A Review of Eikonal Approximation and Recent Applications

Thesis Submitted To

The Department of Mathematics and Natural Sciences, BRAC University
in partial fulfilment of the requirements of the award of the degree of
Bachelor of Science in Physics

By

Arick Shama

Department of Mathematics and Natural Sciences
BRAC University
March, 2016

DECLARATION

I do hereby declare that the thesis titled "A Review of Eikonal Approximation and Recent Applications " is submitted to the Department of Mathematics and Natural Sciences of BRAC University in partial fulfilment of the requirements for the degree Bachelor of Science in Physics. This research is the work of my own and has not been submitted elsewhere for award of any other degree or diploma. Every work that has been used as reference for this work has been cited properly.

_____

Candidate
Arick Shama
ID:06211002

_____

Certified
Dr. Amin Hasan Kazi

Abstract

Eikonal Approximation deals with incidence where scattering of very high energy from a potential with a finite range a, where V is strongly suppressed for r larger than a. $E \gg |V|$ where E = energy of the incoming particles and $k \gg \frac{1}{a}$, meaning $\lambda \ll a$ where, $l_{max} \approx ka \gg 1$. The main contribution to the scattering amplitude therefore comes from partial waves with larger angular quantum number l and λ is restricted to small scattering angle.

The main advantage the Eikonal approximation offers is that the equations reduce to a differential equation with single variable. This reduction to single variable is due to the straight line approximation or the Eikonal approximation which allows choosing the straight line as a distinct direction. Recent Applications portrays that the approximation is still relevant in studying the features of particle physics interactions and optical problems.

Acknowledgement

This dissertation would not have been possible without guidance, patience and assistance of many people. I owe my gratitude to all those people who have made this dissertation possible and because of whom my under-graduation experience has been one that I will never forget.

My deepest gratitude is to my advisor, Dr. AMIN HASAN KAZI. I have been truly privileged to have an advisor who gave me courage to overcome odds and guided me throughout the process.

I express my sincere gratitude and respect to Professor A.A. Ziaud-din Ahmad, Chairperson, Department of Mathematics and Natural Sciences, BRAC University, for allowing me and being patient with me to finish my under graduate thesis.

I am also thankful to Late Prof. Dr. Mofiz Uddin Ahmed, Prof. Dr. Mohammed Arshad Momen and Prof. Dipen Bhattacharya.

My office GIZ has provided me with the opportunity to work on my thesis. My colleagues and friends have always celebrated my every success and encouraged for the next. I am very grateful to all.

Last but not the least, I would like to thank my family: for believing in me and being right beside me.

Contents

Chapter 1

Introduction

Eikonal Approximation is derived from EIKON, the Greek word for image. The approximation originated in Optics. Before the formulation of Maxwells electromagnetic theory, studies of reflection and refraction were done under a branch of physics called Ray Optics where it was assumed that light travelled in straight lines. This assumption holds as long as the size of the obstacle is large compared to the wavelength of light. "Image " is formed by light only in the straight line approximation.

Light scattered from an obstacle is related to its physical properties and hence in principle it is possible to obtain information about the scattering source from an analysis of the scattered light. Thus for many years, the light scattering technique has been used for inferring the size, shape and refractive index of particles in various scientific disciplines. Present areas of interest include bio particles, colloids, macromolecules, optical fibres, plasma diagnostics, atmospheric and astrophysical particles. Because of its simplicity, many instruments have been developed based on this technique for routine measurements in industry. Unfortunately the problems involving the scattering of light by optical scattering source are so complex that exact solutions are unknown except in the simplest and most idealised cases. Even in such cases the solutions are usually complicated and computationally tedious and it has long been desirable to obtain simple approximate formulae that also provide a physical insight into the scattering process. A considerable amount of work has been done in various disciplines towards assessing the usefulness

of a new approximation referred to in the literature either as the eikonal approximation or as the high-energy approximation. The purpose of this paper is to provide a much needed review of this work and discuss the recent applications.

There are times when we can treat a wave as "traveling in a straight line " with "limited scattering. " The eikonal approximation provides the wave equations when this holds as well as the conditions under which we should expect this to hold. The simplest picture is ray optics: although light is a wave, the scattering off of objects large compared to the wavelength of light can be treated as a perturbation. The Born approximation is the equivalent version in scattering of quantum mechanical waves.

Historically the eikonal approximation was born in optics where the term eikonal was introduced by Bruns in 18958.[9] Motivated by optical analogies, this approximation was then widely studied and used in potential and nuclear scattering where it found a number of applications. Its use in quantum field theory enabled one to sum up the high-energy behaviour of a very interesting set of Feynman graphs in a compact and useful manner. In the context of potential scattering the important feature of this approximation is that it was found to be valid in a domain where none of the existing approximations, namely the Born and the WKB, were valid.

The chapters in this dissertation contain summarized descriptions of approximations used in scattering problems namely - Partial wave expansion, Born approximation and WKB approximations, Eikonal approximation and the some recent applications of this approximation of Eikonal calculating final states of deep inelastic scattering from deuteron, Coulomb correction for quasi-elastic scattering, Diamagnetic field plasma, Two-dimensional fermions with long range current-current interaction and atom surface scattering .

Chapter 2

Approximation Used in Scattering

Scattering of one object from another is perhaps our best way of observing and learning about the microscopic world. Indeed it is the scattering of light from objects and the subsequent detection of the scattered light with our eyes that gives us the best information about the macroscopic world. We can learn the shapes of objects as well as some colour properties simply by observing scattered light.

There is a limit to what we can learn with visible light. In Quantum mechanics we know that we cannot discern details of microscopic systems (like atoms) that are smaller than the wavelength of the particle we are scattering. Since the minimum wavelength of visible light is about 0.40 microns, we cannot see atoms or anything smaller even with the use of optical microscopes. The physics of atoms, nuclei, subatomic particles, and the fundamental particles and interactions in nature must be studied by scattering particles of higher energy than the photons of visible light. In experiments on the scattering of a beam of particles, one measures the number of scattered particles falling per unit time on an area dS placed at a distance r from the scattering atoms. Example- A particle with an energy E and impact parameter b, and it immerges at scattering angle θ
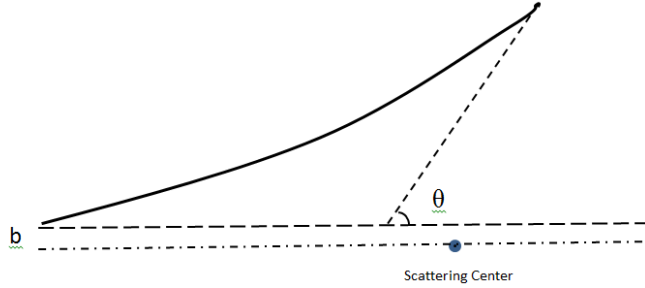
Figure 2.1: Scattering angle center

Ordinarily, the smaller the impact parameter, the greater the scattering angle.In quantum theory we imagine an incident plane wave, $\psi_z = Ae^{ikz}$, travelling in z direction, which encounters a scattering potential, producing an outgoing spherical wave

$$\psi_{r,\theta} \approx A\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\}, for large r \tag{2.1}$$

The wave number k is related to the energy of the incident particle

$$k \equiv \frac{\sqrt{2me}}{\hbar} \tag{2.2}$$

Scattering amplitude $f(\theta)$ gives the probability of scattering in a given direction and is related to the cross section.
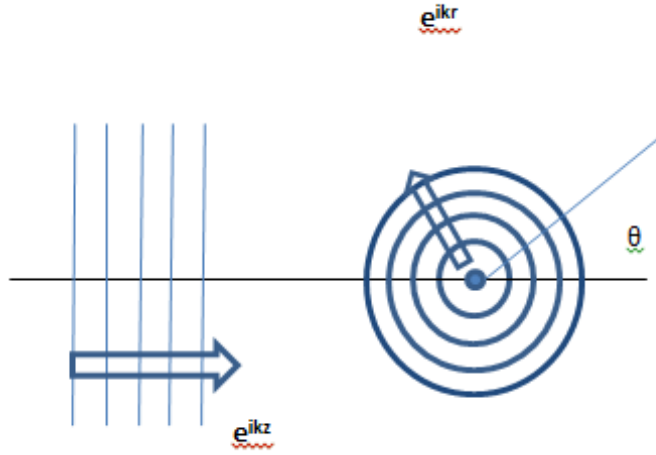
e^{ikr}

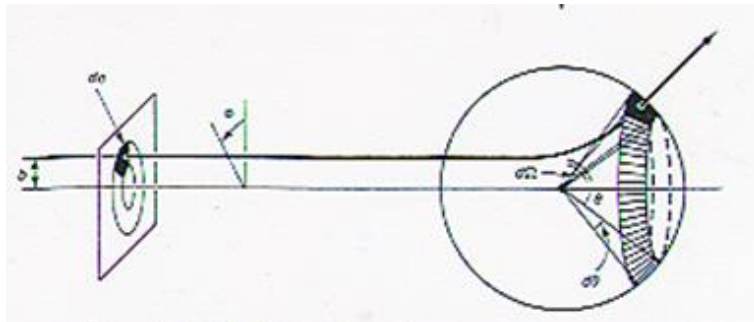e^{ikz}

θ

Figure 2.2: Impact parameter

Figure 2.3: scattering Amplitude

The scattering amplitude is obtained by solving the Schrdinger equation.

2.1 Partial Wave Expansion

[7] The closest we can have to an exact result of the scattering problems is the result obtained by the method of partial wave expansion. The method

breaks down the initial wave function in to infinite sum over angular momentum components labelled by the quantum number l. Each quantum number contributes to the scattering amplitude and is calculated separately. The complete scattering amplitude is then obtained by the summing over all the partial wave scattering amplitudes.

In situation where the semi classical approximation can be applied for given wave number k and scattering angle θ, the sum over l is dominated by values close to the spherical l_0. As it turns out that for most of the potentials (those without adequate initial energy) the contributions from l terms are very much higher the ka diminishes very rapidly, and thus can be neglected. Meaning, only the beams with the sufficient initial energy and those that pass through the scattering length of the potential are deflected. But to look at all possible scattering angles, one needs in general to include all the possible angular momenta and impact parameters and all the values the each l contributes. The situation is further complicated where quantum effects are important, since for a given wave number k and scattering angle θ, a wider range of values of l can give significant contributions. The scattering amplitude defining the differential and total cross section is given for a central problem by infinite sum over partial waves

$$f_\theta = \sum (2l+1) f_e P_e(cos\theta) \tag{2.3}$$

The partial amplitudes f_l are related to the phase shift δ_l displayed by the redial wave function $X_{l(r)}$ in the presence of the potential as compared to the free wave function $X_{0(r)} = r j_l(kr)$ (in absence of potential)

$$f_e = \frac{1}{2ik}(e^{2i\delta_l} - 1) = \frac{1}{k} e^{i\delta_l} sin\delta_l \tag{2.4}$$

The determination of the phase shift δ_l and partial amplitudes f_l requires solving the redial Schrdinger equation. To understand which values of l would make a significant contribution, we have to consider a finite range 'a' beyond which it is strongly suppressed and below which it is significant where l_{max} can be given by

$$l_{max} \simeq ka \tag{2.5}$$

In case of semiclasscal approximation, this corresponds; if $l \gg ka$ then impact parameter $b \gg a$ and the particle cannot be significantly deflected because it never passes the distance r within the range of the potential. For example WKB Approximation.

2.2 WKB Approximation

[3] WKB approximation developed by Wentzel, Keller and Brillouin is a semi classical method to solve Schrdinger equation that does not require the potential to be perturbative to solve a problem. If a potential is considered perturbative add an additional "perturbing " Hamiltonian is added to the calculation representing a weak disturbance to the system. If the disturbance is not too large, the various physical quantities associated with the perturbed system (e.g. its energy levels and eigenstates) can, from considerations of continuity, be expressed as "corrections " to those of the simple system.

In WKB approximation, it only assumes that certain classical quantities having the dimension of action (energy x time) are much larger than Planks constant.

A particle of energy E moving through a region where the potential $V(x)$ is constant, if $E > V$

$$\psi_x = Ae^{\pm ikx} \tag{2.6}$$

The basic idea of WKB approximation, identifies two different x dependence : rapid oscillation and gradual variation in amplitude and wavelength.

The most difficult aspect of the WKB approximation is in the immediate vicinity of a classical turning point. (where $E \approx V$, $\lambda \to \infty$)
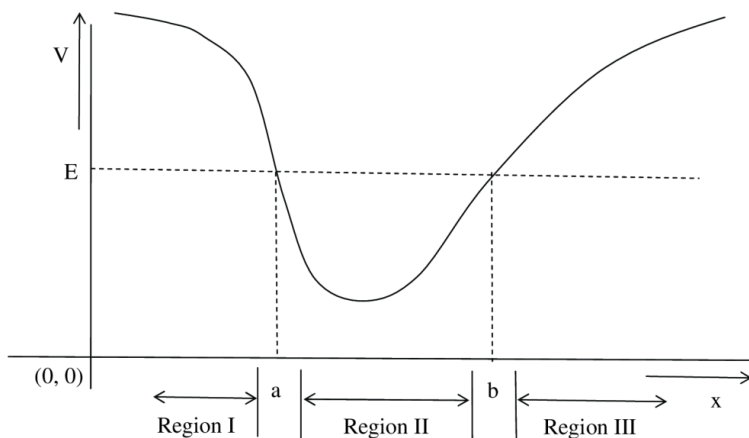

Figure 2.4: WKB turning points

The Schrdinger equation

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_x\psi = E\psi\frac{d^2\psi}{dx^2} = \frac{-\rho^2}{\hbar^2}\psi \tag{2.7}$$

where $\rho(x) = \sqrt{2m[E - V(x)]}$

classically, the particle is confined to a range of x

$$\phi_{(x)} = A_{(}x)e^{i\phi(x)} \tag{2.8}$$

$$\frac{d\phi}{dx} = (A' + iA\phi')e^{i\phi} \tag{2.9}$$

plugging in we get,

$$A'' = A\{(\phi')^2 - \frac{P^2}{\hbar^2}\} \tag{2.10}$$

$$(A^2\phi')^2 = 0 \tag{2.11}$$

$$A^2\phi' = C^2 \tag{2.12}$$

C is a real constant, we assume that the amplitude A varies slowly so that A term is negligible

$$(\phi')^2 = \frac{\rho^2}{\hbar^2} \tag{2.13}$$

$$\phi_{(x)} = \pm\frac{1}{\hbar}\int P(x)dx \tag{2.14}$$

For slowly varying $V(x)$ the first order and the zeroth order approximation gives almost same result

$$|\frac{\partial}{\partial x}k(x)| \ll |k^2(x)| \tag{2.15}$$

The WKB approximation breaks down when E approaches V (classical turning points) in which case the wave vector $k(x)$ approaches 0 but

the derivative does not and there in fact the argument of the approximation does not hold.

Under these circumstances the connection formulas must be applied to tie the regions together on each side of the turning points. For a reasonably smooth potential it may be an adequate approximation to treat a turning point region as one where the potential is increasing linearly with distance over a sufficient range that beyond this point the WKB approximation can be used in both directions.

The solution of Schrdingers equation for a linearly increasing or decreasing potential is well known, it is the Airy function, the solution of the differential equation plotted here at the left-hand turning point

$$\frac{d^2y}{dx^2} + xy = 0 \tag{2.16}$$

Figure 2.5: WKB solution- left hand
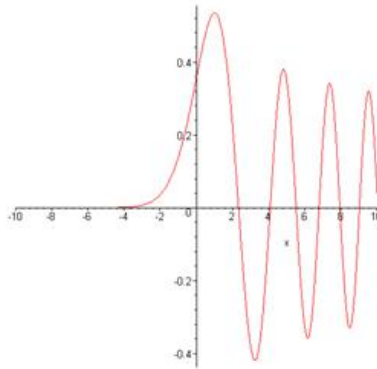
The strategy is to evaluate this function for large x, both positive and negative, so that we can join together the two WKB solutions, valid in the far regions, in a quantitative fashion.

Following Mathews and Walker, the differential equation is most simply solved by taking its Fourier transform. If

$$g(\omega) = \int_{\infty}^{\infty} y(x)e^{-i\omega x dx} \tag{2.17}$$

12

then

$$-\omega^2 g(\omega) + i\frac{dg}{dw} = 0, \, so \, g(\omega) = Ae^{i\frac{\omega^3}{3}} \tag{2.18}$$

Therefore

$$y(x) = A\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \exp[i(\omega x - \frac{\omega^3}{3})] \tag{2.19}$$

2.3 Born Approximation and Weak Coupling Scattering

In cases where the potential is considered to be very small and thus the scattering angle is also small. The effect of the potential can be taken into account in perturbative way. For a small potential the redial function $\chi_{(}r)$ in the presence of a potential differs very little from the radial wave function $\chi_o^l = rj_l(kr)$

The two differential equations satisfying the asymptotic momentum

$$\chi_l''(r) + [k^2 - \frac{l(l+1)}{r^2}]\chi_l(r) = \frac{2m}{\hbar^2}V_{(}r)\chi_l(r) \tag{2.20}$$

$$\chi_l''(r) + [k^2 - \frac{l(l+1)}{r^2}]\chi_l(r) = 0 \tag{2.21}$$

Taking the limit of the total derivative

$$\lim_{n\to\infty}[\chi_l'(r)\chi_l^0(r) - \chi_l^{0'}(r)\chi_l(r)] = \frac{2m}{\hbar^2}\int_0^{+\infty} V(r)\chi_l(r)\chi_l^0(r)dr \tag{2.22}$$

$$\chi_l^0(r) \simeq \frac{1}{k}sin(kr - \frac{l\chi}{2}) \tag{2.23}$$

$$\chi_l^0(r) = \frac{1}{k}\exp^{i\delta_l} sin(kr - \frac{l\pi}{2} + \delta_l) \tag{2.24}$$

$$\lim_{n\to\infty}[\chi_l'(r)\chi_l^0(r) - \chi_l^{0'}(r)\chi_l(r)] = \frac{1}{k}e^{i\delta_l}sin\delta_l = -f_l \tag{2.25}$$

$$-f_l = \frac{2m}{\hbar^2}\int_0^{+\infty} V(r)\chi_l(r)[rj_l(kr)]dr \tag{2.26}$$

13

In case where, the effect of the potential on the radial wave function χ_l is very small. Born approximation allows to take $\chi_l \approx \chi_l^0(r) = r j_l(kr)$ (since the effect of the potential is really small)

$$-f_l = \frac{2m}{\hbar^2} \int_0^{+\infty} V(r)[r j_l(kr)]^2 dr \tag{2.27}$$

An approximation result of phase shift can also be found since $|f_l| \ll 1$, which implies $\delta_l \ll 1$. Therefore, an approximate relation $\delta_l k f_l$ can give the following expression

$$\delta_l \simeq \frac{2m}{\hbar^2} \int_0^{+\infty} V(r)[r j_l(kr)]^2 dr \tag{2.28}$$

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l P_l(cos\theta) \tag{2.29}$$

$$= \frac{2m}{\hbar^2} \int_0^{+\infty} V(r)[\sum_{l=0}^{\infty}(2l+1)[j_l(kr)]^2 P_l(cos\theta)]r^2 dr \tag{2.30}$$

Therefore, momentum that is transferred from the potential is as below

$$\vec{q} = \vec{k}_{sca} - \vec{k}_i nc \tag{2.31}$$

Modulus of the momentum of the particle is same before and after scattering, since the potential tends to zero at infinity. Modulus is related to the scattering angle.

$$q^2 = k_{sca}^2 + k_{inc}^2 - 2\vec{k}_{sca} \cdot \vec{k}_{inc} \tag{2.32}$$

$$= 2k^2(1 - cos\theta) \tag{2.33}$$

$$q = 2k sin(\frac{\theta}{2}) \tag{2.34}$$

For a plane wave $e^{i\vec{q}\cdot\vec{r}} = e^{i\vec{k}_{sca}\cdot\vec{r}} - e^{i\vec{k}_{inc}\cdot\vec{r}}$

14

$$\sum_{l=0}^{\infty}(2l+1)[j_l(kr)]^2 P_l(cos\theta) = \frac{sin(qr)}{qr} \qquad (2.35)$$

$$f(q) \approx \frac{2m}{\hbar^2}\sum_{l=0}^{\infty}V(r)\frac{sin(qr)}{qr}r^2dr \qquad (2.36)$$

$$since,\ \int V(r)e^{-i\vec{q}\cdot\vec{r}}d^3\vec{r} = 4\pi\sum_{l=0}^{\infty}V(r)\frac{sin(qr)}{qr}r^2dr \qquad (2.37)$$

Therefore scattering amplitude in Born Approximation

$$f_q \approx \frac{m}{2\pi\hbar^2}\int V(r)e^{-i\vec{q}\cdot\vec{r}}d^3\vec{r} \qquad (2.38)$$

Differential cross section

$$\delta_q = (\frac{m}{2\pi\hbar^2})^2|\int V(r)e^{-i\vec{q}\cdot\vec{r}}d^3\vec{r}|^2 \qquad (2.39)$$

$$\delta_{tot} = \frac{\pi}{k^2}\int_0^{4k^2}\delta_q^2 dq^2 \qquad (2.40)$$

$f(q)$ is real in the first order expression, the imaginary part only arises at the second order in the perturbative theory corresponding to the leading effect in δ_{tot} is quadratic in potential. The Born approximation does not satisfy the optical theorem but rather is an approximate perturbative way. The first order expression for $f\theta$ is real and an imaginary part can arise only at second order in perturbation theory, corresponding to the fact that the leading effect in δ_{tot} is quadratic in the potential.

Chapter 3

Eikonal Approximation

The simplification deals with incidence where scattering of very high energy from a potential with a finite range a, where V is strongly suppressed for r larger than a. $E \gg |V|$ where E = energy of the incoming particles and $k \gg \frac{1}{a}$, meaning $\lambda \ll a$ where $l_{max} \approx ka >> 1$. The main contribution to the scattering amplitude therefore comes from partial waves with larger angular quantum number l and is restricted to small scattering angle.

The main advantage the Eikonal approximation offers is that the equations reduce to a differential equation with single variable. This reduction to single variable is due to the straight line approximation or the Eikonal approximation which allows choosing the straight line as a distinct direction.[8]The early steps involved in Eikonal approximation are similar to that of WKB approximation.

In Eikonal approximation we may assume $l \gg 1$ and $\ll 1$. This implies that the radial part of the problem is semi classical but the angular part of the problem is dominated by quantum effects.

The semi classical approximation for the phase shift, which is valid for $l \gg 1$

$$\delta_l \simeq \int_{r_0}^{\infty} \sqrt{k^2 - \frac{2m}{\hbar^2}V(r) - \frac{l^2}{r^2}}dr - \int_{r_0}^{\infty} \sqrt{k^2 - \frac{l^2}{r^2}}dr \qquad (3.1)$$

The turning point r_o is dominantly determined by the centrifugal barrier and can be approximately given by $r_o \approx \frac{1}{a}$

Expanding the power of $V_{(r)}$,

$$\delta_l \simeq \frac{m}{\hbar^2} \int_{\frac{l}{k}}^{\infty} V(r)[\sqrt{k^2 - \frac{l^2}{r^2}}]^{-\frac{1}{2}} dr \tag{3.2}$$

$$\simeq \frac{m}{\hbar^2 k} \int_{r_0}^{\infty} V(r)[1 - \frac{r_0^2}{r^2}]^{-\frac{1}{2}} \tag{3.3}$$

Rewriting $r = \sqrt{r_0^2 + z^2}, z = \sqrt{r^2 - r_0^2}, dz = \frac{r}{\sqrt{r - r_0^2}}$

$$\delta_l = \frac{-m}{2\hbar^2 k} \int_{-\infty}^{\infty} V(\sqrt{z^2 + \frac{l^2}{K^2}}) dz \tag{3.4}$$

$$\tag{3.5}$$

In the very high energy scattering the situation is not really semi classical but we may still, in a meaningful way use the semi classical formula where phase of the wave function is given by S/, where S is evaluated on a trajectory that is a straight line with fixed impact parameter b and z going from $-\infty to +\infty$ corresponding to the fact that the scattering angle must be very small

$$\delta_{(b)} \simeq \frac{1}{2} \int_{-\infty}^{\infty} \sqrt{k^2 - \frac{2m}{k^2} V(\sqrt{z^2 + b^2}) dz^{-\frac{1}{2}}} \int_{-\infty}^{\infty} k dz \tag{3.6}$$

$$\simeq \frac{-m}{2\hbar^2 k} \int_{-\infty}^{\infty} V(\sqrt{z^2 + b^2}) dz \tag{3.7}$$

Taking $b \simeq r_0 and \simeq \frac{l}{k}$

$$\delta_l \simeq \delta_b \tag{3.8}$$

The approximate behaviour of Legendre polynomials for small angle $\ll 1$ and large angular momenta $l \gg 1$ is used to derive an approximate expression for scattering amplitude

$$P_l(cos\theta) \simeq J_0(l\theta) \tag{3.9}$$

17

In terms of zeroth order Bessel function $J_o(z)$

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} \exp^{-izcos\psi} d\psi \tag{3.10}$$

Substituting $b \approx \frac{l}{k}$, $\delta_l = \delta_{(}b)$ and $J_o(l\theta) = J_o(k\theta b)$. k is then treated as a large quantity and approximate the infinite sum as an integral over b where $db \approx \frac{1}{k}$

$$f(\theta) \simeq \frac{1}{2ik} \sum_{l \gg 1}^{\infty} (2l + 1)(e^{2i\delta l} - 1)P_l(cos\theta) \tag{3.11}$$

$$\simeq \frac{1}{2ik} \sum_{l \gg 1}^{\infty} (e^{2i\delta l} - 1)J_0(l\theta)l \tag{3.12}$$

$$\simeq ik \int_0^{\infty} (e^{2i\delta_b} - 1)J_0(k\theta b)bdb \tag{3.13}$$

$$f(\theta) \simeq \frac{k}{2\pi i} \int_0^{\infty} \int_0^{2\pi} (e^{2i\delta_b} - 1)e^{-ik\theta cos\psi}bdbd\psi \tag{3.14}$$

In terms of momentum $-q$

$$f(q) \simeq \frac{k}{2\pi i} \int (e^{2i\delta_{(}b)} - 1)e^{-i\vec{q}\cdot\vec{b}}d^2b \tag{3.15}$$

The above equation shows the Eikonal approximation of the scattering amplitude. $\Delta_{(}b)$ phase is semi classical in nature. The approximation satisfies optical theorem exactly.

Thus

$$\delta_{tot} \simeq \frac{4\pi}{k}Imf(0) \qquad\qquad \simeq 4 \int sin^2\delta_b d^2\vec{b} \tag{3.16}$$

In this paper we are going to review the usefulness of the age-old Eikonal approximation. Revisit different applications addressed through this convenient tool. Discuss the achievements and limitations of the approximation. Below are the examples of scattering amplitude found through using Eikonal approximation.

3.1 Example

3.1.1 Barrier potential

Barrier potential is defines as

$$V(\vec{r}) = \begin{cases} V & \text{if } r \text{ is greater than a} \\ 0 & \text{if } r \text{ is less than a} \end{cases}$$

Where $\sqrt{b^2 + z^2} = r^2$,thus integrating z from 0 to $\sqrt{r^2 - b^2}$, We get

$$\xi(t) = -\sqrt{1 - t^2} \tag{3.17}$$

$$f(\theta) = a\frac{1}{i}ka \int_0^1 tdt J_0(tka\theta)[e^{-ika\frac{V}{E}\sqrt{1-t^2}} - 1] \tag{3.18}$$

$$\frac{\sigma_{scatt}}{\pi a^2} = 8 \int_0^1 tdt sin^2[ka\frac{V}{E}\sqrt{1 - t^2}] \tag{3.19}$$

The figure below is δ_{scatt} versus $ka\frac{V}{E}$ graph with comparison between the partial wave method and the obtained Eikonal approximation. Using eqn for $ka = 50$, we have the region of validity to be $1 \ll ka\frac{V}{E} \ll 7$, and for $ka = 750$, we have the region of validity to be $1 \ll ka\frac{V}{E} \ll 27$. These estimates for the valid region are in agreement with the curves obtained through Eikonal approximation and partial wave method.
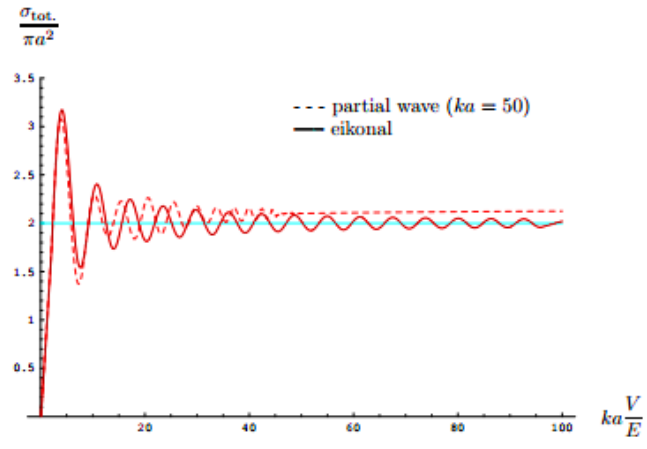
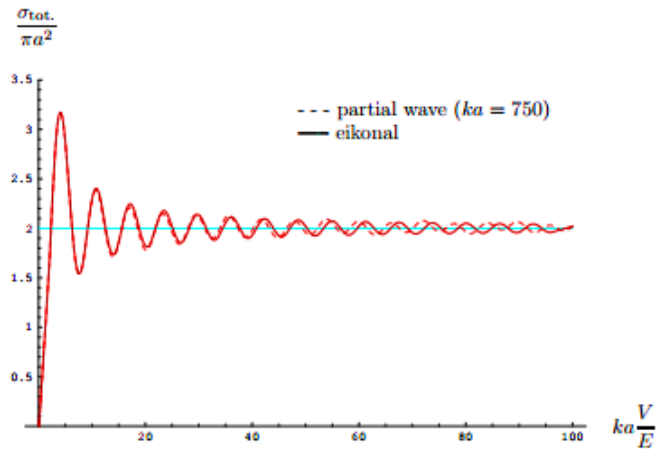Figure 3.1: Results of partial wave and eikonal approximation k=50

Figure 3.2: Results of partial wave and eikonal approximation k=750

3.1.2 Yukawa Potential

Considering Yukawa potential of

$$V(r) = -V\frac{e^{-\frac{r}{a}}}{\frac{r}{a}} \tag{3.20}$$

Where $V = \frac{e^2}{a}$ and $b^2 + z^2 = r^2$

$$\xi(t) = \frac{1}{2}\int_{-\infty}^{\infty} du \frac{e^{-\sqrt{t^2+u^2}}}{\sqrt{t^2+u^2}} = \int_0^{\infty} d\theta e^{-t\cosh\theta} = K_0(t) \tag{3.21}$$

Where $K_o(t)$ is modified Bessel function of order 0

$$f(\theta) = a\frac{1}{i}ka\int_0^{\infty} tdt J_0(tka\theta)[e^{ika\frac{V}{E}K_0(t)} - 1] \tag{3.22}$$

The contribution to $J_o(tka\theta)$ dies off very fast. Therefore the non-zero contribution to the integral for $tka\theta \ll 1$. The limit $t \ll 1$ the modified Bessel function is

$$K_0(t) \approx \ln\frac{2}{t} - \gamma \tag{3.23}$$

Where $\gamma = 0.577$is the Eulers constant. Further we observe that one of the terms in the above equation contributes only at $\theta = 0$ and thus is a delta function. Overall after substituting $\chi = ka\theta$

$$f(\theta) = ika^2\delta(\theta) - ie^{ika\frac{V}{E}(\ln 2 + \ln(ka\theta) - \gamma)}\frac{1}{k\theta^2}\int_0^{\infty} dx J_0(x)x^{1-ika\frac{V}{E}} \tag{3.24}$$

In terms of the gamma functions we have

$$\int_0^{\infty} dx J_n(x)x^{i+2i\alpha} = 2^{2i\alpha+1}\frac{\gamma(1+\frac{n}{2}+i\alpha)}{\gamma(\frac{n}{2})-i\alpha} \tag{3.25}$$

$$|f(\theta)|_{\theta\neq 0} = \frac{1}{2k\frac{\theta}{2}^2}\frac{2}{ka\frac{V}{E}}\frac{\gamma(1-i\frac{2}{ka\frac{V}{E}})}{\gamma(1-i\frac{2}{ka\frac{V}{E}})} \tag{3.26}$$

Where $\frac{1}{2}ka\frac{V}{E} = \frac{e^2}{\hbar k}$ using $V = \frac{e^2}{a}$ and $\hbar k = p = mv$. The result is observed to be similar to the scattering amplitude due to a Coulomb potential for small scattering angles using the approximation $\sin\theta \approx \theta$.

Chapter 4

Some Recent Applications of Eikonal Approximation

4.1 Final-state interactions in inclusive deep-inelastic scattering from the deuteron (2013)[2]

Deuterons are composed of a proton and a neutron, is a stable particle. As an atom, it is called deuterium and as an isotope of hydrogen it has an abundance of 1.5×10^-4 compared to 0.99985 for ordinary hydrogen. The stability of deuterons is remarkable since the free neutron is unstable, undergoing beta decay with a half-life of 10.3 minutes. The measured binding energy of the deuteron is $2.2 MeV$.

In 2013, W. Cosyn, Department of Physics and Astronomy, Ghent University, W. Melnitchouk, Jefferson Lab and M. Sargsian, Department of Physics, Florida International University uses the optical theorem and the properties of high-energy diffractive rescattering, and obtains a general result derived within the generalized eikonal approximation for the final state interaction (FSI) contribution to the inclusive deep inelastic scattering (DIS) deuteron cross section. Building on the knowledge gained from the semi-inclusive analyses, the paper extends the approach to inclusive DIS from the deuteron, over a similar range of Q^2 and W that was covered in the SIDIS

kinematics. The observation from the SIDIS studies that the FSI structure is consistent with diffractive scattering allows the generalized eikonal approximation (GEA) model to be extended to the inclusive DIS reaction through the optical theorem, relating the inclusive cross section to the imaginary part of the forward the imaginary part of the forward $\gamma^* D$ Compton scattering amplitude. The paper concludes that at $\chi > 0.6$ and $Q^2 < 10 GeV2$ the FSI effects can contribute to the deuteron F_D^2 structure function at the level of 25, and should be considered in extractions of the neutron structure function from inclusive deuteron data at low Q^2. At larger Q^2 values ($Q^2 and 10 GeV^2$) in the deep-inelastic region the FSI effects are found to be negligible.

4.2 Coulomb corrections for quasielastic (e, e) scattering (2004)[1]

The inclusive quasielastic scattering process (e, e) -where only the scattered electron is observed in a knockout reaction where the nucleons are hit by the virtual photon emitted by the scattered electron. Inclusive scattering provides information on a number of interesting nuclear properties:

- The width of the quasielastic peak allows a dynamical measurement of the nuclear Fermi momentum.

- The tail of the quasielastic peak at low energy loss and large momentum transfer gives information on high-momentum components in nuclear wave functions

- The integral strength of quasielastic scattering, when compared to sum rules, tells us about the reaction mechanism and eventual modifications of nucleon form factors in the nuclear medium.

- The scaling properties of the quasielastic response allow to study the reaction mechanism.

- Extrapolation of the quasielastic response to A = provides us with a very valuable observable for infinite nuclear matter.

For heavier nuclei, these questions obviously can only be addressed once the Coulomb distortion of the electron waves is properly dealt with.

"Coulomb Corrections for Quasielastic (e,e) Scattering: Eikonal Approximation "(2004)- a paper by Andreas Aste, Kai Hencken, Jrg Jourdan, Ingo Sick and Dirk Trautmann, an approximate treatment of electron CC for inclusive quasi- elastic (e, e) reactions modeled as a nucleon knockout process. In the plane-wave Born approximation (PWBA), the electrons are described as plane Dirac waves, which is a poor approximation for heavy nuclei with strong Coulomb fields. In a better approach, called eikonal distorted wave Born approximation (eDWBA), we use electron waves which are distorted by an additional phase and a change in the amplitude. This phase shift and the modification of the amplitude account for the enhanced momentum and a focusing effect which occurs when the electron approaches the strongly attractive nucleus.

A reliable treatment of Coulomb distortion is needed in particular for a determination of the longitudinal response function and for an extrapolation of nuclear responses to infinite nuclear matter. The eikonal approximation is more transparent and numerically easier to deal with than the exact treatments (solution of the full Dirac equation). At the same time, the eikonal approximation is much more realistic than the effective momentum approximation often employed in the absence of results from exact calculations. The eikonal results for the Coulomb distortion are very close to the results of exact calculations.

4.3   Diamagnetic field-plasma interaction (2007)[4]

O. Keller in his paper Photon wave mechanics in the eikonal limit: Diamagnetic field-plasma interaction(2007) shows microscopic eikonal theory can be established on the basis of photon wave mechanics, i.e. the first-quantized theory of the photon. The papers starting point is the century-old observation of Hamilton that the theory of classical mechanics in the Hamilton-Jacobi formulation shows a formal analogy to the eikonal theory, the basis of geometrical optics.

The diamagnetic field-plasma interaction is of crucial importance for the new eikonal theory, because this interaction dominates at high frequencies. The diamagnetic interaction allows identifying a massive transverse photon (quasi-particle) as "the particle " of the microscopic eikonal theory.

Microscopic Eikonal formed is identical to the one known in macroscopic electrodynamics, without replacing the microscopic field by its macroscopic (locally averaged) value. Macroscopic Electrodynamics incorporates the nonlocal response of matter to first order only (electric plus magnetic dipole response). High-order multi pole responses usually are important in the microscopic theory, and the resulting local-field effects manifest themselves in near-field electrodynamics.

4.4   Two dimensional fermions with long range current-current interaction [5]

D.V Khveshchenko and P.C.E Stamp studied the behaviour of response function of two- dimensional fermions interacting via a long range transverse gauge field in the eikonal approximation in 1994. They observed that an exponentially vanishing wave function renormalization prevents divergences in the density-density correlation function and the pairing susceptibility.
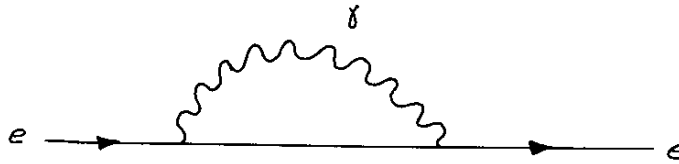

Figure 4.1: Fermion self energy correction

At $\epsilon \approx g^{-6}$ this power law behaviour turns into the exponential

asymptotic of the equation below

$$G(\epsilon, \rho F) \approx \frac{g^{-\frac{3}{2}}}{\epsilon^{\frac{5}{4}}} \exp(-\frac{g^{-3}}{\epsilon^{\frac{1}{2}}}) \tag{4.1}$$

It may seem that the exponential behaviour of the one-particle Green Function in the vicinity of the Fermi surface is an artifact caused by a gauge non-invariance of the object. However the paper goes onto showing that the trace of exponentially decaying Z factor does appear in both gauge invariant and non- invariant response functions which typically receive thei8r singular contributions from momenta close to the Fermi surface. Due to this fact real divergence of susceptibility in both particleparticle and particle-hole channels, which would demonstrate a tendency towards pairing or a formation of charge density wave are not found. The behaviour of one particle Green function is reflected in the experiment conducted, which are sensitive to the behaviour of Fermions near the Fermi energy and thus leading to a dramatic suppression of oscillations in the orbital magnetization in a weak external magnetic field.

The physicists also show that to recover the results of the eikonal approximation, capturing the most relevant features of the long-wavelength dynamics, one has to use the effective bosonic Lagrangian which is purely one dimensional. Using the representation provided by the document, restoration of non-Fermi liquid like properties of the low energy particle-hole subspace of the entire Hilbert space is possible. An example of the application provided - The spectrum of the bosonic collective mode governing particle-hole dynamics, and its contribution to specific heat.

4.5   Atom Surface scattering- Effects of a corrugated attractive well (2000)[6]

The eikonal approximation, which is an extremely useful method of calculating intensities for the scattering of atomic beams from surfaces, is extended to include a periodic corrugation of the leading edge of an attractive square-well potential placed in front of the hard repulsive wall. This provides a method for estimating small effects of corrugation of the attractive physisorption po-

tential on the diffraction spectra. Calculations by J. R. Manson and K.-H. Rieder indicate that the relative phase of the attractive well corrugations, with respect to those of the hard repulsive wall, has a distinctive and characteristic effect on the diffraction intensities.

In this paper the eikonal approximation as applied to elastic atom-surface scattering has been reformulated in terms of the theory of scattering by a phase grating as commonly applied in sound wave or optical wave scattering. This formulation of the eikonal approximation has been used to solve for the diffraction intensities generated by a monoenergetic incident beam of atoms scattering from a hard corrugated wall having an attractive square adsorption well with a corrugated leading edge. This solution is used as a model for estimating the effects of corrugation within the attractive adsorption well and to compare effects of the well corrugation with those of the corrugation of the repulsive part of the potential.

Such a corrugated square-well model is expected to overestimate the effects on the intensity of a more realistic corrugated well potential with the correct $\frac{1}{z^3}$ behaviour of the long-range Van der Waals attraction. However, because of the simplicity of this formalism and the ease of calculations it is expected that this solution will be useful for predicting physical trends, just as the ordinary eikonal approximation is still very useful for obtaining crude theoretical estimates. An even simpler formalism, expressed entirely in terms of Bessel functions, results in the case of purely sinusoidal corrugations for the repulsive wall and leading edge of the well. Several example calculations were carried out, which demonstrate that the corrugation of the leading edge of the square well has an effect on the diffraction intensities that is about 5 percent as strong as that of an equally large corrugation of the repulsive wall. An interesting question that can be answered with this formulation concerns the effect of a well corrugation that is in or out of phase with the corrugation of the repulsive wall. The present calculations show that there is a very characteristic signature of the relative phase of the well corrugation with respect to the corrugation of the repulsive wall. If, when compared to a calculation with an uncorrugated well, the addition of corrugation to the well increases (or decreases) the intensity of a particular diffraction peak, then changing the phase of the well corrugation by 180 will reduce (or increase) the intensity of that same peak.

Chapter 5

Conclusion

The recent applications mentioned earlier projects that eikonal approximation is still relevant today in studying the features of particle physics interactions and optical problems. As seen above eikonal approximation provides with similar results for Coulombs correction for quasielastic (e, e) scattering (2004). Added advantage of eikonal is that even though it has similarities WKB approximation but unlike WKB approximation its variable is not described by the trajectory of the particle; which in general is complicated. Once a special direction can be established for a potential, eikonal approximation can give a very close results to that of manually calculated. A large class of potentials, and for all momentum transfers, each term of the eikonal multiple-scattering series gives the asymptotic value (for large incident wave numbers) of the corresponding term in the Born series. This property, together with the requirement of unitarity, implies that in weak-coupling situations the eikonal approximation is consistently worse than the second Born approximation. For intermediate couplings we find that the eikonal method is remarkably good at all angles for potentials of the Yukawa type. For the case of strong coupling $(|V_0| > E)$ we find that for all potentials studied there is good agreement between exact and eikonal results at small angles.

List of Figures

Bibliography

[1] Aste, A., Hencken, K., Jourdan, J., Sick, I., and Trautmann, D. (2004). Coulomb corrections for quasielastic (e, e) scattering: eikonal approximation. *Nuclear Physics A*, 743(4):259–282.

[2] Cosyn, W., Melnitchouk, W., and Sargsian, M. (2014). Final-state interactions in inclusive deep-inelastic scattering from the deuteron. *Physical Review C*, 89(1):014612.

[3] Griffiths, D. J. (2005). *Introduction to quantum mechanics*. Pearson Education India.

[4] Keller, O. (2008). Photon wave mechanics in the eikonal limit: Diamagnetic field-plasma interaction. *Laser and Particle Beams*, 26(2):287.

[5] Khveshchenko, D. and Stamp, P. (1994). Eikonal approximation in the theory of two-dimensional fermions with long-range current-current interactions. *Physical Review B*, 49(8):5227.

[6] Manson, J. and Rieder, K.-H. (2000). Eikonal approximation in atom-surface scattering: Effects of a corrugated attractive well. *Physical Review B*, 62(19):13142.

[7] Mott, N. and Massey, H. (1965). The theory of atomic collisions, clarendon.

[8] Shajesh, K. (2008). Eikonal approximation. *unpublished study report (Mar 2004), available at¡ http://www. nhn. ou. edu/ shajesh/eikonal*.

[9] Sharma, S. and Somerford, D. (1990). The eikonal approximation revisited. *Il Nuovo Cimento D*, 12(6):719–748.