BRAC University Journal, Vol. II, No. 1, 2005, pp. 83-88

GRAVITATIONAL WAVES IN A MAGNETIZED PLASMA

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ABSTRACT

An overview of gravitational wave and its possible detection while propagating in a plasma is given. The linear propagation of gravitational wave in magnetized plasma is considered. It is found that a × polarized gravitational wave excites Alfve′n modes while a + polarized wave excites magneto-acoustic waves propagating parallel to the gravitational wave. In merging binary pulsars gravitational wave energy can be converted into pulsar wind at a large scale.

Key words: gravitational wave, general theory of relativity, binary pulsars, plasmas.

I. INTRODUCTION

Gravitational waves (GWs) are a concept of Einstein's General Theory of Relativity. These waves originate from the most energetic events in our universe, such as colliding neutron stars, binaries, supernova explosions and gravitational collapses into black holes. They manifest themselves as ripples in space and time, that periodically stretch and compress all present matter and, delay and accelerate the time signals from millisecond pulsars.

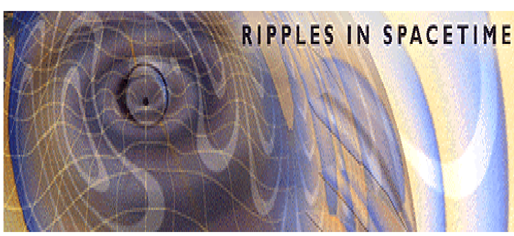

Fig. 1. Gravitational waves as ripples in spacetime

The only evidence, until now, of the existence of gravitational waves is the neutron star binary whose orbital radius is slowly decreasing as a consequence of the radiated gravitational energy. The reason that it is hard to detect gravitational waves (GWs) is that Newton's gravitational constant is extremely small: $G=6.7\times10^{-11} Nm^2 kg^{-2}$. The dimensionless amplitude $\Delta l / l$ of typical gravitational waves reaching the earth is only of the order of 10^{-17}. In other words, a rod of one meter in length will oscillate with an amplitude of a millionth of the radius of hydrogen atom[1]. Still, several direct GW-detectors are being build at present, or have been proposed for the future. The best known of these are Laser Interferometer Ground Observatory, LIGO, and the Laser Interferometer Space Array, LISA. Another approach to detect GWs is to measure relative time displacement $\Delta t / t$, instead of position displacements. Time signals from millisecond pulsars can be measured with high precision. Gravitational waves also may be detected indirectly. It appears that GWs can be converted to electromagnetic waves, viz light, that are more easier to detect.

According to General Theory of Relativity, all forms of energy are equivalent, which in one way to formulate the Equivalence Principle. This reflected by Einstein field equations. Gravitational waves are the first order harmonic solutions for these field equations.

As a result of the Equivalence Principle, not only oscillating matter sources can produce GWs, but oscillating energy densities also can produce them. This means that also reverse process might occur: electromagnetic waves generated by gravitational waves. In other words a gravitational wave passing through a magnetized plasma can excite electromagnetic wave 'photon', which can be detected. In other words, one could indirectly 'watch' gravitational waves in the form of light.

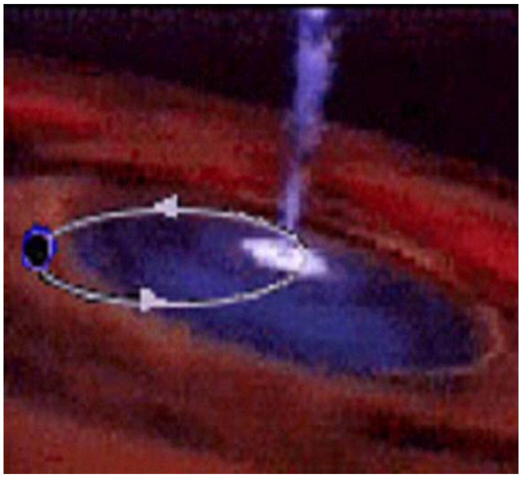

Fig. 2. "Watching" of GWs from binary pulsars

II. GRAVITATIONAL WAVES: GENERAL THEORY OF RELATIVITY

Gravitational waves are defined as propagating perturbations of some flat background. In the General Theory of Relativity, Poisson's equation of gravity is replaced by the equivalent but covariant field equations, introduced by Einstein

Ñ $\phi = 4\pi\rho$ Poisson (1)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad \text{Einstein} \quad (2)$$

or

$$R_{\mu\nu} = \frac{8\pi G}{c^4} (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T). \quad (3)$$

The conceptual measuring of these equations is that localized density distribution curves the space around it. As a result the geodesics of light or particles are deflected in the direction of the center of mass, thus producing the effect of gravitational force. Gravitational wave equation may be derived from the small perturbation $(h_{\mu\nu})$ of the background flat space $(\eta_{\mu\nu})$. In other words, the weak field solutions of Einstein's field equations are obtained by assuming[2, 3]:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \; \left|h_{\mu\nu}\right| << 1$$

$$h^{\lambda}{}_{\mu} = \eta^{\lambda\alpha} h_{\mu\alpha} \text{ and } h \equiv h^{\alpha}_{\alpha} = \eta^{\sigma\lambda} h_{\sigma\lambda} \quad (5)$$

$$R_{\mu\nu} = \Gamma^{\beta}_{\mu\nu,\beta} - \Gamma^{\beta}_{\mu\beta,\nu} + \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}$$

Ñ $\Phi^{\nu}_{\mu} = -\frac{16\pi G}{c^4} T^{\nu}_{\mu}$ (6)

where $\Phi^{\nu}_{\mu} = h^{\nu}_{\mu} - \frac{1}{2}\eta^{\nu}_{\mu} h$ with $\Phi^{\nu}_{\mu,\nu}$=0.

The general solution of Eqn. (6) is the solution of Poisson's equation, i.e.

$$\Phi^{\nu}_{\mu}(r,t) = \frac{4G}{c^4} \int \frac{\left(T^{\nu}_{\mu}\right)_{ret} d^3x'}{\left|\mathbf{r} - \mathbf{r}'\right|} \; . \quad (7)$$

III. EINSTEIN – MAXWELL EQUATIONS

In Lorentz guage, Linearized Einstein Field Equations (EFE) are:

$$G^{\mu\nu} = -\frac{1}{2} h_{\mu\nu} = 8\pi\delta T^{\mu\nu} \quad (8)$$

where $\delta T^{\mu\nu}$ - oscillatory part of Energy-Momentum tensor.

In transverse traceless guage (TT), and GW propagation along *z*-direction, the only independent components of $h^{\mu\nu}$ as:

$h^{xx} = -h^{yy} = h_{+}$ (+ polarization) (9a)

$h^{xy} = h^{yx} = h_{\times}$ (× polarization) (9b)

Then propagation equations for + and × polarizations are:

Ñ $h_{+} = -8\pi\left(\delta T^{xx} - \delta T^{yy}\right)$ (10a)

Ñ $h_{\times} = -8\pi\left(\delta T^{xy} + \delta T^{yx}\right)$ (10b)

GW propagating perpendicular to $B^0_{\times}$ produces $B'_{\times}$, then

Ñ $h_{+}(z,t) = 4B^0_{\times} B'_{\times}(z,t)$, Ñ $h_{\times} = 0$ (11)

From Maxwell's equations (coupled to EFE with conservation of energy and momentum $\nabla_{\mu} T^{\mu\nu} = 0$), one finds wave equations for plasma

Ñ $B'_{\times}(z,t) \propto h_{+}(z,t) B^0_{\times}$. (12)

A. Background Curvature

Einstein field equations describe the curvature of space-time due to the presence of matter and energy.

$$R^{\mu\nu} = 8\pi\left(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\right)$$

$$\frac{1}{R^2} \sim \left(B^0\right)^2.$$

Interaction with magnetic field excites EM waves

$$B' \propto B^0 h_+ kz \tag{13}$$

Therefore,

$$\frac{B'}{h_+} \propto \frac{z}{R} \tag{14}$$

For a neutron star $B^0 \sim 10^8\,\mathrm{T}$, curved space $\sim 10^{10}\,\mathrm{m}$

Hence all GW energy converts to EM energy on a scale length of R.

B. No Coupling to unmagnetized plasma

Stress-energy tensor for homogeneous perfect fluid in rest frame of an observer is

$$\left(u^{\nu} = \left(1 \quad 0 \quad 0 \quad 0\right)\right), \tag{15}$$

$$T^{\mu\nu} = (\rho + P)u^{\mu}u^{\nu} + Pg^{\mu\nu} \tag{16}$$

A linearly polarized GW will produce perturbation in ρ, P and u^i of order of $h_{+,\times}$.

δT^{ij} are all higher order.

Therefore, a GW cannot couple to an unmagnetized perfect fluid in a linearized theory.

C. Space-Time Split

A time like observer moving with 4-velocity u^{μ} perceives space as the 3-dimensional hyper surface orthogonal to u^{μ}, and u^{μ} itself as the time axis.

We can define $u^{\mu}(x^{\nu})$ at each point in space-time as direction of time, and define space as the snapshots of space at constant time.

Subsequently, we can split equations into their space and time components by using parallel and orthogonal projection operators.

$$u^{\mu}_{\nu} = -u^{\mu}u_{\nu},\ H_{\mu\nu} = g_{\mu\nu} + u_{\mu}u_{\nu} \tag{17}$$

with $u^{\mu}_{\nu} + H^{\mu}_{\nu} = \delta^{\mu}_{\nu}$

D. Proper Reference Frame

In Transverse Traceless Frame (TTF)

$$g^{\mu\nu}_{TT} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 + h_+(z,t) & h_\times(z,t) & 0 \\ 0 & h_\times(z,t) & 1 - h_+(z,t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{18}$$

But in Orthogonal Tetrad Frame (OTF) the basis vectors that remain orthogonal in the presence of a TT plane polarized GWs are:

$$\begin{aligned} \vec{e}_0 &= \begin{pmatrix} \frac{\partial}{\partial t} & 0 & 0 & 0 \end{pmatrix} \\ \vec{e}_1 &= \begin{pmatrix} 0 & \left(1 - \frac{h_+}{2}\right)\frac{\partial}{\partial x} & -\frac{h_\times}{2}\frac{\partial}{\partial y} & 0 \end{pmatrix} \\ \vec{e}_2 &= \begin{pmatrix} 0 & -\frac{h_\times}{2}\frac{\partial}{\partial x} & \left(1 + \frac{h_+}{2}\right)\frac{\partial}{\partial y} & 0 \end{pmatrix} \\ \vec{e}_3 &= \begin{pmatrix} 0 & 0 & 0 & \frac{\partial}{\partial z} \end{pmatrix}, \end{aligned} \tag{19}$$

with respect to these basis vectors the metric is $\eta^{\mu\nu}$.

In this frame the linearized Maxwell equation in the presence of GW in a magnetized plasma takes the form:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \vec{v}' = 0 \tag{20}$$

$$\begin{aligned} \omega_{tot}\frac{\partial \mathbf{v}'}{\partial t} &= -\nabla P' + \frac{1}{4\pi}\left(\mathbf{B}^0 \cdot \nabla\right)\mathbf{B}' - \\ &\frac{1}{4\pi}\nabla\left(\mathbf{B}^0 \cdot \mathbf{B}'\right) + \mathbf{B}_0 \frac{\partial(\mathbf{v}' \cdot \mathbf{B}^0)}{\partial t} \\ &- \frac{1}{4\pi}(\mathbf{j}_E \times \mathbf{B}^0_0) \end{aligned} \tag{21}$$

$$\frac{\partial \mathbf{B}'}{\partial t} = (\mathbf{B}^0 \cdot \nabla)\mathbf{v}' - \mathbf{B}^0(\nabla \cdot \mathbf{v}') - \mathbf{j}_B \quad (22)$$

where, $P' = k\rho'^{\gamma}$ $\frac{4}{3} \le \gamma \le \frac{8}{3}$,

$\omega_{tot} = \omega^0 + \frac{|B^0|^2}{4\pi}$ with

$\omega^0 = \mu^0 + \rho^0$ and GW source terms are:

$$\mathbf{j}_E = \frac{B_\times^0}{2}\frac{\partial}{\partial z}\begin{pmatrix} -h_\times \\ h_+ \\ 0 \end{pmatrix}, \mathbf{j}_B = -\frac{B_\times^0}{2}\frac{\partial}{\partial t}\begin{pmatrix} h_+ \\ h_\times \\ 0 \end{pmatrix} \quad (23)$$

We find two waves, namely: non compress ional Alfvén wave:

$$u_A = \frac{\mathbf{B}^0}{\sqrt{4\pi\omega_{tot}}} \text{ (Alfvén speed)} \quad (24)$$

and magneto-accoustic wave:

$$u_m = \sqrt{\frac{\gamma P^0}{\omega_{tot}} + \frac{|B_0|^2}{4\pi\omega_{tot}}} \text{ (Magneto-accoustic speed)} \quad (25)$$

are propagating in the system

By eliminating $\mathbf{B}'$ and P'; the wave equation is

$$\left[\frac{\partial^2}{\partial t^2} - u_m^2 \nabla\nabla\cdot\right]\mathbf{v}' - \left[u_A\frac{\partial^2}{\partial t^2} - (\mathbf{u}_A\cdot\nabla)\nabla\right](\mathbf{v}'\cdot\mathbf{u}_A)$$
$$= (\mathbf{u}_A \cdot \nabla)\mathbf{v}' - \mathbf{u}_A(\mathbf{u}_A\cdot\nabla)\nabla\cdot\mathbf{v}' + \text{GW terms} \quad (26)$$

where GW terms are given by

$$\sqrt{\frac{\omega_{tot}}{4\pi}}\left[\nabla(\mathbf{j}_B\cdot\mathbf{u}_A) - \frac{\partial}{\partial t}(\mathbf{j}_E\times\mathbf{u}_A) - (\mathbf{u}_A\cdot\nabla)\mathbf{j}_B\right] \quad (27)$$

Let GW is propagating along z-direction. $\mathbf{k} = (0 \quad 0 \quad k)$ at an arbitrary angle θ with the ambient magnetic field and considering

$h_{+,\times} \alpha \ e^{i\omega(z-t)}$

Fourier and Laplace transform yield:

$$D\mathbf{v}' \equiv \mathbf{J}'_{GW}$$

$$\begin{pmatrix} \omega^2(1-u_{A\perp}^2)-k^2u_{A_{11}}^2 & 0 & -(\omega^2-k^2)u_{A_{11}}u_{A\perp} \\ 0 & \omega^2-k^2u_{A_{11}}^2 & 0 \\ -(\omega^2-k^2)u_{A_{11}}u_{A\perp} & 0 & \omega^2(1-u_{A\perp}^2)-k^2(u_m^2-u_{A_{11}}^2) \end{pmatrix} = -\frac{c\omega^2 u_{A\perp}}{\omega-k}\begin{pmatrix} u_{A_{11}}h_+ \\ u_{A_{11}}h_\times \\ -u_{A\perp}h_+ \end{pmatrix} \quad (28)$$

A non-trivial solution for the plasma waves requires the determinant of D to vanish**.** Solving for $k = k\ (\omega)$, since ω is fixed by the driving GW , we find six solutions:

$$\omega = \pm K_A u_A \cos\theta = \pm K_A u_{A_{11}} \quad (29)$$

$$\omega = \pm\frac{K_{S,+}}{\sqrt{2}}\sqrt{u_m^2 + c_s^2 u_{A_{11}}^2}\sqrt{1 \pm \sqrt{1-\sigma}} \quad (30)$$

where

$$\sigma(\theta) = \frac{4c_s^2 u_{A_{11}}^2}{\left(u_m^2 + c_s^2 u_{A_{11}}^2\right)^2} \quad (31)$$

Here the first equation represents the forward and backward Alfve′n waves. In the second equation the negative sign represents the magneto-sonic waves:

Here, $u_s = \frac{\omega}{k_s}$ and + sign refers fast magnetosonic waves $u_f = \frac{\omega}{k_f}$.

We see that

i) for $\theta = 0$, (parallel propagation)

$$u_f \to u_A$$
$$u_s \to c_s,$$

ii) for $\theta = \frac{\pi}{2}$ (perpendicular propagation)

$$u_f \to u_m$$
$$u_s \to 0.$$

Solution of inhomogeneous equation is

$$\mathbf{v}' = D^{-1}\mathbf{J}'_{GW} = \frac{\lambda_{ij}(\mathbf{J}'_{GW})_j}{\Lambda} \quad (32)$$

where $\lambda_{ij}(\omega,k)$ and $\lambda(\omega,k)$ are the matrix of cofactors of *D and its* determinant, respectively. These are related by

$$D_{ki}\lambda_{kj} = \delta_{ij}\Lambda\,. \tag{33}$$

Each solution of $\Lambda(\omega,k)=0$ can be identified with a wave mode M with

$$\omega = \omega(k_M) \;\;\&\;\; \omega(-k_M) = -\omega(k_M)$$

The determinant can be factorial into these wave modes:

$$\Lambda(\omega,k) = (\omega^2 - k^2u_{A11}^2)(\omega^2 - k^2u_f^2)(\omega^2 - k^2u_s^2) = 0 \tag{34}$$

$$u_{f,s}^2 = c_s u_{A11}\left(\frac{1\pm\sqrt{1-\sigma}}{\sqrt{\sigma}}\right). \tag{35}$$

The λ_{yy} component couples to the × polarized source term and excites Alfvén waves, viz., perturbations of the magnetic field perpendicular to the background:

$$v'_y(k,\omega) = -\frac{ih_\times u_{A11}u_{A\perp}}{\omega^2 - k^2u_A^2}\frac{\omega+k}{\omega-k} \tag{36}$$

$$B'_y(k,\omega) = -v'_y(k,\omega)\frac{B_x^0}{u_{A11}u_{A\perp}}\frac{\omega+ku_{11}^2}{\omega+k} \tag{37}$$

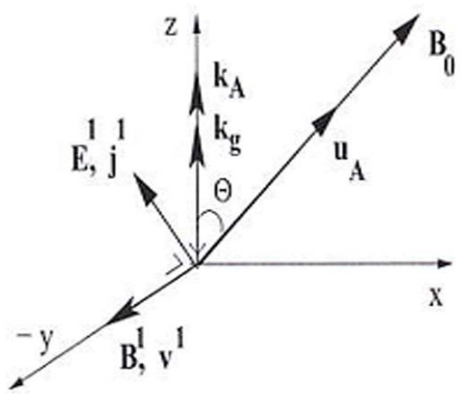

Fig.3. Orientation of the perturbation in the Alfve′n mode.

A + polarized GW excites slow and fast magneto-acoustic waves in the plasma.

The velocity components are:

$$v'_z(k,\omega) = \frac{i}{2}\frac{h_+\omega^3u_{A+}^2}{(\omega^2 - k^2u_f^2)(\omega^2 - k^2u_s^2)}\frac{\omega+k}{\omega-k}$$

$$v'_x(k,\omega) = -\frac{v_z(k,\omega)}{\tan\theta}\left(1-\frac{k^2c_s^2}{\omega^2}\right)$$

The magnetic component can be derived as:

$$\frac{B'_x}{B_0} =' v'_z\sin\theta - v'_x\cos\theta - \frac{\omega}{\omega+k}\frac{1-u_A^2}{u_A^2}\frac{v'_x}{\cos\theta}$$

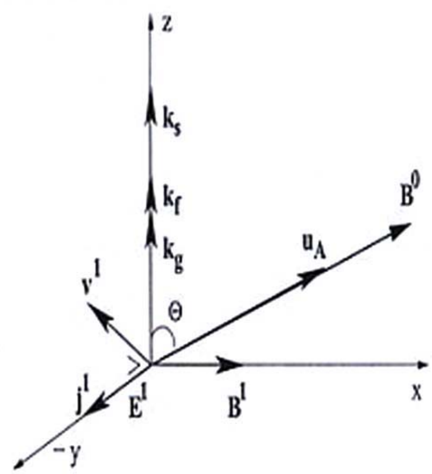

Fig 4. Orientation of the perturbations in the MSW modes

IV. CONCLUSION

We have reviewed the propagation of GWs in magnetized astrophysical plasma. It is shown that a + polarized GW excites magneto-acoustic and a × polarized GW excites Alfvén wave in a uniform magnetic field. The driving force exerted by a GW on test particle is described by the equality

$$\frac{d^2x^i}{dt^2} = -R_{i0j0}x^j = \frac{1}{2}\begin{pmatrix} \ddot{h}_+ & \ddot{h}_\times & 0 \\ \ddot{h}_\times & -\ddot{h}_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Integrating twice w.r.t. time the above equation gives

$$\delta x = \frac{1}{2}\left(h_+ x_0 + h_\times y_0\right), \qquad \delta y = \frac{1}{2}\left(h_\times x_0 - h_+ y_0\right)$$

In the MHD limit test particle are 'glued' to the magnetic field lines, so the magnetic field will exhibit the same behavior. Since GW in *x-y* plane and magnetic field in *x-z* plane, so a + polarized GW excites

$$\delta B_\times \propto \frac{1}{2} h_+ B_x^0, \qquad \delta B_y \propto \frac{1}{2} h_+ B_y^0 = 0$$

and a × polarized GW excites:

$$\delta B_\times \propto \frac{1}{2} h_\times B_y^0 = 0, \qquad \delta B_y \propto \frac{1}{2} h_\times B_x^0 .$$

In a pulsar environment the plasma initially flows out along the open field lines but develops into a force-force wind outside the light cylinder. Here the magnetic field is predominantly perpendicular to the propagation of the wind.

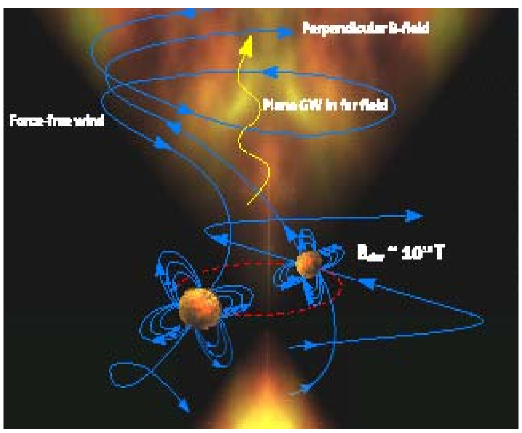

Fig 5. Merging neutron star binary

We see that Alfvén waves are excited by × polarized GW propagating at an angle to the ambient magnetic field. A+ polarized GW excites magneto-acoustic waves propagating parallel to the gravitational waves.

The most effective interaction occurs between the GW and the fast magneto-acoustic waves – as the phase speeds are closer the speed of light. In a merging binary pulsars with the magnetic field ~ 10^{12} T, a total energy of 10^{43} J can be transferred from gravitational waves to the pulsar wind[4,5].

ACKNOWLEDGEMENTS

Correspondence with Professor J. Kujipers of the Department of Astrophysics, University of Nijmegen, The Netherlands is acknowledged.

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