \propto \frac{1}{t^2} \quad ; \quad \rho_{rad} \propto \frac{1}{a^4} \propto \frac{1}{t^{8/3}} \tag{1.6.25}$$

It is apparent that the density of radiation falls off at a faster pace than the density of matter for both of the cases, and we can conclude that radiation domination can not last for all time and therefore is an unstable situation.

1.6.2 Evolution including curvature

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.6.26}$$

An inspection reveals that for a $k < 0$ the right-hand side of the Friedmann equation becomes entirely positive and a universe such as this expands evermore. Moreover, the $\rho_{mat} \propto 1/a^3$ decreases at a faster rate than the k/a^2 term as the universe expands. Consequently, the curvature term ultimately dominates, reducing the Friedmann equation to:

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2} \tag{1.6.27}$$

And thus $a \propto t$; the universe expands at a faster rate.

For $k > 0$, it becomes possible for the universe to not only cease expansion, but to also eventually contract.

1.6.3 Dark Energy (Λ)

It is an innate form of energy inextricable from space. Whilst both the densities of matter and radiation fall off in proportion to the volume of the universe, dark energy(vacuum energy) is curiously distinct in the sense that as the universe expands allowing for new space, the vacuum energy can never dilute, $\dot{\rho}_\Lambda = 0$. In due course, vacuum energy prevails and dominates over the other components. It has a negative effective pressure, which implies that as the universe expands, work is done on the cosmological constant fluid, allowing its energy density to perpetually remain constant. This corresponds to the equation of state,

$$\frac{p}{\rho_\Lambda} = w = -1 \tag{1.6.28}$$

and,

$$\rho_\Lambda \propto a^0 = \text{const} \quad . \tag{1.6.29}$$

$$a(t) \propto e^{H_0 t} \tag{1.6.30}$$

This is known as the de Sitter solution.

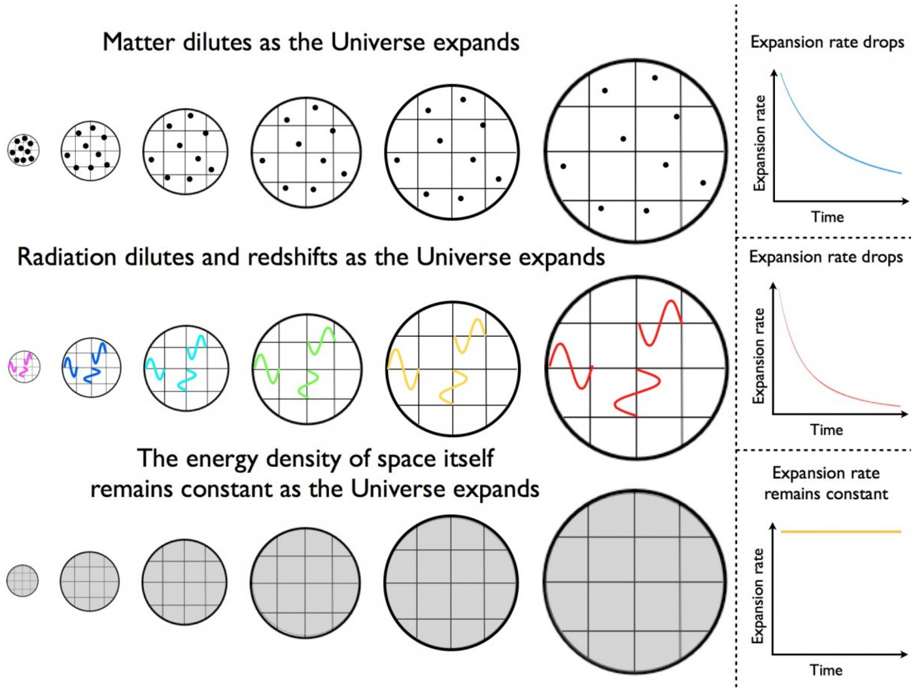


Figure 1.3: Energy densities [3]

Chapter 2

An Overview of the Hot Big Bang

The Hot Big Bang model quite succinctly surmises much of the happenings observed in the universe today.

2.1 Observational Parameters

The particular density for which the universe is flat i.e. $k = 0$ is known to be the **critical density**.

$$\rho_c(t) = \frac{3H^2}{8\pi G} \tag{2.1.1}$$

The critical density varies with time as does H. However, the present day critical density is estimated to be,

$$\begin{aligned} \rho_c(t_0) &= 1.88h^2 \times 10^{-26} \text{ kgm}^{-3} && (2.1.2) \\ &= 2.78h^{-1} \times 10^{11} M_\odot/(h^{-1}Mpc)^3 && (2.1.3) \end{aligned}$$

Since the universe may not be flat, the critical density need not be the actual density of the universe and hence one very convenient way to define the density of the universe is by expressing it relative to the critical density. This ratio is known as the **density parameter** Ω,

$$\Omega(t) \equiv \frac{\rho}{\rho_c} \tag{2.1.4}$$

Alternatively, the Friedmann equation can be written as:

$$H^2 = \frac{8\pi G}{3}\rho_c\Omega - \frac{k}{a^2} = H^2\Omega - \frac{k}{a^2} \tag{2.1.5}$$

or,

$$\Omega - 1 = \frac{k}{a^2H^2} \tag{2.1.6}$$

For our universe:

$$\begin{aligned} H^2 + \frac{k}{a^2} &= \frac{8\pi G}{3}(\rho_{mat} + \rho_{rad}) + \frac{\Lambda}{3} && (2.1.7) \\ 1 + \frac{k}{H^2a^2} &= \frac{8\pi G}{3H^2}(\rho_{mat} + \rho_{rad}) + \frac{\Lambda}{3H^2} && (2.1.8) \end{aligned}$$

It is useful to define the density parameters:

$$\Omega_k = \frac{k}{H^2 a^2} \tag{2.1.9}$$

$$\Omega_{mat} = \frac{8\pi G \rho_{mat}}{3H^2} \tag{2.1.10}$$

$$\Omega_{rad} = \frac{8\pi G \rho_{rad}}{3H^2} \tag{2.1.11}$$

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} \tag{2.1.12}$$

and,

$$\Omega + \Omega_\Lambda - 1 = \frac{k}{a^2 H^2} \tag{2.1.13}$$

A plausible summary of the geometries			
curvature	geometry	density parameter	type of universe
$k > 0$	spherical	$\Omega + \Omega_\Lambda > 1$	Closed
$k = 0$	flat	$\Omega + \Omega_\Lambda = 1$	Flat
$k < 0$	hyperbolic	$\Omega + \Omega_\Lambda < 1$	Open

The total density of the observable universe is very close to ρ_c.

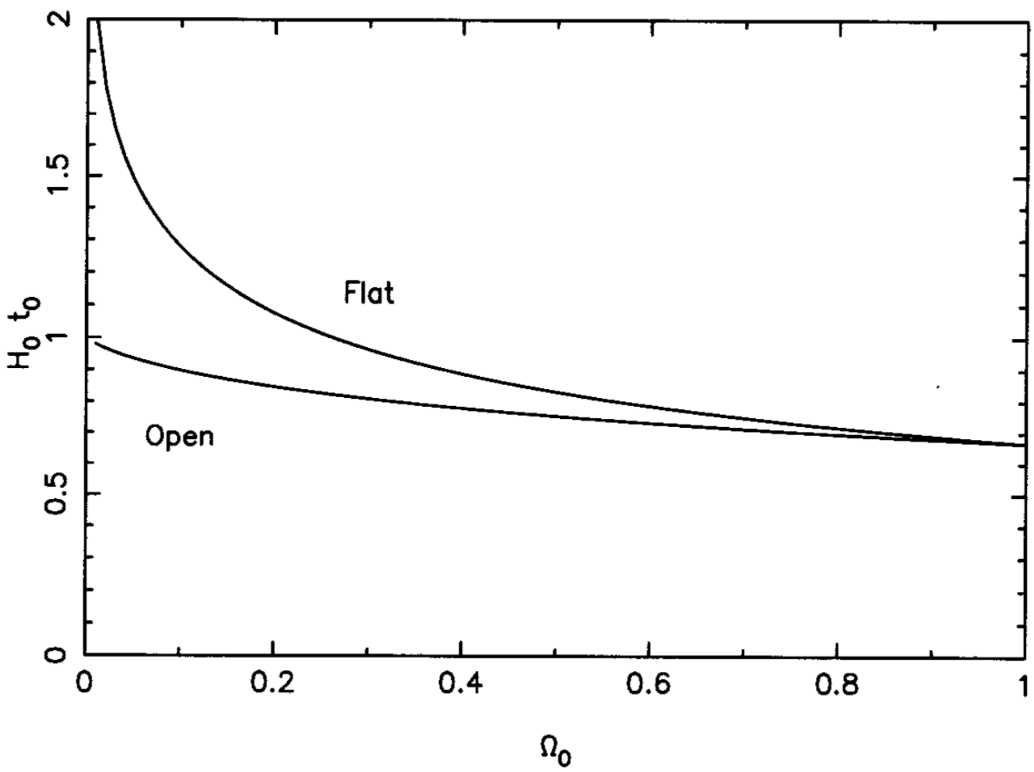


Figure 2.1: Predicted ages for open universes and for flat universes with a cosmological constant.[12]

2.2 Expansion and Redshift

Since everything is moving farther apart now, extrapolating this cosmic expansion to the remote past one arrives at a point of singularity, this is followed by a state of extreme high density and temperature. The Big Bang Cosmology is known as the model of the universe with such a beginning.
The investigation of galactic redshifts proved to be a crucial observational evidence for the cosmic expansion hypothesis.
The redshift z is defined by,

$$z = \frac{\lambda_r - \lambda_e}{\lambda_e} \tag{2.2.1}$$

Let light pass between two infinitesimally separated objects. By Hubble's law, their relative velocity dv shall be

$$dv = H dr = \frac{\dot{a}}{a} dr \tag{2.2.2}$$

Considering a photon emitted at time t_0 with wavelength λ_e and observed at time t_1 with λ_r, the change in wavelength between emission and reception is then, $d\lambda \equiv \lambda_r - \lambda_e$

$$\frac{d\lambda}{\lambda_e} = \frac{dv}{c} = \frac{dv}{(dr/dt)} = \frac{\dot{a}}{a}\frac{dr}{(dr/dt)} = \frac{\dot{a}}{a}dt = \frac{da}{a} \tag{2.2.3}$$

Integrating, we find $\ln\lambda = \ln a +$ const.

$$\lambda \propto a \tag{2.2.4}$$

The Deceleration parameter is defined as

$$q(t) = -\frac{\ddot{a}(t)}{a(t)H^2(t)} \tag{2.2.5}$$

Consider the taylor expansion of the scale factor about the present time, t_0,

$$a(t) = a_0 + \dot{a_0}(t - t_0) + \frac{1}{2}\ddot{a_0}(t - t_0)^2 + ... \tag{2.2.6}$$

By substituting the definitions for the Deceleration parameter and the Hubble parameter into (2.2.6) we then obtain,

$$a(t) = a_0[1 + H_0(t - t_0) - \frac{1}{2}q_0{H_0}^2(t - t_0)^2 + ...] \tag{2.2.7}$$

The redshift is related to $a(t)$ by

$$1 + z = \frac{\lambda_r}{\lambda_e} = \frac{a(t_r)}{a(t_e)} = \frac{a_0}{a} \tag{2.2.8}$$

$$= H_0(t_0 - t) + \left(1 + \frac{q_0}{2}\right){H_0}^2(t_0 - t)^2 + ... \tag{2.2.9}$$

Which when solved for $t_0 - t$ can be written as,

$$(t_0 - t) = (H_0)^{-1}\left[z - \left(1 + \frac{q_0}{2}\right)z^2 + ... \right] \tag{2.2.10}$$

2.3 Big Bang Nucleosynthesis (BBN)

Nuclear fusion reactions in stellar interiors or during supernovae explosions account for the heavier elements present in the universe, however the prevalence of lighter elements suggests that stellar nuclear reactions could not have possibly been the only source of production. However, BBN accurately predicts the light element abundances. Roughly one minute in, all anti-matter and most matter had been destroyed by annihilation, all remnant matter - protons, neutrons, electrons - were fully ionized and dissociated. As the temperature rapidly plummeted to 10 000 000 000 K, protons and neutrons underwent fusion to form heavier atomic nuclei. Photons were free to interact with both nuclei and electrons.

2.4 The Cosmic Microwave Background (CMB)

Another indisputable and pivotal piece of evidence for the Big Bang universe was the detection of the CMB radiation. The total energy density ϵ_{rad} of radiation at temperature T,

$$\epsilon_{rad} \equiv \rho_{rad} c^2 = \alpha T^4 \tag{2.4.1}$$

Since the density of radiation falls off with the expansion of the universe,

$$\rho_{rad} \propto \frac{1}{a^4} \tag{2.4.2}$$

and,

$$\alpha T^4 = \rho_{rad} c^2 \tag{2.4.3}$$

$$T \propto \sqrt[4]{\rho_{rad}} \tag{2.4.4}$$

$$T \propto \frac{1}{a} \tag{2.4.5}$$

i.e. The temperature of the universe falls as it expands. Compared to today's temperature of about 3K, when the universe was about one millionth of its current expanse, the temperature back then is estimated to be a staggering 3 000 000 K. Since the average energy of photons in the thermal distribution was substantially great in that era, any free electrons attempting to latch onto a proton were blasted away by or coupled to a photon of light via Thomson scattering. Thus, atoms did not exist. Eventually the universe expanded and cooled and when it was about one thousandth of its present size, the electrons fell back to their ground states forming atoms that couldn't be ionized with the photons that now had lower energy. The universe had thus lost its opacity and turned utterly transparent at **decoupling**, the photons were free to travel and are observed as the CMB today.

Chapter 3

Inflationary Cosmology

In spite of all its successes, the Hot Big Bang Model brings about a number of upsetting conundrums. The conventional Big Bang theory prerequisites a very fine-tuned set of initial conditions which would have permitted the universe to have evolved to its present state. Indeed that would just mean that the universe as we see it, is just an unconvincing and fanciful accident.

This gave rise to the idea of **cosmological inflation**, not as a replacement theory, but in fact adjunct to the Hot Big Bang Model as it perfectly explains the generic initial conditions of the observable universe. Inflation is described as a period of accelerated expansion at a very early stage.

3.1 Big Bang : Why It's Not So Perfect

3.1.1 The horizon problem

In general relativity the propagation of light is affected by the gravitational potential(metric),

$$ds^2 = dt^2 - a^2(dr^2 + r^2 d\Omega^2) \tag{3.1.1}$$

Assuming there is no scattering, the radial null geodesic of the light particle, $ds^2 = 0$. Therefore,

$$dt = \pm a dr \tag{3.1.2}$$

$$\int \frac{1}{a} dt = \pm \int dr \tag{3.1.3}$$

$$\Delta r = \int \frac{dt}{a(t)} \tag{3.1.4}$$

This is the particle horizon or comoving distance. The physical distance traversed by light may be expressed as $a\Delta r$. $a(t)$ is a function of t, that is to say, for light to have propagated over large expanses, a sufficiently long amount of time must have elapsed.

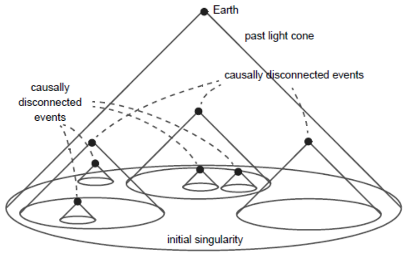


Figure 3.1: Causally disconnected regions

Supposing a radiation dominated evolution down to $a = 0$ (Big Bang singularity), the state of the universe at an early time is affected by numerous disconnected pieces, splitting apart into countless disconnected volumes which have not had time to communicate. For instance, choosing the Planck time, t_p, where $t_p \sim 10^{-43}$

$$\Delta r = \int \frac{dt}{a(t)} = \int \frac{1}{t^{1/2}} dt = 2t^{1/2} \tag{3.1.5}$$

and,

$$\frac{\Delta r(t_0)}{\Delta r(t_p)} = \left(\frac{10^{17} s}{10^{-43} s} \right)^{1/2} = 10^{30} \tag{3.1.6}$$

Which is 10^{90} when translated from the linear scale to the Hubble volume. For so many patches without causal contact, one would ordinarily expect breaks in homogeneity and isotropy however, near-homogeneity of the CMB tells us that it in fact varies by only one part in a 100 000. This is the so-called horizon problem.

3.1.2 The flatness problem

It is known that the universe possesses a total density that lies very close to the critical density, which implies that the universe must be flat/euclidean geometrically. Since the Friedmann equation can be rewritten in the form,

$$|\Omega_{tot}(t) - 1| = \frac{|k|}{a^2 H^2} \tag{3.1.7}$$

Ignoring the effects of the curvature and the cosmological constant term and considering a universe dominated by radiation or matter, we have:

$$a^2 H^2 \propto \frac{1}{t} \qquad \text{radiation domination;}$$

$$a^2 H^2 \propto \frac{1}{t^{2/3}} \qquad \text{matter domination;}$$

or,

$$|\Omega_{tot} - 1| \propto t \qquad \text{radiation domination;}$$

$$|\Omega_{tot} - 1| \propto t^{2/3} \qquad \text{matter domination;}$$

$|\Omega_{tot} - 1|$ is a growing function of time for either of the above cases. For even the slightest deviation, the universe will immediately be increasingly curved. A flat universe seems like an unstable solution of the Friedmann equation. Presently, for our universe to be so close to spatially flat has only one possible implication, it would mean that the universe was exceedingly flat during very early times. An explanation for these fine tuned initial conditions for Ω would be gratifying.

3.1.3 The monopole problem

Modern particle physics predict a variety of relic particles, including *magnetic monopoles* (which play a crucial role in originally instigating inflation). The problem is, these relic particles are expected to be created with a high abundance very early in the universe's history. Radiation density reduces with a^{-4}, much faster than any other type of matter. Even if the early universe had a very small quantity of non-relativistic matter, then its slower dilution in density (compared to radiation) should have rapidly brought it to prominence. However, observations say otherwise. Theories predicting them are irreconcilable with the standard Hot Big Bang model.

3.2 An Inflationary Retrospective

The fine-tuning problems have left us with reservations in regards to the credibility of the Big Bang model, but there exists a most widely accepted explanation to this befuddling puzzle. That comes in the form of cosmic inflation.

3.2.1 Solutions to the Flatness and Horizon Problems

The **acceleration equation**,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{3.2.1}$$

Since inflation is defined as the interval in the course of the evolution of the universe in which the scale factor was accelerating, therefore,

$$\ddot{a} > 0$$

Which inherently means that $\dot{a} = aH$ increases in the inflationary phase and hence the comoving Hubble radius, $(aH)^{-1}$ decreases during inflation. (It is important to draw attention to the subtle distinction between the comoving horizon and the comoving Hubble radius, if particles are set apart by distances greater than the comoving horizon, then they could never have had any interaction in the *past*, but if they are separated by distances greater than the Hubble radius, then they can not communicate in the *present*.) It also becomes immediately obvious that $\rho + 3p < 0$ Consequently this necessitates that the pressure must be negative as density is always assumed to be positive. This corresponds to,

$$a(t) = a_i e^{H(t-t_i)} \tag{3.2.2}$$

For an H which is exactly constant, this would be equivalent to the de Sitter space, purely dominated by vacuum energy with a **constant energy density**.

The particle horizon for a light particle from some initial time t_i when inflation had begun, to some time t may then be expressed as,

$$\begin{aligned}\Delta r = \int_{t_i}^{t} \frac{dt'}{a(t')} &= \int_{t_i}^{t} \frac{1}{a_i e^{H(t'-t_i)}} dt' \\ &= \frac{1}{a_i H_i}(1 - e^{-H(t-t_i)}) \\ &= \frac{1}{a_i H_i}(1 - e^{-N(t)})\end{aligned}$$

$$N(t) = H(t - t_i)$$

N is the number of e-folds.[1] Comoving particle horizon barely varies during inflation, it increases a bit and then it just freezes in place. This is wildly different from radiation or matter domination or any kind of decelerated phase of the universe where the comoving particle horizon grows without restraint. Here instead, it approaches a maximum value $\frac{1}{a_i H_i}$. Subsequently, the physical distance:

$$a \cdot \Delta r = \frac{e^N}{H_i} \tag{3.2.3}$$

An initial Hubble patch $\frac{1}{H_i}$ stretches exponentially, as time passes the physical distance grows and proliferates rapidly. Inflation blows up the universe, light just like everything else is carried along. Be that as it may, the comoving coordinates remain effectively frozen.

Revisiting the **flatness problem**, we know that observationally today, $\Omega_k < 10^{-2}$ and if we extrapolate this very far back in time $|\Omega_k(t_i)|$ must have been very very small. This can be explained by a period of inflation[2],

$$\Omega_k = \frac{-k}{(aH)^2} \tag{3.2.4}$$

Substituting,

$$aH = a_i H_i e^N \tag{3.2.5}$$

We then obtain,

$$\Omega_k = \frac{-k}{(a_i H_i)^2} \cdot e^{-2N} \tag{3.2.6}$$

i.e. $|\Omega_k|$ decreases exponentially and is especially small if N can be a reasonably large number. Which begs the question, exactly how large does N need to be?

We need,

$$|\Omega_k(t_i)| = \frac{1}{(a_i H_i)^2} << 1 \qquad\qquad |\Omega_k(t_0)| = \frac{1}{(a_0 H_0)^2} < 1$$

[1]As N increases, a gets multiplied by that many e

[2]Physically, one could interpret this as $\Omega_k \propto \frac{1}{\left(\frac{a}{1/H}\right)^2} = \frac{(1/H)^2}{a^2} = \left(\frac{\text{size of Hubble patch}}{\text{size of universe}}\right)^2$

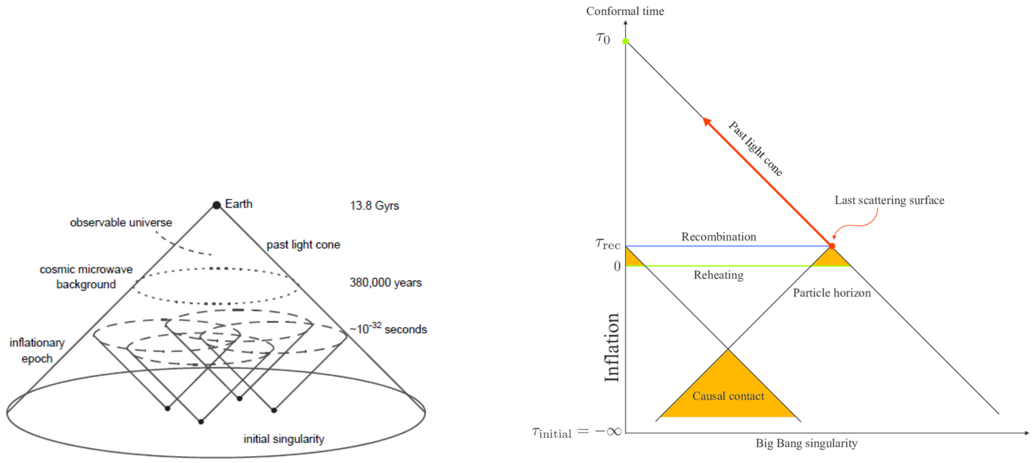


Figure 3.2: Rapid expansion caused very small regions to expand to very large sizes [4]

at some initial time t_i at the beginning of inflation and some time today t_0. Thus:

$$\frac{a_0 H_0}{a_i H_i} > 1 \tag{3.2.7}$$

This offers solutions to both the **flatness** and **horizon problems**, accounting for why there are multitudes of independent horizon volumes in our past and yet why they look so similar and so flat. Inflation solves the **horizon problem** by suggesting that preceding the inflationary period all of the universe was causally connected, and it was during this period that the physical properties levelled out. Inflation then expanded it rapidly, freezing in these properties all over the sky; today faraway regions in the sky appear to be disengaged causally, but in fact were in far greater proximity in the past.
Ergo, the largest scales perceived today $1/H_0$ must be well within the horizon at the beginning of inflation, corresponding to a simple rearrangement of (3.2.7):

$$\frac{1}{a_i H_i} < \frac{1}{a_0 H_0} \tag{3.2.8}$$

and, $\quad a_0 = a_i e^N \left(\frac{a_0}{a_f}\right) \quad$ where, $a_f = a_i e^N \;=\;$ scale factor at end of inflation
Now using (2.4.5), we then obtain:

$$e^N > \frac{T_0}{H_0}\frac{H_i}{T_f} = 10^{27} \tag{3.2.9}$$

$$N > 62 \tag{3.2.10}$$

This adequately solves the **flatness problem**. But in many models, it's not necessary for N to be quite that large. Now, to further go over the particulars of the **horizon problem**, we discuss the behaviour of the particle horizon in a little more detail. Inflation begins at $t = t_i$ and inflation ends at $t = t_f = t_i + \frac{N}{H_i}$, considering a very simple model where right after inflation ends the universe becomes radiation

dominated all the way from t_f to t_0 (As opposed to the real universe where radiation domination is followed by matter domination which eventually gives way to vacuum energy domination). A further supposition for the sake of simplicity is made by presuming that we can actually observe the universe at t_f (Whilst in reality, we only ever observe CMB since decoupling). Reheating marks the end of inflation and we link that to the last scattering.
The conformal time,

$$\tau = \int_{t_i}^{t} \frac{dt'}{a(t')} = \Delta r \tag{3.2.11}$$

(i) Without inflation

$$a = a_i \left(\frac{t}{t_i}\right)^{1/2} \tag{3.2.12}$$

$$\tau = \frac{{t_i}^{1/2}}{a_i} 2t^{1/2} = \frac{1}{H_i a_i}\sqrt{\frac{t}{t_i}} \tag{3.2.13}$$

(ii) With inflation

$$t_i < t < t_f : \qquad a = a_i e^{H(t-t_i)} \tag{3.2.14}$$

$$t > t_f : \qquad a = a_i e^{N}\left(\frac{t}{t_f}\right) \tag{3.2.15}$$

$$t < t_f : \qquad \tau = \frac{1}{a_i H_i}(1 - e^{-N(t)}) \tag{3.2.16}$$

$$t > t_f : \qquad \tau = \frac{1}{a_i H_i} + \frac{2\sqrt{t_i}}{a_i e^N}(\sqrt{t} - \sqrt{t_f}) \tag{3.2.17}$$

If $t \to t_0$, then $\sqrt{t} >> \sqrt{t_f}$ and hence for $t > t_f$,

$$\tau = \frac{1}{a_i H_i} + \frac{2\sqrt{t_i}}{a_i e^N}\sqrt{t} \;\; \simeq \;\; \frac{1}{a_i H_i}\left(1 + e^{-N}\sqrt{\frac{t}{t_i}}\right) \tag{3.2.18}$$

Compared to the amount of conformal time between t_i and t_f, the conformal time between t_f and t_0 is much much less. As we have assumed H_i to be constant during inflation, $a(t)$ becomes infinite at $\tau = 0$. This suggests that inflation will go on for all time, with $\tau = 0$ representing the infinite future, $t \to \infty$. In our actual universe, inflation stops at some finite time, and the conjecture, although valid at early times, falls through towards the end of inflation. So the surface $\tau = 0$ is not the Big Bang, but the end of inflation. Inflation pushes the initial singularity to an arbitrary point far in conformal time $\tau = -\infty$, allowing light cones of two seemingly independent CMB points to be in causal contact in the past.[5]

3.2.2 Relic Particle Abundances

Energy density of the relic particles is reduced much faster than the cosmological constant. During the inflationary era the dramatic expansion causes these particles to be red-shifted away, thereby solving the Monopole problem. (One must be wary of the possibility that reheating may regenerate these unwanted particles via thermal production, but this may be easily avoided if reheating temperature is sufficiently low.)

3.3 Physics of Inflation

At the time of inflation, the early universe expanded exponentially within a tiny fraction of a second. This corresponds to a negative pressure source in Einstein gravity. In this section we acquaint ourselves with the physics underlying inflation.

3.3.1 Scalar Field Dynamics

The scalar field φ is called the **inflaton**, and it is canonically coupled to gravity. This is governed by the action:

$$S = \int d^4x \sqrt{-g}(\mathcal{L}_g + \mathcal{L}_\varphi) \tag{3.3.1}$$

where the Lagrangian of gravitation and the scalar field,

$$\mathcal{L}_g = \frac{R}{16\pi G} \qquad \mathcal{L}_\varphi = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) = \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi) \tag{3.3.2}$$

The variation of the action w.r.t. $g_{\mu\nu}$ allows us to define the energy-momentum tensor,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\varphi}{\delta g^{\mu\nu}} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left(\frac{1}{2}\partial^\sigma\varphi\partial_\sigma\varphi + V(\varphi)\right) \tag{3.3.3}$$

The variation of the action w.r.t. φ gives us the field equation of motion,

$$\frac{\delta S_\varphi}{\delta\varphi} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\varphi) + V_{,\varphi} = 0 \tag{3.3.4}$$

where $V_{,\varphi} = \frac{dV}{d\varphi}$. The same equation could have been obtained by using the components of $T_{\mu\nu}$ for a perfect fluid, assuming homogeneity and isotropy $\varphi = \varphi(t)$, then we can calculate

$$\rho_\varphi \equiv -T^0_0 = \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \qquad \text{(KE + PE)} \tag{3.3.5}$$

$$p_\varphi \equiv \frac{1}{3}T^i_i = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \qquad \text{(KE − PE)} \tag{3.3.6}$$

Therefore the equation of state:

$$w_\varphi \equiv \frac{p_\varphi}{\rho_\varphi} = \frac{\frac{1}{2}\dot{\varphi}^2 - V}{\frac{1}{2}\dot{\varphi}^2 + V} \tag{3.3.7}$$

Acceleration occurs when potential energy dominates over kinetic energy. Substituting (3.3.5) and (3.3.6) into the Friedmann equations[3] we get,

$$H^2 = \frac{1}{3M_{pl}^2}\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right) \tag{3.3.8}$$

$$\dot{H} = -\frac{1}{2M_{pl}^2}\dot{\varphi} \tag{3.3.9}$$

and, $H^2 + \dot{H}^2$:

$$\ddot{\varphi} + 3H\dot{\varphi} = -V_{,\varphi} \tag{3.3.10}$$

This is the **Klein-Gordon** equation.[4]

Condition for inflation to occur,

$$\varepsilon = -\frac{\dot{H}}{H^2} = \frac{1}{M_{pl}^2}\frac{\dot{\varphi^2}}{H^2} = -\frac{d\ln H}{dN} < 1 \tag{3.3.11}$$

So inflation happens for small KE. This sort of inflation is known as the "slow-roll" inflation. For the inflationary phase to sustain for an adequately long amount of time we need the acceleration to be small(so that KE remains small):

$$\eta \equiv -\frac{\ddot{\varphi}}{H\dot{\varphi}} = \varepsilon - \frac{1}{2\varepsilon}\frac{d\varepsilon}{dN} < 1 \tag{3.3.12}$$

The slow-roll approximations enable us to ignore the KE term from (3.3.8) and the acceleration term from the Klein-Gordon equation leading to

$$H^2 \approx \frac{V}{3M_{pl}^2} \tag{3.3.13}$$

$$3H\dot{\varphi} \approx -V_{,\varphi} \tag{3.3.14}$$

The slow roll parameters $\varepsilon, |\eta| < 1$ may be expressed on the shape of the inflationary potential,

$$\varepsilon_V(\varphi) \equiv \frac{M_{pl}^2}{2}\left(\frac{V_{,\varphi}}{V}\right)^2 \tag{3.3.15}$$

$$\eta_V(\varphi) \equiv M_{pl}^2\frac{V_{,\varphi\varphi}}{V} \tag{3.3.16}$$

A breach of the slow-roll conditions ends inflation:

$$\varepsilon(\varphi_f) \equiv 1, \qquad \varepsilon_V(\varphi_f) \approx 1 \tag{3.3.17}$$

[3] $M_{pl}^2 = 1/8\pi G$

[4] where,

$\ddot{\varphi}$: Acceleration

$3H\dot{\varphi}$: Friction

$-V_{,\varphi}$: Force

Therefore, the number of e-folds before inflation ends:

$$N(\varphi) \equiv \int_{a_i}^{a_f} d\ln a = \int_{t_i}^{t_f} H dt \approx \int_{\varphi_i}^{\varphi_f} \frac{1}{\sqrt{2\varepsilon_V}} \frac{|d\varphi|}{M_{pl}} \geq 60 \tag{3.3.18}$$

$N \propto 1/\sqrt{\varepsilon_V}$, hence small ε_V results in large N and the duration of inflation is affected by η_V hence φ_i and φ_f are set farther apart for smaller η_V.

3.3.2 Reheating

In the course of the inflationary phase, the potential energy in the field φ is the most dominant energy component. When $V(\varphi)$ is minimum, inflation comes to an end. The inflaton field rolls to the bottom of the V-hill and then oscillates about the minimum, and the scalar field behaves the same as pressureless matter

$$\frac{d\bar{\rho_\varphi}}{dt} + 3H\bar{\rho_\varphi} = 0 \tag{3.3.19}$$

The inflaton energy then decays due to a coupling of the inflaton field to other particles. (The coupling parameter Υ_φ is dependent on complex and model-dependent physical operations.)

$$\frac{d\bar{\rho_\varphi}}{dt} + (3H + \Upsilon_\varphi)\bar{\rho_\varphi} = 0 \tag{3.3.20}$$

Thereby the universe *reheats* to a sufficiently high temperature for BBN.

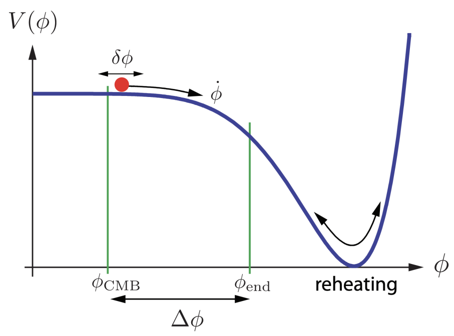


Figure 3.3: The inflaton potential [6]

3.3.3 Models of Inflation

Power-law Inflation

Chaotic inflation is the typical inflationary model. There are numerous models of this type, a lot of which do not necessarily fulfill the condition of a minimum wherein inflation may end, and instead allows inflation to continue perpetually. **Power-law inflation (PLI)** is one such case, with $a(t) \propto t^p$ where p is a constant greater than 1 [7].

PLI is explored shortly.

The total energy density $\rho(t)$ is given by

$$\rho(t) = V(\varphi(t)) + \frac{1}{2}\dot{\varphi}^2(t) + \rho_r(t) \tag{3.3.21}$$

The total pressure is given by

$$p(t) = -V(\varphi(t)) + \frac{1}{2}\dot{\varphi}^2(t) + \frac{1}{3}\rho_r(t) \tag{3.3.22}$$

In our case $\rho_r << V(\varphi)$, thus thermal corrections to the effective potential are inconsequential and we assume that the scalar field has minimal coupling with the geometry.
The time evolution of the model is decided by the following equations

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right] = -3H\dot{\varphi}^2 - \delta \qquad ; \\
&\frac{d}{dt}\rho_r = -4H\rho_r + \delta \qquad ; \\
&H^2 = \frac{8\pi}{3M_{pl}}\left[V(\varphi) + \frac{1}{2}\dot{\varphi}^2 + \rho_r\right] \qquad ;
\end{aligned} \tag{3.3.23}$$

The quantity δ accounts for the formation of the ultra-relativistic particles due to the time variation of φ. The preceding set of equations describes the energy conservation (equation of motion) for φ, the energy conservation equation for radiation, and the Friedmann equation. For the term δ it is presupposed

$$\delta = \Gamma\dot{\varphi}^2 \tag{3.3.24}$$

wherein Γ^{-1} stands for the characteristic time for particle creation by φ, relating to the interactions of φ with other fields. It is then obtained

$$\begin{aligned}
&\dot{\rho}_r + \frac{4}{3}(\Gamma + 3H)\rho_r = -\frac{M_{pl}^2}{4\pi}\Gamma\dot{H} \qquad ; \\
&\dot{\varphi}^2 = -\frac{M_{pl}^2}{4\pi}\dot{H} - \frac{4}{3}\rho_r \qquad ; \\
&V(\varphi) = \frac{M_{pl}^2}{8\pi}(3H^2 + \dot{H}) - \frac{1}{3}\rho_r \qquad ;
\end{aligned} \tag{3.3.25}$$

Given that $V(\varphi)$ is dependent on t solely via φ; when the scale factor is provided, it is then obvious that $\rho_r(t)$, $\varphi(t)$, and $V(\varphi)$ is determinable from (3.3.25). Henceforward the solution to (3.3.25) is considered under the hypothesis,

$$a = a^*(t/t^*)^p \tag{3.3.26}$$

wherein $p > 1$ is a constant, a^* and t^* are arbitrary constants whose value do not appear in any physical quantity. [7] only solves the (3.3.25) during the period when particle creation is insignificant, $\Gamma << 3H$, i.e. $t << t_\Gamma \equiv 3p/\Gamma$. The particle creation process is then assumed to be rapidly *reheating* the universe. The evolution of a system from an initial time t_i with $\varphi(t_i) = \varphi_i \neq 0$ is taken into consideration, when supposedly ρ_r is insignificant w.r.t. the kinetic and potential contributions to

ρ, because a small duration of inflation is sufficient to undermine it. Then, using (3.3.25) it is then obtained,

$$\dot{\varphi}^2 \simeq \frac{M_{pl}^2}{4\pi} p t^2 \tag{3.3.27}$$

which leads to

$$\varphi(t) \simeq \varphi_i \pm \sigma \ln(t/t_i) \quad \text{where,} \quad \sigma = \left(\frac{p}{4\pi}\right)^{1/2} M_{pl} \tag{3.3.28}$$

By plugging in the positive solution into 3.3.25,

$$V(\varphi) \simeq \frac{3p-1}{2} (\sigma/t_i)^2 \exp\left\{-\frac{\varphi - \varphi_i}{\sigma}\right\} \tag{3.3.29}$$

(The negative solution paves way to a potential growing with φ.) It is made obvious that the potential has to be considered just as an approximation of a more intricate potential for the interlude $\varphi_i \lesssim \varphi \lesssim \varphi(t_\Gamma)$. Such potentials are found in Kaluza-Klein and supergravity/superstring models after the implementation of dimensional reduction mechanisms.

Upon further assuming that a slow rolling occurs, $\ddot{\varphi} << 3H\dot{\varphi}$, then $p >> \frac{1}{3}$ is obtained.

To study the evolution of perturbations [7] employs a gauge-invariant approach, the likes of which may be seen in the Bardeen, Steinhardt, and Turner analysis for Standard Inflation[8] (the semantics of which I will begin to explore in the next chapter); the ensuing mass variance at the horizon is perceived to be growing weakly with the scale of the perturbation.

Multi-field Theories

In inflationary model building, a present-day tendency has been to delve into models with more than one scalar field. Within the more extensive category of two and multi-field inflation models, it is pretty common for only one field to be dynamically significant, the standard example is the hybrid inflation. The potential is of the form,

$$V(\varphi, \psi) = \frac{\lambda}{4}\left(\psi^2 - \frac{M^2}{\lambda}\right)^2 + \frac{1}{2}m^2\varphi^2 + \frac{1}{2}\lambda'\varphi^2\psi^2 \tag{3.3.30}$$

When φ^2 is large the field rolls down the channel until it reaches the potential minimum at $\psi = 0$, leading to

$$V \simeq \frac{M^4}{4\lambda} + \frac{1}{2}m^2\varphi^2 \tag{3.3.31}$$

The mass-squared of ψ is negative for $\varphi < \varphi_c \equiv M/g$ and is indicative of instability. Then the field rolls down into one of the true minima at $\phi = 0$ and $\psi = \pm M/\sqrt{\lambda}$.

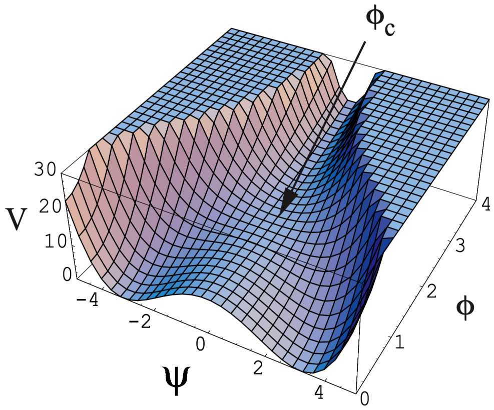


Figure 3.4: The potential for the hybrid inflation model. The field rolls down the channel at $\psi = 0$ until it reaches the critical φ value, then falls off the side to the true minimum at $\varphi = 0$ and $\psi = \pm M/\sqrt{\lambda}$.

Inflation shortly comes to a conclusion after the break in symmetry caused by the rapid rolling of the field ψ. [9]
The number of e-foldings,

$$N \simeq \frac{2\pi M^4}{\lambda m^2 M_{pl}^2} \ln \frac{\varphi_i}{\varphi_c} \tag{3.3.32}$$

φ_i is the initial value of the inflaton and φ_c is the critical value of the inflaton below which $\psi = 0$ becomes unstable.

Chapter 4

Cosmological Perturbations and Inflation

So far we have assumed everything to be perfectly homogeneous. However, inhomogeneities in the form of quantum fluctuations during the course of inflation gives rise to advancements or delays in the local time at which inflation ends. The parts of the universe where the inflation ends earlier, are naturally older. Since the densities of matter and radiation are inversely proportional to time, the older parts of the universe are also less dense. Thus, these quantum fluctuations lead to a patchwork of regions with differing densities, sowing the seeds for cosmological structure formation. [6, 10, 12, 13]

4.1 Cosmological perturbation theory

Henceforward small perturbations are accounted for. All quantities shall now be regarded as,

$$X = \underbrace{\bar{X}}_{\text{background}} + \overbrace{\delta X}^{\text{perturbation}}$$

To ensure that the spatial average of the perturbation is zero, the background and perturbation are set apart.

4.1.1 Metric perturbations

The perturbed FLRW metric may be written as,

$$\mathrm{d}s^2 = a^2(\tau)\left[(1+2A)\mathrm{d}\tau^2 - 2B_i\mathrm{d}x^i\mathrm{d}\tau - (\delta_{ij}+h_{ij})\mathrm{d}x^i\mathrm{d}x^j\right] \tag{4.1.1}$$

where $A(\tau, x^i)$, $B_i(\tau, x^i)$ and h_{ij} are small quantities[1]. It would be prudent to effectuate a *Scalar-Vector-Tensor* (SVT) decomposition, viz. the small quantities may be written as[2],

$$\begin{aligned} &A \to A \\ &B \to \underbrace{\partial_i B}_{\text{scalar}} + \underbrace{B_i^{\mathbf{V}}}_{\text{vector}} \quad \text{where,} \quad \partial^i B_i^{\mathbf{V}} = 0 \end{aligned}$$

[1] $A(\tau, x^i)$ is called the lapse function, and $B_i(\tau, x^i)$ the shift vector.

[2] The bolded $\mathbf{V}$, $\mathbf{T}$ superscripts are being used to denote Vectors and Tensors respectively.

In similar fashion, the symmetric traceless (0, 2)-tensor may be split into the scalar, vector and tensor components,[3]

$$h_{ij} \to \underbrace{2C\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}E}_{\text{scalar}} + \underbrace{2\partial_{(i}E^{\mathbf{V}}_{j)}}_{\text{vector}} + \underbrace{2E^{\mathbf{T}}_{ij}}_{\text{tensor}}$$

and,

$$\partial_{\langle i}\partial_{j\rangle}E \equiv \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E,$$
$$\partial_{(i}E^{\mathbf{V}}_{j)} \equiv \frac{1}{2}\left(\partial_i E^{\mathbf{V}}_j + \partial_j E^{\mathbf{V}}_i\right)$$

where, $\partial^i E^{\mathbf{V}}_i = 0, \partial^i E^{\mathbf{T}}_{ij} = E^{i\mathbf{T}}_i = 0$.

Hence, the metric perturbations constitute,

$$\begin{array}{rclll} 1+1+1+1 & = & 4 & & \textbf{scalars}: A,\ B,\ D,\ E \\ 2+2 & = & 4 & & \textbf{vectors}: B^{\mathbf{V}}_i, E^{\mathbf{V}}_i \\ 2 & = & 2 & & \textbf{tensor}: E^{\mathbf{T}}_{ij} \end{array}$$

10 degrees of freedom.

In first-order perturbation theory, scalar, vector and tensor parts evolve independent of each other and may therefore be treated separately, that is what makes the SVT division so important. The complete evolution of the full perturbation is the linear superposition of the individual parts. The scalar perturbations are responsible for the formation of structure in the universe from small initial perturbations. The vector perturbations decay like a^{-2} in an expanding universe, and are therefore possibly unimportant in cosmology. Tensor perturbations/gravity waves are an important prediction of inflation since, if sufficiently powerful, they have an observable effect on the anisotropy of the CMB. [10]

4.1.2 Gauge choice

The metric perturbations in (4.1.1) are determined by the *gauge choice*[4]. The perturbed metric is described by *specific* spatial coordinates in a *distinct* time slicing of the spacetime. i.e. opting for a different set of coordinates may actually vary the perturbation variables. As a result, fictitious perturbations may be introduced. This unfortunate aspect is confirmed if one were to consider an unperturbed universe with metric

$$\mathrm{d}s^2 = a^2(\tau)(\mathrm{d}\tau^2 - \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j) \tag{4.1.2}$$

With a change of coordinates,

$$x^i \to \tilde{x}^i = x^i + \xi^i(\tau, \mathbf{x}) \Rightarrow \mathrm{d}x^i = \mathrm{d}\tilde{x}^i - \partial_\tau\xi^i\mathrm{d}\tau - \partial_k\xi^i\mathrm{d}\tilde{x}^k \tag{4.1.3}$$

[3]The $\langle\ \rangle$ subscript is used to signify that only the trace-free portion of the object has been taken into consideration.

[4]Choice of coordinates

The line element then takes the form

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - 2\xi_i{}'\mathrm{d}\tilde{x}^i\mathrm{d}\tau - (\delta_{ij} + 2\partial_{(i}\xi_{j)})\mathrm{d}\tilde{x}^i\mathrm{d}\tilde{x}^j + O(\xi^2)\right] \tag{4.1.4}$$

Here, $\xi_i' \equiv \partial_\tau \xi_i$. And, the perturbations introduced are ξ_i' and ξ_i, which are fictitious **gauge modes** that can be easily dismissed by just reverting to the earlier coordinates.

Likewise, we perturb time, $\tau \to \tilde{\tau} = \tau + \xi^0(\tau, \mathbf{x}) \Rightarrow \rho(\tau) \to \rho(\tilde{\tau}) = \rho(\tau + \xi^0(\tau, \mathbf{x})) = \tilde{\rho}(\tau) + \tilde{\rho}'\xi^0$. Here, $\tilde{\rho}'\xi^0 = \delta\rho$ is the fictitious density perturbation that has been induced. It is thus made obvious that even in an unperturbed, perfectly homogeneous universe, an inappropriate gauge choice may result in fake perturbations. Conversely, a real perturbation in the energy density may be removed by choosing the hypersurface of constant time to coincide with the hypersurface of constant energy density[6]. These complications associated with metric perturbations being dependent on the gauge choice are jointly referred to as the *gauge problem*.

4.1.3 Gauge transformations

Recall that a gauge transformation is a first order change in the coordinates. [12]

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(\tau, \mathbf{x}) \qquad \xi^0 \equiv T, \xi^i \equiv L^i = \partial^i L + L_i^{\mathbf{V}} \tag{4.1.5}$$

As can be seen, there is no tensor part. This is because tensor perturbations are *gauge invariant.* [12]

Implementing the transformation law,

$$g_{\mu\nu} = \frac{\partial x'^\alpha}{\partial x^\mu}\frac{\partial x'^\beta}{\partial x^\nu} g'_{\alpha\beta} \tag{4.1.6}$$

The effect to the 00-component of the metric (4.1.1) is then,

$$g_{00} = \frac{\partial \tilde{x}^\alpha}{\partial x^0}\frac{\partial \tilde{x}^\beta}{\partial x^0}\tilde{g}_{\alpha\beta} = \left(\frac{\partial \tilde{x}^0}{\partial x^0}\right)^2 \tilde{g}_{00} = \left(\frac{\partial \tilde{\tau}}{\partial \tau}\right)^2 \tilde{g}_{00} \tag{4.1.7}$$

Substitution of (4.1.5) and (4.1.1) leads to[5],

$$\begin{aligned}
a^2(\tau)(1 + 2A) &= (1 + T')^2 a^2(\tau + T)(1 + 2\tilde{A}) && (4.1.8)\\
&= (1 + 2T' + \dots)(a(\tau) + a'T + \dots)^2((1 + 2\tilde{A}) && (4.1.9)\\
&= a^2(\tau)(1 + 2\mathcal{H}T + 2T' + 2\tilde{A} + \dots) && (4.1.10)\\
\tilde{A} &= A - T' - \mathcal{H}T && (4.1.11)
\end{aligned}$$

In similar fashion, in terms of the SVT decomposition,

$$\begin{aligned}
&\tilde{B} = B + T - L' && \tilde{B}_i^{\mathbf{V}} = B_i^{\mathbf{V}} - L_i'^{\mathbf{V}} \\
&\tilde{C} = C - \mathcal{H}T - \frac{1}{3}\nabla^2 L && && (4.1.11)\\
&\tilde{E} = E - L && \tilde{E}_i^{\mathbf{V}} = E_i^{\mathbf{V}} - L_i^{\mathbf{V}} \quad \tilde{E}_{ij}^{\mathbf{T}} = E_{ij}^{\mathbf{T}}
\end{aligned}$$

[5]where, $\mathcal{H} \equiv a'/a$ is the comoving Hubble parameter

A course of action that can be undertaken to find a way around the *gauge problem* discussed in §**4.1.2** is to work with Gauge invariant quantities. Otherwise known as the *Bardeen variables.*

$$\Psi \equiv A + \mathcal{H}(B - E') + (B - E')' \qquad \Phi_i^{\mathbf{V}} \equiv E_i'^{\mathbf{V}} - B_i^{\mathbf{V}} \qquad E_{ij}^{\mathbf{T}}$$
$$\Phi \equiv -C - \mathcal{H}(B - E') + \frac{1}{3}\nabla^2 E$$

Another alternative is to fix the gauge.[10, 11] There are three popular metric gauges:

Synchronous

$A = B = 0$, time is unperturbed.

Spatially Flat

$C = E = 0$, space is unperturbed.

Longitudinal/Newtonian

The gauge freedom is used to set the scalar perturbations $B = E = 0$. It is obvious from (4.1.11), that this may be achieved by assigning

$$\xi = -E$$
$$\xi^0 = -B + E'$$

And, we have $A \equiv \Psi$ and $C \equiv -\Phi$. Then the metric takes the form,

$$\mathrm{d}s^2 = a^2(\tau)\left[(1 + 2\Psi)d\tau^2 - (1 - 2\Phi)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right] \tag{4.1.12}$$

Newtonian gauge is to be the preferred gauge for studying the origin of large-scale structures.

4.1.4 Matter perturbations

The perturbed stress energy-momentum tensor,

$$\begin{aligned} T_0^0 &= \bar{\rho}(\tau) + \delta\rho \\ T_0^i &= [\bar{\rho}(\tau) + \bar{p}(\tau)]v^i \\ T_j^i &= -[\bar{p}(\tau) + \delta p]\delta_j^i - \Pi_j^i \end{aligned} \tag{4.1.13}$$

Here, v_i is the bulk velocity, $\Pi^i{}_j$ is the anisotropic stress. In a multi-component universe consisting of for e.g. baryons, photons, dark matter et cetera, the sum of the individual energy momentum tensors make up the total energy-momentum tensor. Implying,

$$\delta\rho = \sum_a \delta\rho_a, \qquad \delta p = \sum_a \delta p_a, \qquad q^i = \sum_a q^i_{(a)} \qquad \Pi^{ij} = \sum_a \Pi^{ij}_{(a)} \tag{4.1.14}$$

where, q^i is $(\bar{\rho}+\bar{p})v^i$. Once again, SVT-decomposition is to be implemented, $\delta\rho$ and δp are scalars; q_i has scalar and vector parts;

$$q_i = \partial_i q + q_i^{\mathbf{V}} \tag{4.1.15}$$

Π_{ij} has scalar, vector and tensor parts;

$$\Pi_{ij} = \partial_{\langle i}\partial_{j\rangle}\Pi + \partial_{(i}\Pi^{\mathbf{V}}_{j)} + \Pi^{\mathbf{T}}_{ij} \tag{4.1.16}$$

Under the general gauge transformation,

$$T^{\mu}_{\nu} = \frac{\partial x^{\mu}}{\partial \tilde{x}^{\alpha}}\frac{\partial \tilde{x}^{\beta}}{\partial x^{\nu}}\tilde{T}^{\alpha}_{\beta} \tag{4.1.17}$$

Computing this for the the different components leads to,

$$\delta\tilde{\rho} = \delta\rho - T\bar{\rho}' \tag{4.1.18}$$

$$\delta\tilde{p} = \delta p - T\bar{p}' \tag{4.1.19}$$

$$\tilde{q}_i = q_i + (\bar{\rho}+\bar{p})L'_i \tag{4.1.20}$$

$$\tilde{v}_i = v_i + L'_i \tag{4.1.21}$$

$$\tilde{\Pi}_{ij} = \Pi_{ij} \tag{4.1.22}$$

There are two favourable matter gauges[11]:

Uniform density gauge

$\delta\rho = 0$, spatial slices imitate surfaces of constant density.

Comoving gauge

$q = 0$, scalar momentum density is set to zero.

Gauge invariant quantities can be devised from metric and matter variables, one popular gauge invariant quantity,

$$\Delta \equiv \frac{\delta\rho}{\bar{\rho}} + \frac{\bar{\rho}'}{\bar{\rho}}(v+B) \tag{4.1.23}$$

It may be useful to write $\frac{\delta\rho}{\bar{\rho}} \equiv \delta$, where δ is the dimensionless density contrast.

4.1.5 Equations of motion

In the Newtonian gauge, the metric tensor takes the simple form,

$$g_{\mu\nu} = a^2\begin{pmatrix} 1+2\Psi & 0 \\ 0 & -(1-2\Phi)\delta_{ij}\end{pmatrix} \tag{4.1.24}$$

A tedious calculation of the Christoffel symbols(recall equation 1.3.1) leads to:

$$\begin{aligned}
\Gamma^0_{00} &= \mathcal{H} + \Psi' \\
\Gamma^0_{i0} &= \partial_i\Psi \\
\Gamma^i_{00} &= \delta^{ij}\partial_j\Psi \\
\Gamma^0_{ij} &= \mathcal{H}\delta_{ij} - [\Phi' + 2\mathcal{H}(\Phi+\Psi)]\delta_{ij} \\
\Gamma^i_{j0} &= [\mathcal{H} - \Phi']\delta^i_j \\
\Gamma^i_{jk} &= -2\delta^i_{(j}\partial_{k)}\Phi + \delta_{jk}\delta^{il}\partial_l\Phi
\end{aligned} \tag{4.1.25}$$

Conservation Equations

Now, with the connection coefficients, we may evaluate the constraint equations from[13]

$$\nabla_\mu T^\mu_\nu = 0 \tag{4.1.26}$$
$$= \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\alpha} T^\alpha_\nu - \Gamma^\alpha_{\mu\nu} T^\mu_\alpha \tag{4.1.27}$$

For $\nu = 0$,

$$\partial_0 T^0_0 + \partial_i T^i_0 + \Gamma^\mu_{\mu 0} T^0_0 + \underbrace{\Gamma^\mu_{\mu i} T^i_0}_{O(2)} - \Gamma^0_{00} T^0_0 - \underbrace{\Gamma^0_{i0} T^i_0}_{O(2)} - \underbrace{\Gamma^i_{00} T^0_i}_{O(2)} - \Gamma^i_{j0} T^j_i = 0 \tag{4.1.28}$$

Now, plugging in (4.1.13) and (4.1.25), yields

$$\partial_0(\bar\rho + \delta\rho) + \partial_i q^i + (\mathcal{H} + \Psi' + 3\mathcal{H} - 3\Phi')(\bar\rho + \delta\rho)$$
$$-(\mathcal{H} + \Psi')((\bar\rho + \delta\rho) - (\mathcal{H} - \Phi')\delta^i_j \left[-(\bar p + \delta p)\delta^j_i\right] = 0$$

and,

$$\bar\rho' + \partial_0\delta\rho + \partial_i q^i + 3\mathcal{H}(\bar\rho + \delta\rho) - 3\bar\rho\Phi' + 3\mathcal{H}(\bar p + \delta p) - 3\bar p\Phi' = 0$$

$$\left(0^{th} \text{ order}\right) \quad \bar\rho' = -3\mathcal{H}(\bar\rho + \bar p) \tag{4.1.29}$$
$$\left(1^{st} \text{ order}\right) \quad \partial_\tau \delta\rho = \underbrace{-3\mathcal{H}(\delta\rho + \delta p)}_{\text{dilution due to expansion}} + \underbrace{3\Phi'(\bar\rho + \bar p)}_{\text{perturbed expansion}} - \underbrace{\boldsymbol{\nabla}\cdot q}_{\text{fluid flow}} \tag{4.1.30}$$

The $\left(0^{th}\text{ order}\right)$ equation does not account for perturbations and is merely representative of the conservation of energy in the homogeneous background. The (1^{st} order) equation is the continuity equation for the perturbation in density $\delta\rho$.

For $\nu = i$,

$$\partial_0 T^0_i + \partial_j T^j_i + \Gamma^\mu_{\mu 0} T^0_i + \Gamma^\mu_{\mu j} T^j_i - \Gamma^0_{0i} T^0_0 - \Gamma^0_{ji} T^j_0 - \Gamma^j_{0i} T^0_j - \Gamma^j_{ki} T^k_j = 0 \tag{4.1.31}$$

Now, plugging in (4.1.13) and (4.1.25), yields[6]

$$-\partial_0 q_i + \partial_j \left[-(\bar p + \delta p)\delta^j_i - \Pi^j_i\right] - 4\mathcal{H} q_i - (\partial_j\Psi - 3\partial_j\Phi)\bar p\delta^j_i - \partial_i\Psi\bar\rho$$
$$-\mathcal{H}\delta_{ji} q^j + \mathcal{H}\delta^j_i q_j + \underbrace{\left(-2\delta^j_{(i}\partial_{k)}\Phi + \delta_{ki}\delta^{jl}\partial_l\Phi\right)\bar p\delta^k_j}_{-3\partial_i\Phi\bar p} = 0$$

Consequently,

$$\partial_\tau q_i = -4\mathcal{H} q_i - (\bar\rho + \bar p)\partial_i\Psi - \partial_i\delta p - \partial^j\Pi_{ij} \tag{4.1.32}$$

This is the Euler equation(for a viscous fluid).

---

[6] $T^0_i = -q_i$

Einstein Equations

Now we may proceed onwards with the Einstein equations. We recall the Ricchi tensor and the Einstein tensor are defined as,

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} \tag{4.1.33}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{4.1.34}$$

After plugging in all of the connection coefficients, and computing the Ricchi tensor and the Ricchi scalar, the Einstein tensor in the Newtonian/Longitudinal gauge is eventually found to be,

$$\begin{aligned} G_{00} &= 3\mathcal{H}^2 + 2\nabla^2\Phi - 6\mathcal{H}\Phi' \\ G_{0i} &= 2\partial_i(\Phi' + \mathcal{H}\Psi) \\ G_{ij} &= -(2\mathcal{H}' + \mathcal{H}^2)\delta_{ij} + \partial_i\partial_j(\Phi - \Psi) \\ &\quad + [\nabla^2(\Psi - \Phi) + 2\Phi" + 2(2\mathcal{H}' + \mathcal{H}^2)(\Phi + \Psi) + 2\mathcal{H}\Psi' + 4\mathcal{H}\Phi']\delta_{ij} \end{aligned} \tag{4.1.35}$$

Merging with the stress energy-momentum tensor, the subsequent equations are derived,

(i) For the trace free space-space Einstein equation, $\mu = i$ and $\nu = j$,

$$\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)(\Phi - \Psi) = \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\Pi \tag{4.1.36}$$

and hence, presuming there is little to no anisotropic stress[13], i.e. $\Pi \approx 0$, the Bardeen potentials are equal, i.e. $\Phi = \Psi$.

(ii) For the temporal part, $\mu = 0$ and $\nu = 0$,

$$\left(0^{th} \text{ order}\right) \qquad 3\mathcal{H}^2 = 8\pi G a^2 \bar{\rho} \tag{4.1.37}$$

This is the **Friedmann equation** with $\Lambda = 0$

$$\left(1^{st} \text{ order}\right) \qquad \nabla^2\Phi = 4\pi G a^2\bar{\rho}\delta + 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) \tag{4.1.38}$$

(iii) For the time-space Einstein equation, $\mu = 0$ and $\nu = i$,

$$\left(1^{st} \text{ order}\right) \qquad \Phi' + \mathcal{H}\Phi = -4\pi G a^2 q \tag{4.1.39}$$

(iv) Considering the trace of the space-space Einstein equation, $\mu = i$ and $\nu = j$,

$$\left(0^{th} \text{ order}\right) \qquad 2\mathcal{H}' + \mathcal{H}^2 = -8\pi G a^2 \bar{p} \tag{4.1.40}$$

$$\left(1^{st} \text{ order}\right) \qquad \Phi" + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta p \tag{4.1.41}$$

In this evolution equation for the metric potential, δp is the total pressure perturbation.

Combining (4.1.38) and (4.1.39) gives the **Poisson equation**,

$$\nabla^2\Phi = 4\pi G a^2 \bar{\rho}\Delta \tag{4.1.42}$$

4.2 Initial Conditions

At an adequately early epoch, all scales of interest to present observations were outside the Hubble radius. On super-Hubble scales, the evolution of perturbations becomes very straightforward, particularly for adiabatic initial conditions.

4.2.1 Adiabatic Fluctuations

An important feature of the adiabatic perturbations is that the local state of matter at some spacetime point $(\tau, \mathbf{x})$ of the universe that is perturbed is equivalent to the local state of matter at a slightly different time $\tau + \delta\tau(\mathbf{x})$ in the background universe. Adiabatic perturbations can hence be perceived as a consequence of some parts of the universe leading the evolution compared to other parts by having reheated earlier on.

Adiabatic density perturbations,

$$\delta\rho_a(\tau, \mathbf{x}) \equiv \bar{\rho}_a(\tau + \delta\tau(\mathbf{x})) - \bar{\rho}_a(\tau) = \bar{\rho}_a' \delta\tau(\mathbf{x}) \tag{4.2.1}$$

where the local time perturbation is the same for all species a. So we have,

$$\delta\tau = \frac{\delta\rho_a}{\bar{\rho}_a'} = \frac{\delta\rho_b}{\bar{\rho}_b'} \quad \text{for all species } a \text{ and } b \tag{4.2.2}$$

Now, using $\bar{\rho}_a' = -3\mathcal{H}(1 + w_a)\bar{\rho}_a$, this can be written as

$$\frac{\delta_a}{1 + w_a} = \frac{\delta_b}{1 + w_b} \quad \text{for all species } a \text{ and } b \tag{4.2.3}$$

And hence, all matter components ($w_m = 0$) have equivalent fractional perturbations and all radiation perturbations ($w_r = \frac{1}{3}$) follow,

$$\delta_r = \frac{4}{3}\delta_m \tag{4.2.4}$$

Since the total density perturbation,

$$\delta\rho \equiv \sum_a \bar{\rho}_a \delta_a$$

This implies that whichever species dominates the background, i.e. carries the dominant energy density $\bar{\rho}_a$, dominates the fluctuations(as all δ_a's are similar). At the time of inflation, the energy density is dominated by the inflaton perturbation $\delta\varphi$ which will have spatially varying fluctuations. Thus there will be local differences in the time when inflation comes to an end, leading to distinct regions of space being subject to varying degrees of inflation. The histories of the local expansions induce the differences in the local densities following inflation. [7]

[7] The fluctuations to the inflaton field are a natural consequence of handling inflation quantum mechanically.

4.2.2 Curvature Perturbations

If the equation of state of the background is constant, only then is Φ, the gravitational potential, constant on super-Hubble scales. Whenever the equation of state transitions, for instance from inflation to radiation domination or from radiation to matter domination, so will Φ. It would be prudent to adopt an alternative perturbation variable that even in these more generalized circumstances remains the same on large scales . The *comoving curvature perturbation* is a variable of this sort. $\mathcal{R}$ is gauge-invariant.

$$\mathcal{R} = -\Phi + \frac{\mathcal{H}}{\bar{\rho}+\bar{p}}\delta q \tag{4.2.5}$$

wherein $\delta^0_j \equiv -\partial_j \delta q$ Expressing the intial conditions in terms of this *comoving curvature perturbation* enables us to readily relate to the predictions made by **inflation** to the fluctuations in the primordial plasma. [13] At horizon crossing, we make a shift from the inflaton fluctuation $\delta\varphi$ to $\mathcal{R}$. However, we consider a spatially flat gauge(The reason for which is discussed in §4.3). Thus for the unpertubed spatial part of the metric, we have, $\Phi = 0$. The perturbed momentum density is given by,

$$\delta T^0_j = g^{0\mu}\partial_\mu\varphi\partial_j\delta\varphi = \bar{g}^{00}\partial_0\bar{\varphi}\partial_j\delta\varphi = \frac{\bar{\varphi}'}{a^2}\partial_j\delta\varphi \tag{4.2.6}$$

and, for $\bar{\rho}+\bar{p} = a^{-2}(\bar{\dot{\varphi}}')^2$, we get,

$$\mathcal{R} = -\frac{\mathcal{H}}{\bar{\dot{\varphi}}'}\delta\varphi = -H\frac{\delta\varphi}{\bar{\dot{\varphi}}} = -H\delta t \tag{4.2.7}$$

i.e. The curvature perturbation is introduced by the time delay at the end of inflation.

4.2.3 Statistics of Cosmological Perturbations

Quantum mechanics during **inflation** describes the CMB fluctuations in individual directions as opposed to identifying the temperature fluctuation in one particular direction. The ensemble average of the fluctuations is given by the two-point correlation function,

$$\langle\mathcal{R}(x)\mathcal{R}(x')\rangle \equiv \xi_{\mathcal{R}}(x,x') = \xi_{\mathcal{R}}(|x-x'|) \tag{4.2.8}$$

The Fourier transform of $\mathcal{R}$ supplies,

$$\langle\mathcal{R}(k)\mathcal{R}^*(k')\rangle = \frac{2\pi^2}{k^3}\Delta^2_{\mathcal{R}}(k)\delta_D(k-k') \tag{4.2.9}$$

Here $\Delta^2_{\mathcal{R}}$ is the *power spectrum* of $\mathcal{R}$.

$$\Delta^2_{\mathcal{R}} = \left(\frac{H^2}{2\pi\dot{\varphi}}\right) \tag{4.2.10}$$

The inflaton satisfies, $\ddot{\varphi} + 3H\dot{\varphi} + V' = 0$. And during slow-roll, we can ignore $\ddot{\varphi}$, and hence, $\dot{\varphi} \approx \frac{V'}{3H}$. Therefore, $\Delta^2_{\mathcal{R}}$ may also be written as,

$$\Delta^2_{\mathcal{R}} = \frac{1}{24\pi^2}\frac{1}{\varepsilon_V}\frac{V}{M^4_{pl}} \tag{4.2.11}$$

The spectrum is very nearly described by a power law, $\Delta^2_{\mathcal{R}} = A_s(k/k_*)^{n_s-1}$, with the spectral index,

$$n_s - 1 \equiv \frac{d\ln\Delta^2_{\mathcal{R}}}{d\ln k} = \frac{d\ln\Delta^2_{\mathcal{R}}}{d\ln aH} \approx \frac{d\ln\Delta^2_{\mathcal{R}}}{d\ln a} = \frac{d\ln\Delta^2_{\mathcal{R}}}{Hdt} = -6\varepsilon_V + 2\eta_V \qquad (4.2.12)$$

This elucidates the amplitude of curvature perturbations and the spectral index in terms of the inflaton potential shape.[8]

4.3 Inflaton fluctuations: Classical

During inflation, the perturbations are dictated by $\delta\varphi$ and $\delta g_{\mu\nu}$, the inflaton perturbations and the metric perturbations respectively. In a general gauge, a perturbation in the inflaton field means a perturbation of the energy-momentum tensor.

$$\delta\varphi \Longrightarrow \delta T_{\mu\nu}$$

A perturbation in the energy-momentum tensor thereby suggests a perturbation of the metric. viz. the Einstein equations,

$$\delta T_{\mu\nu} \Longrightarrow \left[\delta R_{\mu\nu} - \frac{1}{2}\delta(g_{\mu\nu}R)\right] = 8\pi G\delta T_{\mu\nu} \Longrightarrow \delta g_{\mu\nu}$$

Concurrently, any perturbation of the metric interjects a backreaction on the evolution of $\delta\varphi$ by way of the perturbed Klein-Gordon equation of the inflaton field,

$$\delta g_{\mu\nu} \Longrightarrow \delta\left(\partial_\mu\partial^\mu\varphi + \frac{\partial V}{\partial\varphi}\right) = 0 \Longrightarrow \delta\varphi$$

i.e. these are tightly coupled, we conclude by this method of reasoning

$$\boxed{\delta\varphi \Longleftrightarrow \delta g_{\mu\nu}}$$

We begin this section by looking at the classical dynamic of the inflaton fluctuation. The inflaton action,

$$S = \int \mathrm{d}\tau\mathrm{d}^3x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right] \qquad (4.3.1)$$

where, $g \equiv \det(g_{\mu\nu})$. We require the action at quadratic order in fluctuation for us to study the linearised dynamics. However, looking into the quadratic action for the couple fluctuations $\delta\varphi$ and $\delta g_{\mu\nu}$ may prove to be unnecessarily tedious, a trouble that we may be exempted from if were to opt for a more favourable choice of gauge. We work with the *spatially flat gauge*, wherein the freedom in the choice of coordinates allowed us to set the unperturbed spatial metric, $g_{ij} = -a^2\delta ij$. The perturbations are dictated by $\delta\varphi$ and $\delta g_{0\mu}$. The perturbations $\delta g_{0\mu}$ are subdued compared to the inflaton fluctuations by factors of the slow-roll parameter ε; when $\varepsilon \to 0, \delta_{0\mu}$ vanishes. Thus at leading order, we may perturb the inflaton field separately from other fluctuations.[13]

[8]Here, ε_V and η_V are as defined in (3.3.15) and (3.3.16).

We obtain from (4.3.1), for the unperturbed FLRW metric,

$$S = \int \mathrm{d}\tau \mathrm{d}^3x \left[\frac{1}{2}a^2((\varphi')^2 - (\nabla\varphi)^2) - a^4 V(\varphi)\right] \tag{4.3.2}$$

The perturbed inflaton field may be written as,

$$\varphi(\tau, x) = \bar{\varphi}(\tau) + \frac{f(\tau, x)}{a(\tau)} \tag{4.3.3}$$

Plugging in (4.3.3) into (4.3.2) and separating the terms with two factors of f leads to,

$$S_{(2)} = \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x \left[(f')^2 - (\nabla f)^2 - 2\mathcal{H}ff' + (\mathcal{H}^2 - a^2 V_{,\varphi\varphi})f^2\right] \tag{4.3.4}$$

Integrating $ff' = \frac{1}{2}\partial_\tau(f^2)$ by parts,

$$\begin{aligned} S_{(2)} &= \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x \left[(f')^2 - (\nabla f)^2 + (\mathcal{H}' + \mathcal{H}^2 - a^2 V_{,\varphi\varphi})f^2\right] && (4.3.5)\\ &= \frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3x \left[(f')^2 - (\nabla f)^2 + \left(\frac{a''}{a} - a^2 V_{,\varphi\varphi}\right) f^2\right] && (4.3.6)\end{aligned}$$

During slow-roll inflation,

$$\frac{V_{,\varphi\varphi}}{H^2} \approx \frac{3M_{pl}^2 V_{,\varphi\varphi}}{V} = 3\eta_V \ll 1$$

And because, $a' = a^2 H$, with $H \approx const.$ during inflation, we have,

$$\frac{a''}{a} \approx 2a'H = 2a^2H^2 \gg a^2 V_{,\varphi\varphi}$$

Therefore, we can drop $V_{,\varphi\varphi}$

$$S_{(2)} = \int \mathrm{d}\tau \mathrm{d}^3x \, \frac{1}{2}\left[(f')^2 - (\nabla f)^2 + \left(\frac{a''}{a}\right) f^2\right] \tag{4.3.7}$$

This suggests the following equation of motion

$$\boxed{f_k'' + \left(k^2 - \frac{a''}{a}\right) f_k = 0} \qquad f_k(\tau) \equiv \int \frac{\mathrm{d}^3x}{(2\pi)^{3/2}} f(\tau, x) e^{-ik\cdot x} \tag{4.3.8}$$

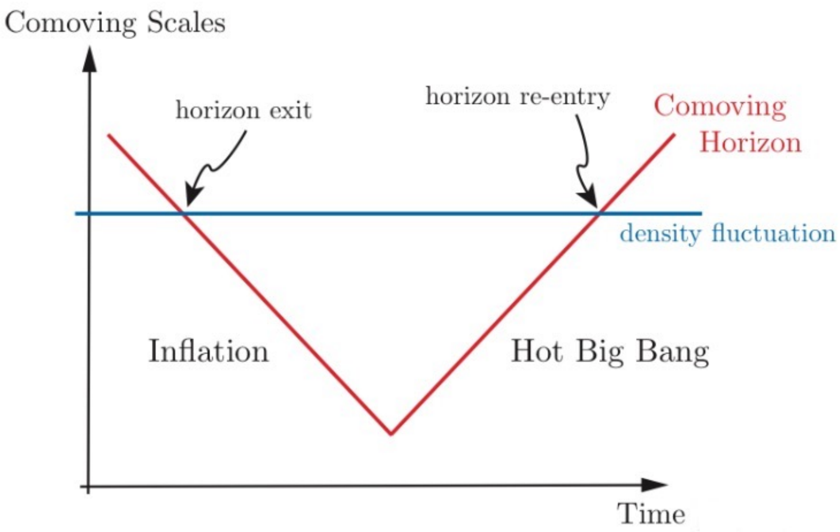
Which is sometimes knows as the *Mukhanov-Sasaki* equation. We have $\frac{a''}{a} \approx 2\mathcal{H}^2 = \frac{2}{\tau^2}$ in a quasi-de Sitter background. Thereby, the *Mukhanov-Sasaki* equation reduces to

$$f_k'' + \left(k^2 - \frac{2}{\tau^2}\right) f_k = 0$$

During inflation, the comoving Hubble radius $\mathcal{H}^{-1} = (aH)^{-1}$ shrinks ($a \approx -1/(H\tau)$ and τ the conformal time progresses from $-\infty$ to 0). The *Mukhanov-Sasaki* equation then becomes

$$f_k'' + k^2 f_k \approx 0 \quad \text{(at early times, for } |\tau| \gg k^{-1}\text{)} \tag{4.3.9}$$

which is simply the equation of a *simple harmonic oscillator*. The quantum fluctuations of these oscillators brings forth the emergence of structure in the universe.



Creation and evolution of perturbations in the inflationary universe. Fluctuations are created quantum mechanically on subhorizon scales. While comoving scales, k^{-1}, remain constant the comoving Hubble radius during inflation, $(aH)^{-1}$, shrinks and the perturbations exit the horizon. Causal physics cannot act on superhorizon perturbations and they freeze until horizon re-entry at late times.

Figure 4.1: [6]

Chapter 5

Conclusion

This thesis provided an introductory account to inflationary cosmology with a culminating focus targeted towards corroborating inflation as a model for the origin of structure. There is much scope for elaboration and improvement, by delving more deeply into the study of the models of inflation and by quantising the inflaton fluctuations. Structure formations are thought to be a consequence of these fluctuations. Through ages, and under the effects of gravity, these matter fluctuations eventually formed galaxies, stars and all celestial bodies as we know them.

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