Supersymmetric Gauge Theory: An Application of Group Theory in High Energy Physics and Starobinsky Model of Cosmological Inflation

Thesis submitted in partial fulfilment of the requirements for the degree of
Bachelor of Science in Physics

by

Ipshita Bonhi Upoma

Department of Mathematics and Natural Sciences
BRAC University

Declaration

I, hereby, declare that the thesis titled **Supersymmetric Gauge Theory: An Application of Group Theory in High Energy Physics and Starobinsky Model of Cosmological Inflation** is submitted to the Department of Mathematics and Natural Sciences of BRAC University in partial fulfillment of the requirements for the degree of Bachelor of Science in Physics. This is a work of my own and has not been submitted elsewhere for award of any other degree or diploma. Every work that has been used as reference for this thesis has been cited properly.

Candidate

Ipshita Bonhi Upoma

ID: 11311002

Certified by

Dr. Mahbub Majumdar

Supervisor

Professor

Department of CSE

BRAC University, Dhaka

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ABSTRACT

This thesis discusses the general role of symmetry in high-energy physics. It concentrates on the role of fermionic symmetry generators, which generate the symmetry known as supersymmetry. An extensive discussion of spinors in D-dimensions is given and the necessity of spinors is explained from a group theoretical point of view. The supersymmetric transformation rules of the fields are explained and Lagrangians for simple theories are explained. We generalize to $SU(n)$ gauge theories and give a discussion of confinement in supersymmetric gauge theories. We also apply supersymmetry in the Starobinsky model of cosmological inflation by constructing a NSWZ model equivalent to the R^2 Starobinsky model. A parametrically quadratic function, $x(t)$ providing favourable conditions for inflation is found for field, x in terms of a string modulus, t.

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Chapter 1

Introduction

A symmetry is a transformation on a physical system that can be applied without changing the physical observables of the system. We observe symmetry in Quantum Field Theory. We can classify symmetries as discrete and continuous symmetries or global and local symmetries. For elementary particles two kinds of symmetry can be observed.

- **Internal symmetry:** Symmemtries that correspond to transformations of the different fields in a field theory are internal symmetries. For a space-time independent theory we get a global symmetry. A local symmetry is observed, otherwise.

- **Space-time symmetry:** Space-time symmetries are transformations on fields which change the space-time coordinates. General coordinate transformations defining general relativity are local symmetries. Lorentz and Poincaré transformation are global symmetries defining special relativity.

Supersymmetry

Supersymmetry (SUSY) is a space-time symmetry where a mapping between particles or fields of integer spin (bosons) and particles or fields of half-integer spin can be applied.

Fermions are particles which are constrained by the Pauli Exclusion Principle. Fermions include quarks and leptons. Boson not being constrained by Pauli Exclusion principle shows different physical properties than those of fermions. Bosons include photons and the mediators of all other interactions. To understand high energy physics at TeV scale, a mathematical framework which explains the relation between these two types of particles is needed. Supersymmetry allows us to unify bosons and fermions despite their different physical properties.

By the action of a SUSY generator, Q on a fermionic particle we may get the bosonic super-partner of that particle. Similarly, we get the fermionic super-partner when the generator acts on a bosonic particle.

$$Q|\text{Fermion}\rangle = |\text{Boson}\rangle \text{ and vice versa} \tag{1.1}$$

Supersymmetry is a space-time symmetry. When the supersymmetry operator changes a particle to its super-partner, what it actually does is to alter the spin of the particle. That

means by an action of SUSY operator on a particle, the space-time properties of the particle are being changed.

If supersymmetry is realized in nature, every one-particle should have a super-partner. So, instead of single particle states, super-multiplets of particle states have to be observed. It is believed that the super-partners of the elementary particles are to be observed in high energy physics of TeV scale. Many experiments in LHC and other laboratories are designed to find these particles.

Particles belonging to the same super-multiplet have different spins but same mass and quantum numbers. This happens because the SUSY generator, Q commutes with the translations and quantum number but does not commute with Lorentz generators.

A supersymmetric field theory describes a set of fields and a Lagrangian that exhibits such a symmetry for those fields. Thus, it provides a description for the particles and interactions between them.

As all the particles predicted in supersymmetry are not observed in nature, it is assumed that at low energy physics the supersymmetry is broken. The standard model of QFT can explain interactions at low energy physics. To extend our understanding of nature at high energy physics a supersymmetric extension of standard model is done. At low energy, as supersymmetry is broken, the concept of vacua in supersymmetry and the appearance of goldstino particle by Goldstine theorem is introduced.

By providing a unified description of bosons and fermions, SUSY may provide a natural framework to formulate a theory where unification of all known interactions can be explained.

Motivation

Supersymmetry may become one of the most useful theories which answers various unsolved questions of physics.

By extending standard model to a minimally supersymmetric standard model, the fundamental interactions of particles can be described at high energy physics. Thus, problems like confinement and mass gap in high energy physics, naturalness and hierarchy problem, strong coupling problem can be solved. A Grand Unification Theory (GUT) which unifies all known interactions may be derived. So, gravity at a quantum level, at a scale smaller than the Planck's scale can be explained. A natural framework for inflation model for early universe cosmology may also be found by applications of supersymmetry and supergravity. As new super-particles are predicted, the dark matter particles can be explained.

Supersymmetry can also be found useful in solving strong coupling problems. As supersymmetric theory has some renormalization properties, it can be used to put more constraints on Quantum Chromo-Dynamics and thus, exact solutions to strong coupling effects may be found. By using *localization* of supersymmetry, infinite path integrals can be reduced to finite path integrals as well as to simplify the complex integral problems. Using

the property of *holomorphy*, computing some non-perturbative contributions to Lagrangian is possible.

This work is an attempt to explore the basic formalism and apply supersymmetry in high energy physics and Starobinsky cosmological inflation.

Chapters 2, 3, 4, 5, 6 discuss the formalism and applications of supersymmetry. Chapter 7 is a discussion of applications of supersymmetry in high energy physics considering the MSSM and supersymmetry breaking. Chapter 8 is an introduction on the Starobinsky model of cosmological inflation where we also showed the identical features in different conditions of this model with the other inflation models.

Several lecture notes by reknowned professors and physicists were used to go through the basic concepts of supersymmetry. Many books were used to understand the mathematical concepts and to formulate the algebra for supersymmetry. Lecture notes by Quevedo, Krippendorf, and Schlotterer (2010) and Bertolini (2012) provided discussions on most of the topics that were covered in this paper. So, most of the chapters in this paper are based on these two lecture notes. Along with that, several other lecture notes, books and articles by reknowned physicists around the world have improved my insight on supersymmetry. To understand the construction of fermions and spinors with a Lie algebra perspective Georgy (1999) has been very helpful. Whereas, the new properties that add to spinors with addition of new dimensions is discussed from the concepts gained from notes by Lambert (2014) and book by Ammon and Erdmenger (2015). Mathematical concepts from books by Müller-Kirsten and Wiedemann (2009) and Aitchison (2007) have helped to construct supersymmetry algebra, superspace and superfields. Notes by Argyres (2001) and Bajc (2009) have provided with an extensive discussion on construction of the Lagrangians for superfields for both Abelian and non-Abelian field strengths. Along with the lecture notes and books described previously, lecture notes by Shirman (2009) aided the understanding of Wess-Zumino model, supersymmetry breaking and construction of MSSM. "Notes on Supersymmetry" (2012) also guided comprehensively on MSSM, particles in MSSM with their coupling and the necessity of R-parity.

To understand the concept of applying localization field in supersymmetry for computing exact results of QFT article by Rovelli (1999) and Hosomichi (2015) and notes by Terashima (2005) have also been very helpful. The prediction of supersymmetric particles as candidate for dark matter is discussed based on Lahanas (2006).

Reports by Bechtle, Plehn, and Sander (2015) has provided the information on the experiments run in LHC which was aimed at search for supersymmetric particle and establishment of MSSM in high energy physics.

In addition to the above mentioned notes and books, the book on weak-scale supersymmetry by Baer and Tata (2006), lecture notes by Hollywood (2008), article on by Kobayashi and Sasaki (2005), Ellis, Nanopoulos, and Olive (2013), Kehagias, Dizgah, and Riotto (2013), have expanded the ideas on supersymmetry and its application in different fields.

In Chapter 9 of this thesis, we worked on the Starobinsky model of cosmological inflation and showed that supersymmetry being connected to this model can provide a more natural framework to the inflation model. Supergravity is necessary to combine supersymmetry with this model. For this, we reduced the no-scalar supersymmetric Wess-Zumino model(NSWZ) to a case where it becomes equivalent to the Starobinsky model of cosmological inflation. As NSWZ model is a supersymmetric realization of inflation it includes supergravity and therefore, the new NSWZ model embeds supergravity into the Starobinsky model of inflation.

We found a parametric function for the field, x, in terms of a String modulus, t in the NSWZ model, in Chapter 10. We have shown that when instead of being an independent field, x is parametrically quadratic in terms of String modulus, t, it provides surprisingly good conditions for inflation. This function, $x(t)$, produces a single real field with double well which can be the vacuum states and also generates a very flat potential. Both of these scenarios being very important conditions for the rise of inflation make this model with parametrically quadratic, x a favourable model of inflation.

Chapter 2

Lie Groups Defining Supersymmetry

2.1   Lie groups

A continuous group with elements that can be parameterized by d parameters is named as a Lie Group. For d real numbers that vary continuously, we have a d-dimensional manifold. Every point in the parameter space can be described by a Cartesian co-ordinate system of d-orthogonal axes. The topology of that parameter space can be described by the topology of the group.

Finally, we can define that a group formed by infinite number of elements which are analytic functions of d parameters is a Lie Group.

If an element g(x), of the Lie group is parameterized by d parameters, $x = (x_1, x_2, x_3, ...x_d)$ which at $x = 0$ gives us the identity element $g(x)|_{x=0} = e$ then for any element in some neighbourhood of the identity, the group elements can be described as:

$$g(x) = e^{ix_a X_a} \qquad \text{for } a = (1, 2, 3.....d) \tag{2.1}$$

A d-dimensional vector space is formed with all the linear combinations of X_a, $x_a X_a$. Here, x_a is the basis of this vector space. When the generators form an algebra which operates under the commutator algebra, the vector space becomes the lie algebra.

$$[X_a, X_b] = X_a X_b - X_b X_a \tag{2.2}$$

Let, $x_a X_a$ and $x_b X_b$ be two different linear combination of generators. Now, as the exponential of these combinations form a representation of the group, the product of the exponential should be some exponential of some generators

$$e^{ix_a X_a} e^{ixbX_b} = e^{i\delta_c X_c} \tag{2.3}$$

As the generator X_a has the properties of group g, the vector product of two generators X_a and X_b will give another vector in our vector space that is spanned by the X_a. So, we get

$$[X_a, X_b] = if_{ab}^c X_c \tag{2.4}$$

2.1.1 Properties of Lie algebra

The commutation relation given in equation 2.4 is the Lie algebra of the group. The Lie algebra has to be antisymmetric and it should also have a derivative property. As the generators are hermitian:

$$X_a^\dagger = X_a \tag{2.5}$$

f_{ab}^c, which are the structure constants describing the group operation law, shows the antisymmetric property.

$$f_{ab}^c = -f_{ba}^c \tag{2.6}$$

So,

$$[X_a, X_b] = -[X_b, X_a] \tag{2.7}$$

The derivative property of the Lie algebra can be described by the Jacobi identity. This can be written as:

$$[X_a, [X_b, X_c]] = [[X_a, X_b], X_c] + [X_b, [X_a, X_c]] \tag{2.8}$$

2.2 Spinors and SO(n) algebra

The three generators of SU(2) is represented by J_i. The commutation relations are described by $[J_i, J_j] = i\xi_{ijk}Jk$. This is the definition of the su(2) algebra. As the generators are hermitian $J_i^\dagger = J_i$ these representations are unitary representations of su(2).

In Quantum mechanics, for $|\gamma\rangle$ denoting an eigenstate of J_3 to the eigenvalue γ:

$$
\begin{align}
J_3|\gamma\rangle &= \gamma|\gamma\rangle \tag{2.9} \\
\langle\gamma|\gamma\rangle &\neq 0 \tag{2.10} \\
J_\pm &= J_1 \pm iJ_2 \tag{2.11} \\
[J_3, J_\pm] &= \pm J_\pm \tag{2.12} \\
[J_+, J_-] &= 3J_3 \tag{2.13} \\
J_3 J_\pm|\gamma\rangle &= J_\pm|\gamma\rangle(\gamma \pm 1) \tag{2.14}
\end{align}
$$

This results to either

$$J_\pm|\gamma\rangle = 0 \tag{2.15}$$
$$\text{or}$$
$$J_\pm|\gamma\rangle = |\gamma \pm 1\rangle \tag{2.16}$$

So, J_+ is either a raising operator or an annihilator which results $J_+|\gamma\rangle = 0$. Similarly, J_- is either a lowering operator or annihilates $J_-|\gamma\rangle = 0$.

If the highest weight state $|\gamma\rangle$ is defined by $J_+|j\rangle = 0$; $J_3|j\rangle = j|j\rangle$; $\langle j|j\rangle = 1$. If J_- is applied on state $|j\rangle$ k times then

$$
\begin{align}
(J_-)^k|j\rangle &= N_k|j - k\rangle \tag{2.17} \\
\langle j - k|j - k\rangle &= 1 \tag{2.18} \\
|N_k|^2 &= \frac{k!(2j)!}{(2j - k)!} \tag{2.19}
\end{align}
$$

Angular momentum, J_3 results that finite dimensional irreducible representations can be labelled by j, such that $2j = 0, 1, 2...$ and we obtain a $(2j + 1)$ dimensional representation. The Lie algebra describes the covering group of SU(2). For a rotation by 2π in the 1-2 plane

$$e^{i2\pi J_3}|j, m\rangle = (-1)^{2m}|j, m\rangle \tag{2.20}$$

If $2j$ is odd, $2m$ is odd. Then for a rotation of 2π we do not get back to the same element here. There is a flip by -1. This represents spin(n).

Spin(n) is the covering group of SO(n). For $j = integers$ we get tensor representation of SO(n) and for half-odd-integers, we get spinor representations of SO(n).

2.3 Fermions and Clifford numbers

From the Dirac Equation of Motion for electrons, we get,

$$(\gamma^\mu \partial_\mu - M)\psi = 0 \tag{2.21}$$
$$(\gamma^\mu \partial_\mu + M)(\gamma^\mu \partial_\mu - M)\psi = 0 \tag{2.22}$$

According to the Mass-shell condition:

$$E^2 - P^2 - m^2 = 0 \tag{2.23}$$

gives us the Klein-Gordon Equation

$$(\partial^2 - m^2)\psi = 0 \tag{2.24}$$

Taking $m = M$ and

$$\partial_\mu \partial_\nu \psi = \partial_\nu \partial_\mu \psi \tag{2.25}$$

results to

$$P_1^2 + P_2^2 + P_3^2 + P_4^2 = \{\gamma_1 P_1 + \gamma_2 P_2 + \gamma_3 P_3 + \gamma_4 P_4\} \tag{2.26}$$

To support this, equation 2.22 requires Clifford Algebra,

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu$$
$$= 2\delta_{\mu\nu} \tag{2.27}$$

For a 4-dimensional spinor space $\mu, \nu = 1, 2, 3, 4$

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}\mathbf{1} \tag{2.28}$$
$$\Rightarrow [\gamma_i, \gamma_j \gamma_k] = \{\gamma_i, \gamma_j\}\gamma_k - \gamma_j\{\gamma_i, \gamma_k\} \tag{2.29}$$
$$= 2\delta_{i[j}\gamma_{k]} \tag{2.30}$$

$$\gamma_i \gamma_j = \frac{1}{2}\{\gamma_i, \gamma j\} + \frac{1}{2}[\gamma_i, \gamma_j] \tag{2.31}$$

$$= \delta_{ij}\mathbf{1} + \frac{1}{2}\gamma_{[i}\gamma_{j]} \tag{2.32}$$

In this space γ's are completely antisymmetrized.

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For a n-dimensional spinor space, if $n = 2m$ (even) or $n = 2m + 1$ (odd) there are 2m numbers of $2m \times 2m$ γ matrices.

For,
$$\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \sigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

$$
\begin{aligned}
\gamma_{2s-1} &= \rho_s \\
&= (\tau\otimes)^{s-1}\rho(\otimes 1_2)^{m-s} \\
\gamma_{2s} &= \sigma_s \\
&= (\tau\otimes)^{s-1}\sigma(\otimes 1_2)^{m-s} \\
\gamma_{2m+1} &= (-i)^m\gamma_1\gamma_2......\gamma_{2m} \\
&= \tau(\otimes\tau)^{m-1}
\end{aligned}
$$

(2.33)

(2.34)

(2.35)

For, $n = 2m$ the γ_{2m+1} matrix anticommutes with all γ matrices.

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}1_2; \qquad i,j = 1,2,3...2m+1 \tag{2.36}$$

Where as, for $n = 2m + 1$ the γ_{2m+1} matrix commutes with all γ matrices, so, for odd n, we can not get γ_{2m+1}.

Properties of fermions

As the γ_{2m+1} matrix described in equation 2.35 is always unitary and Hermitian matrix, it must have eigenvalues of ± 1.

A projection operator,

$$P_{\pm} = \frac{1}{2}(1 \pm \gamma_{2m+1}) \tag{2.37}$$

can project on spaces where $\gamma_{2m+1} = \pm 1$. So, a spinor Ψ can be written uniquely as

$$\Psi = \Psi_+ + \Psi_- \tag{2.38}$$

where $\Psi_{\pm}$ has eigenvalue of ± 1.

Due to the inequivalent eigenvalues of Ψ_+ and Ψ_- there is no such similarity transformation which can transform +1 into -1. So, these two semispinors live in two different spaces and thus the Pauli exclusion principle is followed for fermions.

2.4 Spinor representation

The equation 2.30 from the previous section results to

$$[\Gamma_{ij}, \gamma_k] = -i\gamma_i\delta_{j]k} \tag{2.39}$$

This means that the Clifford numbers γ_k transform as the components of an n-vector under SO(n) rotations generated by Γ_{ij}. We can define,

$$
\begin{aligned}
\Gamma_{ij} &= -\frac{1}{4}\gamma_{[i}\gamma_{j]}\Gamma_{ij} \\
&= -\Gamma_{ji}\left[\Gamma_{ij}, \Gamma_{mn}\right] \\
&= -i(\Gamma_{j[m}\delta_{n]i} - \Gamma_{i[m}\delta_{n]j})
\end{aligned}
\tag{2.40}
$$

Γ_{ij} are a representation of the spinor representation.

2.4.1 Properties of spinors

A spinor is an object that transforms in a spinor representation of the Lorentz group. Under a finite Lorentz transformation generated by $\omega^{\mu\nu}$, spinors transform as,

$$
\psi_a \longrightarrow (e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}})_\alpha{}^\beta \psi_\beta \qquad \text{left-handed spinor} \tag{2.41}
$$

$$
\bar{\chi}^{\dot{a}} \longrightarrow (e^{-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{\chi}^{\dot{\beta}} \qquad \text{right-handed spinor} \tag{2.42}
$$

This can be defined as

$$
\delta\Psi_a = \frac{1}{4}\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta \Psi_\beta \tag{2.43}
$$

Considering a finite Lorentz transformation, for an infinitesimal rotation by an angle θ in the (x^1, x^2) plane:

$$
\delta\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \theta\begin{pmatrix} x^0 \\ -x^2 \\ x^1 \\ x^3 \end{pmatrix} \tag{2.44}
$$

$$
\omega_{12} = -\omega^{21} = \theta; \qquad M^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.45}
$$

We can obtain a finite rotation by exponentiating M^{12}

$$
x^\mu \longleftarrow (e^{\omega^{\lambda p}M_{\lambda p}})^\mu{}_\nu x^\nu \tag{2.46}
$$

Using the formula:

$$
\begin{aligned}
e^{i\theta} &= \cos\theta + i\sin\theta e^{\theta M_{12}} \\
&= \cos\theta + M_{12}\sin\theta
\end{aligned}
\tag{2.47}
$$

For a rotation by $\theta = 2\pi$, $e^{2\pi M_{12}} = 1$. But, under such a rotation, a spinor will transform differently. We know,

$$
\begin{aligned}
\delta\psi &= \frac{1}{4}\omega^{\mu\nu}\gamma_{\mu\nu}\psi \\
&= \frac{1}{2}\theta\gamma_{12}\psi\psi & (2.48)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2}\theta\gamma_{12}\psi\psi &\longrightarrow e^{\frac{1}{2}\theta\gamma_{12}}\psi \\
&= \cos\frac{\theta}{2} + \gamma_{12}\sin\frac{\theta}{2} & (2.49)
\end{aligned}
$$

So, for $\theta = 2\pi$, $\psi \longrightarrow -\psi$. This means, a spinor under a rotation by 2π gets a minus sign.

2.4.2 Spinors in different dimensions

In different dimensions, each dimension adds some features to the spinors. Depending on the number of dimensions, n, we get different γ matrices.

We already know that in even dimensions ($n = 2m$) we get an extra Hermitian γ-matrix, γ_{2m+1}. As the γ_{2m+1} matrix is Hermitian and traceless, we get a basis of eigenvectors with eigenvalues ± 1. This eigenvalues represent the chirality of the spinor.

Weyl Spinor   A spinor with a definite γ_{n+1} eigenvalue is called a Weyl spinor.

Majorana Spinor   If the γ matrices are purely real meaning that the γ matrices are self-conjugate then they represent Majorana spinors or in other words Real spinors.

n=1

$n = 1 = 2(0) + 1$ and $(\gamma_1)^2 = -1$ which means $\gamma_1 = \pm i$
As γ_1 is complex, in a one-dimensional space, where the only dimension is time, there are no Majorana spinor.

n=2

$n = 2 = 2(1)$ So, $m = 1$ To construct the γ matrices we use the equations from sec:2.3

$$
\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.50}
$$

$$
\gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2.51}
$$

as the space is even dimensional, we get a $\gamma_{2m+1} = \gamma_3$ matrix.

$$
\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.52}
$$

In this case, we have, self-conjugate and symmetric matrices.

$$\begin{aligned} \gamma_1{}^2 &= \gamma_2{}^2; \\ \gamma_1\gamma_2 &= \gamma_2\gamma_1 \end{aligned} \tag{2.53}$$

So, the spinors are symplectic. Due to the presence of real matrix, we have Majorana spinors. As γ_3 have eigenvalue of ± 1 we also get Weyl spinors. So, we can have Majorana spinors and Weyl spinors simultaneously. So, we may have presence of Majorana-Weyl spinors.

n=3

For $n = 3 = 2(1) + 1$ we have $m = 1$. Here, we have the same γ matrices as a 2-dimensional space described in equation 2.50. However, as this is a case of odd dimensional space, we do not get a $\gamma_{2m+1} = \gamma_3$ matrix with eigenvalue of ± 1. Thus, there is no presence of Weyl spinor.

We can have only Majorana Spinors in three dimensions.

n=4

$n = 4 = 2(2)$ So, we get $m = 2$ and therefore four 2×2 γ matrices.

$$\gamma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \qquad \gamma_4 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

$$\gamma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\tag{2.54}$$

The γ matrices are self-conjugate and symmetric. So, we have symplectic spinors and thus, find Majorana spinor.

In four dimension we can get either Majorana or Weyl spinors but not both.

Chapter 3

Supersymmetry Algebra and Representations

3.1 Lorentz and Poincaré groups

3.1.1 Properties of Lorentz group

The Lorentz group SO(4) satisfies the following relation

$$\Lambda^T \eta \Lambda = \eta \tag{3.1}$$

where η is the Minkowski space metric tensor.

$$\eta_{\mu\nu} = diag(+,-,-,-)$$

The Lorentz group has six generators, the generator J_i of rotations and K_i of Lorentz boosts. Here, J_i are Hermitian and K_i are anti-Hermitian. These two generators are expressed as

$$J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}; \quad K_i = M_{0i}; \tag{3.2}$$

The Lorentz generators follow the commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k; \tag{3.3}$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k; \tag{3.4}$$

$$[K_i, K_j] = i\epsilon_{ijk}J_k; \tag{3.5}$$

By introducing a complex linear combinations of the generators J_i and K_i, we can construct representation of Lorentz algebra

$$J_i^{\pm} = \frac{1}{2}(J_i \pm ik_i) \tag{3.6}$$

$J_i^{\pm}$ are hermitian. Expressing eq:3.3 in terms of $J_i^{\pm}$ we get

$$\begin{aligned}
\left[J_i^{\pm}, J_j^{\pm}\right] &= i\epsilon_{ijk}J_k^{\pm} \tag{3.7} \\
\left[J_i^{\pm}, J_j^{\pm}\right] &= 0 \tag{3.8}
\end{aligned}$$

From this, we can understand that the Lorentz algebra is equivalent to SU(2) algebras.

$$SO(4) \simeq SU(2) \times SU(2)$$

In the Minkowski space complex conjugation interchanges the two SU(2)s. To satisfy that all rotation and boost parameters are real, all the J_i and K_i have to be imaginary.

$$\therefore (J_i^{\pm})^* = -J_i^{\mp} \tag{3.9}$$

Therefore, the Lorentz algebra changes into

$$SO(1,3) \simeq SU(2) \times SU(2)^*$$

A four-vector notation for the Lorentz generators in terms of an anti-symmetric tensor $M_{\mu\nu}$ is introduced. A four-dimensional matrix representation for the $M^{\mu\nu}$ is

$$(M^{\mu\nu})^{\mu}{}_{\nu} = i \left(\eta^{\mu\nu} \delta^{\rho}{}_{\nu} - \eta^{\rho\mu} \delta^{\sigma}{}_{\nu} \right) \tag{3.10}$$

$M_{\mu\nu} = -M_{\nu\mu}$, $M_{0i} = K_i$ and $M_{ij} = \epsilon_{ijk} J_k$ for $\mu = 0, 1, 2, 3$. In terms of $M_{\mu\nu}$, the Lorentz algebra reads

$$[M_{\mu\nu}, M_{\rho\sigma}] = i \left(\eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\rho} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} \right) \tag{3.11}$$

The Lorentz transformations act on four-vectors as

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{3.12}$$

The Poincaré group corresponds to the basic symmetries of special relativity. It acts on a space-time coordinates.

$$x'^{\mu} = \underbrace{\Lambda^{\mu}{}_{\nu}}_{Lorentz} x^{\nu} + \underbrace{a^{\mu}}_{translation} \tag{3.13}$$

The Poincaré group is the Lorentz group augmented by the space time translation generator, P_{μ}. Expressing this algebra in terms of the generators P_{μ}, $M_{\mu\nu}$ gives

$$[P_{\mu}, P_{\nu}] = 0; \tag{3.14}$$
$$[M_{\mu\nu}, P_{\rho}] = -i\eta_{\rho\mu} P_{\nu} + i\eta_{\rho\nu} P_{\mu} \tag{3.15}$$

3.2   Coleman-Mandula Theorem

Coleman-Mandula Theorem: In 1967, Coleman and Mandula proved a theory and showed that in a generic quantum field theory, under a number of assumptions like locality, causality, positivity of energy, finiteness of number of particles etc..., the only possible continuous symmetries of the S-matrix were the ones generated by Poincaré group generators, P_{μ} and $M_{\mu\nu}$ plus some internal symmetry group G commuting with them. Here, G is a semi-simple group times the Abelian factors.

$$[G, P_{\mu}] = [G, M_{\mu\nu}] = 0$$

So, it can be said that the S-matrix has a product structure of

Poincaré $\times$ Internal Symmetries

Due to the Poincaré symmetries with generators P_μ, $M_{\mu\nu}$ and the internal symmetry group with generators , B_l which are related to some conserved quantum numbers like electric charges, isopin, etc, we get the full symmetry algebra,

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 & (3.16)\\
[M_{\mu\nu}, M_{\rho\sigma}] &= -i\eta_{\mu\rho}M_{\nu\sigma} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\nu\rho}M_{\mu\sigma} & (3.17)\\
[M_{\mu\rho}, P_\rho] &= -i\eta_{\rho\mu}P_\nu + i\eta_{\rho\nu}P_\mu & (3.18)\\
[P_\mu, B_l] &= 0 & (3.19)\\
[M_{\mu\nu}, B_l] &= 0 & (3.20)
\end{aligned}
$$

Here, f_{lm}^n are structure constants and the last two commutation relations show that the full symmetry algebra is the direct product of the Poincaré algebra and the algebra, G spanned by the scalar bosonic generators B_l

$$\mathrm{ISO}(1,\ 3) \times \mathrm{G}$$

The Coleman-Mandula Theorem assumes that the symmetry algebra involves only commutators. Haag, Lopuszanski, and Sohnius introduced Graded algebra which is the extension of the Lie algebra for supersymmetry; one which includes the anti-commutators in addition to commutators.

3.3 Graded algebra

The concept of graded algebra is introduced to have a supersymmetric extension of the Poincar'e algebra. Let O_a be the operators of a Lie algebra, then

$$O_a O_b - (-1)^{\eta_a \eta_b} O_b O_a = iC^e{}_{ab} O_e s \qquad (3.21)$$

here,

$$
\eta_a = \begin{cases} 0: & O_a \quad \text{bosonic} \quad \text{generator} \\ 1: & O_a \quad \text{fermionic} \quad \text{generator} \end{cases} \qquad (3.22)
$$

For supersymmetry, there are two types of generators; the Poincar'e generators $P_{\mu\nu}$ and $M_{\mu\nu}$ and the spinor generators Q_α^I and $\bar{Q}_{\dot\alpha}^I$, where $I = 1, 2, 3....N$. To explain simple supersymmetry we use $N = 1$. When $N > 1$ extended supersymmetry is explained. We find the following commutation relations between

- the Poincaré generator and the spinor generator $[Q_\alpha, M^{\mu\nu}]$ and $[Q_\alpha, P^\mu]$

- two spinor generators $\{Q_\alpha, Q_\beta\}$ and $\{Q_\alpha, \bar{Q}_{\dot\beta}\}$

- a spinor generator and a internal symmetry generator T_i, $[Q_\alpha, T_i]$

a) $[\mathbf{Q}_\alpha, \mathbf{M}^{\mu\nu}]$

The spinor generator Q_α transforms under the exponential of the Lorentz, $SL(2, \mathbb{C})$ generators $\sigma^{\mu\nu}$

$$Q'_\alpha = exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha{}^\beta Q_\beta \tag{3.23}$$

$$= \left(1 - \frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha{}^\beta Q_\beta \tag{3.24}$$

but Q_α also acts as an operator transforming under Lorentz transformations $U = exp(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})$ to

$$Q'_\alpha = U^\dagger Q_\alpha U$$

$$\approx (1 + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})Q_\alpha(1 - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}) \tag{3.25}$$

Comparing the two expressions for Q'_α upto first order in $\omega_{\mu\nu}$,

$$Q_\alpha - \frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta = Q_\alpha - \frac{i}{2}\omega_{\mu\nu}(Q_\alpha M^{\mu\nu} - M^{\mu\nu}Q_\alpha) + \mathcal{O}(\omega^2) \tag{3.26}$$

$$\Rightarrow (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta = Q_\alpha M^{\mu\nu} - M^{\mu\nu}Q_\alpha \tag{3.27}$$

$$\therefore [Q_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta \tag{3.28}$$

b) $[\mathbf{Q}_\alpha, \mathbf{P}^\mu]$

The Jacobi identity for P^μ, P^ν and Q_α is

$$[P^\mu, [P^\nu, Q_\alpha]] + [P^\nu, [Q_\alpha, P^\mu]] + [Q_\alpha, [P^\mu, P^\nu]] = 0 \tag{3.29}$$

From (3.14), $[P^\mu, P^\nu] = 0$

$$\therefore [P^\mu, [P^\nu, Q_\alpha]] + [P^\nu, [Q_\alpha, P^\mu]] = 0 \tag{3.30}$$

Let us define $[Q_\alpha, P^\mu]$ with free indices μ, α which will be linear in Q

$$[Q_\alpha, P^\mu] = c(\sigma^\mu)_{\alpha\dot\alpha}\bar{Q}^{\dot\alpha} \tag{3.31}$$

$$\Rightarrow [P^\mu, Q_\alpha] = -c(\sigma^\mu)_{\alpha\dot\alpha}\bar{Q}^{\dot\alpha} \tag{3.32}$$

By taking adjoints using $(Q_{\dot\alpha})^\dagger = \bar{Q}_{\dot\alpha}$ and $(\sigma^\mu\bar{Q})_\alpha{}^\dagger = (Q\sigma^\mu)_{\dot\alpha}$ we get,

$$[\bar{Q}^{\dot\alpha}, P^\mu] = c^*(\bar{\sigma})^{\dot\alpha\beta}Q_\beta \tag{3.33}$$

Using these in (3.30) we get,

$$\begin{aligned}
[P^\mu, [P^\nu, Q_\alpha]] + [P^\nu, [Q_\alpha, P^\mu]] &= [P^\mu, -c(\sigma^\nu)_{\alpha\dot\alpha}\bar{Q}^{\dot\alpha}] + [P^\nu, c(\sigma^\mu)_{\alpha\dot\alpha}\bar{Q}^{\dot\alpha}] \\
&= -c(\sigma^\nu)_{\alpha\dot\alpha}[P^\mu, \bar{Q}^{\dot\alpha}] + c(\sigma^\mu)_{\alpha\dot\alpha}[P^\nu, \bar{Q}^{\dot\alpha}] \\
&= c.c^*(\sigma^\nu)_{\alpha\dot\alpha}(\bar{\sigma}^\mu)^{\dot\alpha\beta}Q_\beta - cc^*(\sigma^\mu)_{\alpha\dot\alpha}(\bar{\sigma}^\mu)^{\dot\alpha\beta}Q_\beta \\
&= |c|^2(\sigma^\nu)_{\alpha\dot\alpha}(\bar{\sigma}^\mu)^{\dot\alpha\beta}Q_\beta - |c|^2(\sigma^\mu)_{\alpha\dot\alpha}(\sigma^\nu)^{\dot\alpha\beta}Q_\beta \\
&= |c|^2(\sigma^\nu\sigma^\mu - \sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\beta Q_\beta = 0
\end{aligned}$$

Now, as $(\sigma^\nu\bar{\sigma}^\mu - \sigma^\mu\bar{\sigma}^\nu) \neq 0$ for general Q_β the equation given above can hold only for $c = 0$

$$\therefore [Q_\alpha, P^\mu] = [\bar{Q}^{\dot\alpha}, P^\mu] = 0 \tag{3.34}$$

c) $\{\mathbf{Q}_\alpha, \mathbf{Q}_\beta\}$

Due to the index structure, this commutator relationship looks like:

$$\{Q_\alpha, Q_\beta\} = k(\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu} \tag{3.35}$$

Now, $\{Q_\alpha, Q^\beta\}$ commutes with P^μ, but $k(\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu}$ does not. So, the relationship (3.35) can be true only if $k = 0$.

$$\therefore \{Q_\alpha, Q_\beta\} = 0 \tag{3.36}$$

d) $\{\mathbf{Q}_\alpha, \bar{\mathbf{Q}}_{\dot{\beta}}\}$

Due to the index structure, we get an ansatz.

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = t(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \tag{3.37}$$

where, t is a non-zero constant. By convention, t is set to be 2.

$$\therefore \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \tag{3.38}$$

The symmetry transformations $Q_\alpha \bar{Q}_\beta$ give the effect of translation. For $|B\rangle$ being a Bosonic state and $|F\rangle$ a Fermionic state, we get,

$$Q_\alpha |F\rangle = |B\rangle,$$

$$\begin{aligned} \bar{Q}_{\dot{\beta}} |B\rangle &= |F\rangle \\ \Rightarrow Q\bar{Q} : |B\rangle &\longrightarrow |B \text{ (translated)}\rangle \end{aligned}$$

e) $[\mathbf{Q}_\alpha, \mathbf{T_i}]$

In general, this commutator vanishes but for U(1) automorphism of the supersymmetry.

$$Q_\alpha \longrightarrow e^{i\lambda} Q_\alpha, \qquad \bar{Q}_{\dot{\alpha}} \longrightarrow e^{-i\lambda} \bar{Q}_{\dot{\alpha}} \tag{3.39}$$

So, if R is a U(1) generator then,

$$[Q_\alpha, R] = Q_\alpha, \qquad [\bar{Q}_{\dot{\alpha}}, R] = -\bar{Q}_{\dot{\alpha}} \tag{3.40}$$

For extended supersymmetry algebra with supercharges Q_α^I, we can add some central charges Z^{IJ} consistent with the Jacobi identities and Coleman Mandula Theorem. Z^{IJ} is a Lorentz scalar, so, it commutes with all other generators.

$$\begin{aligned} \{Q_\alpha^I, \bar{Q}_{\dot{\alpha}}^J\} &= 2\sigma_{\alpha\dot{\alpha}}^\mu P_\mu \delta^{IJ} \tag{3.41} \\ \{Q_\alpha^I, Q_\beta^J\} &= \epsilon_{\alpha\beta} Z^{IJ} \tag{3.42} \\ [Q_\alpha, Z^{IJ}] &= 0 \tag{3.43} \\ [P_\mu, Z^{IJ}] &= 0 \tag{3.44} \\ [M_{\mu\nu}, Z^{IJ}] &= 0 \tag{3.45} \end{aligned}$$

The anti-commutation relations imply that $Z^{IJ} = -Z^{JI}$. Due to this relation, when $N = 1$, the central charge, Z^{IJ} is zero.

3.4 Representations of supersymmetry algebra

The supersymmetry algebra includes the Casimir operators of the Poincare algebra. The Poincare algebra has two Casimir operators.

$$P^2 = P_\mu P^\mu \quad and \quad W^2 = W_\mu W^\mu \tag{3.46}$$

These two operators commute with all the generators. Here, $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}$ is the Pauli-Lubanski vector. Casimir operators are used to classify irreducible representations of a group. These representations of Poincare group are called particles. These particles can be massive of massless particle.

Massive particle: We consider, a massive particle with mass, m at the rest frame $P_\mu = (m, 0, 0, 0)$. In this frame, $P^2 = m^2$. We know that, $W_\mu P^\mu = 0$. Therefore, at the rest frame, $W_0 = 0$. So, in the rest frame $W_\mu = (0, \frac{1}{2}\epsilon_{i0jk}mM^{jk})$. From this, we can get, $W^2 = -m^2\bar{J}^2$. We can say that massive particles are distinguished by their mass and their spin.

Massless particle: Massless particles have $P^2 = 0$ and $W^2 = 0$. In the rest frame, $P_\mu = (E, 0, 0, E)$. This implies that $W^\mu = M_{12}P^\mu$. The two operators are proportional for a massless particle. The constant of proportionality is the helicity, $M_{12} = \pm j$. These representations have a fixed spin and the different states are distinguished by their energy and the sign of their helicity.

Super-multiplet: An irreducible representation of the supersymmetry algebra is called a super-particle. A super-particle corresponds to a collection of particles where these particles are related by the action of supersymmetry generators Q_α^I and $\bar{Q}_{\dot\alpha}^I$. Their spins differ by units of half. As the super-particles are multiplet of different particles, they are sometimes called super-multiplets.

3.4.1 Properties of supersymmetry algebra

1. The Poincare algebra has two Casimir operators, P^2 and W^2. Compared to this the supersymmetry algebra has only one Casimir, P^2. As $M_{\mu\nu}$ does not commute with the supersymmetry generators, W^2 is not a Casimir operator for this algebra. So, a super-multiplet can contain particle having the same mass having different spins.

   As we can not observe the mass degeneracy between bosons and fermions in known particle spectra, we can imply that if there is any supersymmetry in nature then it must be broken.

2. In a supersymmetric theory, the energy of a space is always greater than or equal to zero. We use the supersymmetry algebra on an arbitrary state $|\phi\rangle$ we get,

$$\begin{aligned}
\langle\phi|\{Q_\alpha^I, Q_{\dot\alpha}^I\}|\phi\rangle &= 2\sigma_{\alpha\dot\alpha}^\mu\langle\phi|P_\mu|\phi\rangle\delta^{IJ} \\
&= \langle\phi|(Q_\alpha^I(Q_\alpha^I)^\dagger + (Q_\alpha^I)^\dagger Q_\alpha^I)|\phi\rangle \quad (taking \quad \bar{Q}_{\dot\alpha}^I = (Q_\alpha^I)^\dagger)
\end{aligned}$$

Now, from the positivity of the Hilbert Space,

$$||(Q_\alpha^I(Q_\alpha^I)^\dagger||^2 + ||(Q_\alpha^I)^\dagger Q_\alpha^I)||^2 \geq 0 \tag{3.47}$$

We know, $Tr\sigma^\mu = 2\delta^{\mu 0}$, now, we take (3.47) and sum over $\alpha = \dot\alpha = 1, 2$ and get

$$4\langle\phi|P_0|\phi\rangle \geq 0 \tag{3.48}$$

as anticipated.

3. A super-multiplet contains an equal number of bosonic and fermionic degrees of freedom. $N_B = N_F$. Defining a fermion number operator.

$$(-1)^{N_F} = \begin{cases} -1 & \text{fermionic} \quad \text{state} \\ +1 & \text{bosonic} \quad \text{state} \end{cases} \tag{3.49}$$

N_F is twice the spin, $N_F = 2s$. When acting on a bosonic particle this produces

$$(-1)^{N_F}|\mathrm{B}\rangle = |\mathrm{B}\rangle \tag{3.50}$$

Where as, acting on a fermionic state this results to

$$(-1)^{N_F}|\mathrm{F}\rangle = -|\mathrm{F}\rangle \tag{3.51}$$

3.4.2 Simple supersymmetry representation

As discussed in section 3.3, for simple supersymmetry, the spinor generator, Q_α^I and $Q_{\dot\alpha}^I$ has $I = \mathcal{N} = 1$. Both massless and massive super-multiplets can be constructed by this algebra.

Massless super-multiplets:

We know from section 3.3 that for massless representation, the central charges, $Z^{IJ} = 0$. From 3.36 and 3.42 we know that all Q's and $\bar{Q}$'s commute among themselves. To construct the irreducible representation the following steps are followed.

1. In the rest frame, $P_\mu = (E, 0, 0, E)$ we get,

$$\sigma^\mu P_\mu = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix} \tag{3.52}$$

Using 3.52 in 3.41 we get,

$$\{Q_\alpha^I, \bar{Q}_{\dot\beta}^J\} = \begin{pmatrix} 0 & 0 \\ 0 & 4E \end{pmatrix}_{\alpha\dot\beta} \delta^{IJ} \tag{3.53}$$

$$\Rightarrow \{Q_1^I, \bar{Q}_{\dot 1}^J\} = 0 \tag{3.54}$$

From this equation, we get,

$$\langle\phi|\{Q_1^I, \bar{Q}_{\dot 1}^I\}|\phi\rangle = 0 \tag{3.55}$$

which results to $Q_1^I = Q_{\dot 1}^I = 0$. Then, we have only Q_1^I and $Q_{\dot 1}^I$, hence, only half of the generators exist in this case.

2. From the non-trivial generators, we define,

$$a_I = \frac{1}{\sqrt{4E}} Q_2^I; \qquad (a_I)^\dagger = \frac{1}{\sqrt{4E}} \bar{Q}_{\dot{2}}^I \tag{3.56}$$

For a set of N creation and N annihilation operators, these operators a_I and $(a_I)^\dagger$ satisfy the following anticommutation relations.

$$\begin{aligned} \{a_I, a_J{}^\dagger\} &= \delta^{IJ} & (3.57) \\ \{a_I, a_J\} &= 0 & (3.58) \\ \{a_I{}^\dagger, a_J{}^\dagger\} &= 0 & (3.59) \\ && (3.60) \end{aligned}$$

Since,

$$\begin{aligned} \left[M_{12}, Q_2^I\right] &= i(\sigma_{12})_2{}^2 Q_2^I & (3.61) \\ &= -\frac{1}{2} Q_2^I & (3.62) \\ \left[M_{12}, \bar{Q}_{\dot{2}}^I\right] &= \frac{1}{2} \bar{Q}_{\dot{2}}^I \quad \text{and} \quad J_3 = M_{12} & (3.63) \end{aligned}$$

the operator Q_2^I or a_I lowers the helicity of half unit and $\bar{Q}_{\dot{2}}^I$ or $(a_I)^\dagger$ rises the helicity of half unit.

3. As $m = 0$, the state will carry some helicity λ_0. We start from the Clifford vacuum, where

$$a_I|\lambda_0\rangle = 0 \tag{3.64}$$

4. The super-multiplet is obtained by creation operators, $(a_I)^\dagger$ acting in $|\lambda_0\rangle$

$$|\lambda_0\rangle, a_I{}^\dagger|\lambda_0\rangle \equiv |\lambda_0 + \frac{1}{2}\rangle_I, \qquad a_I{}^\dagger a_J{}^\dagger \equiv |\lambda_0 + 1\rangle_{IJ}, ... \tag{3.65}$$

$$... \quad a_1{}^\dagger a_2{}^\dagger ....a_N{}^\dagger|\lambda_0\rangle \equiv |\lambda_0 + \frac{N}{2}\rangle \tag{3.66}$$

As the Clifford vacuum has helicity λ_0, the highest state representation has the highest helicity $\lambda = \lambda_0 + \frac{N}{2}$.

Due to the anti-symmetry in $I \leftrightarrow J$, at helicity level $\lambda = \lambda_0 + \frac{k}{2}$, we have,

$$\text{number of states with helicity,} \quad \lambda_0 + \frac{k}{2} = \binom{N}{k} \tag{3.67}$$

Where, $k = 0, 1, 2, ...N$ the total number of states in the irrep will be

$$\sum_{k=0}^{N} \binom{N}{k} = 2^N = (2^{N-1})_B + (2^{N-1})_F \tag{3.68}$$

Where, half of the states being bosons have integer helicity and the other half of them being fermions have half integer helicity.

5. As CPT flips the sign of helicity, in order to be CPT invariant, the helicity has to be distributed symmetrically around zero, i.e CPT conjugates of the constructed particles must be obtained. This is not needed if the super-multiplet is self-CPT conjugate.

- **Matter (or chiral) multiplet**

$$\lambda_0 = 0 \longrightarrow \left(0, +\frac{1}{2}\right) \oplus \left(-\frac{1}{2}, 0\right) \tag{3.69}$$

This representation have the same degrees of freedom as one Weyl-fermion and one complex scalar.

$\lambda = 0$ scalar	$\lambda = \frac{1}{2}$ fermion
squark	quark
slepton	lepton
Higgs	Higgsino

- **Gauge (or vector) multiplet**

$$\lambda_0 = \frac{1}{2} \longrightarrow \left(+\frac{1}{2}, +1\right) \underset{CPT}{\oplus} \left(-1, -\frac{1}{2}\right) \tag{3.70}$$

This has the degrees of freedom of one vector and one Weyl-fermion. This representation helps to describe gauge fields in a supersymmetric theory. Quarks and leptons are accommodated in these multiplets.

$\lambda = \frac{1}{2}$ fermion	$\lambda = 1$ boson
photino	photon
gluino	gluon
Wino, Zino	W, Z

- **Gravitino and graviton multiplets**

$$\lambda_0 = 1 \longrightarrow \left(+1, +\frac{3}{2}\right) \underset{CPT}{\oplus} \left(-\frac{3}{2}, -1\right) \tag{3.71}$$

The degrees of freedom are those of a spin $\frac{3}{2}$ particle and one vector. In a $N = 1$ supersymmetric theory, a gravitino multiplet can occur if and only if it is supersymmetric partner, graviton appears.

$$\lambda_0 = \frac{3}{2} \longrightarrow \left(+\frac{3}{2}, +2\right) \underset{CPT}{\oplus} \left(-2, \frac{3}{2}\right) \tag{3.72}$$

Graviton multiplet has helicity 2.

$\lambda = \frac{3}{2}$ fermion	$\lambda = 2$ boson
gravitino	graviton

Massive super-multiplet

Although the steps to construct a massive super-multiplet representation is almost the same as the massless representation, there are some significant differences.

Considering a state with mass, m in its rest frame, $P_\mu = (m, 0, 0, 0)$. As there are full sets of 2N creation and 2N annihilation operators, we get

$$\{Q_\alpha^I, \bar{Q}_{\dot{\beta}}^J\} = 2m\delta_{\alpha\dot{\beta}}\delta^{IJ} \tag{3.73}$$

And instead of helicity, we take spin. A Clifford vacuum is defined by m, j and $j(j+1)$ is the eigenvalue of J^2.

Construction of a massive super-multiplet in simple supersymmetry algebra:
Using the oscillator algebra, we find the annihilation and creation operators.

$$a_{12} = \frac{1}{\sqrt{2m}} Q_{1,2}, \qquad a_{12}{}^\dagger = \frac{1}{\sqrt{2m}} \bar{Q}_{1,2} \tag{3.74}$$

Here, $a_1{}^\dagger$ lowers the spin by half unit, on the other hand $a_2{}^\dagger$ rises it by half unit.
A Clifford vacuum is a state with mass, m and spin, j_0. This state can be annihilated by
both a_1 and a_2. Massive representations are constructed by acting creation operators on the
Clifford vacuums.

• **Matter multiplet**

$$j = 0 \longrightarrow \left(-\frac{1}{2}, 0, 0', +\frac{1}{2} \right) \tag{3.75}$$

The number of degrees of freedom are the same as the massless case. But, here, we do not
need the addition of CPT conjugates.
This multiplet is made of a massive complex scalar and a massive Majorana spinor.

• **Gauge (or Vector) multiplet**

$$j = \frac{1}{2} \longrightarrow \left(-1, 2 \times -\frac{1}{2}, 2 \times 0, 2 \times +\frac{1}{2}, 1 \right) \tag{3.76}$$

We have, degrees of freedom of those of one massive vector, one massive Dirac-fermion and
one massive real scalar.

3.4.3 Extended supersymmetry

Algebra of extended supersymmetry: For extended supersymmetry, the spinor
generators get an additional label, $I, J = 1, 2, ...\mathscr{N}$. The algebra is slightly different from
the $N = 1$ algebra. Here, we have, (3.41) and (3.42) and anti-symmetric central charges
$Z^{IJ} = -Z^{JI}$ commuting with all the generators.

$$[Z^{IJ}, P^\mu] = [Z^{IJ}, M^{\mu\nu}] = [Z^{IJ}, Q_\alpha^I] = [Z^{IJ}, Z^{KL}] = [Z^{IJ}, T_a] = 0 \tag{3.77}$$

The existence of central charges, gives rise to the extended supersymmetry.

Massless representation for extended supersymmetry

N=2 Supersymmetry

• **Gauge (or Vector) multiplet:**

$$\lambda_0 = 0 \longrightarrow \left(0, +\frac{1}{2}, +\frac{1}{2}, +1 \right) \underset{CPT}{\oplus} \left(-1, -\frac{1}{2}, -\frac{1}{2}, 0 \right) \tag{3.78}$$

The degrees of freedom are those of one vector, two Weyl-fermions and one complex
scalar.
This $N = 2$ multiplet can be decomposed in terms of one $N = 1$ vector and one $N = 1$
chiral multiplet.

- **Matter multiplet (or Hypermultiplet)**

$$\lambda_0 = -\frac{1}{2} \longrightarrow \left(-\frac{1}{2}, 0, 0, +\frac{1}{2}\right) \underset{CPT}{\oplus} \left(-\frac{1}{2}, 0, 0, +\frac{1}{2}\right) \tag{3.79}$$

This can also be decomposed in terms of $N = 1$ chiral multiplets.

- **Gravitino multiplet:**

$$\lambda_0 = -\frac{3}{2} \longrightarrow \left(-\frac{3}{2}, -1, -1, -\frac{1}{2}\right) \underset{CPT}{\oplus} \left(+\frac{1}{2}, +1, +1, +\frac{3}{2}\right) \tag{3.80}$$

We get, the degrees of freedom of a spin $\frac{3}{2}$ particle, two vectors and a Weyl-fermion.

- **Graviton multiplet:**

$$\lambda_0 = -2 \longrightarrow \left(-2, -\frac{3}{2}, -\frac{3}{2}, -1\right) \underset{CPT}{\oplus} \left(+1, \frac{3}{2}, \frac{3}{2}, +2\right) \tag{3.81}$$

In this case, the degrees of freedom are those of a graviton, two gravitinos, one vector which is referred as graviphoton.

N=4 Supersymmetry

$$\lambda_0 = -1 \longrightarrow \left(-1, 4 \times -\frac{1}{2}, 6 \times 0, 4 \times +\frac{1}{2}, +1\right) \tag{3.82}$$

The degrees of freedom are those of one vector, four Weyl-fermions, three complex scalars.

This single $N = 4$ multiplet has states with helicity, $\lambda < 2$. It can be made of $N = 2$ vector multiplet, $N = 2$ hypermultiplet and their CPT conjugates or one $N = 1$ vector multiplet, three $N = 1$ chiral multiplets and their CPT conjugates.

N=8 Supersymmetry

Maximum multiplet:

$$
\begin{aligned}
1 \times \lambda &= \pm 2 \\
8 \times \lambda &= \pm \frac{3}{2} \\
28 \times \lambda &= \pm 1 \\
56 \times \lambda &= \pm \frac{1}{2} \\
70 \times \lambda &= \pm 0
\end{aligned}
$$

Massive representation for extended supersymmetry

As the central charge matrix Z^{IJ} is anti-symmetric with U(N) rotation, we can put it in a standard block diagonal form.

$$
Z^{IJ} = \begin{pmatrix}
0 & q_1 & 0 & 0 & 0 & \cdots \\
-q_1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & q_2 & 0 & \cdots \\
0 & 0 & -q_2 & 0 & 0 & \cdots \\
0 & 0 & 0 & \ddots & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & q_{\frac{N}{2}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & -q_{\frac{N}{2}} & 0
\end{pmatrix}
\tag{3.83}
$$

For K of the q_i being equal to 2m, there are $2N - 2K$ creation operators and $2^{2(N-k)}$ states.

$$
k = 0 \Rightarrow (2^{2N-1})_B + (2^{2N-1})_F \qquad = \qquad 2^{2N} \quad \text{states,}
$$
$$
\text{long multiplets} \tag{3.84}
$$

$$
0 < k < \frac{N}{2} \Rightarrow (2^{2(N-k)-1})_B + (2^{2(N-k)-1})_F \qquad = \qquad 2^{2(N-k)} \quad \text{states,}
$$
$$
\text{short multiplets} \tag{3.85}
$$

$$
k = \frac{N}{2} \Rightarrow (2^{N-1})_B + (2^{N-1})_F \qquad = \qquad 2^{N} \quad \text{states,}
$$
$$
\text{ultra-short multiplets}
$$
$$
\tag{3.86}
$$

N=2 Supersymmetry

• **Long multiplets**

Gauge (or Vector) multiplet:

$$
j = 0 \longrightarrow \left(-1, 4 \times -\frac{1}{2}, 6 \times 0, 4 \times +\frac{1}{2}, 1 \right)
\tag{3.87}
$$

The degrees of freedom are those of a massive vector, two Dirac-fermions and five real scalars. The number of degrees of freedom is equal to a massless $N = 2$ vector multiplet and a massless $N = 2$ hyper-multiplet.

• **Short multiplets**

Matter multiplet:

$$
j = 0 \longrightarrow \left(2 \times -\frac{1}{2}, 4 \times 0, 2 \times +\frac{1}{2} \right)
\tag{3.88}
$$

The degrees of freedom correspond to one massive Dirac fermion and two massive complex scalars. The number of degrees of freedom equals to those of a massless hyper multiplet.

Vector multiplet

$$j = \frac{1}{2} \longrightarrow \left(-1, 2 \times -\frac{1}{2}, 2 \times +\frac{1}{2}, +1 \right) \tag{3.89}$$

The degrees of freedom correspond to one massive vector, one Dirac fermion and one real scalar.

N=4 Supersymmetry

We can get ultra-short multiplets in $N = 4$ massive supersymmetry. This will consist of a massive vector, two Dirac fermions and five real scalars.

Chapter 4

Superspace and Superfields

4.1 Supersymmetric field theory

In Chapter 3, we discussed supersymmetry representations on states, where the representations were in terms of multiplets of states. Now, to explain supersymmetric field theories the representation of supersymmetry has to be constructed in terms of multiplets of fields.

4.1.1 Constructing supersymmetric representation for supersymmetric field theory

In this section, we build a $N = 1$ supersymmetry. In the previous chapter we took Clifford Vacuum, $|\lambda_0\rangle$ as the ground state of the supersymmetric representation.

To explain supersymmetric field, we start with a field $\phi(x)$. Supersymmetry generators acting on this field will generate new fields belonging to the same representation

$$[\bar{Q}_{\dot{\alpha}}, \phi(x)] = 0 \tag{4.1}$$

Let us consider $\phi(x)$ to be a scalar field. If $\phi(x)$ is real, the Hermitian conjugate of (4.1) will be

$$[Q_\alpha, \phi(x)] = 0 \tag{4.2}$$

Using Jacobi identity for $(\phi, Q, \bar{Q})$ we get

$$[\phi(x), \{Q_\alpha, \bar{Q}_{\dot{\alpha}}\}] + \{Q_\alpha, [\bar{Q}_{\dot{\alpha}}, \phi(x)]\} + \{\bar{Q}_{\dot{\alpha}}, [\phi(x), Q_\alpha]\} = 0 \tag{4.3}$$

Using equation (3.41) and (4.1) we get,

$$[\phi(x), 2\sigma^\mu P_\mu] = 0 \tag{4.4}$$

$$\Rightarrow 2\sigma^\mu[\phi(x), P_\mu] = 0 \tag{4.5}$$

$$\Rightarrow [P_\mu, \phi(x)] \sim \delta_\mu \phi(x)$$

$$= 0 \tag{4.6}$$

This would imply that, $\phi(x)$ is constant and therefore not a field. Therefore, $X_{\dot{\alpha}\beta}$ is a space-time derivative of the scalar field, ϕ.

Using generalized Jacobi identity on $(\phi(x), Q, Q)$ we get

$$[\phi(x), \{Q_\alpha, Q_\beta\}] + \{Q_\alpha, [Q_\beta, \phi(x)]\} - \{Q_\beta, [\phi(x), Q_\alpha]\} = 0 \tag{4.7}$$

In a $N = 1$ supersymmetry, central charge, $Z^{IJ} = 0$.
Using (3.42) we get,

$$\{Q_\alpha, Q_\beta\} = 0 \tag{4.8}$$

so

$$\{Q_\alpha, [Q_\beta, \phi]\} - \{Q_\beta, [\phi, Q_\alpha]\} = 0$$
$$\{Q_\alpha, \psi_\beta\} - \{Q_\beta, -\psi_\alpha\} = 0$$
$$\{Q_\alpha, \psi_\beta\} + \{Q_\beta, \psi_\alpha\} = 0$$
$$F_{\alpha\beta} + F_{\beta\alpha} = 0 \tag{4.9}$$
$$F_{\alpha\beta} = -F_{\beta\alpha} \tag{4.10}$$

This means, $F_{\alpha\beta}$ is antisymmetric under $\alpha \leftrightarrow \beta$. So, $F_{\alpha\beta}$ is antisymmetric and F is a new scalar field.

So, we take $\phi(x)$ to be complex. In this case, the hermitian conjugate of (4.1) can not be obtained to be the same. So we get,

$$[Q_\alpha, \phi(x)] \equiv \psi_\alpha(x) \tag{4.11}$$

Where, ψ_α is a new field which belongs to the same supersymmetry representation. As ϕ is a scalar field, *psi* is a Weyl-spinor. Let us assume,

$$\{Q_\alpha, \psi_\beta(x)\} = F_{\alpha\beta}(x) \tag{4.12}$$
$$\{\bar{Q}_{\dot\alpha}, \psi_\beta(x)\} = X_{\dot\alpha, \beta}(x) \tag{4.13}$$

Using (4.11) in the Jacobi identity for $(\phi, Q, \bar{Q})$

$$[\phi(x), \{Q_\beta, \bar{Q}_{\dot\alpha}\}] + \{Q_\beta, [\bar{Q}_{\dot\alpha}, \phi(x)]\} - \{\bar{Q}_{\dot\alpha}, [\phi(x), Q_\beta]\} = 0$$
$$2\sigma^\mu_{\beta\dot\alpha}[\phi(x), P_\mu] - \{\psi_\beta(x), \bar{Q}_{\dot\alpha}\} = 0$$
$$X_{\dot\alpha\beta} = \{\psi_\beta(x), \bar{Q}_{\dot\alpha}\} \tag{4.14}$$
$$= 2\sigma^\mu_{\beta\dot\alpha}[P_{\mu,\phi(x)}] \tag{4.15}$$
$$\sim \delta_\mu\phi \tag{4.16}$$

To find the resulting fields of actions by supersymmetry generators on F, we assume

$$[Q_\alpha, F] = \lambda_\alpha \tag{4.17}$$
$$[\bar{Q}_{\dot\alpha}, F] = \bar{\chi}_{\dot\alpha} \tag{4.18}$$

Enforcing Jacobi identity on (ψ, Q, Q) we get,

$$[\psi_\alpha, \{Q_\alpha, Q_\beta\}] + \{Q_\alpha, [Q_\beta, \psi_\alpha]\} - \{Q_\beta, [\psi_\beta, Q_\alpha]\} = 0$$
$$\{Q_\alpha, F_{\beta\alpha}\} - \{Q_\beta, F_{\beta\alpha}\} = 0$$
$$\{Q_\alpha, F_{\beta\alpha}\} + \{Q_\beta, F_{\alpha\beta}\} = 0$$
$$\lambda_\alpha + \lambda_\alpha = 0$$
$$\text{so,} \quad \lambda_\alpha = 0 \tag{4.19}$$

Using Jacobi identity in $(\psi, Q, \bar{Q})$ we get,

$$
\begin{aligned}
\left[\psi, \{Q_\alpha, \bar{Q}_{\dot\alpha}\}\right] + \{Q_\alpha, [\bar{Q}_{\dot\alpha}, \psi]\} - \{\bar{Q}_{\dot\alpha}, [\psi, Q_\alpha]\} &= 0 \\
[\psi, 2\sigma^\mu P_\mu] + \{Q_\alpha, X_{\dot\alpha\beta}\} - \{\bar{Q}_{\dot\alpha}, F_{\alpha\beta}\} &= 0 \\
2\sigma^\mu [\psi, P_\mu] - \bar{\chi}_{\dot\alpha} &= 0 \\
\bar{\chi}_{\dot\alpha} &= 2\sigma^\mu [P_\mu, \psi] && (4.20) \\
\bar{\chi}_{\dot\alpha} &= 2\sigma^\mu \delta_\mu \psi && (4.21)
\end{aligned}
$$

which means, $\bar{\chi}_{\dot\alpha}$ is proportional to a space-time derivative of the field ψ. So, new field are not being generated any more. Finally, the multiplet of fields we get here, is

$$
(\phi, \psi, F) \tag{4.22}
$$

ϕ being a scalar field, due to the only presence of particles of spin-0 and spin-$\frac{1}{2}$, the constructed multiplet is a matter multiplet. It is called a Chiral or Wess-Zumino multiplet.

4.1.2 Formulation of supersymmetric field theory

To construct a supersymmetric field theory, we need a set of multiplets and a Lagrangian made out of the desired field content. Unless the Lagrangian transforms as a total space-time derivative under the supersymmetric transformations, the theory can not be supersymmetric. For this, the action constructed for this Lagrangian

$$
\mathcal{S} = \int d^4 x \mathcal{L} \tag{4.23}
$$

has to be a supersymmetric invariant.

In ordinary space-time supersymmetry is not manifest. So, the usual space-time Lagrangian is a difficult formulation to construct the supersymmetry algebra.

Supersymmetric field theories involve some supersymmetry generators which are associated with the extra space-time symmetries. This can be defined on an extension of Minkowski space, known as superspace. Supersymmetry Lagrangian can be easily constructed in this extended space.

The extension of ordinary space-time is done by adding 2+2 anticommuting Grassmann coordinates, θ_α and $\bar{\theta}_\alpha$ which are associated to the supersymmetry generators, Q_α and $\bar{Q}_{\dot\alpha}$. The Minkowski space-time, labelled with the coordinates, x^μ associated to the general P_μ are thus, extended to a eight-coordinate superspace labelled by $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot\alpha})$. Many hidden properties of supersymmetry field theory, along with many classical and quantum properties of supersymmetry can become manifest in superspace.

4.2 Superspace

4.2.1 Groups and cosets

Every continuous group, G defines a manifold, $\mathcal{M}_G$ by

$$
\begin{aligned}
\Lambda : G &\longrightarrow \mathcal{M}_G; && (4.24) \\
\{g = e^{i\alpha_a T^a}\} &\longrightarrow \alpha_a && (4.25)
\end{aligned}
$$

Here, the dimension of G and the dimension of $\mathcal{M}_G$ are same.

We can define a coset G/H where $g \in G$ is identified with $g \cdot h \quad \forall \quad h \in H$,

For example:

$$\text{If } G = U_1(1) \times U_2(1) \ni g \;\; = \;\; e^{i(\alpha_1 Q_1 + \alpha_2 Q_2)}$$
$$H = U_1(1) \ni h \;\; = \;\; e^{i\beta Q_1}$$

Then, the coset, $G/H = \frac{U_1(1) \times U_2(1)}{U_1(1)}$ and the identification gh will be:

$$gh \;\; = \;\; e^{i((\alpha_1 + \beta)Q_1 + \alpha_2 Q_2)}$$
$$= \;\; e^{i(\alpha_1 Q_1 + \alpha_2 Q_2)}$$
$$= \;\; g$$

So, $G/H = U_2(1)$ where α_2 contains the effective information.

In general, a coset, $\frac{\mathcal{M}_{so(n+1)}}{SO(n)} = S^n$.

4.2.2 Minkowski space and Poincare group

For a Poincare group, ISO(1, 3) and Lorentz group, SO(1, 3) a four-dimensional coset-space can be defined as the Minkowski space, $M_{1,3} = \frac{ISO(1,3)}{SO(1,3)}$. The Poincare group is an isometry group of this Minkowski space.

Each point in this space has a unique representative which is a translation and can be parameterized by a coordinate x^μ.

$$x^\mu \longleftrightarrow e^{a^\mu P_\mu} \tag{4.26}$$

4.2.3 Defining superspace

A superspace can be defined similarly to a coset Minkowski space. But in the case of superspace, the Poincare group has to be extended into a super-Poincare group.

From the previous discussion,

$$Minkowski = \frac{Poincare}{Lorentz} = \frac{\{\mathcal{W}^{\mu\nu}, a^\mu\}}{\mathcal{W}^{\mu\nu}} \tag{4.27}$$

which simplifies to a translation that can be identified to a Minkowski space by $a^\mu = x^\mu$.

As the group is the exponent of the algebra, to extend the Poincare group to a super-Poincaré group, we describe the supersymmetry algebra in terms of commutators. These commutators are defined as Grassmann variables. Grassman variables commute with fermionic generators and anticommute with bosonic generators.

$N = 1$ superspace is defined as a coset

$$\frac{Super Poincare}{Lorentz} = \frac{\{\mathcal{W}^{\mu\nu}, a^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}}\}}{\{\mathcal{W}^{\mu\nu}\}} \tag{4.28}$$

The anticommutation relation of the Grassmann variables are

$$\{\theta^\alpha, \theta^\beta\} \quad = \quad 0 \tag{4.29}$$

$$\{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\} \quad = \quad 0 \tag{4.30}$$

$$\{\theta^\alpha, \bar{\theta}_{\dot{\beta}}\} \quad = \quad 0 \tag{4.31}$$

Using the Grassmann parameters, the anticommutator relations for $Q_\alpha, \bar{Q}^{\dot{\beta}}$ can be reduced to commutators:

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} \quad = \quad 2\,(\sigma^\mu)_{\alpha\dot{\alpha}}\,P_\mu \tag{4.32}$$

$$\left[\theta^\alpha Q_\alpha, \bar{\theta}^{\dot{\beta}}\bar{Q}_{\dot{\beta}}\right] \quad = \quad 2\theta^\alpha\,(\sigma^\mu)_{\alpha\dot{\beta}}\,\bar{\theta}^{\dot{\beta}}P_\mu \tag{4.33}$$

$$\text{and}\quad \left[\theta^\alpha Q_\alpha, \theta^\beta Q_\beta\right] \quad = \quad \left[\bar{\theta}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\bar{Q}_{\dot{\beta}}\right] = 0 \tag{4.34}$$

A supersymmetry algebra in terms of only commutators can be obtained in this way and thus, the super-Poincare group can be obtained by exponentiating this Lie algebra.

A general element, g of super-Poincare group can be written as:

$$g = e^{i(\mathcal{W}^{\mu\nu}M_{\mu\nu} + a^\mu P_\mu + \theta^\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}})} \tag{4.35}$$

A point in superspace can be identified with a coset representative by a super-translation through the one-to-one map

$$\left(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right) \longleftrightarrow e^{x^\mu P_\mu} e^{(\theta Q + \bar{\theta}\bar{Q})} \tag{4.36}$$

The Grassmann numbers $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ can be resembled as coordinates of the superspace.

4.3 Superfields

Superfields are fields which are functions of the superspace coordinates $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. We know, θ_α and $\bar{theta}_{\dot{\alpha}}$ anticommutes.

$$\theta_\alpha \quad = \quad -\theta_\beta\theta_\alpha \tag{4.37}$$

$$\text{So, } \theta_\alpha\theta_\alpha \quad = \quad 0 \tag{4.38}$$

$$\text{and } \theta_\alpha\theta_\beta\theta_\gamma \quad = \quad 0 \tag{4.39}$$

where, $\alpha = \beta$.

Therefore, the most general scalar superfield, $S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$ can be expanded like Taylor expansion, in powers of $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$.

$$S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) = \varphi(x) + \theta\psi(x) + \bar{\theta}\bar{x}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}N(x) + (\theta\sigma^\mu\bar{\theta})V_\mu(x)$$
$$+ (\theta\theta)\bar{\theta}\bar{\lambda}(x) + (\bar{\theta}\bar{\theta})\theta\rho(x) + (\theta\theta)(\bar{\theta}\bar{\theta})D(x) \tag{4.40}$$

We see that each entries in (4.40) is a field. So, it can be said that a superfield is a multiplet of ordinary fields.

4.3.1 Properties of Grassmann variables

Supersymmetric Lagrangians are constructed in terms of superfields. As these superfields have to interact with each other in terms of different mathematical operations, the properties of Grassmann variables are discussed in this section.

We start by taking one single variable θ and expand an analytic function of θ as a power series.

$$f(\theta) = \sum_{k=0}^{\infty} = f_0 + f_1\theta + f_2\theta^2$$

As the higher power terms of θ blow up to zero and we get a general linear function, $f(\theta)$.

$$f(\theta) = f_0 + f_1\theta$$

Taking the derivative, we get

$$\frac{df}{d\theta} = f_1$$

Assuming that there are no boundary terms, we define the integrals.

$$\int d\theta \frac{df}{d\theta} := 0$$

$$\Rightarrow \int d\theta = 0 \tag{4.41}$$

We define integrals over θ such that a non-trivial result is obtained.

$$\int d\theta := 1$$

$$\Rightarrow \delta\theta = \theta \tag{4.42}$$

We find that the integral over a function $f(\theta)$ is equal to it's derivative.

$$\begin{aligned}
\int d\theta f(\theta) &= \int d\theta (f_0 + f_1\theta) \\
&= f_0 \int d\theta + f \int \theta d\theta \\
&= f_1(1) \tag{4.43} \\
&= \frac{df}{d\theta} \tag{4.44}
\end{aligned}$$

Definition of spinors of Grassmann numbers:

The squares of θ^α, $\bar{\theta}_{\dot{\alpha}}$ the spinors of Grassmann numbers are defined as

$$\begin{aligned}
\theta^\alpha \theta_\alpha &= \theta\theta \\
\Rightarrow \theta^\alpha \theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta \tag{4.45} \\
\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} &= \bar{\theta}\bar{\theta} \\
\Rightarrow \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} &= \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}\bar{\theta} \tag{4.46}
\end{aligned}$$

Derivatives of these spinors are similar to Minkowski coordinates

$$\frac{\partial \theta^\beta}{\partial \theta^\alpha} = \delta_\alpha{}^\beta$$

$$\Rightarrow \frac{\partial \bar{\theta}^{\dot\beta}}{\partial \bar{\theta}^{\dot\alpha}} = \delta_{\dot\alpha}{}^{\dot\beta} \tag{4.47}$$

The multi-integrals are defined by

$$
\begin{aligned}
\int d\theta^1 \int d\theta^2 \theta^2 \theta^1 &= \int d\theta^1 \int d\theta^2 (\frac{1}{2}\epsilon^{12}\theta\theta) \\
&= \frac{1}{2}\int d\theta^1 \int d\theta^2 \theta\theta \\
&= \frac{1}{2}\int d\theta^1 \int (\theta\theta)d\theta^2 \\
&= \frac{1}{2}\int d\theta^1 \left[\theta\int \theta d\theta^2 + \theta\int \theta d\theta^2\right] \\
&= \frac{1}{2}\int d\theta^1 \left[\theta\cdot 1 + 1\cdot\theta\right] \\
&= \frac{1}{2}\int d\theta^1 2\theta \\
&= \int \theta d\theta^1 \\
&= 1
\end{aligned}
\tag{4.48}
$$

So, we can define,

$$\frac{1}{2}\int d\theta^1 \int d\theta^2 = \int d^2\theta \tag{4.49}$$

$$\int d^2\theta(\theta\theta) = 1 \tag{4.50}$$

$$\int d^2\theta \int d^2\bar{\theta}(\theta\theta)(\bar{\theta}\bar{\theta}) = 1 \tag{4.51}$$

This can also be written in terms of ϵ

$$d^2\theta = -\frac{1}{4}d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta} \tag{4.52}$$

$$d^2\bar{\theta} = \frac{1}{4}d\bar{\theta}^{\dot\alpha} d\bar{\theta}^{\dot\beta} \epsilon_{\dot\alpha\dot\beta} \tag{4.53}$$

Integration and differentiation can be identified as

$$\int d^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta} \tag{4.54}$$

$$\text{and}\quad \int d^2\bar{\theta} = -\frac{1}{4}\epsilon^{\dot\alpha\dot\beta}\frac{\partial}{\partial\bar{\theta}^{\dot\alpha}}\frac{\partial}{\partial\bar{\theta}^{\dot\beta}} \tag{4.55}$$

4.3.2 Transformation of the general scalar superfield

A general scalar field, $\varphi(x^\mu)$ is a function of space-time coordinates x^μ which transforms under Poncaré translations. φ can be treated as an operator which changes by a translation with parameter, a_μ.

$$\varphi \longrightarrow e^{ia_\mu P^\mu}\varphi(x^\mu) \; = \; \varphi(x^\mu - a^\mu) \tag{4.56}$$
$$\Rightarrow P^\mu = -i\delta_\mu \tag{4.57}$$

Here, P is a representation of an abstract operator P^μ which acts on $\mathscr{F}$. Comparing these two transformations to first order in a_μ:

$$\begin{aligned}
(1 - ia_\mu P^\mu)\,\varphi\,(1 + ia_\mu P^\mu) &= (1 - ia_\mu P^\mu) \\
\varphi\,(1 + ia_\mu P^\mu) - ia_\mu P^\mu \varphi\,(1 + ia_\mu P^\mu) &= (1 - ia_\mu P^\mu) \\
\varphi\,(ia_\mu P^\mu) - ia_\mu P^\mu \varphi - (ia_\mu P^\mu)(ia_\mu P^\mu) &= -ia_\mu P^\mu \\
i\,[\varphi, a_\mu P^\mu] &= -ia_\mu P^\mu \varphi + ia_\mu P^\mu \varphi ia_\mu P^\mu \\
&= -ia_\mu P^\mu \varphi\,(1 - a_\mu P^\mu) \\
&= -ia_\mu P^\mu \varphi \\
&= -ia^\mu P_\mu \varphi \\
\Rightarrow i\,[\varphi, a_\mu P^\mu] &= -ia^\mu\,(-i\delta_\mu)\,\varphi \\
&= -a^\mu \delta_\mu \varphi \\
i\,[\varphi, a_\mu P^\mu] &= -a^\mu \delta_\mu \varphi \tag{4.58}
\end{aligned}$$

As a field operator, scalar superfield, $S(x^\mu, \theta_\alpha, \bar{\theta}_\alpha)$, transforms under super-Poincaré translation.

$$\begin{aligned}
S\left(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right) &\longrightarrow e^{-i\epsilon Q + \bar{\epsilon}\bar{Q}} S e^{i\epsilon Q + \bar{\epsilon}\bar{Q}} \\
\Rightarrow S\left(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\right) &\longrightarrow S\left(x^\mu + \delta x^\mu, \theta_\alpha + \delta\theta_\alpha, \bar{\theta}_{\dot{\alpha}} + \delta\bar{\theta}_{\dot{\alpha}}\right) \\
&= e^{-i(\epsilon Q + \bar{\epsilon}\bar{Q})} e^{-i(x^\mu P^\mu + \theta_\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}} + \bar{Q}_{\dot{\alpha}})} S e^{i(\epsilon Q + \bar{\epsilon}\bar{Q})} e^{-i(x^\mu P^\mu + \theta_\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}} + \bar{Q}_{\dot{\alpha}})}
\end{aligned}$$

Now, we evaluate, $e^{i(\epsilon Q + \bar{\epsilon}\bar{Q})} e^{i(x^\mu P^\mu + \theta_\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}} + \bar{Q}_{\dot{\alpha}})}$

$$\begin{aligned}
e^{i(\epsilon Q + \bar{\epsilon}\bar{Q})} e^{i(x^\mu P^\mu + \theta_\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}} + \bar{Q}_{\dot{\alpha}})} &= e^{ix^\mu P_\mu + i(\epsilon + \theta)Q + i(\bar{\epsilon} + \bar{\theta})Q - \frac{1}{2}[\bar{\theta}\bar{Q}, \epsilon Q] - \frac{1}{2}[\theta Q, \bar{\epsilon}\bar{Q}]} \\
&= e^{ix^\mu P_\mu + i(\epsilon + \theta)Q + i(\bar{\epsilon} + \bar{\theta})Q - \frac{1}{2}(-2\epsilon\sigma^\mu\bar{\theta}P_\mu) - \frac{1}{2}(2\theta\sigma^\mu\bar{\epsilon}P_\mu)} \\
&= e^{ix^\mu P_\mu + i(\epsilon + \theta)Q + i(\bar{\epsilon} + \bar{\theta})Q + \epsilon\sigma^\mu\bar{\theta}P_\mu) - \theta\sigma^\mu\bar{\epsilon}P_\mu)} \\
&= e^{i(x^\mu + i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta})P_\mu + i(\epsilon + \theta)Q + i(\bar{\epsilon} + \bar{\theta})\bar{Q}}
\end{aligned}$$

Now, under Poincaré translation

$$\begin{aligned}
S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) &\longrightarrow e^{-i(\epsilon Q + \bar{\epsilon}\bar{Q})} S e^{i(\epsilon Q + \bar{\epsilon}\bar{Q})} \\
\Rightarrow S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) &\longrightarrow S\left(x^\mu + \delta x^\mu, \theta_\alpha + \delta\theta_\alpha, \bar{\theta}_{\dot{\alpha}} + \delta\bar{\theta}_{\dot{\alpha}}\right) \\
&= e^{-i(\epsilon Q + \bar{\epsilon}\bar{Q})} e^{-i(x^\mu P_\mu + \theta_\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}}\bar{Q}_{\dot{\alpha}})} S e^{i(\epsilon Q + \bar{\epsilon}\bar{Q})} e^{i(x^\mu P_\mu + \theta_\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}}\bar{Q}_{\dot{\alpha}})} \\
&= e^{-i(\epsilon Q + \bar{\epsilon}\bar{Q})} e^{-i(x^\mu P_\mu + \theta_\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}}\bar{Q}_{\dot{\alpha}})} S e^{i(x^\mu + i\theta_\alpha\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta}_{\dot{\alpha}})P_\mu + i(\epsilon_\alpha + \theta_\alpha)Q + i(\bar{\epsilon}_{\dot{\alpha}} + \bar{\theta}_{\dot{\alpha}})\bar{Q}}
\end{aligned}$$

Again, as a Hilbert vector S transforms as

$$S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) = e^{-i(\epsilon Q + \bar{\epsilon}\bar{Q})}$$

Comparing

$$S(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}) \longrightarrow S\left(x^\mu + \delta x^\mu, \theta_\alpha + \delta\theta_\alpha, \bar{\theta}_{\dot{\alpha}} + \delta\bar{\theta}_{\dot{\alpha}}\right) \tag{4.59}$$

$$= S e^{i(x^\mu + i\theta_\alpha \sigma^\mu \bar{\epsilon} - i\epsilon \sigma^\mu \bar{\theta}_{\dot{\alpha}})P_\mu + i(\epsilon_\alpha + \theta_\alpha)Q + i(\bar{\epsilon}_{\dot{\alpha}} + \bar{\theta}_{\dot{\alpha}})\bar{Q}} \tag{4.60}$$

$$= S(x^\mu + i\theta_\alpha \sigma^\mu \bar{\epsilon} - i\epsilon \sigma^\mu \bar{\theta}_{\dot{\alpha}}, \epsilon_\alpha + \theta_\alpha, \bar{\epsilon}_{\dot{\alpha}} + \bar{\theta}_{\dot{\alpha}})\bar{Q} \tag{4.61}$$

Finally, we get

$$\delta x = i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta} \tag{4.62}$$

$$\delta\theta_\alpha = \epsilon_\alpha \tag{4.63}$$

$$\delta\bar{\theta}_{\dot{\alpha}} = \bar{\epsilon}_{\dot{\alpha}} \tag{4.64}$$

The ϵ_α and $\bar{\epsilon}_{\dot{\alpha}}$ needs to be consistent with the supersymmetry algebra $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} \sim P_\mu$ as the space-time translation is generated by the supersymmetry transformation.
We also get,

$$Q_\alpha = -i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu{}_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\frac{\partial}{\partial x^\mu} \tag{4.65}$$

$$= -i\partial_\alpha - \sigma^\mu{}_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}_\mu \tag{4.66}$$

As $\bar{Q}_{\dot{\alpha}} = Q_\alpha^\dagger$ we get,

$$\bar{Q}_{\dot{\alpha}} = i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\frac{\partial}{\partial x^\mu} \tag{4.67}$$

$$= i\bar{\partial}_{\dot{\alpha}} + \theta^\beta\sigma^\mu{}_{\beta\dot{\alpha}}\partial_\mu \tag{4.68}$$

$$\delta S = i\left[S, \epsilon Q, \bar{\epsilon}\bar{Q}\right] \tag{4.69}$$

$$= i(\epsilon Q + \bar{\epsilon}\bar{Q})S \tag{4.70}$$

From this, the explicit terms for the change in different parts of S can be obtained. From (4.40) we got the expansion of the general scalar superfield. Now, when a supersymmetry action acts on a scalar superfield, S

$$\delta S = i(\epsilon Q + \bar{\epsilon}\bar{Q})S(x, \theta, \bar{\theta}) \tag{4.71}$$

$$= i(\epsilon^\alpha Q_\alpha + \bar{Q}_{\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}})S(x, \theta, \bar{\theta}) \tag{4.72}$$

$$= \left[\epsilon^\alpha(\frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu{}_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu) + (-\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\beta\sigma^\mu{}_{\beta\dot{\alpha}}\partial_\mu)\bar{\epsilon}^{\dot{\alpha}}\right]S(x, \theta, \bar{\theta}) \tag{4.73}$$

When this acts on a scalar field, we can find the individual components of $S(x, \theta, \bar{\theta})$ by comparing order of θ and $\bar{\theta}$. Finally,

$$\delta\varphi = \epsilon\psi + \bar{\epsilon}\bar{\chi} \tag{4.74}$$

$$\delta\psi = 2\epsilon M + \sigma^\mu\bar{\epsilon}(i\partial_\mu\varphi + V_\mu) \tag{4.75}$$

$$\delta\bar{\chi} = 2\bar{\epsilon}N - \epsilon\sigma^\mu(i\partial_\mu\varphi - V_\mu) \tag{4.76}$$

$$\delta M = \bar{\epsilon}\bar{\lambda} - \frac{1}{2}\partial_\mu\psi\sigma^\mu\bar{\epsilon} \tag{4.77}$$

$$\delta N = \epsilon\rho + \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar{\chi} \tag{4.78}$$

$$\delta V_\mu = \epsilon\sigma_\mu\bar{\lambda} + \rho\sigma_\mu\bar{\epsilon} + \frac{i}{2}(\partial^\nu\psi\sigma_\mu\bar{\sigma}_\nu\epsilon - \bar{\epsilon}\bar{\sigma}_\nu\sigma_\mu\partial^\nu\bar{\chi}) \tag{4.79}$$

$$\delta\bar{\lambda} = 2\bar{\epsilon}D + \frac{i}{2}(\bar{\sigma}^\nu\sigma^\mu\bar{\epsilon})\partial_\mu V_\nu + i\bar{\sigma}^\mu\bar{\epsilon}\partial_\mu M \tag{4.80}$$

$$\delta\rho = e\epsilon D - \frac{i}{2}(\sigma^\nu\bar{\sigma}^\mu\epsilon)\partial_\mu V_\nu + i\sigma^\mu\bar{\epsilon}\partial_\mu N \tag{4.81}$$

$$\delta D = \frac{i}{2}\partial_\mu(\epsilon\sigma^\mu\bar{\lambda} - \rho\sigma^\mu\bar{\epsilon}) \tag{4.82}$$

4.3.3 Properties of superfields

The product of the two superfields S_1 and S_2 is also a superfield.

$$\delta(S_1, S_2) = i\left[S_1 S_2, \epsilon Q + \bar{\epsilon}\bar{Q}\right] \tag{4.83}$$

$$= iS_1\left[S_2, \epsilon Q + \bar{\epsilon}\right] + i\left[S_1, \epsilon Q + \bar{\epsilon}\right]S_2 \tag{4.84}$$

$$= i(\epsilon Q + \bar{\epsilon}\bar{Q})S_1 S_2 \tag{4.85}$$

Linear combination of two superfields are also superfield $\partial_\mu S$ is a superfield. $\partial_\alpha S$ is not a superfield but D_α is a superfield.

$$i\partial_\alpha(\epsilon Q + \bar{\epsilon}\bar{Q})S \neq i(\epsilon Q + \bar{\epsilon}\bar{Q})\partial_\alpha S$$

as $(\partial_\alpha, \epsilon Q + \bar{\epsilon}\bar{Q}) \neq 0$.
Defining a covariant derivative,

$$D_\alpha = \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \tag{4.86}$$

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu \tag{4.87}$$

This satisfies that

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot{\beta}}\} = \{\bar{D}_{\dot{\alpha}}, Q_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0 \tag{4.88}$$

$$\therefore [D_\alpha, \epsilon Q + \bar{\epsilon}\bar{Q}] = 0 \tag{4.89}$$

Therefore, $D_\alpha S$ is a superfield.
The anticommutation relations of the super covariant derivatives are

$$\{D_\alpha, \bar{D}_{\dot{\beta}}\} = -2i(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu \tag{4.90}$$

$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \tag{4.91}$$

4.3.4 Reduced superfields

A general superfield is often a product of superfields. When supersymmetry invariant constraints are acted on S, we can obtain it's subsets. This will form some reduced set of superfields carrying the representation of supersymmetry algebra.

Some of the superfields are:

- Chiral Superfield such that $\bar{D}_{\dot{\alpha}}\phi = 0$

- Anti-chiral superfield such that $D_{\alpha}\phi = 0$

- Vector superfield such that $V = V^{\dagger}$

- Linear superfield such that $DDL = 0$ and $L = L^{\dagger}$

4.4 Chiral superfields

From the definition of covariant derivatives, D_{α}, $\bar{D}_{\dot{\alpha}}$ which anti-commute with the supersymmetry generator Q_{α}, $\bar{Q}_{\dot{\alpha}}$, we can imply that

$$\delta_{\epsilon,\bar{\epsilon}}(D_{\alpha}S) = D_{\alpha}(\delta_{\epsilon,\bar{\epsilon}})S \tag{4.92}$$

So, S is a superfield following (4.40) and therefore, $D_{\alpha}S$ is also a superfield. So, we can impose $D_{\alpha}S = 0$ as a supersymmetry invariant constraint to reduce the number of components of Y. As D_{α} commutes with Q and $\bar{Q}$, $D_{\alpha}S$ and $\bar{D}_{\dot{\alpha}}S$ are both superfields. Similarly, $D_{\mu}S$ is also superfield since D_{μ} commutes with Q and $\bar{Q}$. A Chiral superfield ϕ is a superfield such that

$$\bar{D}_{\dot{\alpha}}\phi = 0 \tag{4.93}$$

and an anti-chiral superfield ψ is a superfield such that

$$D_{\alpha}\psi = 0 \tag{4.94}$$

If ϕ is Chiral $\bar{\phi}$ has to be anti-chiral. So, a chiral superfield can not be real and $\bar{\phi} = \phi$ has to be anti-chiral. So, a chiral superfield cannot be real and $\bar{\phi} \neq \phi$.

4.4.1 General expression for chiral superfield

To construct a general expression for Chiral superfield, we define some new coordinates.

$$y^{\mu} = x^{\mu} + i\theta\sigma^{\mu}\bar{\theta} \tag{4.95}$$
$$\bar{y}^{\mu} = x^{\mu} - i\theta\sigma^{\mu}\bar{\theta} \tag{4.96}$$

From this we get,

$$\bar{D}_{\dot{\alpha}}\theta_{\beta} = \bar{D}_{\dot{\alpha}}y^{\mu} = 0 \tag{4.97}$$
$$D_{\alpha}\bar{\theta}_{\beta} = D_{\alpha}\bar{y}^{\mu} = 0 \tag{4.98}$$

From (4.93) we can show that, ϕ does not depend on $\bar{\theta}^{\dot{\alpha}}$ and only depends on θ

$$\phi(y^\mu, \theta^\alpha) = \varphi(y^\mu) + \sqrt{2}\theta\psi(y^\mu) + \theta\theta F(y^\mu) \tag{4.99}$$

A Chiral field corresponds to a multiplet of states. φ represents a scalar part (Squark, Slepton, Higgs). ψ is a particle with half-spin (Quark, Lepton, Higgsino) and F is an auxiliary field.

Taylor expanding (4.99) on z results to

$$\phi(x, \theta, \bar{\theta}) = \varphi(x) + \sqrt{2}\theta\psi(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi(x) + \theta\theta F(x)$$
$$- \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\varphi(x) \tag{4.100}$$

Similarly, an anti-Chiral superfield will be

$$\bar{\phi}(x, \theta, \bar{\theta}) = \bar{\phi}(\bar{y}^\mu) + \sqrt{2}\bar{\theta}\bar{\psi}(\bar{y}^\mu) - \bar{\theta}\bar{\theta}\bar{F}(\bar{y}) \tag{4.101}$$
$$= \bar{\varphi}(x) + \sqrt{2}\bar{\theta}\bar{\psi}(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi(x) + \bar{\theta}\bar{\theta}\bar{F}(x)$$
$$+ \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi}(x) - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\bar{\varphi}(x) \tag{4.102}$$

4.4.2 Supersymmetry transformation of chiral field

Under a supersymmetry transformation, a Chiral superfield and an antichiral superfield transform as follows

$$\delta_{\epsilon, \bar{\epsilon}}\phi = i(\epsilon Q + \bar{\epsilon}\bar{Q})\phi \tag{4.103}$$
$$\delta_{\bar{\epsilon}, \epsilon}\bar{\phi} = i(\bar{\epsilon}\bar{Q} + \epsilon Q)\bar{\phi} \tag{4.104}$$

Writing the supersymmetry generators, Q_α, $\bar{Q}_{\dot{\alpha}}$ as the differential operators in the $(y^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$ coordinate system we get,

$$Q'_\alpha = -i\partial_\alpha \tag{4.105}$$
$$\bar{Q}'_{\dot{\alpha}} = i\bar{\partial}_{\dot{\alpha}} + 2\theta^\alpha\sigma^\mu{}_{\alpha\dot{\alpha}}\frac{\partial}{\partial y^\mu} \tag{4.106}$$

Using this two definition in equation (4.103) and (4.104) results in

$$\delta_{\epsilon, \epsilon}\phi = (\epsilon^\alpha\partial_\alpha + 2i\theta^\alpha\sigma^\mu{}_{\alpha\dot{\beta}}\bar{\epsilon}^{\dot{\beta}}\frac{\partial}{\partial y^\mu})\phi \tag{4.107}$$

$$= \sqrt{2}\epsilon\psi - 2\epsilon\theta F + 2i\theta\sigma^\mu\bar{\epsilon}(\frac{\partial}{\partial y^\mu}\phi + \sqrt{2}\theta\frac{\partial}{\partial y^\mu}\psi) \tag{4.108}$$

$$= \sqrt{2}\epsilon\psi + \sqrt{2}\theta(-\sqrt{2}\epsilon F + \sqrt{2}i\sigma^\mu\bar{\epsilon}\frac{\partial}{\partial y^\mu}\phi) - \theta\theta(-i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\frac{\partial}{\partial y^\mu}\psi) \tag{4.109}$$

Now, er arrive to the final expression of the supersymmetry variation of the different components of the Chiral superfield, ϕ.

$$\delta\phi = \sqrt{2}\epsilon\psi \tag{4.110}$$
$$\delta\psi_\alpha = \sqrt{2}i(\sigma^\mu)_{\dot{\alpha}}\partial_\mu\bar{\varphi} - \sqrt{\bar{\epsilon}_{\dot{\alpha}}}\bar{F} \tag{4.111}$$
$$\delta F = -i\sqrt{2}\partial_\mu\bar{\psi}\sigma^\mu\epsilon \tag{4.112}$$

Here, δF is a total derivative.

4.4.3 Properties of chiral superfield

1. The product of Chiral superfield are also superfield.

2. Any holomorphic function $f(\phi)$ where ϕ is Chiral, is also an Chiral superfield.

3. $\phi = \bar{\phi}$ is anti-Chiral for a Chiral superfield ϕ.

4.5 Vector (or Real)superfield

A real or vector superfield is defined such that

$$V = V^{\dagger} \tag{4.113}$$

Due to this reality condition, the supersymmetry invariant projection saves the vector field, V^{μ} in the general expression. Thus, we get some gauge interactions.
The most general vector superfield,

$$V(x, \theta, \bar{\theta}) = V^{\dagger}(x, \theta, \bar{\theta}) \tag{4.114}$$

has the form

$$V(x, \theta, \bar{\theta}) = C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \frac{i}{2}(\theta\theta)(M(x) + iN(x)) - \frac{i}{2}\bar{\theta}\bar{\theta}(M(x) - iN(x))$$

$$+ \theta\sigma^{\mu}\bar{\theta}V_{\mu}(x) + i\theta\theta\bar{\theta}(-i\bar{\lambda}(x) + \frac{i}{2}\bar{\sigma}^{\mu}\partial_{\mu}\chi(x))$$

$$- i\bar{\theta}\bar{\theta}\theta(i\lambda(x) - \frac{i}{2}\sigma^{\mu}\partial_{\mu}\bar{\chi}(x)) + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})(D - \frac{1}{2}\partial_{\mu}\partial^{\mu}C) \tag{4.115}$$

This superfield has $8_B + 8_F$ degrees of freedom.

Upon supersymmetry gauge transformation, the off-shell degrees of freedom reduces to $4_B + 4_F$ where as the on-shell degrees of freedom become $2_B + 2_F$. We see that, the on-shell case resembles the massless vector multiplet of states. The 8 bosonic components are C,M,N,D,V_{μ} and the 4+4 fermionic components are χ_{α} and λ_{α}.

For a Chiral superfield, Λ, $i(\Lambda - \Lambda^{\dagger})$ will be a vector superfield. The components of this vector superfield will be

$$C = i(\varphi - \varphi^{\dagger}) \tag{4.116}$$

$$\chi = \sqrt{2}\psi \tag{4.117}$$

$$\frac{1}{2}(M + iN) = F \tag{4.118}$$

$$v_{\mu} = -\partial_{\mu}(\varphi + \varphi^{\dagger}) \tag{4.119}$$

$$\lambda = D = 0 \tag{4.120}$$

A generalized gauge transformation to vector fields can be defined as

$$V \longrightarrow V - \frac{i}{2}(\Lambda - \Lambda^{\dagger}) \tag{4.121}$$

An ordinary gauge transformation for the vector component V will be

$$V_{\mu} \longrightarrow V_{\mu} + \partial_{\mu}[Re\varphi] \tag{4.122}$$

$$=: V_{\mu} - \partial_{\mu}\alpha \tag{4.123}$$

Some of the components of V will gauge away by choosing φ, ψ, F within Λ.

4.5.1 Wess-Zumino gauge

When the components of V, $C = \chi = M = N = 0$ the Wess-Zumino Gauge is obtained. This can be found by choosing the components φ, ψ, F of Λ.

In a Wess-Zumino gauge, a vector superfield is expressed as

$$V_{wz}(x, \bar{\theta}, \bar{\theta}) = (\theta \sigma^\mu \bar{\theta})V_\mu(x) + (\theta\theta)(\bar{\theta}\bar{\lambda}(x)) + (\bar{\theta}\bar{\theta})(\theta\lambda(x)) + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D(x) \qquad (4.124)$$

V_μ corresponds to gauge particles (γ, $W^\pm$, Z, gluon, λ and $\bar{\lambda}$ to gauginos), D is an auxiliary field.

Off-shell, we get $4_B + 4_F$ degrees of freedom, due to the gauge invariance. When the equation of motion for the auxiliary field, D, spinor λ and vector, V^μ is imposed, we get $2_B + 2_F$ degrees of freedom, on-shell.

According to the definition of Wess-Zumino gauge, we have no restriction on V^μ. So, supersymmetry gauge transformations can be performed on Wess-Zumino gauge with gauge parameters,

$$\varphi = \bar{\varphi}, \quad \psi = 0, \quad F = 0 \qquad (4.125)$$

Powers of V_{wz} are given by,

$$V_{wz}^2 = \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})V^\mu V_\mu \qquad (4.126)$$

$$V_{wz}^{2+n} = 0 \quad \text{for} \quad \text{all} \quad n \in N \qquad (4.127)$$

As, $V_{wz} \longrightarrow V'_{wz}$ under supersymmetry, Wess-Zumino gauge does not commute with supersymmetry. Thus, Wess-Zumino gauge is not supersymmetric. When a supersymmetry transformation acts on a Wess-Zumino gauge, a new field will be obtained and this field will not be a Wess-Zumino gauge.

4.5.2 Abelian field strength superfield

Under local U(1) with charge q and local parameter $\alpha(x)$, a non-supersymmetric complex scalar field, ϕ which is coupled to a gauge field, V_μ via co-variant derivative $D_\mu = \delta_\mu - iqV_\mu$ will transform as

$$\varphi(x) \longrightarrow e^{iq\alpha(x)}\varphi(x) \qquad (4.128)$$

$$V_\mu(x) \longrightarrow V_\mu(x) + \partial_\mu\alpha(x) \qquad (4.129)$$

When supersymmetry is taken into account, we work on Chiral superfields, ϕ and vector superfield vector.

Upon imposing the transformation properties,

$$\phi \longrightarrow e^{iq\Lambda}\phi \qquad (4.130)$$

$$V \longrightarrow V - \frac{i}{2}(\Lambda - \Lambda^\dagger) \qquad (4.131)$$

we get gauge invariant $\phi^\dagger e^{2qV}\phi$.

We can construct a gauge invariant quality out of ϕ and V.

Λ is the Chiral superfield which defines the generalized gauge transformations. If ϕ is Chiral, then $e^{iq\Lambda}\phi$ will also be Chiral.

An Abelian field strength in a non-supersymmetric analogy is defined as

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \tag{4.132}$$

For supersymmetry, it will be

$$W_\alpha := -\frac{1}{4}(\bar{D}\bar{D})D_\alpha V \tag{4.133}$$

This will result to a field which is Chiral as well as invariant under generalized gauge transformation.

Properties of Abelian field strength superfield :

Chirality: When acted on a superfield, a right handed super-covariant derivative $\bar{D}_{\dot{\alpha}}W_\alpha$ can be re-written as $\epsilon^{\dot{\beta}\dot{\gamma}}\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}\bar{D}_{\dot{\gamma}}$.

As the $\bar{D}$ anti-commute, the expression $\bar{D}_{\dot{\alpha}}$, $\bar{D}_{\dot{\beta}}$, $\bar{D}_{\dot{\gamma}}$ are totally anti-symmetrized. Due to the restriction on the index value of $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ indices, the anti-symmetric rank three tensor vanishes,

$$T_{[\dot{\alpha},\dot{\beta},\dot{\gamma}]} = 0 \tag{4.134}$$

So, we can say,

$$\bar{D}_{\dot{\alpha}}W_\alpha = -\frac{1}{4}\epsilon^{\dot{\beta}\dot{\gamma}}\bar{D}_{[\dot{\alpha}}\bar{D}_{\dot{\beta}}\bar{D}_{\dot{\gamma}]}(D_\alpha V) = 0 \tag{4.135}$$

Invariance: As $\Lambda^\dagger$ is anti-Chiral, it will not be contributing to the transformation law of V. Since, $\delta_\mu\Lambda$ is a Chiral superfield, the anti-commutator $\{D_\alpha, \bar{D}_{\dot{\beta}}\} = -2i(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu$ results to

$$\delta W_\alpha = \frac{i}{8}\epsilon^{\dot{\beta}\dot{\gamma}}\bar{D}_{\dot{\beta}}\{\bar{D}_{\dot{\gamma}}, D_\alpha\}\Lambda \tag{4.136}$$

$$= -\frac{1}{4}(\sigma^\mu)_{\alpha\dot{\gamma}}\bar{D}^{\dot{\gamma}}(\partial_\mu\Lambda) \tag{4.137}$$

$$= 0 \tag{4.138}$$

under a transformation of

$$V \longrightarrow V - \frac{1}{2}(\Lambda - \Lambda^\dagger) \tag{4.139}$$

If V is re-written in a shifted co-ordinate system

$$y_\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} \tag{4.140}$$

Where,

$$\theta\sigma^\mu\bar{\theta}V_\mu(x) = \theta\sigma^\mu\bar{\theta}V_\mu(y) - \frac{1}{2}\theta^2\bar{\theta}^2\partial_\mu V^\mu(y) \tag{4.141}$$

then, the super-covariant derivatives simplifies to

$$D_\alpha = \delta_\alpha + 2i(\sigma^\mu\bar{\theta})_\alpha\partial_\mu \tag{4.142}$$

$$= \bar{D}_\alpha \tag{4.143}$$

$$\bar{D}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} \tag{4.144}$$

Using this new co-ordinate system to write a new expression for V_{wz}, and using the super-covariant derivative of equation (4.142), we can obtain W_α in components. This can be expressed as

$$W_\alpha(y,\theta) = \lambda_\alpha(y) + \theta_\alpha D(y) + (\sigma^{\mu\nu})_\alpha F_{\mu\nu}(y) - i(\theta\theta)(\sigma)_{\alpha\dot{\beta}}\partial_\mu\bar{\lambda}^{\dot{\beta}}(y) \tag{4.145}$$

4.5.3 Non-Abelian field strength

When the supersymmetry U(1) gauge theories are generalized to non-Abelian theories, they are generalized to non-Abelian gauge groups. The gauge degrees of freedom will take the values which are in the associated Lie algebra spanned by hermitian generator T^α

$$\Lambda = \Lambda_a T^a \tag{4.146}$$

$$V = V_a T^a \tag{4.147}$$

$$\left[T^a, T^b\right] = i f^{abc} T_c \tag{4.148}$$

Similar to the Abelian case, we expect that $\phi^\dagger e^{2qV} \phi$ will be invariant under gauge transformation $\phi \longrightarrow e^{iq\Lambda}\phi$. However, a non-linear transformation law $V \longrightarrow V'$ is obtained due to the non-commutative nature of Λ and V.

$$e^{2qV'} = e^{iq\Lambda^\dagger} e^{2qV} e^{-iq\Lambda} \tag{4.149}$$

$$\Rightarrow V' = V - \frac{i}{2}(\Lambda - \Lambda^\dagger) - \frac{iq}{2}\left[V, \Lambda + \Lambda^\dagger\right] \tag{4.150}$$

Under unitary transformations the field-strength tensor $F_{\mu\nu}$ of non-supersymmetric Yang-Mills theories transform to $U F_{\mu\nu} U^{-1}$.

Similarly, to obtain a gauge covariant quantity, we can define

$$W_\alpha := -\frac{1}{8q}(\bar{D}\bar{D})(e^{-2qV} D_\alpha e^{2qV}) \tag{4.151}$$

Under gauge transformations $e^{2qV} \longrightarrow e^{iq\Lambda^\dagger} e^{2qV} e^{-iq\Lambda}$, we assume to get a transformed field strength superfield

$$
\begin{aligned}
W'_\alpha &= -\frac{1}{8q}(\bar{D}\bar{D})(e^{iq\Lambda} e^{-2qV} e^{-iq\Lambda^\dagger} D_\alpha e^{iq\Lambda^\dagger} e^{2qV} e^{-iq\Lambda}) \\
&= -\frac{1}{8q} e^{iq\Lambda}(\bar{D}\bar{D})\left(e^{-2qV} D_\alpha(e^{2qV} e^{-iq\Lambda}\right) \\
&= e^{iq\Lambda}\left(-\frac{1}{8q}(\bar{D}\bar{D} e^{-2qV} D_\alpha e^{2qV}) - \frac{1}{8q} e^{-iq\Lambda}(\bar{D}\bar{D} D_\alpha e^i q\Lambda)\right) \\
&= e^{iq\Lambda} W_\alpha e^{-iq\Lambda} - \frac{1}{8q} e^{-iq\Lambda}(\bar{D}\bar{D} D_\alpha e^i q\Lambda)
\end{aligned}
\tag{4.152}
$$

Now, we know from the anti-commutation relation of D_α and $\bar{D}_{\dot\beta}$

$$
\begin{aligned}
\{D_\alpha, \bar{D}_{\dot\beta}\} &= -2i(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu \\
\Rightarrow \{D_\alpha, \bar{D}_{\dot\beta}\}e^{iq\Lambda} &= -2i(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu e^{iq\Lambda} \\
\Rightarrow D_\alpha \bar{D}_{\dot\beta} e^{iq\Lambda} + \bar{D}_{\dot\beta} D_\alpha e^{iq\Lambda} &= 0 \\
\Rightarrow \bar{D}_{\dot\beta} D_\alpha e^{iq\Lambda} &= 0 \\
\therefore (\bar{D}\bar{D}) D_\alpha e^{iq\Lambda} &= 0
\end{aligned}
$$

Replacing this in equation (4.152) we get,

$$W'_\alpha = e^{iq\Lambda} W_\alpha e^{-iq\Lambda} \tag{4.153}$$

So, transformation law for W_α under $e^{2qV} \longrightarrow e^{iq\Lambda^\dagger} e^{2qV} e^{-2q\Lambda}$ is

$$W_\alpha \longrightarrow e^{iq\Lambda} W_\alpha e^{-iq\Lambda} \tag{4.154}$$

In Wess-Zumino gauge, For

$$
\begin{align}
F_{\mu\nu}^\alpha &:= \delta_\mu V_\nu^a - \delta_\nu V_\mu^a + q f^a{}_{bc} V_\mu^b V_\nu^c \tag{4.155} \\
D_\mu \bar\lambda^a &:= \delta_\mu \lambda^a + q V_\mu^b \bar\lambda^c f_{bc}^a \tag{4.156}
\end{align}
$$

the supersymmetric field strength can be evaluated as

$$
\begin{align}
W_\alpha^a(y,\theta) &= -\frac{1}{4}(\bar D \bar D) D_\alpha (V^\alpha(y,\theta,\bar\theta) + i V^b(y,\theta,\bar\theta) V^c(y,\theta,\bar\theta) f_{bc}^a) \tag{4.157} \\
&= \lambda_\alpha^a(y) + \theta_\alpha D^\alpha(y) + (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}^\alpha(y) - i(\theta\theta)(\sigma^\mu)_{\alpha\dot\beta} D_\mu \bar\lambda^{\alpha\dot\beta}(y) \tag{4.158}
\end{align}
$$

Chapter 5

Supersymmetric Lagrangians and Actions

A supersymmetry transformation action transforms the highest order component of a superfield into a total derivative. For a general scalar superfield, the highest order term is $(\theta\theta)(\bar\theta\bar\theta)D(x)$ and under supersymmetry transformation this term changes into

$$\delta D = \frac{i}{2}\partial_\mu(\epsilon\sigma^\mu\bar\lambda - p\sigma^\mu\bar\epsilon) \tag{5.1}$$

Here, δD is a total derivative.

Due to this property, a space-time integral of this quantity will be invariant under supersymmetric transformation.

Superfields ϕ, V and W_α includes the particles of standard model. To determine supersymmetric couplings of this superfields, Lagarangians which will be invariant under supersymmetry transformations are constructed. A supersymmetric action integral is defined as

$$A \;:=\; \int d^4x \int d^4\theta \mathcal{L} \tag{5.2}$$

$$=\; \int d^4x \int d^2\theta \int d^2\bar\theta \mathcal{L} \tag{5.3}$$

Here, $\mathcal{L}$ is a supersymmetry Lagrangian density.

5.1 Chiral superfield Lagrangian

The highest term component of the Chiral superfield is F. So, under supersymmetry transformation, we get,

$$\delta F = i\sqrt{2}\bar\epsilon\bar\sigma^\mu\partial_\mu\psi \tag{5.4}$$

So, the most general Lagrangian for a chiral superfield, ϕ is

$$\mathcal{L} = K(\phi,\phi^\dagger)\big|_D + (W(\phi))\big|_F + h.c) \tag{5.5}$$

$K(\phi,\phi^\dagger)$ is a real function of ϕ and $\phi^\dagger$, named Kähler potential. $W(\phi)$ is a holomorphic function of the Chiral superfield, so, $W(\phi)$ is a Chiral superfield which is the superpotential.

In equation (5.5) the $|_D$ and $|_F$ refers to the D-term and the F-term of the corresponding superfield.

To satisfy the condition of the renormalizable theory, the Lagrangian must have dimensionality 4. We know, a Chiral field is constructed of one scalar, one spinor and another scalar component and the dimensionality of the spinor is the same as a standard fermion.

$$[\psi] = \frac{3}{2}$$

The dimensionality of the superfield is same as its scalar component.

$$[\phi] = \varphi = 1$$

From the expansion of (4.100) in section 4.4 we get,

$$
\begin{aligned}
[\theta] &= -\frac{1}{2} \quad \text{and} \\
[F] &= 2
\end{aligned}
$$

Satisfying the normalizability theory, we take

$$
\begin{aligned}
K &= \phi^\dagger \phi \text{ and} \\
W &= \alpha + \lambda\phi + \frac{m}{2}\phi^2 + \frac{g}{3}\phi^3
\end{aligned}
$$

Replacing K and W in (5.5) will result to

$$\mathcal{L} = \phi^\dagger\phi\big|_D + \left(\left(\alpha + \lambda\phi + \frac{m}{2}\phi^2 + \frac{g}{3}\phi^2\right)\big|_F + h.c\right) \tag{5.6}$$

Now, only the D-term of $\phi^\dagger\phi$ will be included in the Lagrangian.
From the previous chapter we got,

$$
\begin{aligned}
\phi &= \varphi + \sqrt{2}\theta\psi + (\theta\theta)F + i(\theta\sigma^\mu\bar{\theta})\partial_\mu + \varphi - \frac{(\theta\theta)(\bar{\theta}\bar{\theta})}{4}\partial_\mu\partial^\mu\varphi - \frac{i\theta\theta}{\sqrt{2}}\partial_\mu\psi\sigma^\mu\bar{\theta} \\
\bar{\phi}^\dagger &= \varphi^*\sqrt{2}\bar{\theta}\bar{\psi} + \bar{\theta}\bar{\theta}F^* - i(\theta\sigma^\mu\bar{\sigma})\partial_\mu\varphi^* - \frac{(\theta\theta)(\bar{\theta}\bar{\theta})}{4}\partial_\mu\partial^\mu + \frac{i\theta\bar{\theta}}{\sqrt{2}}\theta\sigma^\mu\partial_\mu\bar{\psi}
\end{aligned}
$$

For the D-term of K, we only take the $(\theta\theta)(\bar\theta\bar\theta)$ component of $\phi\phi^\dagger$

$$\phi^\dagger\phi = ((\varphi + \sqrt{2}\theta\psi + (\theta\theta)F + i(\theta\sigma^\mu\bar\theta)\partial_\mu + \varphi - \frac{(\theta\theta)(\bar\theta\bar\theta)}{4}\partial_\mu\partial^\mu\varphi - \frac{i\theta\theta}{\sqrt{2}}\partial_\mu\psi\sigma^\mu\bar\theta)$$

$$(\varphi^*\sqrt{2}\bar\theta\bar\psi + \bar\theta\bar\theta F^* - i(\theta\sigma^\mu\bar\sigma)\partial_\mu\varphi^* - \frac{(\theta\theta)(\bar\theta\bar\theta)}{4}\partial_\mu\partial^\mu + \frac{i\theta\bar\theta}{\sqrt{2}}\theta\sigma^\mu\partial_\mu\bar\psi)$$

$$\supset (\theta\theta)(\bar\theta\bar\theta)\left[-\frac{1}{4}\varphi^*\partial_\mu\partial^\mu\varphi - \frac{1}{4}\varphi\partial_\mu\partial^\mu\varphi^* + |F|^2\right]$$
$$+ (\theta\sigma^\mu\bar\theta)(\theta\sigma\bar\theta)\partial_\nu\varphi\partial_\mu\phi^* - i\bar\theta\bar\psi(\theta\theta)\partial_\mu\psi\sigma^\mu\bar\theta + i(\bar\theta\bar\theta)(\theta\sigma^\mu\partial_\mu\bar\psi)(\theta\psi)$$
$$= (\theta\theta)(\bar\theta\bar\theta)\left[-\frac{1}{4}\varphi^*\partial_\mu\partial^\mu\varphi - \frac{1}{4}\varphi\partial_\mu\partial^\mu\varphi^* + |F|^2\right]$$
$$+ \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)\partial^\mu\varphi\partial_\mu\varphi^* + i\bar\theta^{\dot\alpha}\bar\psi_{\dot\alpha}(\theta\theta)\partial_\mu\psi^\beta(\sigma)_{\beta\dot\beta}\bar\theta^{\dot\beta} + i(\bar\theta\bar\theta)\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu\bar\psi^{\dot\alpha}\theta^\beta\psi_\beta$$
$$= (\theta\theta)(\bar\theta\bar\theta)\left[-\frac{1}{4}\varphi^*\partial_\mu\partial^\mu\varphi - \frac{1}{4}\varphi\partial_\mu\partial^\mu\varphi^* + |F|^2\right]$$
$$+ \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)\partial^\mu\varphi\partial_\mu\varphi^* + \frac{i}{2}\epsilon^{\dot\alpha\dot\beta}(\bar\theta\bar\theta)\bar\psi_{\dot\alpha}(\theta\theta)\partial_\mu\psi^\beta(\sigma^\mu)_{\beta\dot\beta} + \frac{i}{2}(\bar\theta\bar\theta)(\theta\theta)\epsilon^{\alpha\beta}(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu\bar\psi^{\dot\alpha}\psi_\beta$$
$$= (\theta\theta)(\bar\theta\bar\theta)\left[-\frac{1}{4}\varphi^*\partial_\mu\partial^\mu\varphi - \frac{1}{4}\varphi\partial_\mu\partial^\mu\varphi^* + |F|^2 + \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi^* \quad \frac{i}{2}\partial_\mu\psi(\sigma^\mu)\bar\psi - \frac{i}{2}\psi(\sigma^\mu)\partial_\mu\bar\psi\right]$$
$$= (\theta\theta)(\bar\theta\bar\theta)\left[|F|^2 + \partial^\mu\varphi\partial_\mu\varphi^* - i\psi(\sigma^\mu)\partial_\mu\bar\psi\right] + \quad \text{total} \quad \text{derivatives}$$

Therefore, the $(\theta\theta)(\bar\theta\bar\theta)$ of K is the corresponding D-term of the superfield.

$$\phi^\dagger\phi|_D = \partial^\mu\phi^*\partial_\mu - i\bar\psi\bar\sigma^\mu\partial_\mu\psi + FF^* \tag{5.7}$$

Again, the F term of the superpotential, W will be included in the Lagrangian. We have, Taylor expansion of $W[\phi]$ around $\phi = \varphi$

$$W(\phi) = W(\varphi) + (\phi - \varphi)\frac{\partial W}{\partial\phi} + \frac{1}{2}(\phi - \varphi)^2\frac{\partial^2 W}{\partial\varphi^2} \tag{5.8}$$

Here, the $(\phi - \varphi)$ term refers to $\theta\theta F$ and $(\phi - \varphi)^2$ refers to $\theta\psi(\theta\psi)$ term. So, by taking the $(\theta\theta)$ term out of this expansion, we get the F terms of $W(\phi)$. Now, assuming

$$W(\phi) = a + \lambda\phi + \frac{m}{2}\phi^2 + \frac{g}{3}\phi^3 \tag{5.9}$$

$$\frac{\partial W}{\partial\phi} = \lambda + m\phi + g\phi^2 \tag{5.10}$$

$$\frac{\partial^2 W}{\partial\phi^2} = m + 2g\theta \tag{5.11}$$

Now, evaluating the $(\theta\theta)$ term from, $W = \frac{m}{2}\phi^2 + \frac{g}{3}\phi^3$

$$\frac{m}{2}\phi^2 = \frac{m}{2}\Big(\varphi + \sqrt{2}\theta\psi + (\theta\theta)F + i(\theta\sigma^\mu\bar\theta)\partial_\mu\varphi - \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\partial_\mu\partial^\mu\varphi - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi\sigma^\mu\bar\theta\Big)$$

$$\Big(\varphi + \sqrt{2}\theta\psi + (\theta\theta)F + i(\theta\sigma^\mu\bar\theta)\partial_\mu\varphi - \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\partial_\mu\partial^\mu\varphi - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi\sigma^\mu\bar\theta\Big)$$

Taking only the terms that will result to $(\theta\theta)$ terms

$$\frac{m}{2}\phi^2 \supset (\varphi + \sqrt{2}\theta\psi + (\theta\theta)F)(\varphi + \sqrt{2}\theta\psi + (\theta\theta)F)$$

$$= \frac{m}{2}\Big((\theta\theta)(\varphi F + F\varphi) + 2\theta^\alpha\psi_\alpha\theta^\beta\psi_\beta\Big)$$

$$= \frac{m}{2}\Big((\theta\theta)(2\varphi F) - 2\theta^\alpha\theta^\beta\psi_\alpha\psi_\beta\Big)$$

$$= \frac{m}{2}\Big((\theta\theta)(2\varphi F) - 2\frac{1}{2}(\theta\theta)\epsilon^{\alpha\beta}\psi\psi\Big)$$

$$= m(\theta\theta)(\varphi F - \frac{1}{2}(\psi\psi))$$

And

$$\frac{1}{3}g\phi^3 \supset \frac{g}{3}(\varphi + \sqrt{2}\theta\psi + (\theta\theta)F)(\varphi + \sqrt{2}\theta\psi + (\theta\theta)F)(\varphi + \sqrt{2}\theta\psi + (\theta\theta)F)$$

Here, we again took the terms that will result to $(\theta\theta)$ terms

$$\supset \frac{g}{3}\Big((\theta\theta)(\varphi^2 F + \varphi F\varphi + F\varphi^2) + 2\varphi(3\theta^\alpha\psi_\alpha\theta^\beta\psi_\beta)\Big)$$

$$\supset \frac{g}{3}\Big((\theta\theta)(\varphi^2 F + \varphi F\varphi + F\varphi^2) - 2\varphi(3\theta^\alpha\theta^\beta\psi_\alpha\psi_\beta)\Big)$$

$$= g(\theta\theta)\big(\varphi^2 - \varphi(\psi\psi)\big)$$

Now,

$$W(\phi) = \frac{1}{2}m\phi^2 + \frac{g}{3}\phi^3$$

$$= m(\theta\theta)\big(\varphi F - \frac{1}{2}(\psi\psi)\big) + g(\theta\theta)\big(\varphi^2 F - \varphi(\psi\psi)\big)$$

$$= (\theta\theta)(m\varphi + g\varphi^2)F - \frac{1}{2}(\theta\theta)(\psi\psi) - g(\theta\theta)(\psi\psi)\varphi$$

$$= (\theta\theta)(m\varphi + g\varphi^2) - \frac{1}{2}(\theta\theta)(\psi\psi)(m + 2g\varphi)$$

$$= (\theta\theta)\frac{\partial W}{\partial\varphi} - \frac{1}{2}(\theta\theta)\frac{\partial^2 W}{\partial\phi^2}(\psi\psi)$$

$$W(\phi)|_D = (\frac{\partial W}{\partial\phi}F + h.c) - (\frac{1}{2}\frac{\partial W^2}{\partial\phi^2} + h.c) \tag{5.12}$$

Using (5.7) and (5.12) in (5.6), we finally, get the Lagrangian.

$$\mathcal{L} = \partial^\mu\phi^*\partial_\mu\phi - i\bar\psi\bar\sigma^\mu\delta_\mu\psi + FF^* + (\frac{\partial W}{\partial\phi}F + h.c) - (\frac{1}{2}\frac{\partial W^2}{\partial\phi^2} + h.c) \tag{5.13}$$

This is known as the Wess-Zumino model.

The part of Lagrangian which depends in the auxiliary field, F

$$\mathcal{L}_{(F)} = FF_* + \frac{\partial W}{\partial \phi}F + \frac{\partial W^*}{\partial \varphi^*}F^* \tag{5.14}$$

This equation says that the field, F does not propagate. So, we obtain,

$$\frac{\delta S_{(F)}}{\delta F} = F^* + \frac{\partial W}{\partial \phi} = 0$$

$$\Rightarrow F^* = -\frac{\partial W}{\partial \phi} \tag{5.15}$$

$$\frac{\delta S_{(F)}}{\delta F^*} = F + \frac{\partial W^*}{\partial \phi^*} = 0$$

$$\Rightarrow F = -\frac{\partial W^*}{\partial \phi^*} \tag{5.16}$$

Replacing F and F^* with the values in (5.15) and (5.16) in (5.14) we find,

$$\mathcal{L} = (-\frac{\partial W}{\partial \varphi})(-\frac{\partial W^*}{\partial \varphi^*}) + (\frac{\partial W}{\partial \varphi})(-\frac{\partial W^*}{\partial \varphi^*}) + (\frac{\partial W^*}{\partial \varphi^*})(-\frac{\partial W^*}{\partial \varphi^*})$$

$$\mathcal{L} = -\left|\frac{\partial W}{\partial \phi}\right|^2$$

$$= -V_{(F)}(\varphi) \tag{5.17}$$

This Lagrangian defines the scalar potential. It is a positive definite scalar potential, $V_{(F)}(\varphi)$.

5.2 Abelian vector superfield Lagrangian

Lagrangian for Chiral superfield describes spin-0 and spin-$^1/_2$ particles. Lagrangian of a supersymmetric Abelian gauge theory will describe spin-1 particles. To construct an Abelian vector superfield, we introduce gauge invariance to Kahler potential under supersymmetry. In general, with supersymmetry, Kahler Potential, $K = \phi^\dagger \phi$ is not invariant under

$$\phi \longrightarrow e^{iq\Lambda}\phi$$
$$\phi^\dagger \phi \longrightarrow \phi^\dagger e^{iq\Lambda - \Lambda^\dagger}\phi$$

for Chiral Λ.

We introduce a field V, such that

$$K = \phi^\dagger e^{2qV}\phi$$
$$V \longrightarrow V - \frac{i}{2}(\Lambda - \Lambda^\dagger)$$

So, under general gauge transformation invariance of K is obtained.

A kinetic term for V with coupling τ is introduced

$$\mathcal{L} = f(\Phi)(W^\alpha W_\alpha)|_F + \text{h.c} \tag{5.18}$$

For general $f(\Phi)$, this term is not renormalizable but when $f(\Phi)$ is a constant, $f = \tau$, it is renormalizable. For the non-renormalizable term, $f(\Phi)$ is known as the gauge kinetic

function. For renormalizable super Q.E.D, $f = \tau = \frac{1}{3}$.

An extra term name "Fayet Illipoulous term" can be added to $\mathcal{L}$, in supersymmetric terms.

$$\begin{aligned} \mathcal{L}_{FI} &= \varepsilon V_D \\ &= \frac{1}{2}\varepsilon D \end{aligned}$$

where, ε is a constant.

In non-Abelian gauge theory, the gauge fields and their corresponding D-term transforms under the gauge group, so the $\mathcal{L}_{FI}$ term can not exist. Due to the chargeless-ness of the gauge fields in U(1) theory, the FI term is invariant. So, it can be said that the FI term exists only for the Abelian gauge theories.

A renormalizable Lagrangian of super QED:

$$\mathcal{L} = (\Phi^\dagger e^{2qV}\Phi)\big|_D + (W(\Phi)\big|_F + \text{h.c}) + \varepsilon V\big|_D \tag{5.19}$$

Now,

$$\begin{aligned} (\Phi^\dagger e^{2qV}\phi)\big|_D &= F^*F + \partial_\mu\phi\partial^\mu\phi^* + i\bar{\psi}\bar{\sigma}^\mu\delta_\mu\psi \\ &+ qV^\mu\big(\bar{\psi}\bar{\sigma}_\mu\psi + i\varphi^*\partial_\mu\varphi - i\varphi\partial_\mu\varphi - i\varphi\partial_\mu\varphi^*\big) \\ &+ \sqrt{2}q(\varphi\bar{\lambda}\bar{\psi} + \varphi^*\lambda\psi) + q(D + qV_\mu V^\mu)|\varphi|^2 \end{aligned}$$

Due to Wess-Zumino Gauge $V^{n\geq 3} = 0$,

$$\therefore (\Phi^\dagger e^{2qV}\Phi)\big|_D = qD|\varphi|^2$$

We take, W^α to be a Chiral, So, $W^\alpha W_\alpha$ has to be scalar field. So, we use only the $(\theta\theta)$ terms to get the F-part of $W^\alpha W_\alpha$

$$\begin{aligned} \frac{1}{4}W^\alpha W_\alpha|_F &= \frac{1}{4}(\theta\theta)(-2i\lambda^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar{\lambda}^{\dot\alpha} + D^2) - \frac{1}{16}(\sigma^\mu\bar{\sigma}^\nu\theta)^\alpha(\sigma^\rho\bar{\sigma}^\lambda\theta)_\alpha F_{\mu\nu}F_{\rho\lambda} \\ &+ \frac{i}{4}D\theta^\alpha(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu} \end{aligned}$$

First, we evaluate $-\frac{1}{16}(\sigma^\mu\bar{\sigma}^\nu\theta)^\alpha(\sigma^\rho\bar{\sigma}^\lambda\theta)_\alpha F_{\mu\nu}F_{\rho\lambda}$ term,

$$\begin{aligned} -\frac{1}{16}(\sigma^\mu\bar{\sigma}^\nu\theta)^\alpha(\sigma^\rho\bar{\sigma}^\lambda\theta)_\alpha F_{\mu\nu}F_{\rho\lambda} &= -\frac{1}{16}\epsilon_{\alpha\beta}(\sigma^\mu\bar{\sigma}^\nu\theta)^\alpha(\sigma^\rho\bar{\sigma}^\lambda\theta)^\beta F_{\mu\nu}F_{\rho\lambda} \\ &= -\frac{1}{16}\epsilon^{\alpha\beta}(\sigma^\mu)_{\alpha\dot\alpha}(\bar{\sigma}^\nu)^{\dot\alpha\gamma}\theta_\gamma(\sigma^\rho)_{\beta\dot\beta}(\bar{\sigma}^\lambda)^{\dot\beta\delta}\theta_\delta F_{\mu\nu}F_{\rho\lambda} \\ &= -\frac{1}{32}(\theta\theta)\text{Tr}\{\sigma^\mu\bar{\sigma}^\nu\sigma^\lambda\bar{\sigma}^\rho\}F_{\mu\nu}F^{\mu\nu} \end{aligned}$$

Here,

$$\begin{aligned} \text{Tr}\{\sigma^\mu\bar{\sigma}^\nu\sigma^\lambda\bar{\sigma}^\rho\} &= 2i\epsilon^{\mu\nu\lambda\rho} + 2\eta^{\mu\nu}\eta^{\lambda\rho} - 2\eta^{\mu\lambda}\eta^{\nu\rho} + 2\eta^{\mu\rho}\eta^{\nu\lambda} \\ \text{So}, -\frac{1}{16}(\sigma^\mu\bar{\sigma}^\nu\theta)^\alpha(\sigma^\rho\bar{\sigma}^\lambda\theta)_\alpha F_{\mu\nu}F_{\rho\lambda} &= -\frac{1}{16}(\theta\theta)\epsilon^{\mu\lambda\tau\rho}F_{\mu\lambda}F_{\rho\tau} - \frac{1}{8}(\theta\theta)F_{\mu\nu}F^{\mu\nu} \end{aligned}$$

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Then, we evaluate the term $\frac{i}{4}D\theta^\alpha(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu}$,

$$
\begin{aligned}
\frac{i}{4}D\theta^\alpha(\sigma^\mu\bar{\sigma}^\nu\theta)_\alpha F_{\mu\nu} &= \frac{i}{4}DF_{\mu\nu}\theta^\alpha\theta^\gamma(\sigma^\mu)_{\alpha\dot\alpha}(\bar{\sigma}^\nu)^{\dot\alpha\beta}\epsilon_{\beta\gamma}F_{\mu\nu} \\
&= -\frac{i}{8}DF_{\mu\nu}(\theta\theta)\epsilon^{\alpha\gamma}(\sigma^\mu)_{\alpha\dot\alpha}(\bar{\sigma}^\nu)^{\dot\alpha\beta}\epsilon_{\beta\gamma} \\
&= \frac{i}{8}DF_{\mu\nu}(\theta\theta)\epsilon^{\alpha\gamma}(\sigma^\mu)_{\alpha\dot\alpha}(\bar{\sigma}^\nu)^{\dot\alpha\beta}\epsilon_{\beta\gamma} \\
&= \frac{1}{8}DF_{\mu\nu}(\theta\theta)(\sigma^\mu)_{\alpha\dot\alpha}(\bar{\alpha}^\nu)^{\alpha\dot\alpha} \\
&= \frac{i}{4}DF_{\mu\nu}(\theta\theta)\eta^{\mu\nu} \\
&= 0
\end{aligned}
$$

Now, rewriting $\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}$ as $\overline{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda}$, we obtain,

$$
\frac{1}{4}W_\alpha W^\alpha|_F = -\frac{i}{2}\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{4}D^2 - \frac{1}{8}F_{\mu\nu}F^{\mu\nu} + \frac{i}{8}\overline{F}_{\mu\nu}F^{\mu\nu} \tag{5.20}
$$

If, $f(\Phi)$ is real, then the term $\overline{F}_{\mu\nu}$ vanishes otherwise, it becomes a total derivative. So, for a Q.E.D choice, $f = \frac{1}{4}$, the kinetic terms for the vector superfields are given by

$$
\begin{aligned}
\mathcal{L}_{kin} &= \frac{1}{4}W^\alpha W_\alpha\big|_F + \quad \text{h.c} \tag{5.21} \\
&= \frac{1}{2}D^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\bar{\lambda} \tag{5.22}
\end{aligned}
$$

With the FI contribution $\varepsilon V\big|_D = \frac{1}{2}\varepsilon D$, the collection of the D terms in $\mathcal{L}$ is

$$
\mathcal{L}_{(D)} = qD|\varphi|^2 + \frac{1}{2}D^2 + \frac{1}{2}\varepsilon D \tag{5.23}
$$

will result to

$$
\begin{aligned}
\frac{\delta\mathcal{S}}{\delta D} &= 0 \\
\Rightarrow D &= -\frac{\varepsilon}{2} - q|\varphi|^2
\end{aligned}
$$

Substituting those back into (5.23) we obtain,

$$
\begin{aligned}
\mathcal{L}_{(D)} &= -\frac{1}{8}\left(\varepsilon + 2q|\varphi|^2\right)^2 \\
&=: -V_{(D)}(\varphi) \tag{5.24}
\end{aligned}
$$

a positive semi-definite scalar potential $V_{(D)}(\varphi)$. So, this with the potential for Chiral superfield from previous section produces the total potential.

$$
\begin{aligned}
V(\varphi) &= V_{(F)}(\varphi) + V_{(D)}(\varphi) \\
&= \left|\frac{\partial W}{\partial\varphi}\right|^2 + \frac{1}{8}\left(\varepsilon + 2q|\varphi|^2\right)^2 \tag{5.25}
\end{aligned}
$$

5.3 Action as a superspace integral

Generally, in a non-supersymmetric physics, the relationship between $\mathcal{S}$ and $\mathcal{L}$ is

$$\mathcal{S} = \int d^4x \mathcal{L} \tag{5.26}$$

From the definition of Grassmann Variables,

$$\int d^2\theta(\theta\theta) = 1 \tag{5.27}$$

$$\int d^4\theta(\theta\theta)(\bar{\theta}\bar{\theta}) = 1 \tag{5.28}$$

So (5.19) can be written as

$$\mathcal{L} = \int d^4\theta \quad K + \left(\int d^2\theta \quad W + \quad h.c \right) + \left(\int d^2 \quad \theta W^\alpha W_\alpha + \quad h.c \right) \tag{5.29}$$

So, the most general action for a supersymmetric Lagrangian can be expressed as

$$\mathcal{S}\left[K(\Phi_i^\dagger, e^{2qV}, \Phi_i), W(\Phi_i), f(\Phi_i), \varepsilon \right] = \int d^4x \quad \int d^4 \quad \theta(K + \varepsilon V)$$
$$\int d^4x \int d^2\theta(W + fW^\alpha W_\alpha + \quad h.c) \tag{5.30}$$

5.4 Non-Abelian field strength superfield Lagrangian

In section 4.5.3 vector superfield with non-Abelian field strength vector is discussed and the supersymmetric field strength for Wess-Zumino gauge is described by equation (4.158). In this section, we are going to obtain the Lagrangian for a supersymmetric non-Abelian gauge theory.

A non-Abelian field strength adds a covariant derivative on a gaugino, λ.

The general expression for the gauge field strength is described by

$$W_\alpha = T^a W_\alpha^a \tag{5.31}$$

$$\mathrm{Tr}(T^a T^b) = C\delta^{ab} \tag{5.32}$$

To introduce these terms, the gauge kinetic term is normalized by

$$\frac{1}{16q^2C} \left[\mathrm{Tr}(W^\alpha W_\alpha) \right]_{\theta\theta} + h.c$$

For the non-Abelian field strength Lagrangian, the Fayet-Illiopoulos term from the Abelian case is either gauge invariant or zero. So this term is omitted for this case. Thus, the Lagrangian for a supersymmetric non-Abelian gauge theory

$$\mathcal{L} = \left[\phi_i^\dagger e_{ij}^{2qV} \phi_j \right]_{\theta\theta\bar{\theta}\bar{\theta}} + \left(\left[W(\phi_i) + \frac{1}{16q^2} W^{a\alpha} W_\alpha^a \right]_{\theta\theta} + h.c \right) \tag{5.33}$$

Using this in the Lagrangian for Abelian field strength vector, we may find a general expression for the Lagrangian of a non-Abelian field strength vector superfield.

$$
\begin{aligned}
\mathcal{L} = {}& (D^\mu \phi_i)^* D_\mu \phi_i + \bar{\psi}_i i\bar{\sigma}^\mu D_\mu \psi_i + |F|^2 \\
& - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\lambda}^a i\bar{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^a D^a \\
& - \left(\frac{\partial W}{\partial \phi_i} F_i \frac{1}{2} \frac{\partial^2 W}{\partial \phi_i \partial \phi_j}(\phi)\psi_i\psi_j + h.c \right) \\
& + qD^a \phi_i^*(T^a)_{ij}\phi_j + iq\sqrt{2}\phi_i^*\lambda^a(T^a)_{ij}\psi_j - iq\sqrt{2}\bar{\psi}_i\bar{\lambda}^a(T^a)_{ij}\phi_j
\end{aligned}
\tag{5.34}
$$

where,

$$
\begin{aligned}
D_\mu \phi_i &= \partial_\mu \psi_i + iqv^a_\mu(T^a)_{ij}\phi_j & (5.35) \\
D_\mu \psi_i &= \partial_\mu \psi_i + iqv^a_\mu(T^a)_{ij}\psi_j & (5.36) \\
D_\mu \lambda^a &= \partial_\mu \lambda^a - qf_{abc}v^b_\mu \lambda^c & (5.37) \\
F^a_{\mu\nu} &= \partial_\mu v^a_\nu - \partial_\nu v^a_{mu} - qf_{abc}v^b_\mu v^c_\nu & (5.38)
\end{aligned}
$$

The potential $V = V(\phi_i, \phi_i^*)$ is the sum of the F-terms and the D-terms of the superfield.

$$
\begin{aligned}
V_F &= \sum_i |F_i|^2 \\
&= \sum_i \left|\frac{\partial W}{\partial \phi_i}\right|^2 & (5.39) \\
V_D &= \sum_a \frac{1}{2} D^a D^a \\
&= \frac{q^2}{2}\left(\phi_i^*(T^a)_{ij}\phi_j\right)^2 & (5.40)
\end{aligned}
$$
$$\tag{5.41}$$

So, the potential, V is expressed by

$$
V(\phi_i, \phi_i^*) = \sum_i \left|\frac{\partial W}{\partial \phi_i}\right|^2 + \frac{q^2}{2}\left(\phi_i^*(T^a)_{ij}\phi_j\right)^2
\tag{5.42}
$$

By integrating the auxiliary fields, we get,

$$
\begin{aligned}
F_i^* &= \frac{\partial W}{\partial \phi_i} & (5.43) \\
D^a &= -q\phi^*(T^a)_{ij}\phi_j & (5.44)
\end{aligned}
$$

and using (5.42) we get the final result for the Lagrangian for non-Abelian field strength vector superfield.

$$
\begin{aligned}
\mathcal{L} = {}& (D^\mu \phi_i)^* D_\mu \phi_i + \bar{\psi}_i i\bar{\sigma}^\mu D_\mu \psi^i - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\lambda}^a i\bar{\sigma}^\mu D_\mu \lambda^a \\
& - \left(\frac{1}{2} \frac{\partial^2 W}{\partial \phi_i \partial \phi_j}\psi_i\psi_j - iq\sqrt{2}\phi_i^*\lambda^a(T^a)_{ij}\psi_j + h.c \right) - V
\end{aligned}
\tag{5.45}
$$

Chapter 6

Supersymmetry Breaking

At energies of order 10^2 GeV or lower, mass degeneracy in the elementary particle does not occur. So, it can be said that for supersymmetry to be realized in nature, it has to be broken in low energy. At some scale M_s such that $E < M_s$, supersymmetry is broken and the theory only behaves symmetrically when $E > M_s$. Supersymmetry can be broken in two ways.

Spontaneous SUSY breaking: The theory is supersymmetric with a scalar potential admitting supersymmetry breaking vacua. In such a vacua, an energy scale determined by a non-vanishing VEV of order M_s is introduced. This is the scale of SUSY breaking. In standard model, $M_{ew} \approx 10^3$ GeV, defines the basic scale of mass for the particles of the standard model. Through Yukawa couplings, the electroweak gauge bosons and the matter fields obtain their mass from this symmetry breaking.

Explicit SUSY breaking: The Lagrangian may contain some terms which do not manifest supersymmetry. So, to preserve the supersymmetric theories, these terms has positive mass dimension.

Under finite and infinitesimal group elements, the fields φ_i of gauge theories transform as

$$\varphi_i \quad \longleftrightarrow \quad \left(e^{i\alpha^a T^a}\right)_i{}^j \varphi_j \tag{6.1}$$

$$\delta\varphi_i \quad = \quad i\alpha^a (T^a)_i{}^j \varphi_j \tag{6.2}$$

If the vacuum state (φ_{vac}), transforms in a non-trivial way, i.e

$$(\alpha^a T^a)_i{}^j (\varphi_{vac})_j \neq 0 \tag{6.3}$$

the gauge symmetry is broken.

Let, $\varphi = Pe^{i\mathbb{\sqsubset}}$ in complex polar coordinates of $U(1)$, then infinitesimally

$$\delta\varphi \quad = \quad i\alpha\varphi \tag{6.4}$$

$$\Rightarrow \delta P \quad = \quad 0 \text{ and} \tag{6.5}$$

$$\delta\mathfrak{v} \quad = \quad \alpha \tag{6.6}$$

$\delta\mathfrak{v} = \alpha$ corresponds to a Goldstone boson. Similarly, SUSY breaks when the vacuum state $|vac\rangle$ satisfies

$$Q_\alpha |vac\rangle \neq 0 \tag{6.7}$$

When the anti-commutation relation $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2(\sigma^\mu)_{\alpha\dot\beta}P_\mu$ is contracted with $(\bar{\sigma}^\nu)^{\dot\beta\alpha}$, we get

$$
\begin{aligned}
(\bar{\sigma}^\nu)^{\dot\beta\alpha}\{Q_\alpha, \bar{Q}_{\dot\beta}\} &= 2(\bar{\sigma}^\nu)^{\dot\beta\alpha}2(\sigma^\mu)_{\alpha\dot\beta}P_\mu \\
&= 4\eta^{\mu\nu}P_\mu \\
&= 4P^\nu
\end{aligned}
\tag{6.8}
$$

For $\nu = 0$, $\bar{\sigma}^0 = \mathbb{1}$ and

$$
\begin{aligned}
(\bar{\sigma}^0)^{\dot\beta\alpha}\{Q_\alpha, \bar{Q}_{\dot\beta}\} &= \sum_{\alpha=1}^{2}(Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha) \\
&= 4P^0 \tag{6.9}\\
&= 4E \tag{6.10}
\end{aligned}
$$

This implies that,

- As $Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha$ is positive definite, $E \geq 0$ for any state.

- $\langle vac|Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha|vac\rangle \geq 0$, so the energy, E is strictly positive. In broken SUSY, $E \geq 0$.

6.1 Vacua in supersymmetric theories

The vacuum energy is zero if and only if the vacuum preserves supersymmetry. So, non-supersymmetric vacua corresponds to minima of the potential which are not zero. So, the SUSY is broken on positive energy vacua.

In a SUSY-Gauge theory four possible states are possible.

1. Both gauge symmetry and supersymmetry are broken at the minima.

2. Both gauge symmetry and supersymmetry are preserved at the minima.

3. The minima preserves the gauge symmetry and breaks SUSY.

4. The minima preserves the SUSY and breaks gauge symmetry.

If supersymmetric vacua is present then it has to be the global minima of the potential.

Supersymmetry vacua is described by all possible set of scalar field VEVs satisfying the D-term and F-term equations to be zero.

$$
\bar{F}^i(\phi) = 0 \qquad D^a(\bar{\phi}, \phi) = 0
\tag{6.11}
$$

Supersymmetry breaks for a set of VEVs, where equation (6.11) does not hold and the minima of the potential, $V_{min} \geq 0$.

On a supersymmetric vacua the supersymmetry variations of the fermion field vanishes. Due to Lorentz invariance, on a vacuum any fields except for the scalar field, VEV and its

derivative vanishes. Applying the laws for the transformations of the field components of chiral and vector superfield, we get the following equations for the vacuum state,

$$\delta\langle\phi^i\rangle \quad = \quad 0 \tag{6.12}$$
$$\delta\langle F^i\rangle \quad = \quad 0 \tag{6.13}$$
$$\delta\langle\psi^i_\alpha\rangle \quad \sim \quad \epsilon_\alpha\langle F_i\rangle \tag{6.14}$$
$$\text{and} \tag{6.15}$$
$$\delta\langle F^a_{\mu\nu}\rangle \quad = \quad 0 \tag{6.16}$$
$$\delta\langle D^a\rangle \quad = \quad 0 \tag{6.17}$$
$$\delta\langle\lambda^a_\alpha\rangle \quad \sim \quad \epsilon_\alpha\langle D^a\rangle \tag{6.18}$$

In a generic vacuum, the supersymmetric variations of the fermions is proportional to F and D-terms. According to definition, a supersymmetric vacuum state is SUSY invariant. So, from the above equation it can be implied that the variations of fermions being equivalent to the F and D-terms is zero.

6.2 The Goldstone theorem and the goldstino

According to Goldstone theorem, when a global symmetry is spontaneously broken, a massless mode is present in the spectrum, called the Goldstone field. This field has quantum numbers related to the broken symmetry. As supersymmetry is a fermionic symmetry, the Goldstone field is going to be a spin-$\frac{1}{2}$ Majorana fermion. This field is called the goldstino.

6.3 F-term breaking

To explain the F-term breaking, we consider a case of chiral superfield. The most general renormalizable Lagrangian for this case would be

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta}\bar{\Phi}_i\Phi^i + \int d^2\theta W(\Phi^i) + \int d^2\bar{\theta}\bar{W}(\bar{\Phi}_i) \tag{6.19}$$

where,

$$W(\Phi^i) = a_i\Phi^i + \frac{1}{2}m_{ij}\Phi^i\Phi^j + \frac{1}{3}g_{ijk}\Phi^i\Phi^j\Phi^k \tag{6.20}$$

The equation of motions for the auxiliary fields can be written as

$$\bar{F}_i(\phi) \quad = \quad \frac{\partial W}{\partial\phi^i}$$
$$= \quad a_i + m_{ij}\phi^j + g_{ijk}\phi^j\phi^k \tag{6.21}$$

The potential can be written as

$$V(\phi,\bar{\phi}) = \sum_i \left|a_i + m_{ij}\phi^j + g_{ijk}\phi^j\phi^k\right|^2 \tag{6.22}$$

The transformation laws under supersymmetry for components of a chiral superfield, ϕ are

$$\delta\varphi \quad = \quad \sqrt{2}\epsilon\psi \tag{6.23}$$
$$\delta\psi \quad = \quad \sqrt{2}\epsilon F + i\sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu\varphi \tag{6.24}$$
$$\delta F \quad = \quad i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi \tag{6.25}$$

So, for a supersymmetry to be broken one of $\delta\varphi$, $\delta\psi$, $\delta F \neq 0$. But to preserve Lorentz invariance, we must have

$$\langle\psi\rangle = \langle\partial_\mu\varphi\rangle = 0 \tag{6.26}$$

as they both transform under some Lorentz group. So, the supersymmetry breaking condition is

$$\cancel{SUSY} \Leftrightarrow \langle F\rangle \neq 0 \tag{6.27}$$

Supersymmetry breaks if there is a set of VEVs for which all F-terms will not vanish. This means that for a supersymmetry to be broken, it is necessary to have some a_i which has non-zero values.

As only the fermionic term of the superfield Φ will change

$$\delta\varphi = \delta F = 0 \tag{6.28}$$
$$\delta\psi = \sqrt{2}\epsilon\langle F\rangle \neq 0 \tag{6.29}$$

Here, ψ is a Goldstone fermion or the goldstino.

The F-term of the scalar potential is given by

$$V_{(F)} = (K^{-1})_i{}^j \frac{\partial W}{\partial\phi^i}\frac{\partial\bar{W}}{\partial\bar{\phi}_j} \tag{6.30}$$

So, supersymmetry breaking will happen only for a positive vacuum expectation value.

$$SUSY \Leftrightarrow \langle V_{(F)}\rangle > 0 \tag{6.31}$$

6.4   D-term breaking

In a generic theory with both chiral and vector superfields, if the F-I term does not exist then the supersymmetry breaking is manifest due to the F-term dynamics. If the F-term goes to zero, the D-terms can be set to zero by using global gauge invariance. In order to consider a case where D-term breaking occurs without any influence of F-term, we consider Abelian gauge factor where FI terms are included.

In this case, we will consider two massive chiral superfield with opposite charge coupled to a single U(1) factor, which includes the F-I term in the Lagrangian.

$$\mathcal{L} = \frac{1}{32\pi}\text{Im}\left(\tau\int d^2\theta W^\alpha W_\alpha\right) + \int d^2\theta d^2\bar{\theta}\left(\xi V + \bar{\Phi}_+ e^{2qV}\bar{\Phi}_+ + \bar{\Phi}_- e^{-2qV}\bar{\Phi}_-\right)$$
$$+ m\int d^2\theta\Phi_+\Phi_- + h.c \tag{6.32}$$

Under gauge transformation, the two chiral superfields transform as

$$\Phi_\pm \to e^{\pm iq\Lambda}\Phi_\pm \tag{6.33}$$

The auxiliary fields have the equation of motion

$$\bar{F}_\pm = m\phi_\mp \tag{6.34}$$
$$D = -\frac{1}{2}\left[\xi + 2q\left(|\phi_+|^2 - |\phi_-|^2\right)\right] \tag{6.35}$$

Due to the F-I parameter, ξ all the auxiliary field equations can not be satisfied. This results to broken symmetry.

The scalar potential,

$$V = \frac{1}{8}\left[\xi + 2q\left(|\phi_+|^2 - |\phi_-|^2\right)\right]^2 + m^2\left(|\phi_+|^2 + |\phi_-|^2\right) \tag{6.36}$$

$$= \frac{1}{8}\xi^2 + \left(m^2 - \frac{1}{2}q\xi\right)|\phi_-|^2 + \left(m^2 + \frac{1}{2}q\xi\right)|\phi_+|^2 + \frac{1}{2}e^2\left(|\phi_+|^2 - |\phi_-|^2\right)^2 \tag{6.37}$$

The vacuum structure and the low energy dynamics depends on the sign of $\left(m^2 - \frac{1}{2}q\xi\right)$.

- **$m^2 > \frac{1}{2}q\xi$**   All terms in potential are positive and the minimum of V is at $\langle\phi_\pm\rangle = 0$, where, $V = \frac{1}{8}\xi^2$. Here, supersymmetry is broken and gauge symmetry is preserved. The only auxiliary field which gets a VEV is D. Thus, here a pure D-term breaking occurs.

  The two fermions which belong to the two chiral superfield have mass, m. So, they can form a massive Dirac fermion. The two scalar fields, ϕ_+ has mass $\sqrt{m + \frac{1}{2q\xi}}$ and ϕ_- have mass $\sqrt{m - \frac{1}{2q\xi}}$. Due to gauge symmetry preservation, the photon A_μ remains massless and as the supersymmetry is broken λ remain massless which is the goldstino.

- **$m^2 < \frac{1}{2}q\xi$**   As the sign of the mass term for ϕ_- is negative, the minimum of the potential is at $\langle\phi_+\rangle = 0$ and $\langle\phi_-\rangle = \sqrt{\frac{\xi}{2q} - \frac{m^2}{q^2}} \equiv h$. The minimum potential here is $V = \frac{1}{8}\xi^2 - \frac{1}{2}e^2 h^4$. So, both supersymmetry and gauge symmetry are broken and both the D-term and F-term are also broken.

  Due to the Yukawa couplings, three fermions mix and form a goldstino and two other particles with equal mass of $m_{\bar{\psi}_\pm} = \sqrt{q\xi - m^2}$.

Chapter 7

Supersymmetry in High Energy Physics

The standard model of QFT is used to describe all known particles and interactions in four-dimensional space at low energies. To form a model for which the particles can be described in high energy physics, a supersymmetric extension of the standard model is introduced. This is the minimally supersymmetric extension of the standard model, or in short called the MSSM. As at low energy, supersymmetry is not observed, we say that if supersymmetry is present in nature, it must be broken at the energy scale of 1 TeV.

7.1 The MSSM

In standard model, matter is chiral. So, the Left-handed chiralities and the right handed chiralities transform under different representations of the gauge group. The field of standard model includes spin-0 Higgs field, spin-1/2 quark and lepton fields. In MSSM, these fields are assigned to Chiral and gauge supermultiplets and will generate mass by Higgs interactions and SUSY-breaking.

Under supersymmetry transformation, the $SU(3)_c$, $SU(2)_L$, $U(1)$ do not change their quantum numbers. This implies that the SM fields and their assigned partners must have the same quantum numbers as $SU(3)_c \times SU(2)_L \times U(1)$. So, in MSSM, we have vector fields transforming under $SU(3)_c \times SU(2)_L \times U(1)_Y$ and chiral superfields which represent

	Left-Handed	Right-handed	
Quarks	$Q_i = (3, 2, -^1/_6)$	$\bar{U}_i^c = (\bar{3}, 1, {}^2/_3)$	$d_i^c = (\bar{3}, 1, -^1/_3)$
Leptons	$L_i = (1, 2, {}^1/_2)$	$\bar{q}_i^c = (1, 1, -1)$	$\bar{v}_i^c = (1, 1, 0)$

The vector multiplets include new fermions named gauginos and higgsinos

$$W^{\pm} = (A_w^{\pm}, \lambda_W^{\pm}, D^{\pm}) \tag{7.1}$$

$$W^0 = (A_w^0, \lambda_W^0, D^0) \tag{7.2}$$

$$A = (A, \lambda, D) \tag{7.3}$$

presence of gaugino do not effect the cancellation of gauge anomalies due to their vectorial coupling. But a single Higgsino running in a triangle loop contributes to $Y_H{}^3 = +1^3$ which

provides the gauge anomaly. To make this term vanish a second Higgs doublet is introduced such that

$$Y_{H_1}{}^3 + Y_{H_2}{}^3 = (+1)^3 + (-1)^3 = 0 \tag{7.4}$$

Finally,

Higgs	$SU(3) \times SU(2) \times U(1)$
H_1	$(1, 2, {}^1/_2)$
H_2	$(1, 2, -{}^1/_2)$

7.2 Interaction

In standard model, the three gauge couplings running in its spectrum do not meet at a single point at higher energies. But, in MSSM, these three different couplings meet at a single point, at large E. This provides a scope for supersymmetric gauge coupling unification.

To avoid the breaking of charge and color, we take the F-I term, ξ to be zero. We also need the Higgs to break $SU(2) \times U(1)_{(Y)} \longrightarrow U(1)_{em}$. The standard Yukawa coupling should give mass to up-quarks, down-quarks and leptons. So, the superpotential, W is given by

$$W = y_1 Q H_2 \bar{u}^c + y_2 Q H_1 \bar{d}^c + y_3 L H_1 \bar{q}^c + \mu H_1 H_2 + W_{\cancel{BL}} \tag{7.5}$$

The first three terms correspond to standard Yukawa couplings and the fourth term is a mass term for the two Higgs field.

$$W_{\cancel{BL}} = \lambda_1 L L \bar{q}^c + \lambda_2 L Q \bar{d}^c + \lambda_1 \bar{u}^c \bar{d}^c \bar{d}^c + \mu' L H_2 \tag{7.6}$$

The $\cancel{BL}$ terms break in baryon or lepton number. Standard models preserve baryon and lepton numbers but due to the couplings of $\cancel{BL}$ term this conservation is violated. So, to forbid these coupling, another symmetry, R-parity is imposed. This is defined as

$$
\begin{aligned}
R \quad &:= \quad (-1)^{3(B-L)2s} \\
&= \quad
\begin{cases}
+1 : & \text{all observed particle} \\
-1 : & \text{superparticles}
\end{cases}
\end{aligned} \tag{7.7}
$$

The $W_{\cancel{BL}}$ terms are forbidden by this.

7.3 Supersymmetry breaking in MSSM

If supersymmetry is spontaneously broken in MSSM, it follows the condition

$$
\begin{aligned}
S\mathrm{Tr}M^2 \quad &= \quad \mathrm{Tr}(-1)^F M^2 \\
&= \quad \mathrm{Tr}M^2_{scalar} - \mathrm{Tr}M^2_{fermions}
\end{aligned}
$$

The particles in MSSM couples very weakly than the SM and the effects of SUSY breaking is weakly mediated.

The low energy effective Lagrangian in the observable sectors develops the theory of the soft

supersymmetry breaking.

This Lagrangian has contribution

$$\mathcal{L} = \mathcal{L}_{SUSY} + \mathcal{L}_{\cancel{SUSY}}$$

Here,

$$\mathcal{L}_{\cancel{SUSY}} = \underbrace{m_0{}^2 \varphi^* \varphi}_{\text{scalar masses}} + (\underbrace{M_\lambda \lambda \lambda}_{\text{gaugino masses}} + h.c) + (A\varphi^3 + h.c) \tag{7.8}$$

M_λ, $m_0{}^2$, and A are the soft breaking terms that determine the amount by which supersymmetry is expected to be broken.

7.4 Comparison of QCD and SYM in terms of confinement and mass Gap

In a non-Abelian gauge theory, strong coupling appears at low energy when the group remains unbroken. Due to this strong coupling, confinement and mass-gap appears in this theory.

7.4.1 QCD, the theory of strong interaction

Confinement: The value of coupling increases as the energy decreases. Thus, at very low energy, the strong energies become so strong that the quarks can not separate. Due to the confinement, at strong coupling, quarks and anti-quarks are bind into pairs and these color singlet bilinears form a condensate which fill the vacuum.

$$\langle q_L^i q_R^j \rangle = \triangle \delta^{ij} \tag{7.9}$$

having $\triangle \sim \Lambda_{QCD}^3$.

Generation of mass-gap: $\triangle \delta^{ij}$ is invariant under a diagonal SU(3). SU(3) is a subgroup of the original $SU(3)_R \times SU(3)_L$ group. Therefore, the global symmetry group G_F has a chiral supersymmetry breaking.

$$SU(3)_L \times SU(3)_R \times U(1)_B \longrightarrow SU(3)_D \times U(1)_B$$

The quark condensates break eight global symmetries. So, eight Goldstone boson is expected. By experiment, it is observed that the eight goldstone bosons correspond to eight pseudoscalar mesons named pions. These are $\pi^{0,\pm}$, $k^{0,-}$, $\bar{k}^{0,+}$, η. We have,

$$\pi^+ = u\bar{d} \qquad \pi^- = d\bar{u} \qquad \pi^0 = d\bar{d} - u\bar{u} \qquad k^0 = s\bar{d}$$
$$k^- = \bar{u}s \qquad \bar{k}^0 = \bar{s}d \qquad \bar{K}^+ = \bar{s}u \qquad \eta = u\bar{u} + d\bar{d} - 2s\bar{s}$$

If $U(1)_A$ were not anomalous, a ninth meson, η' meson would appear. This meson would result to a shift in the phase of condensate. As the $\mathbb{Z}_{2F}$ symmetry is broken into $\mathbb{Z}_2$, massive quarks appear. Due to this massive quark, a mass gap is generated in QCD.

7.4.2 SYM, supersymmetric gauge interactions without matter fields

In SYM, we shall observe strict confinement, a mass gap. Due to the anomaly of $U(1)_A$ a massive η' particle similar to QCD also appears due to the a chiral symmetry breaking occurring same as the case of QCD. SYM has multiple isolated vacua.

The structure of the (on- shell) SYM Lagrangian

$$\mathcal{L}_{SYM} = -\text{Tr}\Big[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\lambda}\slashed{D}\lambda\Big] \tag{7.10}$$

Strict confinement: The gauginos transform in the adjoint representation that is in the same N-ality class of the singlet representation. So, the gauginos do not break the flux tubes. Due to this, the QCD quarks has no effect on the confinement and SYM can enjoy strict confinement.

Mass-gap: Gauginos have R-charge equal to one. So, the $U(1)_R$ symmetry can be broken to $\mathbb{Z}_{2n}$ at the quantum level. Due to this property and the anomaly of $U(1)_R$ symmetry, the symmetry will be similar to the symmetry of QCD. As in the vacuum, the gaugino bilinears get a non-vanishing VEV, SYM enjoys chiral symmetry breaking. So we have,

$$\langle\lambda\lambda\rangle \sim \Lambda^3 e^{w\pi i k/N}, \qquad k = 0, 1, 2...N - 1 \tag{7.11}$$

which breaks $\mathbb{Z}_{2n} \to \mathbb{Z}_2$. Thus, we get N isolated vacua. All of these vacua are $\mathbb{Z}_2$ symmetric and related by $\mathbb{Z}_N$ rotations. All of these vacua are related by $\mathbb{Z}_N$ rotations.

Due to the symmetry breaking η' meson appears. As the η' is the phase of the condensate, mass-gap is generated in SYM. However, due to the isolated vacua the mass-gap is dynamical.

Chapter 8

Starobinsky Model of Cosmic Inflation

The general linear representation for cosmic inflation is given by the action

$$S = \int d^4x \sqrt{-g} R \tag{8.1}$$

The chaotic inflation model includes an arbitrary function of R, to describe the non-linear representation by the action

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} f(R) \tag{8.2}$$

For the Starobinsky model of cosmic inflation, this function is defined as

$$f(R) = R - \frac{R^2}{6M^2} \tag{8.3}$$

So, the Starobinsky model of cosmic inflation is described by

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(M_p{}^2 R + \frac{1}{6M^2} R^2 \right) \tag{8.4}$$

This theory includes a graviton and a scalar degree of freedom.
The linear representation of the action of this model which includes the scalar degree of freedom can be described by

$$S = \int d^4x \sqrt{-g} \left(\frac{M_p{}^2}{2} R + \frac{1}{M} R\psi - 3\psi^2 \right) \tag{8.5}$$

This is also the linear expression of the action for Starobinsky model. The equivalent scalar field version of the Starobinsky model can be found by conformal transformation in the Einstein frame

$$S = \int d^4x \sqrt{-g} \left[\frac{M_p{}^2}{2} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{3}{4} M_p^4 M^2 (1 - exp(-\sqrt{\tfrac{2}{3}} \phi / M_p)^2 \right] \tag{8.6}$$

This represents the Starobinsky model where the extra scalar degree of freedom is manifest. The inflaton here, is the scalar, spin-0 metric. In this theory, the scalar potential is

$$V(\phi) = \frac{3}{4} M^2 M_p^4 (1 - exp(-\sqrt{\tfrac{2}{3}} \phi / M_p)^2 \tag{8.7}$$

During inflation (at large values of ϕ) the dynamic is dominated by this vacuum energy,

$$V(\phi) = \frac{3}{4}M_p{}^2 M^2 \tag{8.8}$$

This results to scale invariance. For finite values of ϕ, the scale invariance is not exact. So, this violation is measured by the slow-roll parameters.
From 8.6, we get inflation with scalar tilt and tensor to tensor ratio

$$n_s - 1 ~\sim~ -\frac{2}{N} \tag{8.9}$$

$$r ~\sim~ \frac{12}{N^2} \tag{8.10}$$

The additional $\frac{1}{N}$ in r with respect to scalar tilt shows the consistency of this theory with Planck data by predicting the tiny amount of gravitational wave.

8.1 Higgs inflation as Starobinsky model

The Higgs inflation can be described by the form

$$S_{HI} = \int d^4x \sqrt{-g}\Big[\frac{M_p{}^2}{2}R + \xi H^\dagger H R - \partial_\mu H^\dagger \partial^\mu H - \lambda(H^\dagger H - V^2)^2\Big] \tag{8.11}$$

where H is the SM Higgs doublet and v is its vacuum expectation value.
In the unitary gauge, $H = \frac{h}{\sqrt{2}}$ and for $h^2 >> v^2$, inflation exists where, $\xi^2\lambda \sim 10$. This can be described by the action

$$S_{HI} = \int d^4x \sqrt{-g}\Big(\frac{M_p{}^2}{2}R + \frac{1}{2}\xi h^2 R - \frac{\lambda}{4}h^4\Big) \tag{8.12}$$

Here, the Higgs field becomes an auxiliary field.

$$\xi h R = \lambda h^3 ~=~ 0$$
$$\text{and} ~~ h^2 ~=~ \frac{\xi R}{\lambda} \tag{8.13}$$

In the Starobinsky model, if we take

$$M^2 = \frac{\lambda}{3\xi^2} \tag{8.14}$$

then the Higgs inflation can be identified as the Starobinsky model

$$S_{HI} = \int d^4x \sqrt{-g}\Big(\frac{M_p{}^2}{2}R + \frac{\xi^2}{4\lambda}R^2\Big) \tag{8.15}$$

The vacuum energy, driving the inflation is

$$V_{HI} = \frac{\lambda}{4\xi^2}M_p^4 \tag{8.16}$$

The Higgs inflation and the Starobinsky inflation theory differs by the kinetic term of Higgs inflation in Einstein frame

$$\Delta\mathcal{L} = \frac{1}{2}\frac{1}{1 + \xi h^2/M_p{}^2}\partial_\mu h\partial^\mu h \tag{8.17}$$

The number of e-folds till the end of the inflation is related to h by

$$N \sim \frac{6\xi h^2}{8M_p{}^2} \tag{8.18}$$

So, the ratio of the slow-roll parameters is given by

$$\frac{\epsilon_{HI}}{\epsilon_s} = \frac{8N\xi}{1 + \frac{4}{3}N8M\xi}$$

$$\sim 1 - \frac{10^{-5}}{6\lambda} \tag{8.19}$$

The difference of the corresponding reheating temperatures of this two inflation leads to the value of spectral index at the level of 10^{-3}.

8.2 Universal attractor model as Starobinsky model

The general form of the non-minimal coupling of the universal attractor model is

$$S_{att} = \int d^4x\sqrt{-g}\Big[\frac{1}{2\Omega(\phi)R} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V_j(\phi)\Big] \tag{8.20}$$

Here,

$$\Omega(\phi) = M_p^2 + \xi f(\phi)$$
$$V_J = f(\phi)^2$$

As the dynamic is completely dominated by the potential, we can describe the action by

$$S_{att} = \int d^4x\sqrt{-g}\Big[\frac{M_p{}^2}{2}R + \frac{1}{2}\xi f(\phi)R - f(\phi)^2\Big] \tag{8.21}$$

The scalar field equation admits two solutions,

$$f' = 0$$
$$f = \frac{1}{4}\xi R$$

So,

$$S_{att} = \int d^4x + \sqrt{-g}\Big[\frac{M_p{}^2}{2}R + \frac{\xi^2}{16}R^2\Big] \tag{8.22}$$

For $M^2 = \frac{4}{3\xi^2}$, 8.22 is a Starobinsky model.
The vacuum energy which drives the inflation is

$$V_{att} = \frac{3}{4}M^2 M_p{}^4$$

$$= \frac{M_p{}^4}{\xi^2} \tag{8.23}$$

Universal attractor model and the Starobinsky model differs by their kinetic terms

$$\Delta\mathcal{L} = -\frac{1}{2}\sqrt{-g}\frac{\partial_\mu\phi\partial^\mu\phi}{1 + \xi f/M_p{}^2} \tag{8.24}$$

The number of e-folding is related to ϕ as

$$N \sim \frac{3\xi\phi^n}{4M_p^n}$$

So, the ratio of the slow-roll parameters are given by

$$\frac{\epsilon_{att}}{\epsilon_s} \sim 1 - \frac{N^{2/n-1}}{2n^2\xi^2\xi^{2/n}}\left(\frac{4}{3}\right)^{2/n} \tag{8.25}$$

This value is negligible as it deviates from the unity by 10^{-3}

8.3 Higher-dimensional Starobinsky model descendants

The higher-dimensional generalization of the Starobinsky model can be described by the action

$$S = \int d^d x \sqrt{-g}\left(\frac{M_*{}^{d-2}}{2}R + aR^b\right) \tag{8.26}$$

Here, R is the (4+d) dimensional Ricci-scalar, M_* is the corresponding Planck mass and a and b are dimensionless parameters. By introducing an auxiliary field ϕ, the higher dimensional theory can be linearized in the scalar curvature.

$$S = \int d^d x \sqrt{-g}\left(\frac{M_*{}^{d-2}}{2}R + \omega\phi^2 R - \phi^{2b/b-1}\right) \tag{8.27}$$

Here,

$$\omega = \frac{b}{b-1}\left((b-1)a\right)^{1/b}$$

By conformal transformation, where $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ and $\Omega^{d-2} = \left(1 + \frac{2\omega\phi^2}{M_*{}^{d-2}}\right)^{-1}$ we get,

$$S = \int d^4 x \sqrt{-g}\left(\frac{M_*{}^{d-2}}{2}R - \frac{1}{2}(d-1)(d-2)M_*{}^{d-2}(\partial_\mu log\Omega)^2 - V_0\langle(\Omega^{2-d}-1)\Omega^{(b-1)d/b}\rangle^{b/b-1}\right) \tag{8.28}$$

Here,

$$V_0 = \frac{M_*{}^{b(d-2)/b-1}}{2\omega^{b/b-1}} \tag{8.29}$$

If we take, $d-2 = \left(\frac{b-1}{b}\right)d$ or $b = \frac{d}{2}$ and parameterize Ω as $log\Omega = -\frac{1}{\sqrt{(d-1)(d-2)}}\frac{\psi}{M_*{}^{(d-2)/2}}$ we get,

$$S = \int d^d x \sqrt{-g}\left[\frac{M_*{}^{d-2}}{2}R - \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - V_0(1 - exp(-\sqrt{\frac{d-2}{d-1}})\frac{\psi}{M_*{}^{(d-2)/2}})^{d/d-2}\right] \tag{8.30}$$

By a dimensional reduction in a d-4 torus, T^{d-4} having volume, V_{d-4} and identifying

$$\chi = V_{d-4}^{\frac{1}{2}}\psi$$
$$V_{d-4}M_*^{d-2} = M_p^2$$

we arrive to the four-dimensional action,

$$S = \int d^4x\sqrt{-g}\Big[\frac{M_p^2}{2}R - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V_0(1 - exp(-\sqrt{\frac{d-2}{d-1}}\frac{\chi}{M_p}))^{\frac{d}{d-2}}\Big] \tag{8.31}$$

The potential of this generalized Starobinsky model is

$$V = V_0\big(1 - exp(\alpha\frac{\phi}{M_p})\big)^\beta \tag{8.32}$$

For this potential,

$$n_s \sim 1 - \frac{2}{N} \tag{8.33}$$

$$r \sim \frac{8}{\alpha^2 N^2} \tag{8.34}$$

The conformally invariant SO(1, 1) two-field model of conformal inflation is described by Lagrangian

$$\mathcal{L} = \sqrt{-g}\Big[\frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{\chi^2}{12}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi\frac{\phi^2}{12}R - \frac{\lambda}{4}(\phi^2 - \chi^2)^2\Big] \tag{8.35}$$

The Lagrangian is invariant under SO(1, 1), rotations of (ϕ, χ). The gauge fixings by going to Einstein frame $\chi^2 - \phi^2 = 6M_p^2$ or to the Jordan frame $\chi = \sqrt{6}M_p^2$ will give,

$$\mathcal{L} = \sqrt{-g}\big(\frac{M_p^2}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - 9\lambda M_p^4\big) \tag{8.36}$$

Ignoring the kinetic term and leaving the auxiliary fields, ϕ and χ by integration, we get

$$\mathcal{L} = \sqrt{-g}\frac{1}{144\lambda}R^2 \tag{8.37}$$

This is the Starobinsky model for $M_p \rightarrow \infty$ limit. Like the linear representation, this propagates a graviton and a scalar.

$$\mathcal{L} = \sqrt{-g}(\varphi R - 36\lambda\varphi^2) \tag{8.38}$$

integrating out the φ term we get the R^2 theory of 8.37

8.4 Starobinsky inflation and supersymmetry

By embedding supersymmetry in the theory for Starobinsky inflation, the theoretical context of inflation can be connected to particle physics. The upper-limit on R implies that the energy scale during the inflation must be much smaller than the Planck scale. Supersymmetry allows to maintain this constraint naturally, without any fine-tuning.
To combine supersymmetry with inflation, supergravity is needed. In the early-universe

scenario, an effective inflationary potential that varies slowly over a large range of inflation field must exist. This can happen for a no-scalar supersymmetric Wess-Zumino model, which is consistent with the Planck data for $N = 50-60$ e-folds. For $\lambda = \frac{\mu}{3}$ in Planck units and upon a conformal transformation, this Wess- Zumino model is equivalent to Starobinsky R^2 model.

Although, the global supersymmetry is broken, a local supersymmetry must exist in the inflationary phase. This connects the inflationary phase with a re-normalization group flow of ultra-violet(UV) to the infra-red(IR) of a constrained Chiral scalar superfield of supergravity. This gives rise to the Goldstino supermultiplet. As the supersymmetry breaking is directly related to the gravitino mass, gravitino condensation occurs. In the next chapter, supergravity is embedded into inflation model of cosmology.

Chapter 9

Embedding Supergravity into Inflation model

The simplest globally symmetric model is the Wess-Zumino model with a chiral superfield Φ. This model has the superpotential,

$$W = \frac{\hat{\mu}}{2}\Phi^2 - \frac{\lambda}{3}\Phi^3 \tag{9.1}$$

From [13], when the imaginary part of the scalar component of Φ vanishes, the Wess-Zumino model reduces to

$$V = A\phi^2(v - \phi)^2 \tag{9.2}$$

Here, for small value of λ, this model will yield a Planck-compatible inflation. To consider early-universe cosmology, gravity should be included in this global symmetry. So, we introduce supergravity by constructing a locally supersymmetric model.

By considering an inflation superfield together with a modulus field T which is embedded in an $SU(2,1)/SU(2) \times U(1)$ no-scale supergravity sector, we can show the equivalence of the simplest globally supersymmetric model and a no-scale supergravity (no-scale Wess-Zumino) model. It can be shown that, for specific value of $\frac{\hat{\mu}}{\lambda}$ this model is compatible to Planck-data.

For the no-scalar case with non-compact $SU(N,1)/SU(N) \times U(1)$ symmetry, a kinetic term and the effective potential term for the $N = 1$ supergravity can be found.

The scalar field can be described as the combination $G = K + lnW + lnW^*$ where, K is a hermitian Kähler function and W is a holomorphic superpotential. For a Kähler metric $K_i^{j*} \equiv \partial^2 K/\partial\phi^i\phi_j^*$ the kinetic term is given by

$$\mathcal{L}_{KE} = K_i^{j*}\partial_\mu\phi^i\partial_\mu\phi_j^* \tag{9.3}$$

The effective potential is

$$V = e^G\left[\frac{\partial G}{\partial\phi^i}K_{j*}^i\frac{\partial G}{\partial\phi_j^*} - 3\right] \tag{9.4}$$

Here, K_{j*}^i is the inverse of Kahler metric.

For minimal no-scale $SU(2,1)/SU(2) \times U(1)$ case, the Kähler function is

$$K = -3ln(T + T^* - \frac{|\phi|^2}{3}) \tag{9.5}$$

As we consider the complex scalar fields, ϕ and a modulus field, T, we find new expression for the kinetic term and the potential term.

$$\mathcal{L}_{KE} = (\partial_\mu \phi^*, \partial_\mu T^*) \left(\frac{3}{(T + T^* - \frac{|\phi|^2}{3})^2} \right) \begin{pmatrix} \frac{(T+T^*)}{3} & -\frac{\phi}{3} \\ -\frac{\phi^*}{3} & 1 \end{pmatrix} \begin{pmatrix} \partial^\mu \phi \\ \partial^\mu T \end{pmatrix} \tag{9.6}$$

and the effective potential,

$$V = \frac{\hat{V}}{((T + T^* - \frac{|\phi|^2}{3})^2} \tag{9.7}$$

$$\text{for} \quad \hat{V} = \left| \frac{\partial W}{\partial \phi} \right|^2$$

T field has a vacuum expectation value $2 < ReT >= c$ and $< ImT >= 0$. Ignoring the kinetic mixing between the T and ϕ fields we find,

$$\mathcal{L}_{eff} = \frac{c}{\left(c - \frac{|\phi|^2}{3} \right)^2} |\partial_\mu \phi|^2 - \frac{\hat{V}}{\left(c - \frac{|\phi|^2}{3} \right)^2} \tag{9.8}$$

Here, this is the minimal Wess-Zumino superpotential for the inflation field. Taking $\phi = \sqrt{3}c \tanh \frac{\chi}{\sqrt{3}}$, where $\xi = \frac{x+iy}{\sqrt{2}}$ we get,

$$\mathcal{L}_{eff} = \frac{1}{2} \sec^2(\sqrt{2/3}y) \left((\partial_\mu x)^2 - (\partial_\mu y)^2 \right)$$
$$- \mu^2 \frac{exp(-\sqrt{2/3}x)}{2} \sec^2(\sqrt{2/3}y) \left(\cosh(\sqrt{2/3}x) - \cos(\sqrt{2/3}y) \right) \tag{9.9}$$

Absorbing the VEV of T into the mass which is defined to be $\tilde{\mu} = \mu\sqrt{\frac{c}{3}}$.
During inflation, x is large and we have $m_y = \mu/\sqrt{3}$. At the end of inflation, $x = 0$ which gives $m_y = \mu/\sqrt{6}$.

To get the minimal kinetic terms in terms of x and y, we expand the Lagrangian about $y = 0$. Finally, we get the potential for the real part of the inflation

$$V = \mu^2 e^{-\sqrt{2/3}x} \sinh^2(x/\sqrt{6}) \tag{9.10}$$

This is the potential for the NSWZ model for $\lambda \sim \frac{\mu}{3}$ in Planck limits.
We see that, the potential in 9.10 and the values of n_s and r for $\frac{\lambda}{3}$ is equivalent to the inflation of $R + R^2$ model proposed in Starobinsky inflation model.

Einstein-Hilbert action containing on R^2 contribution, where, R is the Ricci-scalar curvature.

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left(R + \frac{R^2}{6M^2} \right) \tag{9.11}$$

where, $M << M_p$ is some mass-scale. This theory includes supergravity by considering gravity with an additional scalar field. Considering transformation,

$$\tilde{g}_{\mu\nu} = \left(1 + \frac{\varphi}{3M^2} \right) g_{\mu\nu}$$

and redefining field as

$$\varphi' = \sqrt{\frac{3}{2}}ln\big(1 + \frac{\varphi}{3M^2}\big)$$

we obtain the equation of action

$$S = \frac{1}{2}\int d^4x\sqrt{-\tilde{g}}\bigg[\tilde{R} + (\partial_\mu\varphi')^2 - \frac{3}{2}M^2\big(1 - exp(-\sqrt{2/3}\varphi')\big)^2\bigg] \qquad (9.12)$$

this action corresponds to a potential

$$V = \frac{3}{4}M^2\big(1 - exp(-\sqrt{2/3}\varphi')\big)^2 \qquad (9.13)$$

is equivalent to 9.10 to the NSWZ model in the real direction. This is the same potential for the Higgs inflation and other inflation models. When $c =< (T + T^*) >= 1$, we see that $M^2 = \frac{\mu^2}{3}$ can be identified to μ^2. This implies that the Starobinsky mass, M is directly related to the NSWZ mass μ in the superpotential of the Wess-Zumino model in 9.1. Thus, NSWZ model shows that inflation models can include no-scale supergravity.

The next chapter is aimed at finding a parametric function of field x in terms of string modulus, t for which we can construct an NSWZ model that can contribute to cosmological inflation.

Chapter 10

Modelling the Starobinsky Potential: When Does Inflation Arise?

From the previous chapter, we got equation for the potential for the NSWZ model for $\lambda \sim \frac{\mu}{3}$ in Planck limits.

$$V = \mu^2 e^{\sqrt{\frac{2}{3}}x} \sinh^2(x/\sqrt{6})$$

For $\lambda = \frac{\mu}{3}$, the VEV would be

$$
\begin{aligned}
A_s &= \frac{V}{24\pi^2 \epsilon} \\
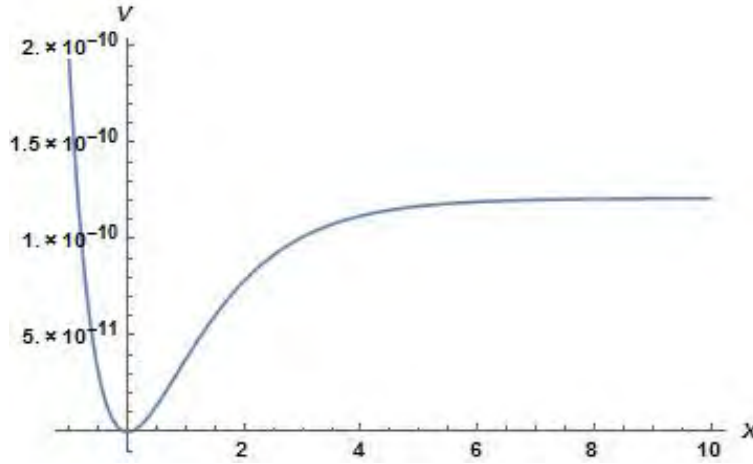&= \frac{\mu^2}{8\pi^2} \sinh^4(x/\sqrt{6})
\end{aligned}
\tag{10.1}
$$

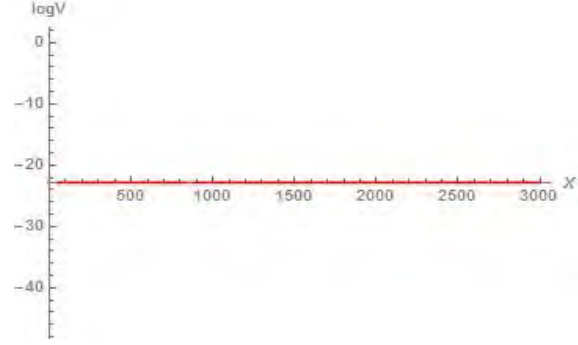Figure 10.1: The potential V in the NSWZ model for small x

We see from the graph in Fig:10.1 that for x near to zero there is a potential drop and a local minima is observed. Then the potential increases with the increment of x and reaches a stable value at approximately $x = 5.3$.

For large values of x, we plot graphs of $logV$ vs x and observe in Fig:10.2 that for large values of x, the potential does not change as we depicted in Fig:10.1.

For this case of inflation, the value of x is fixed by requiring $N = 50 - 60$ e-folds.
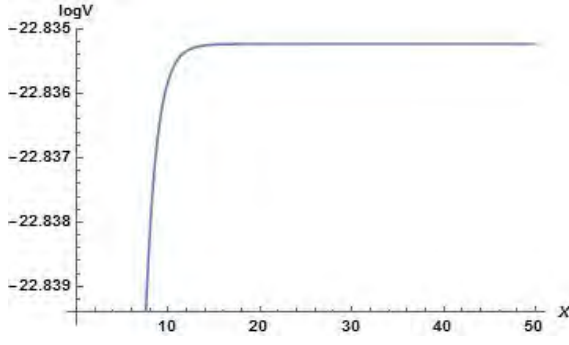
(a) Change in potential V in NSWZ model for x greater than 10

(b) Change in potential V in NSWZ model for x going to infinity

Figure 10.2: The potential V in NSWZ model for large x

Having, $N = 55$, $x = 5.35$ like Ellis et al. (2013). From the graph in Fig:10.1, it can be observed that the potential at $x = 5.35$ reaches a plateau and we would expect a cosmological inflation at this point.

The slow-roll parameter, ϵ measures the slope of the potential, V and η measures the curvature of the potential. If we chose, $N = 55$ e-folds of inflation, then μ is fixed to be 2.2×10^{-5}. In this limit, the slow-roll inflation parameters are

$$\epsilon = \frac{1}{3}\text{csch}^2(x/\sqrt{6})e^{-\sqrt{x^2/3}}, \tag{10.2}$$

$$\eta = \frac{1}{3}\text{csch}^2(x/\sqrt{6})\left(2e^{-x\sqrt{2/3}} - 1\right) \tag{10.3}$$

The equations derived in 10.2 and 10.3 produces similar results as the standard equations for ϵ and η which are defined as

$$\epsilon = \frac{m_{pl}}{16\pi}\left(\frac{V'}{V}\right)^2 \tag{10.4}$$

$$\eta = \frac{m_{pl}}{8\pi}\left(\frac{V''}{V}\right) \tag{10.5}$$

From the graph in Fig:10.3a plotted for equation 10.2, a large value of ϵ is observed for x near to zero. However, with the increase of x, exponential decay in the value of ϵ is observed. For very large x, ϵ is observed to be near to zero.

Observing the graph in Fig:10.3b plotted for equation 10.3, we find that η initially has a large value. With the increase of x, a large drop in η is observed. At $x = 1.345$ we arrive to a global minima where $\eta = -\frac{1}{9}\text{csch}[\frac{\log 3}{2}]^2 \sim -0.333$. As x increases from 1.345, we observe a logarithmic rise in η again. However, we can see from this graph that for $x > 1.344$, η is very small and always negative.
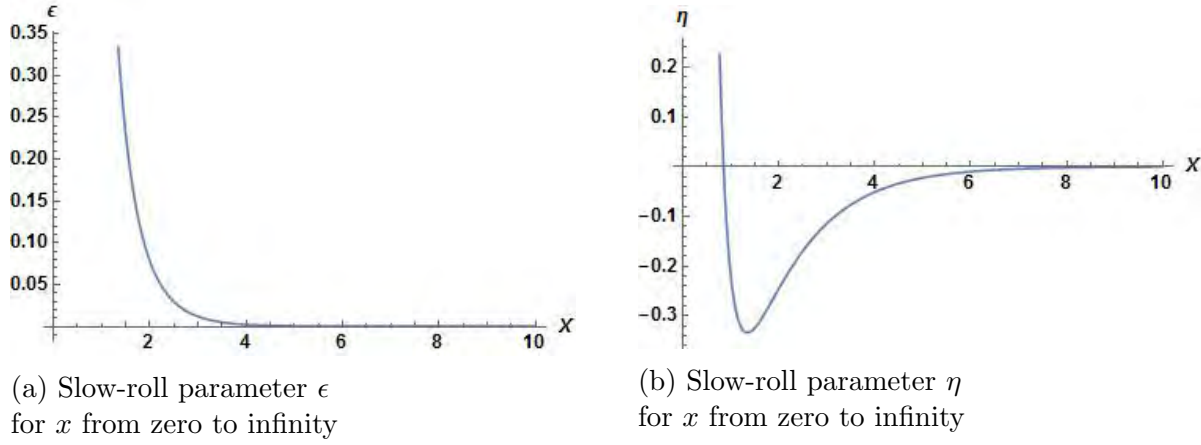
(a) Slow-roll parameter ϵ
for x from zero to infinity

(b) Slow-roll parameter η
for x from zero to infinity

Figure 10.3: Slow-roll parameters of inflation changing with x

10.1   Modelling the slow-roll inflation parameters for x as different parametric functions

So far, we have observed the slow-roll inflation parameters,η and ϵ by choosing x to be independent field that is not determined by other dynamics. When this model is embedded in a String theory, x will be determined by string theoretic considerations. For example, in some cases, String theory will force x to be parametrically dependent on a string modulus, t as in $x(t)$. In this section, we investigate these parameters for x as different mathematical functions and determine a function for x which results to good approximation of η and ϵ. Assuming,

- $x = ae^t$ where x is an exponential function of t

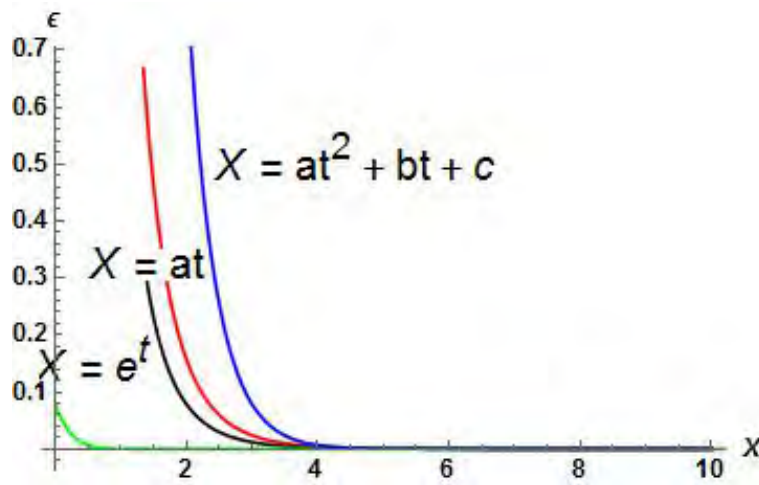- $x = at^2 + bt + c$ where x is a quadratic function of t

Figure 10.4: Comparison of slow-roll parameter ϵ for x as different parametric functions of t

From the graphs in Fig:10.4 and Fig:10.5, we see that both $x = e^t$ and $x = at^2 + bt + c$ results to similar curves for ϵ and η.
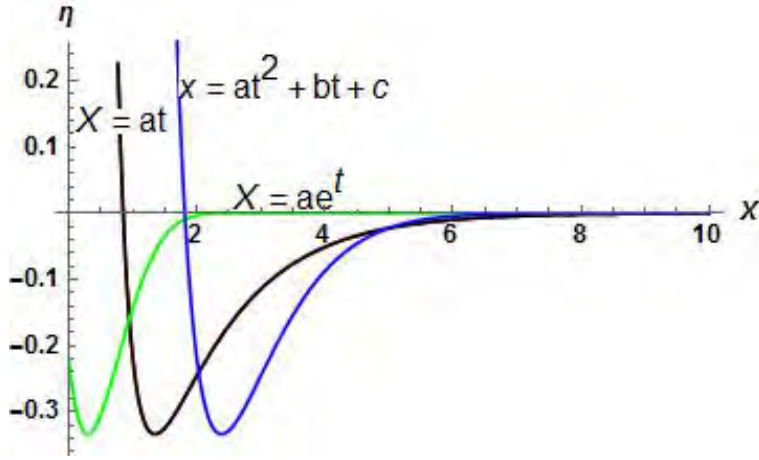
Figure 10.5: Comparison of slow-roll parameter η for x as different mathematical functions of t

The global minima determined for ϵ from equation 10.2 is $\epsilon = 2.91708 \times 10^{-60}$ at $x = 84.123$. For $x = e^t$, we find that the minima, $\epsilon = 0$ is at $x = 252.9$. While for $x = at^2 + bt + c$ results a minima at $\epsilon = 7.22998 * 10^{-57}$ for $x = 79.337$. So, the quadratic function for x yields to a better result for ϵ.

Again for η, from equation 10.3 we get a minima of $\eta = -0.333$ for $x = 1.34552$. While, taking $x = e^t$ produces a minima of $\eta = -0.333$ at $x = 2.691$, $x = at^2 + bt + c$ gives a minima of $\eta = -0.333$ at $x = 1.34552$.

From the above discussion, we find that x as a quadratic function yields best results for ϵ and η.

The slow-roll expressions for the tensor-to-scalar ratio, r and spectral index, n_s for the scalar perturbations are found in terms of slow-roll inflation parameters, ϵ and η.
For this we need the value of x, which is fixed by $N = 50 - 60$ e-folds. The nominal choice of $N = 55$ yields $x = 5.35$. Using this in equation 10.2 and 10.3 we get the value of ϵ and η.

$$
\begin{aligned}
\epsilon &= 0.000219683 \text{ and} \\
\eta &= -0.016895
\end{aligned}
$$

Using these values for

$$
\begin{aligned}
\text{spectral index, } n_s &= 1 - 4\epsilon + 2\eta \\
&= 0.965331 \\
\text{scalar-to-tensor ratio, } r &= 10\epsilon \\
&= 0.00219683
\end{aligned}
$$

These results satisfy the Planck data ("Planck 2013 results. XVI. Cosmological parameters" (2014))

10.2 Evaluating potential, V when x is parametrically quadratic in terms of a string modulus, t

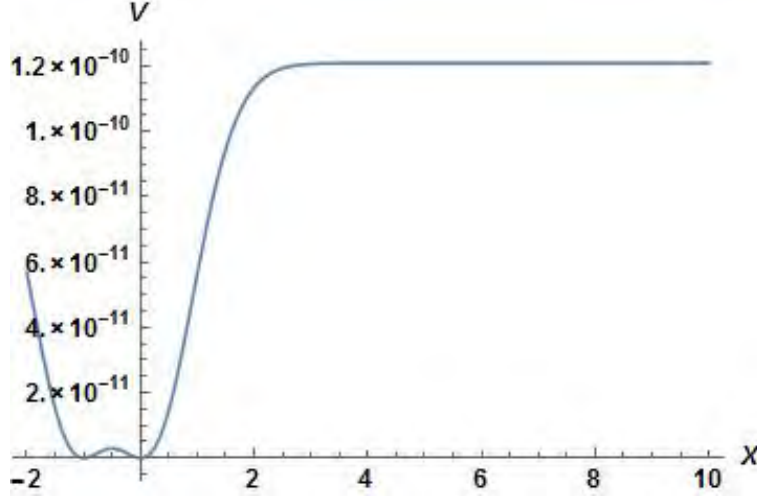For, $x = at^2 + bt + c$, a good approximation for the potential is also observed.

Figure 10.6: The Potential V in NSWZ model when x is a quadratic function of t

From the graph in Fig:10.6 we observe that when x is a quadratic function, two vacuum are produced. This provides a single real field with double well and thus contributes to better slow-roll inflation.

We compare the graphs for the potential with x being linear and quadratic functions of t in Fig:10.7 We see that the curves show similar drop and rise in potential. However, x when
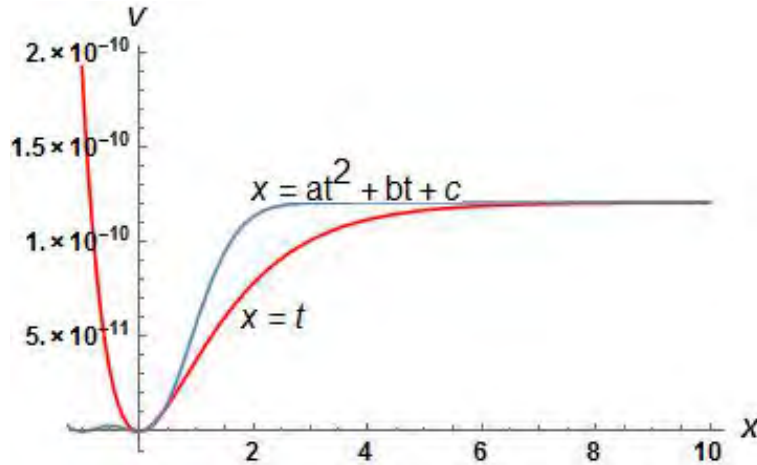

Figure 10.7: Comparison between the potential V in NSWZ model for x as a quadratic function and x as a linear function

parametrically quadratic produces a very flat potential, which is another promising scenario for the rise of inflation. This warrants for further investigation on the effects of field x being a quadratic function of string modulus, t.

Chapter 11

Conclusion

Supersymmetry is a comparatively new field in theoretical physics which may provide a natural framework leading to a theory, where unification of all known interactions can be explained. This theory extends the standard model of particles in QFT to MSSM and introduces scope for unification of bosons and fermions. Along with framework for the grand unification theory, supersymmetry might also provide new approaches to solve a lot other modern physics research problems.

In recent researches of QFT, supersymmetric gauge field theory is playing a vital role. It introduced a new field called supersymmetric localization in Quantum field theory which allows us to get exact results in various computational problems of QFT. SUSY gauge theory implements the mathematical concept of localization and reaches to a lot of formulas to approach strong coupling dynamics of gauge theories.

As the action functional, S is invariant under supersymmetry, the path integral which measures the expectation value of the observable, goes to zero. Hence, the observable is invariant under transformation. To reduce the difficulty of path integrals over complicated spaces, integral over a space with continuous symmetry is expressed as sums of contributions of the symmetry invariant points. This allows the computation of an infinite dimensional path integral of QFT to be simplified into a finite dimensional one. In many QFT problems, complicated spaces arise as moduli space of solutions to the field equations. The integrals over this moduli spaces may provide a useful low-energy approximation to the original path integral. By using the localization based fermionic symmetry, we can deform the theory, to provide us with an exact result of the moduli space.

Localization in supersymmetry can be used to reduce infinite path integrals to finite ones or to simplify integrals of complicated moduli spaces.

Supersymmetry along with particle physics may also answer some of the biggest questions of astrophysics, with one of them being the existence of dark matter. As supersymmetry extends the standard model to MSSM, many theorists tend to believe that dark matters might be some supersymmetry particles that are yet to be discovered through experiment. These supersymmetry particles are thought to be weakly interacting massive particles since they could be thermally stable and be abundant in the condition of early universe. It is also computed that the relic abundance for the annihilation cross section, if a particle of

weak-scale interactions exists, is equal to the one for existence of a stable particle.

For the existence of supersymmetric dark matter, the R-parity must be conserved. No special selection rules can be followed if R-parity is broken, as this will result to violation of baryon-lepton number conservation and will also restrict the existence of cold dark matter. As the lightest super particles (LSP) is stable under R-parity conservation, only the WIMP of LSP are taken as the dark matter candidate. As dark matters cannot scatter light, the LSP has to be chargeless, preserving the relic abundance, the possible WIMP candidates for dark matter maybe neutralino or gravitino. In most theories, neutralino which is a linear combination of supersymmetric partners of photon, Z^0, and Higgs bosons is assumed to be the LSP of dark matters.

Standard Model including the Higgs boson has so far proved to be correct as it is able to explain the observations made from experiments in LHC and other high-energy experiments. It also, provides unification of all gauge couplings except for gravity. However, for a wide range of energies, this model seems to be incomplete as it can not explain a lot of phenomena like dark matter, matter-antimatter asymmetry and the inconsistency of grand-unification of three gauge couplings at large scale. On the other hand, supersymmetric extension can answer these questions by including super-particles that are partner to every existing particles in SM. The experimental proof of supersymmetry can be established if these additional partner particles are found. Experiments run in LHC, ATLAS, CERN are designed to find proofs that show that the properties and the effects on precision measurements predicted by SUSY is present in high energy physics. Thus, the existence of SUSY particle will also be proved. So far, no experimental data has supported the existence of such particles but a definite answer is expected to be found when these experiments are run for energy of order TeV.

Supersymmetry can lead to more natural framework of inflationary model. In this thesis, we presented the Starobinsky model of inflation to be identical with the other inflation models. By combining supersymmetry with the Starobinsky model of inflation, supergravity is introduced to this inflation model. A no-scalar supersymmetric Wess-Zumino model offers a scenario where the effective inflationary potential may vary slowly over a large range of inflation field. When $\lambda = \frac{\mu}{3}$ in Planck units, this model upon a conformal transformation becomes equivalent to the R^2 Starobinsky model of inflation and thus, supergravity can be realized in the Starobinsky model.

We found that the Starobinsky potential as a realization of no-scalar supergravity, gives rise to inflation when the field, $x = 5.35$ and at this point, the slow-roll parameters, ϵ and η results to values consistent with Planck data.

In this thesis, we showed that when the field, x is parametrically quadratic in a string modulus, t, a single real field with double well is produced and also found a very flat effective potential for the NSWZ model. This scenario is very promising for the rise of inflation. This deserves further investigation on the quadratic relationship of x with the string modulus, t.

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