From Local Operators to Extended Objects: Higher-Form Symmetries and Axion Quality

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B.Sc. in Physics

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Declaration

It is hereby declared that

1. The thesis submitted is my/our own original work while completing degree at Brac University.

2. The thesis does not contain material previously published or written by a third party, except where this is appropriately cited through full and accurate referencing.

3. The thesis does not contain material which has been accepted, or submitted, for any other degree or diploma at a university or other institution.

4. We have acknowledged all main sources of help.

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Abstract

The central conceptual shift underlying this thesis is the transition from viewing symmetries as acting only on local point operators to recognizing that symmetries can act on extended objects such as lines and higher-dimensional defects. This generalized perspective reorganizes the structure of gauge theories and provides a natural language for non-local constraints that are invisible in the traditional framework. We begin by reformulating ordinary (0-form) global symmetries in terms of topological symmetry defect operators (SDOs), emphasizing that their action on charged operators is understood purely by topological linking. We then develop the corresponding formalism for 1-form symmetries, where the charged objects are line operators. Using intersection theory and the topological nature of SDOs, we show why continuous 1-form symmetry groups are necessarily abelian and analyze their action on Wilson and 't Hooft lines. In this framework, four-dimensional Maxwell theory provides a satisfying explanation about the massless $4d$ photon: the Coulomb phase can be interpreted as a phase with spontaneous breaking of a continuous electric 1-form symmetry, with the photon appearing as the associated Goldstone mode.

With these structural tools in place, we turn to the strong CP problem and the axion. After reviewing the origin of the QCD θ-angle and the Peccei–Quinn mechanism, we focus on the axion quality problem, which revolves around the question of why the axion shift symmetry can be extraordinarily well preserved despite general expectations that quantum gravity effects violate global symmetries. The central application of this thesis is a higher-dimensional gauge theory realization of the axion in which the axion arises as a Kaluza–Klein zero mode of a higher-dimensional $U(1)$ gauge field. In this setting, the would-be axion shift symmetry is embedded into an electric higher-form symmetry, sharply constraining which operators can contribute to an axion potential. We show how introducing electrically charged matter breaks the continuous electric 1-form symmetry to a discrete subgroup and how non-perturbative wrapped worldline configurations of heavy charged particles generate the allowed axion potential with exponential suppression. Thus, higher-form symmetry controls the structure of the axion potential while non-local dynamics controls its suppressed nature, providing exponentially good control over the axion quality problem.

Keywords: Generalized global symmetries; higher-form symmetries; symmetry defect operators; linking number; intersection theory; 1-form symmetry; Wilson lines; 't Hooft lines; spontaneous symmetry breaking; Coulomb phase; strong CP problem; Peccei–Quinn mechanism; axion; axion quality problem; Kaluza–Klein compactification; extra dimensions; worldline instantons; non-perturbative effects.

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Chapter 1

Introduction

Symmetry has always served as one of the most powerful guiding principles in physics. Symmetry provides a systematic way to classify physical laws, restrict allowed dynamics, and uncover conserved quantities. The development of group theory in the twentieth century firmly established symmetry as the mathematical backbone of modern theoretical physics, which plays a central role in QFT and particle physics.

Historically, our understanding of symmetry has been built almost entirely around local, point-like operators. In this traditional framework, continuous global symmetries act on these local operators and, through Noether's theorem, lead to conserved currents and charges. Alongside these global symmetries, gauge symmetries arise as a different structure: rather than representing physical transformations of states, gauge symmetries encode a redundancy in our description. Physical observables must be gauge invariant, and the gauge symmetry itself does not correspond to an observable conserved charge. This distinction between global symmetries and gauge redundancies is one of the foundational pillars of modern quantum field theory.

For many decades, this point-like operator viewpoint dominated our conception of symmetry. However, it gradually became clear that certain physical phenomena, particularly those involving non-local operators, could not be fully captured within the traditional framework. This limitation was addressed by the seminal work of David Gaiotto, Anton Kapustin, Nathan Seiberg, and Brian Willett [7], which introduced the modern concept of generalized global symmetries, now commonly known as higher-form symmetries. What distinguishes higher-form symmetries from the traditional framework of symmetry is that they act on extended objects which are described by nonlocal operators such as line operators, surface operators, and higher-dimensional defects.

In this generalized framework, ordinary global symmetries become 0-form symmetries, which act on point operators. A 1-form symmetry acts on line operators such as Wilson and 't Hooft lines, a 2-form symmetry acts on surface operators, and so on. These symmetries are generated not by local charges, but by topological defect operators supported on extended submanifolds. This shift from point-like to extended charged objects leads to radically new physical interpretations. One striking consequence is that familiar phases of gauge theories, such as the Coulomb phase, can be reinterpreted as phases of spontaneous breaking of higher-form symmetries. Even the photon itself may be viewed as a Goldstone mode of a spontaneously broken 1-form symmetry, a perspective that would have been inaccessible in the traditional symmetry framework.

The power of higher-form symmetries lies not only in classification but also in dynamical constraint. Because these symmetries act on non-local operators and are sensitive to topological linking, they impose sharp restrictions on which non-perturbative effects are allowed. This has opened a new pathway for addressing long-standing problems in quantum field theory and particle physics. One such problem is the axion quality problem, which concerns the stability of the axion shift symmetry against uncontrolled ultraviolet corrections.

The axion was originally introduced as part of the Peccei–Quinn mechanism to solve the strong CP problem in quantum chromodynamics (QCD). Although the Peccei–Quinn symmetry is designed to be exact at the classical level, generic quantum gravity effects are expected to violate global symmetries. The central question of axion quality is therefore: why is the axion shift symmetry so exceptionally well preserved? Recent developments have shown that higher-form symmetries, particularly when realized through extra-dimensional gauge theories, can provide a natural and technically robust explanation for this protection.

This thesis is devoted to the systematic construction of higher-form symmetries, showing how non-local effects have a natural home in this new framework and how these effects could help us arrive at an elegant solution to the strong CP problem. The remainder of the thesis is structured as follows:

Chapter 2 reviews ordinary symmetries in classical and quantum field theory. We derive Noether currents, define conserved charges, and establish the Ward identities that encode symmetry at the quantum level.

Chapter 3 reformulates 0-form symmetries in terms of topological symmetry defect operators (SDOs). We show how symmetries act by linking with operators and how charges acquire a purely topological interpretation.

Chapter 4 introduces 1-form symmetries, which act on line operators rather than point operators. We develop the intersection theory and linking formalism needed to describe their action, prove that all continuous 1-form symmetries are necessarily abelian, and study their realization in four-dimensional Maxwell theory. We also interpret the Coulomb phase as spontaneous breaking of a 1-form symmetry.

Chapter 5 generalizes the construction to p-form symmetries, establishing their Ward identities and the universal structure of their symmetry defect operators and charged extended operators.

Chapter 6 develops spontaneous symmetry breaking for both 0-form and higher-form symmetries. We review Goldstone's theorem in the ordinary case and its generalized counterpart for p-form symmetries, providing the conceptual bridge between symmetry and low-energy excitations.

Chapter 7 turns to the strong CP problem and axions. We review the origin of the θ-angle in QCD, the Peccei–Quinn mechanism, the emergence of the axion as a pseudo-Goldstone boson, and the formulation of the axion quality problem.

Chapter 8 presents the central application of this thesis: the interpretation of the axion as a Kaluza–Klein zero mode of a higher-dimensional gauge field, the emergence of associated higher-form symmetries, and the way in which dynamical charged matter and symmetry gauging modify these symmetries. We show how wrapped worldline effects generate controlled axion potentials. providing a solution to the axion quality problem.

Chapter 9 summarizes the main results and outlines future directions.

In summary, this thesis aims to demonstrate that higher-form symmetries provide not only a refined classification of quantum field theories but also a powerful dynamical principle capable of addressing deep problems in particle physics. By extending the notion of symmetry beyond point-like operators to extended topological structures, we gain a new and unifying perspective on gauge theories, phases of matter, and the remarkable robustness of the axion.

Chapter 2

Aspects of Ordinary Symmetries

2.1 Symmetries in Classical Field Theory

In this section, we will discuss how Noether's theorem allows us to see the relationship between continuous global symmetries and the corresponding conserved currents.

Given an action $S[\Phi]$ depending on a set of fields Φ, we assume that the action is invariant (up to a total derivative term, which we shall ignore) under an infinitesimal transformation of the fields

$$\Phi_i \to \Phi_i' = \Phi_i + \alpha^a \delta_a \Phi_i \tag{2.1.1}$$

where α^a are constant (global) parameters, which, based on our assumption, will define a classical symmetry.

To obtain the corresponding conserved current, we let the global parameters α^a to be local, i.e., $\alpha^a \to \alpha^a(x)$ for which the transformation (2.1.1) will generically no longer be a symmetry of the action.

The induced variation of the action must be proportional to $\partial_\mu \alpha^a(x)$, and the variation should vanish once restricted to constant global parameters α^a. Therefore, we can write the variation as

$$\delta S = \int d^d x J_a^\mu(\Phi_i(x), \partial_\mu \Phi_i(x), \ldots) \partial_\mu \alpha^a(x), \tag{2.1.2}$$

with J_μ^a being some local function of the fields. Now we note that when the fields are on-shell, any field configuration is a stationary point of the action, and when evaluated on-shell, the action is invariant under any infinitesimal transformation of the fields. Then

$$\delta S = \int d^d x J_a^\mu \partial_\mu \alpha^a(x) = 0. \tag{2.1.3}$$

Doing an integration by parts and dropping the boundary term by assuming that the fields vanish at infinity, we get

$$-\int d^dx\alpha^a(x)\partial_\mu J_a^\mu = 0. \tag{2.1.4}$$

And since this holds for arbitrary $\alpha^a(x)$ we reach the conclusion that J_a^μ are conserved currents, and we get the conservation law

$$\partial_\mu J_a^\mu = 0 \tag{2.1.5}$$

There is an associated conserved charge that is defined as an integral of J_a^0 over a constant time slice.

$$Q_a(t) = \int d^{d-1}x J_a^0(x,t) \tag{2.1.6}$$

The conservation of Q_a follows from (2.1.5)

$$\partial_0 Q_a = \int d^{d-1}x\partial_0 J_a^0 = -\int d^{d-1}x\partial_i J_a^i = -\int d^{d-2}xn_i J_a^i = 0, \tag{2.1.7}$$

where we have used Stokes theorem. Here n_i is the outward pointing unit normal on the boundary at spatial infinity, and we have assumed that the fields (and hence the current J_a^i) vanish sufficiently fast at spatial infinity so that the boundary integral vanishes.

We note that we can add any term of the form $\partial_\nu B^{\mu\nu}$ with anti-symmetric indices on $B^{\mu\nu}$ and define a new current

$$\tilde{J}_a^\mu = J_a^\mu + \partial_\nu B_a^{\mu\nu}, \tag{2.1.8}$$

which are still conserved and lead to the same charges.

The conserved charges are the generators of infinitesimal transformations. They do so via the Poisson brackets

$$\delta_a\Phi_i = \{\Phi_i, Q_a\}. \tag{2.1.9}$$

2.2 Symmetries In Quantum Field Theory

2.2.1 Canonical Formalism

In qft, the operator of concern, when we consider acting with symmetry, is an unitary operator. For continuous symmetries,

$$U = \exp\,(i\epsilon_a Q_a). \tag{2.2.1}$$

These unitary operators act on field by conjugation

$$\phi \to \phi' = U\phi U^\dagger, \tag{2.2.2}$$

with the infinitesimal transformation being

$$\delta_a \phi = i[Q_a\ , \phi]. \tag{2.2.3}$$

2.2.2 Ward Identities in Quantum Field Theory

We need a quantum version of our classical conservation laws. As we shall see, these are given by the Ward identities. We start with the partition function

$$Z[J] = \int \mathcal{D}\phi_a \exp\left(iS[\phi_a] + i\int d^dx J\phi_a\right) \tag{2.2.4}$$

here $J(x)$ is the background source for ϕ. In this section only, we will denote the current as N^μ so as to avoid confusing it with the source. Consider the following transformation

$$\phi_a \to \phi'_a = \phi_a + \alpha(x)\delta\phi_a \tag{2.2.5}$$

The fields are integration variables, and so we shall consider this as a change of variables in the partition function

$$Z[J] = \int \mathcal{D}\phi'_a \exp\left(iS[\phi'_a] + i\int d^dx J\phi'_a\right) \tag{2.2.6}$$

We know from equation (2.1.4) that

$$S[\phi'_a] = S[\phi_a] - \int d^dx \alpha(x)\partial_\mu N^\mu_a \tag{2.2.7}$$

Using equations (2.2.5) and (2.2.7) and expanding to the leading order in α, we can write (2.2.6) as

$$\begin{aligned} Z[J] &= \int \mathcal{D}\phi'_a \exp\left(iS[\phi_a] + i\int d^dx J\phi_a\right)\exp\left(-i\int d^dx\alpha(x)(\partial_\mu N^\mu_a - J\delta\phi_a)\right) \\ &\approx \int \mathcal{D}\phi'_a \exp\left(iS[\phi_a] + i\int d^dx J\phi_a\right)\left[1 - i\int d^dx\alpha(x)\left(\partial_\mu N^\mu_a - J\delta\phi_a\right)\right] \end{aligned} \tag{2.2.8}$$

We now need an extra condition that will help us relate equations (2.2.4) and (2.2.8) namely that (2.2.5) be a symmetry of the measure. If it is not a symmetry of the measure, then we would have to account for eventual anomalies. But we will be doing this derivation assuming the absence of anomalies and so we require

$$\mathcal{D}\phi_a = \mathcal{D}\phi'_a \tag{2.2.9}$$

We can now relate equations (2.2.4) and (2.2.8) and notice that the first term in (2.2.8) is just (2.2.4). Then we have

$$\int \mathcal{D}\phi_a \exp\left(iS[\phi_a] + i\int d^dx J\phi_a\right)\left[\int d^dx\alpha(x)\left(\partial_\mu N_a^\mu - J\delta\phi_a\right)\right] = 0 \tag{2.2.10}$$

Since this is true for all $\alpha(x)$ we get

$$\int \mathcal{D}\phi_a \exp\left(iS[\phi_a] + i\int d^dx J\phi_a\right)\left(\partial_\mu N_a^\mu - J(x)\delta\phi_a\right) = 0. \tag{2.2.11}$$

We can now extract the Ward identities from the equation above. Setting $J = 0$ gives us

$$\langle \partial_\mu N_a^\mu \rangle = 0. \tag{2.2.12}$$

To get insertions, we can differentiate with respect to $J(x')$ and then set $J = 0$

$$\frac{-i\delta}{\delta J(x')}\left[\int \mathcal{D}\phi_a \exp\left(iS[\phi_a] + i\int d^dx J(x)\phi_a(x)\right)\left(\partial_\mu N_a^\mu - J(x)\delta\phi_a\right)\right] = 0. \tag{2.2.13}$$

And so we get

$$\begin{aligned}\int \mathcal{D}\phi_a \exp\left(iS[\phi_a] + i\int d^dx J(x)\phi_a(x)\right)\left(\int d^dx\delta(x-x')\phi_a(x)\right)\left(\partial_\mu N_a^\mu - J(x)\delta\phi_a\right)\\ +i\int \mathcal{D}\phi_a \exp\left(iS[\phi_a] + i\int d^dx J(x)\phi_a(x)\right)\delta(x-x')\delta\phi_a = 0\end{aligned} \tag{2.2.14}$$

Setting $J = 0$ would give us

$$\partial_\mu\langle N^\mu(x)\phi(x')\rangle = -i\delta(x-x')\langle\delta\phi_a\rangle \tag{2.2.15}$$

To get multiple insertions, we would need successive differentiations.

If x does not coincide with any insertion points,

$$\partial_\mu\langle N^\mu(x)\phi(x^1)\ldots\phi(x^n)\rangle = 0 \quad \text{for } x \neq x^i \tag{2.2.16}$$

For every insertion point with which x coincides, we pick up a term proportional to $\delta\phi$ like in (2.2.15).

These expressions are collectively called Ward identities. They have the interpretation that the correlators of $\partial_\mu N^\mu$ with other operators that are charged under the symmetry associated with the current vanish as long as its position does not coincide with the insertion points of other operators.

Chapter 3

0-Form Symmetries

3.1 Symmetries as Topological operators

A Noether current is associated with a symmetry, and it satisfies the continuity equation

$$\partial_\mu J^\mu = 0. \tag{3.1.1}$$

We can think of the Noether Current as a component of a 1-form

$$J = J_\mu dx^\mu. \tag{3.1.2}$$

It's Hodge dual is a $(d-1)$-form

$$\star J \equiv \frac{1}{(d-1)!} J_\mu \epsilon^\mu{}_{\mu_1 \cdots \mu_{d-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{d-1}}. \tag{3.1.3}$$

Then (3.1.1) can be rewritten as

$$d \star J = 0 \tag{3.1.4}$$

since

$$\begin{aligned} d \star J &= \frac{1}{(d-1)!} (\partial_\nu J_\mu) \epsilon^\mu_{\mu_1 \cdots \mu_{d-1}} dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{d-1}} \\ &= \frac{1}{(d-1)!} (\partial_\nu J_\mu) \epsilon^\mu_{\mu_1 \cdots \mu_{d-1}} \epsilon^{\nu \mu_1 \cdots \mu_{d-1}} dx^0 \wedge \cdots \wedge dx^{d-1} \\ &= (\partial_\mu J^\mu) dx^0 \wedge \cdots \wedge dx^{d-1} = 0, \end{aligned} \tag{3.1.5}$$

where

$$\epsilon^\mu{}_{\mu_1 \cdots \mu_{d-1}} \epsilon^{\nu \mu_1 \cdots \mu_{d-1}} = (d-1)! \ \delta^{\mu\nu}. \tag{3.1.6}$$

Equation (3.1.4) implies that $\star J$ is a closed form.

The Noether charge can be written as

$$Q(\Sigma_{d-1}) = \int_{\Sigma_{d-1}} \star J = \int_{\Sigma_{d-1}} \frac{1}{(d-1)!} J_\mu \ \epsilon^{\mu}_{\mu_1 \ldots \mu_{d-1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{d-1}}, \tag{3.1.7}$$

where Σ_{d-1} is a closed $(d-1)$-dimensional submanifold.
If we use the Ward identity

$$\partial_\mu \langle J^\mu(x) \mathcal{O}(y) \rangle = -i\delta^{(d)}(x-y) \langle \delta \mathcal{O}(y) \rangle, \tag{3.1.8}$$

and integrate it over Σ_d bounded by Σ_{d-1}, then using Stokes theorem the left hand side gives us

$$\int_{\Sigma_d} \langle d \star J \mathcal{O}(y) \rangle = \int_{\Sigma_{d-1}} \langle \star J \mathcal{O}(y) \rangle = \langle Q(\Sigma_{d-1}) \mathcal{O}(y) \rangle. \tag{3.1.9}$$

Using this to rewrite (3.1.8), we get

$$\langle Q(\Sigma_{d-1}) \ \mathcal{O}(y) \rangle = -i \int_{\Sigma_d} d^d x \delta^{(d)}(x-y) \langle \delta \mathcal{O}(y) \rangle. \tag{3.1.10}$$

Here, the number $\int_{\Sigma_d} d^d x \delta^{(d)}(x-y)$ can either take the value 0 or 1 depending on whether y is inside Σ_d or not. We will define this as a *linking number* of Σ_{d-1} and y.

$$\mathrm{Link}(\Sigma_{d-1}, y) = \int_{\Sigma_d} d^d x \ \delta^{(d)}(x-y). \tag{3.1.11}$$

We can then write (3.1.10) as

$$\langle Q(\Sigma_{d-1}) \mathcal{O}(y) \rangle = -i \ \mathrm{Link}(\Sigma_{d-1}, y) \langle \delta \mathcal{O}(y) \rangle. \tag{3.1.12}$$

Here, both the link number and the charge are topological in the sense that they do not change under *deformations.*The deformation of the surface Σ_{d-1} would have no effect on the link number as long as the deformations do not cross the point y, which means that if $y \in \Sigma_d$ the deformations should not be such that $y \notin \Sigma_d$ anymore. The charge is also topological and we see that if we deform Σ_d into $\Sigma'_d = \Sigma_d \cup \Sigma_{d_0}$ and make sure that $y \notin \Sigma_{d_0}$. This gives us

$$\langle Q(\Sigma_{d-1} + \partial \Sigma_{d_0}) \ \mathcal{O}(y) \rangle = \int_{\Sigma_d \cup \Sigma_{d_0}} \langle d * J \mathcal{O}(y) \rangle = \int_{\Sigma_d} \langle d * J \mathcal{O}(y) \rangle + \int_{\Sigma_{d_0}} \langle d * J \mathcal{O}(y) \rangle. \tag{3.1.13}$$

Here, since $y \notin \Sigma_{d_0}$ we can use the conservation law to set $\int_{\Sigma_{d_0}} \langle d * J \mathcal{O}(y) \rangle = 0$ and we get

$$\langle Q(\Sigma_{d-1} + \partial\Sigma_{d_0})\mathcal{O}(y)\rangle = \int_{\Sigma_d} \langle d * J\mathcal{O}(y)\rangle = \langle Q(\Sigma_{d-1})\mathcal{O}(y)\rangle. \tag{3.1.14}$$

By formulating $Q(\Sigma_{d-1})$ in this way, we see that the conservation law is translated into the fact that $Q(\Sigma_{d-1})$ is topological.

we also note that the link number defined in (3.1.11) can in general be established for any spacetime dimension d between a point and a closed $(d-1)$ manifold surrounding it. Recall that we defined

$$\mathrm{Link}(\Sigma_{d-1}, y) = \int_{\Sigma_d} d^d x \delta^{(d)}(x-y). \tag{3.1.15}$$

We are integrating over the whole d-manifold Σ_d. If $y \in \Sigma_d$, it gives us 1. So, we can always link a closed $(d-1)$-manifold like S^{d-1} that surrounds a point y, which means y is a point in the interior of S^{d-1}.

For a global symmetry element $g \in G$, the QFT has a unitary operator U_g, which acts on the Hilbert space. By definition of a symmetry of the theory $U_g\ \mathcal{O}(x)\ U_g^\dagger$ is again an operator in the theory, transformed under g. A true symmetry means that the time evolution is unchanged when conjugated by U_g. Time evolution is generated by the Hamiltonian H. So, the condition is

$$U_g\ H\ U_g^\dagger = H, \tag{3.1.16}$$

which is the invariance of the Hamiltonian under the symmetry. Because if the Hamiltonian is changed, then the symmetry would not persist under time evolution: the same transformation at different times would act differently. Therefore,

$$[U_g, H] = 0 \tag{3.1.17}$$

Now recall the Heisenberg equation of motion for any operator $\mathcal{O}(t)$

$$\partial_t\ \mathcal{O}(t) = i\left[H, \mathcal{O}(t)\right]. \tag{3.1.18}$$

Then

$$\partial_t\ U_g = i\left[H, U_g\right] = 0. \tag{3.1.19}$$

So U_g is time independent: the symmetry defect operator does not evolve over time. We have $U_g(t) = U_g(t')$ for all t and t'. By conjugation, it transforms a local operator $\mathcal{O}(x)$ into another local operator $\mathcal{O}'(x)$. For example, in $\Sigma_{d=1} = \mathbb{R}$, $U(t)$ is the symmetry operator defined at a point $t \in \mathbb{R}$. Since $[U, H] = 0$ U can move along the time axis as long as it does not cross $\mathcal{O}(t)$.

Now in d-dimensions we can insert U which is a codimension-1 operator along a codimension-1 manifold Σ_{d-1}. We denote this by $U(\Sigma_{d-1})$. We can generalize the

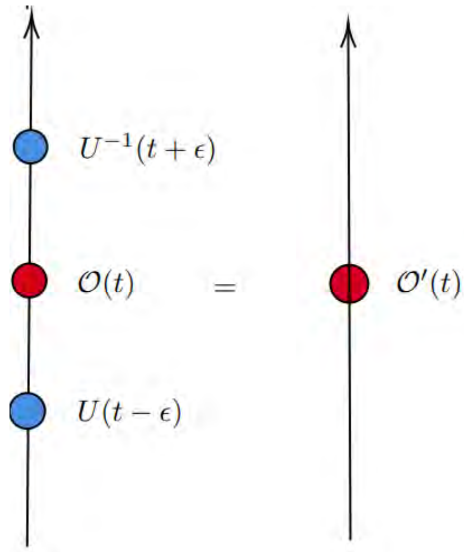

Figure 3.1. Symmerty operator insertion in 1D

relation $U(t) = U(t')$ for all t and t' by noticing the fact that for a fixed time t, we can choose Σ_{d-1} to be a spatial slice, where Σ_{d-1} can be identified with $U(t)$. So $U(t) = U(t')$ would imply that we have

$$U(\Sigma_{d-1}) = U(\Sigma'_{d-1}), \tag{3.1.20}$$

if Σ'_{d-1} is a topological deformation of Σ_{d-1}. This describes the fact that U is a topological operator. To be precise, Σ_d is an oriented cobordism.

Oriented cobordism: Given any two compact, oriented,$d-1$ dimensional manifolds Σ_1 and Σ_2, an oriented cobordism $M : \Sigma_1 \to \Sigma_2$ is a compact, oriented, d dimensional manifold along with smooth orientation preserving embeddings $f_1 : -\Sigma_1 \to \partial M$and $f_2 : \Sigma_2 \to \partial M$, (where $-\Sigma_1$ is Σ_1 with the orientation reversed) such that $f_1 \cup f_2 : -\Sigma_1 \cup \Sigma_2 \to \partial M$ is a diffeomorphism.

In our setup

$$\Sigma'_{d-1} - \Sigma_{d-1} = \partial\Sigma_d, \tag{3.1.21}$$

we can describe the action of the symmetry operator on a local operator in two ways. First, we try to generalize the conjugation mentioned before, i.e.

$$U(t)\mathcal{O}(x)U^{-1}(t) = \mathcal{O}'(x) \tag{3.1.22}$$

by inserting U on Σ_{d-1} and Σ'_{d-1}, where the codimension-1 manifolds can be thought of as a spatial slice at a particular time t, that is, we let $\Sigma_{d-1} = \Sigma_t$ and $\Sigma'_{d-1} = \Sigma_{t'}$. By writing

$$U(\Sigma_{d-1})\mathcal{O}(x) = \mathcal{O}'(x)U(\Sigma'_{d-1}), \tag{3.1.23}$$

and taking the limit $\Delta t \to 0$, we get back (3.1.22).

Another way of thinking about this uses the concept of linking discussed earlier. Here we insert the symmetry operator along a S^{d-1} which as discussed before links to a point x. a local operator is placed on that point. Then, by continuously deforming the surface, we make sure that $U(S^{d-1})$ no longer links with x. But this process will transform the local operator located at x due to (3.1.23) and we get

$$U(S^{d-1})\mathcal{O}(x) = \mathcal{O}'(x). \tag{3.1.24}$$

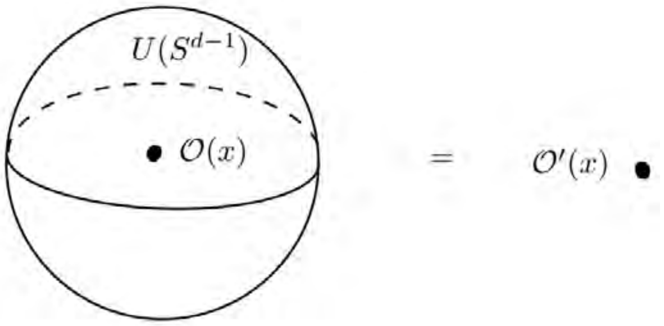

Figure 3.2. Action of Symmetry Operator

To see how (3.1.22) and (3.1.24) are equivalent, consider the 2-dimensional case, for example,

In general, the form of a symmetry operator depends on the representation. All 0-form symmetries form a symmetry group, which we denote by $G^{(0)}$, with group multiplication defined by

$$U_g(\Sigma_{d-1})U_{g'}(\Sigma_{d-1}) = U_{gg'}(\Sigma_{d-1}). \tag{3.1.25}$$

If we think of a specific time slice Σ_{d-1}, then the symmetry operators have a Hilbert space representation. We also have another representation of the group due to the vector space structure of the local operator at a point. If we denote

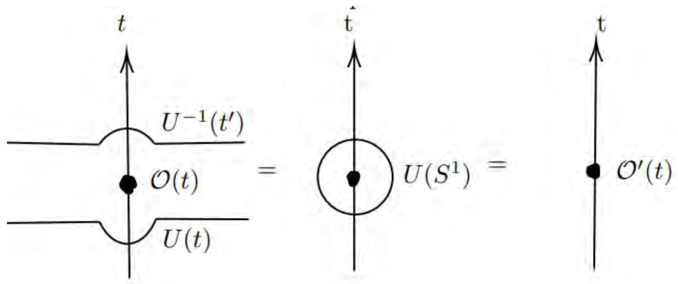

Figure 3.3. Action of Symmetry Operator

$$U_g\left(S^{d-1}\right)\mathcal{O}(x) = g\cdot\mathcal{O}(x), \tag{3.1.26}$$

then using (3.1.25) we get

$$g\cdot(g'\cdot\mathcal{O}(x)) = (gg')\cdot\mathcal{O}(x). \tag{3.1.27}$$

3.2 Example: 0-form symmetry group $U(1)$

To summarize the concepts discussed above, let us consider the case when the the 0-form symmetry group is $U(1)$. Here, $U_g(t)$ takes the form

$$U_g(t) = \exp\left(i\alpha Q(t)\right), \quad g = e^{i\alpha}\in U(1). \tag{3.2.1}$$

Since $\partial_t Q = 0$ we have $U_g(t) = U_g(t')$.

When we insert U_g on a submanifold Σ_{d-1} it takes the form

$$U_g(\Sigma_{d-1}) = \exp\left(i\alpha\int_{\Sigma_{d-1}}\star J\right). \tag{3.2.2}$$

As a consequence of $\Sigma'_{d-1}-\Sigma_{d-1} = \partial\Sigma_d$, we see that $U_g(\Sigma_{d-1})$ is indeed topological.

$$U_g(\Sigma'_{d-1}) = U_g(\Sigma_{d-1})\times\exp\left(i\alpha\int_{\Sigma'_{d-1}-\Sigma_{d-1}=\partial\Sigma_d}\star J\right). \tag{3.2.3}$$

where using Stokes' theorem we can set

$$\exp\left(i\alpha\int_{\Sigma_d} d\star J\right) = 0 \tag{3.2.4}$$

to get

$$U_g(\Sigma'_{d-1}) = U_g(\Sigma_{d-1}). \tag{3.2.5}$$

The insertion of a local operator $\mathcal{O}(x)$ makes the ward identity

$$\mathcal{O}(x)\ d\star J = q\ \delta^{(d)}(x)\ \mathcal{O}(x) \tag{3.2.6}$$

If we compute the action of the symmetry operator on $\mathcal{O}(x)$ we get

$$\begin{aligned} U_g\left(S^{d-1}\right)\mathcal{O}(x) &= \exp\left(i\alpha\int_{S^{d-1}}\star J\right)\mathcal{O}(x)\\ &= \exp\left(i\alpha\int_{D^d} d\star J\right)\mathcal{O}(x)\\ &= \exp\left(i\alpha\int_{D^d} q\delta^d(x)\right)\mathcal{O}(x). \end{aligned} \tag{3.2.7}$$

Here, since S^{d-1} and x are linked,

$$\int_{D^d} q\delta^d(x) = q, \tag{3.2.8}$$

and so

$$U_g(S^{d-1})\mathcal{O}(x) = \exp\left(iq\alpha\right)\mathcal{O}(x). \tag{3.2.9}$$

Chapter 4

1-Form Symmetries

4.1 1-Form Symmetry as an Abelian Symmetry Group

To get accustomed to higher form symmetries, let us first see the construction of 1-form symmetries. Here we will construct them as a continuous abelian symmetry group $G^{(1)}$ that acts on 1-dimensional line operators. Recall that every compact, connected abelian lie group is a torus, i.e., the direct product of n copies of the group $U(1)$. Therefore, $G^{(1)} \cong U(1)^N$. For our purposes, we will consider the case $G^{(1)} = U(1)$. In the following section, we will show why $G^{(1)}$ is abelian.

As before a 2-form current is associated to a continuous 1-form symmetry

$$J_2 = \frac{1}{2!} J_{\mu\nu} dx^\mu \wedge dx^\nu, \tag{4.1.1}$$

where J_2 is a conserved current. We express the conservation law as

$$\partial_\mu J^{[\mu\nu]}(x) = 0 \quad \text{or} \quad d \star J_2 = 0, \tag{4.1.2}$$

where $\star J_2$ is a $d-2$ form. Here, the conservation of $J^{\mu\nu}(x)$ is encoded in the fact that the 2-form current is (co)-closed. We can define the conserved charge as an integral of $\star J_2$ over a codimension-2 manifold Σ_{d-2}.

$$Q = \int_{\Sigma_{d-2}} * J_2 \tag{4.1.3}$$

We define the symmetry defect operator by *exponentiating* the conserved charge

$$U_g(\Sigma_{d-2}) = e^{i\lambda \int_{\Sigma_{d-2}} \star J_2} \quad , \quad g = e^{i\lambda}, \quad \lambda \in S^1. \tag{4.1.4}$$

For the symmetry defect operators to form a group, they must obey a group multiplication law among other group axioms.

The group multiplication takes the form

$$U_{g_1}(\Sigma_{d-2}) \cdot U_{g_2}(\Sigma_{d-2}) = \exp\left(i(\lambda_1 + \lambda_2) \int_{\Sigma_{d-2}} \star J_2 \right). \tag{4.1.5}$$

4.2 Topological Property

As in the case for 0-form symmetries, the symmetry defect operators or SDOs being topological means that the unitary operators are independent of the submanifold we insert it into. We can show it by considering two codimension-2 submanifolds Σ_{d-2} and Σ'_{d-2}, where Σ'_{d-2} is obtained by topologically deforming Σ_{d-2} which means they are homotopic, $\Sigma_{d-2} \sim \Sigma'_{d-2}$ and

$$\Sigma'_{d-2} - \Sigma_{d-2} = \partial \Sigma_{d-1}. \tag{4.2.1}$$

Here Σ_{d-1} is the surface swept out by the deformation. Now we can consider the product of $U_g(\Sigma'_{d-2})$ and $U_{g^{-1}}(\Sigma_{d-2})$

$$\begin{aligned} U_g(\Sigma'_{d-2}) \cdot U_{g^{-1}}(\Sigma_{d-2}) &= \exp\left\{ i\lambda \left(\int_{\Sigma'_{d-2}} \star J_2 - \int_{\Sigma_{d-2}} \star J_2 \right) \right\} \\ &= \exp\left\{ i\lambda \int_{\Sigma_{d-1}} d \star J \right\} \\ &= \mathbb{1}. \end{aligned} \tag{4.2.2}$$

Thus we have

$$U_g(\Sigma_{d-2}) = U_g(\Sigma'_{d-2}). \tag{4.2.3}$$

4.3 SDO Acting on Line Operators

Let us now try to understand the action of 1-form symmetries on 1-dimensional operators. These 1-dimensional operators that are charged under 1-form symmetries are called line operators. They are denoted as

$$L_q(\gamma), \tag{4.3.1}$$

where q is the charge of the line operator under $G^{(1)} \cong U(1)$ and γ is the curve that parameterizes the line operator. Before going any further, we need some concepts that will be crucial for our discussion from here on out, namely, we will need to know about transversality and oriented intersection theory.

4.3.1 Transversality and Oriented Intersection Theory

To understand these concepts, we shall closely follow [1, 2].

Consider a finite d-dimensional manifold M_d and two of its submanifolds N_s and W_t of dimensions s and t, respectively.

Definition(Transversality) N_s and W_t are said to be transversal or intersect transversally if $\forall p \in N_s \cap Y_t$,

$$T_pN_s \oplus T_pY_t \subseteq T_pM_d \,, \qquad s + t \leq d, \tag{4.3.2}$$

and denoted as

$$N_s \pitchfork Y_t \tag{4.3.3}$$

Next, we will discuss the linking number. Let M_d be compact, oriented and closed, i.e. $\partial M_d = \varnothing$. Let N_s and X_t be oriented submanifolds such that, $s+t = d-1$. We assume that N_s and X_t are non-intersecting and homotopically trivial, meaning they can be boundaries of manifolds of one higher dimension. Then we can introduce Y_{t+1} as a submanifold of M_d, such that $\partial Y_{t+1} = X_t$. Then $N_s \pitchfork Y_{t+1}$ so that N_s and Y_{t+1} only intersect at some finite number of points, i.e. $N_s \cap Y_{t+1} = \{p_i\}$. Therefore, at each p_i,

$$T_{p_i}N_s \oplus T_{p_i}Y_{t+1} = T_{p_i}M_d. \tag{4.3.4}$$

We recall that we assumed M_d to have an orientation, and (4.3.4) also induces an orientation on M_d. Then, for each p_i, we define $\mathrm{sign}(p_i)$ such that

$$\mathrm{sign}(p_i) = \begin{cases} +1 & \text{induced orientation of } M_d \text{ at } p_i \text{ same as original} \\ -1 & \text{induced orientation of } M_d \text{ at } p_i \text{ opposite to original} \end{cases} \tag{4.3.5}$$

From here we can define a topological invariant called the linking number.

Definition(Linking Number) The linking number of N_s and Y_t can be defined as

$$\mathrm{Link}(N_s, X_t) = \sum_i \mathrm{sign}(p_i)\,; \qquad t = d - s - 1. \tag{4.3.6}$$

Back to SDO Acting on Line Operators Now going back to our case of 1-form symmetries, we see that we can define the linking number as follows

$$\begin{aligned} \mathrm{Link}(N_s, X_t) &= \int_{M_d} \delta^{(d-t-1)}(p \in Y_{t+1}) \wedge \delta^{(d-s)}(p \in N_s) \\ &= \int_{Y_{t+1}} \delta^{(d-s)}(p \in N_s) \\ &= \int_{N_s} \delta^{(d-t-1)}(p \in Y_{t+1}), \end{aligned} \tag{4.3.7}$$

where $\delta^{(t)}(x \in \gamma)$ denotes a t-form that integrates to 1 on any manifold transverse to γ and zero otherwise.

Finally, we can discuss the action of 1-form symmetry on line operators. Since the line operators in (4.3.1) are charged under 1-form symmetries as discussed in (4.1.4), they have the following Ward identity [2]

$$d\star J_2(x)L_q(\gamma) = q\delta^{(d-1)}(x\in\gamma)L_q(\gamma), \tag{4.3.8}$$

where $\delta^{(d-1)}(x\in\gamma)$ is a $(d-1)$-form that integrates to 1 on any manifold transverse to γ and zero otherwise.

Now we can link the curve γ with a $(d-2)$-dimensional manifold Σ_{d-2}, and wrap a symmetry defect operator on the manifold, i.e. $U_g(\Sigma_{d-2})$. Next, as in the case of the 0-form symmetry, we topologically deform $\Sigma_{d-2}\to\Sigma'_{d-2}$, and in doing so Σ_{d-2} would cross γ. Due to the Ward identity in (4.3.8), when $U_g(\Sigma_{d-2})$ intersects $L_q(\gamma)$, it generates a phase

$$\langle U_g(\Sigma_{d-2})L_q(\gamma)\rangle = e^{i\lambda q\mathrm{Link}(\Sigma_{d-2},\gamma)}\langle L_q(\gamma)U_g(\Sigma'_{d-2})\rangle. \tag{4.3.9}$$

We can show this explicitly. Recall that

$$U_g(\Sigma_{d-2}) = e^{i\lambda\int_{\Sigma_{d-2}}\star J_2} \quad , \quad g=e^{i\lambda}, \quad \lambda\in S^1. \tag{4.3.10}$$

We can use this to write the L.H.S of (4.3.9) as

$$\langle\exp\left(i\lambda\int_{\Sigma_{d-2}}*J_2\right)L_q(\gamma)\rangle. \tag{4.3.11}$$

Now exactly following our discussion on linking number, we can define a $(d-1)$-dimensional manifold Y_{d-1} such that

$$\partial Y_{d-1}=\Sigma_{d-2}, \tag{4.3.12}$$

and rewrite (4.3.11)

$$\begin{aligned}\langle\exp\left(i\lambda\int_{\partial Y_{d-1}}*J_2\right)L_q(\gamma)\rangle &= \langle\exp\left(i\lambda\int_{Y_{d-1}}d*J_2\right)L_q(\gamma)\rangle\\ &= \langle\exp\left(i\lambda\int_{Y_{d-1}}q\delta^{(d-1)}(x\in\gamma)\right)L_q(\gamma)\rangle,\end{aligned} \tag{4.3.13}$$

where we have used Stokes theorem and the Ward identity in (4.3.8). Since $Y_{d-1}\pitchfork\gamma$, we can define the linking number of Σ_{d-2} and γ as in (4.3.7)

$$\mathrm{Link}(\Sigma_{d-2},\gamma) = \int_{Y_{d-1}}\delta^{(d-1)}.(x\in\gamma) \tag{4.3.14}$$

Now we start deforming $\Sigma_{d-2} \to \Sigma'_{d-2}$ and notice that in the process, (4.2.1) comes into play. We deform it to the point where Σ'_{d-2} and γ do not link. Therefore, we can write the action as

$$e^{i\lambda q \mathrm{Link}(\Sigma_{d-2},\gamma)} \langle L_q(\gamma) U_g(\Sigma'_{d-2}) \rangle. \tag{4.3.15}$$

We can produce the same result by wrapping the symmetry defect operator on S^{d-2}

$$U_g(S^{d-2}) L_q(\gamma) = e^{i\lambda q\,\mathrm{Link}(S^{d-2},\gamma)} L_{q^\cdot}(\gamma) \tag{4.3.16}$$

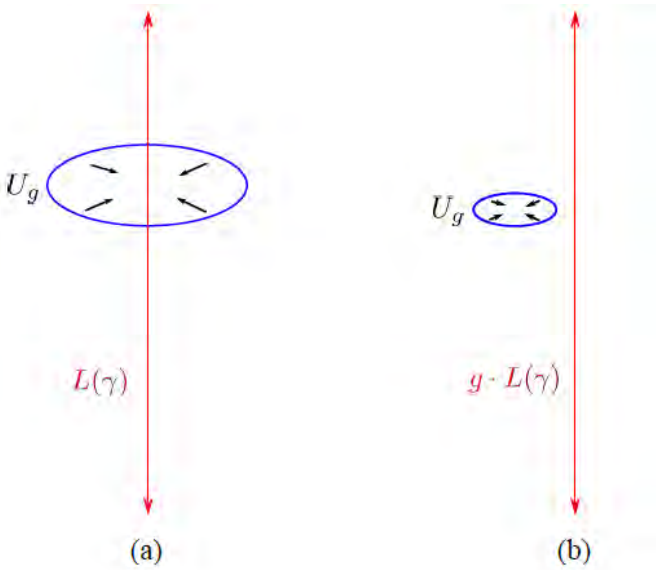

Figure 4.1. 1-form SDO U_g acting on line operator $L(\gamma)$: in (a) U_g wraps $L(\gamma)$. (b) shows the effect of U_g contracting through $L(\gamma)$. The original figure is from [2]

Recall that the dimensions of the two submanifolds that link add up to $(d-1)$. Then a line in d-dimensions can only link to a $(d-2)$-dimensional manifold. A $(d-2)$-dimensional manifold corresponds to a 2-form current J_2. That is why the current for a 1-form symmetry has to be a 2-form conserved current. See figure 4.1.

Next, we will try to show why $G^{(1)}$ is abelian.

4.4 $G^{(1)}$ Must be Abelian

Since the 1-form symmetry operators are associated with a $(d-2)$ dimensional manifold, there exists a locally transverse plane due to which we can shift the symmetry operators by smooth topological deformations, as shown in the figure below. See figure 4.2. So, there cannot be a well defined notion of ordering these SDOs, and we have

$$U_{g_1}(\Sigma_{d-2}) \cdot U_{g_2}(\Sigma_{d-2}) \;\cong\; U_{g_2}(\Sigma_{d-2}) \cdot U_{g_1}(\Sigma_{d-2}). \tag{4.4.1}$$

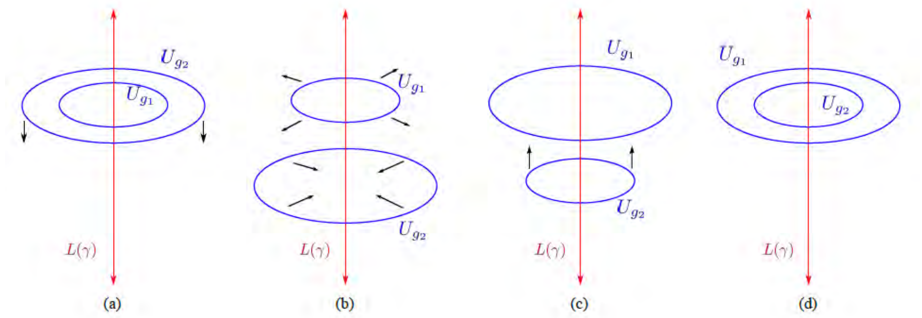

Figure 4.2. The figure shows why 1-form symmetries must be abelian. In (a) there is a configuration, where both SDOs are inserted on Σ_{d-2}, and have a natural radial ordering. (b)-(d) illustrate a series of topological deformations of the SDOs, where they do not cross the line operator $L(\gamma)$ that they are linking, and at the end their order is exchanged. During this process the SDOs remain topological and so the product of SDOs must be abelian. This figure is adapted from [2] .

4.5 $U(1)$ Maxwell Theory

Let us look at an example where we can have an in depth discussion on the structure of 1-form global symmetries. We will consider $U(1)$ pure Maxwell theory in $4d$ with gauge field $A_{(1)}$ with coupling e. The action is given by

$$S[A_{(1)}] = \frac{1}{2e^2} \int_{M_4} F_{(2)} \wedge \star F_{(2)} \tag{4.5.1}$$

Here, the gauge field is a 1-form

$$A_{(1)} = A_\mu dx^\mu, \tag{4.5.2}$$

and the field strength is a 2-form

$$F_{(2)} = dA_{(1)}\,, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.5.3}$$

Then $F_{(2)}$ is closed, which is the Bianchi identity

$$dF_{(2)} = 0. \tag{4.5.4}$$

The equation of motion can be derived by varying S with respect to $A_{(1)}$

$$
\begin{aligned}
\delta S &= \frac{1}{2e^2}\int_{M_4} (\delta F \wedge \star F + F \wedge \star \delta F) \\
&= \frac{1}{e^2}\int_{M_4} \delta F \wedge \star F = \frac{1}{e^2}\int_{M_4} \delta(dA) \wedge \star F \\
&= \frac{1}{e^2}\int_{M_4} d(\delta A) \wedge \star F \\
&= \frac{1}{e^2}\int_{M_4} d(\delta A \wedge \star F) - \frac{1}{e^2}\int_{M_4} \delta A \wedge d(\star F) \\
&= -\frac{1}{e^2}\int_{M_4} \delta A \wedge d(\star F) = 0,
\end{aligned}
\tag{4.5.5}
$$

where in the fourth line we have used $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha}\alpha \wedge d\beta$ and $\deg(\delta A) = 1$ so we get $d(\delta A \wedge \star F) = d(\delta A) \wedge \star F - \delta A \wedge d(\star F)$. Then, in the last line we assumed $\delta A|_{\partial M_4} = 0$ and by Stokes theorem the boundary term vanishes. Since δA is arbitrary, we get the equation of motion

$$
d \star F_{(2)} = 0. \tag{4.5.6}
$$

The gauge redundancy is denoted by

$$
A_{(1)} \to A_{(1)} + d\lambda_{(0)}\,, \qquad \lambda \sim \lambda + 2\pi. \tag{4.5.7}
$$

Instead of $\lambda(x)$ being a real valued function, it is a circle valued function of spacetime. With these gauge transformations defined, A_μ is a $U(1)$ gauge field. We require that the gauge field A is fixed on the boundary

$$
A|_{\partial M} = A_{bdy}. \tag{4.5.8}
$$

Then the gauge transformation that changes $A|_{\partial M}$ would violate the boundary condition. So, to preserve it, we must require $d\lambda|_{\partial M} = 0$. This means that λ must be constant along the boundary, which we take to be zero. Therefore, we adopt the boundary condition

$$
\lambda|_{\partial M} = d\lambda|_{\partial M} = 0 \tag{4.5.9}
$$

If we use the Dirac argument, we see that there is a connection between the periodicity of λ and the allowed fluxes

$$
\frac{1}{2\pi}\int_{\Sigma_2} F_{(2)} \in \mathbb{Z}. \tag{4.5.10}
$$

Here Σ_2 is any closed, oriented manifold like S^2. To see why this true, let us take a crude example of the Dirac monopole. Consider $\mathbb{R}^3$ with the origin removed (this is where the monopole sits). Topologically, this is homotopic to S^2, which is going to be the base manifold for our bundle. Let us cover S^2 by two charts H_+ and H_-, which are the two hemispheres and choose gauge potentials A_+ and A_-, respectively to be defined there. And also $F = dA_\pm$. The structure group will be $U(1)$. So we are constructing a principal bundle $P(S^2, U(1))$, where the fibres make the set of all possible $U(1)$ gauge symmetry elements, which is to say, the fibres are identical to the structure group itself. Our bundle locally looks like $H_+ \times U(1)$ and $H_- \times U(1)$ with bundles coordinates $(\theta, \phi, e^{i\lambda_+})$ and $(\theta, \phi, e^{i\lambda_-})$, respectively, where $0 \leq \theta < \pi$ and $0 \leq \phi < 2\pi$. The transition functions h will connect the fibres from one chart to another.

$$e^{i\lambda_-} = h\ e^{i\lambda_+} \tag{4.5.11}$$

The transition functions are elements of $U(1)$ and are defined on the equator $H_+ \cap H_- = S^1$, $\ h(\phi) \in U(1)$. The transition functions must glue the fibres together exactly when we complete a full turn around the equator. Therefore, our only choice is

$$h = e^{in\lambda}, \qquad n \in \mathbb{Z}. \tag{4.5.12}$$

The integers n are the winding numbers. When λ varies from $0 \to 2\pi$, the group $U(1)$ is wound around n times. The compatibility condition

$$A_+ = h^{-1} A_- h + h^{-1} dh \tag{4.5.13}$$

in our Abelian case for $U(1)$ becomes

$$A_+ = A_- + i n d\lambda. \tag{4.5.14}$$

If we do the integral

$$\begin{aligned}\frac{1}{2\pi}\int_{S^2} F &= \frac{1}{2\pi}\left[\int_{H_+} dA_+ + \int_{H_-} dA_-\right] \\ &= \frac{1}{2\pi}\left[\int_{\partial H_+} A_+ + \int_{\partial H_-} A_-\right] = \frac{1}{2\pi}\int_{S^1}(A_+ - A_-) \\ &= \frac{1}{2\pi}\int_0^{2\pi} d\varphi \cdot n = n,\end{aligned} \tag{4.5.15}$$

where we have used Stokes theorem and also absorbed a factor of i in the difference of the fields.

4.6 Construction of 1-Form symmetries

Returning to our original discussion, we know from equation (4.5.4) and (4.5.6)

$$d\star F_{(2)}=0\,,\quad dF_{(2)}=0, \tag{4.6.1}$$

we can think of these as the conservation equation of two 2-form currents (upto a choice of normalization),

$$J_2^e=\frac{1}{e^2}F_{(2)}\,,\qquad J_2^m=\frac{1}{2\pi}\star F_{(2)} \tag{4.6.2}$$

Before we go any further, let us see another presentation of free Maxwell theory that is equally valid.

4.6.1 S-duality

We will start with the action

$$S=\frac{1}{2e^2}\int_{M_4}F_{(2)}\wedge\star F_{(2)} \tag{4.6.3}$$

This time we will not impose the constraints $dF_{(2)}=0$ and $\frac{1}{2\pi}\int_{\Sigma_2}F_{(2)}\in\mathbb{Z}$ by writing $F_{(2)}=dA_{(1)}$ directly. Instead, we will impose them using a Lagrange multiplier $\tilde{A}_{(1)}$:

$$S[F_{(2)},\tilde{A}_{(1)}]=\frac{1}{2e^2}\int_{M_4}F_{(2)}\wedge\star F_{(2)}+\frac{1}{2\pi}\int_{M_4}F_{(2)}\wedge d\tilde{A}_{(1)}, \tag{4.6.4}$$

so that we get $dF_{(2)}=0$ from the equation of motion of $\tilde{A}_{(1)}$. If we look at the equation of motion of $F_{(2)}$ we get

$$\delta_{F_{(2)}}S=\frac{1}{e^2}*F_{(2)}-\frac{1}{2\pi}d\tilde{A}_{(1)}=0, \tag{4.6.5}$$

and so we get

$$\star F_{(2)}=\frac{e^2}{2\pi}d\tilde{A}_{(1)}\,,\qquad F_{(2)}=-\frac{e^2}{2\pi}\star d\tilde{A}_{(1)}. \tag{4.6.6}$$

Substituting this into (4.6.4) we get the action $\tilde{S}$

$$\begin{aligned}\tilde{S}&=-\frac{e^2}{8\pi^2}\int_{M_4}\star d\tilde{A}_{(1)}\wedge d\tilde{A}_{(1)}-\frac{e^2}{4\pi^2}\int_{M_4}\star d\tilde{A}_{(1)}\wedge d\tilde{A}_{(1)}\\&=-\frac{e^2}{4\pi^2}\int_{M_4}\star d\tilde{A}_{(1)}\wedge d\tilde{A}_{(1)}\\&=\frac{e^2}{4\pi^2}\int_{M_4}d\tilde{A}_{(1)}\wedge\star d\tilde{A}_{(1)}\end{aligned}\quad. \tag{4.6.7}$$

Considering $\tilde{A}_{(1)}$ as the dual $U(1)$ gauge field, we can define the dual field strength as

$$\tilde{F}_{(2)} = d\tilde{A}_{(1)}, \tag{4.6.8}$$

and write the dual action

$$\tilde{S} = \frac{1}{\tilde{e}^2} \int_{M_4} \tilde{F}_{(2)} \wedge \star \tilde{F}_{(2)} \,, \qquad \tilde{e}^2 = \frac{4\pi^2}{e^2}. \tag{4.6.9}$$

Furthermore, similar to our previous argument,

$$\frac{1}{2\pi} \int_{\Sigma_2} \tilde{F}_{(2)} \in \mathbb{Z}. \tag{4.6.10}$$

Back to the construction of 1-Form Symmetries Recall that we had two 2-form currents in (4.6.2). J_2^e implies that we have a $U(1)$ 1-form symmetry called the *electric 1-form symmetry.* Consider the exponential of the dual 2-form current $\star J_2^e$ integrated over a closed 2-manifold. This is the corresponding symmetry defect operator

$$U_g^e(\Sigma_2) = \exp\left\{ i\lambda \oint_{\Sigma_2} \star J_2^e \right\} = \exp\left\{ \frac{i\lambda}{e^2} \oint_{\Sigma_2} \star F \right\}. \tag{4.6.11}$$

We can see that $\frac{1}{e^2}\int_{\Sigma_2} \star F$ measures the charge enclosed by Σ_2. We also have charge quantization $\oint_{\Sigma_2} \star F \in \mathbb{Z}$. To see this, we only need the fact that for a $U(1)$ 1-form symmetry the parameter is periodic, i.e. $\lambda \sim \lambda + 2\pi$ and

$$U_g^e(\Sigma_2; \lambda + 2\pi) = U_g^e(\Sigma_2; \lambda) \exp\left\{ i\frac{2\pi}{e^2} \int_{\Sigma_2} \star F \right\}. \tag{4.6.12}$$

For the symmetry to be well defined, the extra factor must be the identity

$$\exp\left\{ i\frac{2\pi}{e^2} \int_{\Sigma_2} \star F \right\} = 1 \implies \frac{1}{e^2} \int_{\Sigma_2} \star F \in \mathbb{Z}. \tag{4.6.13}$$

There is also a dual 1-form symmetry due to J_2^m, which is called a *magnetic 1-form symmetry.* The symmetry defect operator is formed by integrating the dual current over a 2-manifold

$$U_g^m(\Sigma_2) = \exp\left\{ i\lambda \oint_{\Sigma_2} \star J_2^m \right\} = \exp\left\{ i\lambda \oint_{\Sigma_2} \frac{F}{2\pi} \right\}. \tag{4.6.14}$$

Here, $\frac{1}{2\pi}\oint_{\Sigma_2} F$ measures the magnetic flux passing through Σ_2. From (4.5.15) we know that $\oint_{\Sigma_2} F = 2\pi\mathbb{Z}$.

Now we need to discuss about the operators that are charged under this symmetry. First imagine a point particle living in $(d-1)$-dimensional space carrying a charge, a 0-form charge to be exact. Now when we integrate the 0-form charge over the entire spatial slice Σ_{d-1}, we necessarily cross the location of that particle because we are integrating over all space. So, it would have to pick up the charge of that particle.(The conservation law $\dot{Q} = 0$ has the consequence that the charge $Q_{\Sigma_{d-1}}$ is independent of the choice of time slice Σ_{d-1}. Previously we have seen that $Q_{\Sigma_{d-1}}$ counts the number of particle worldlines passing Σ_{d-1}, and because of $\dot{Q} = 0$, the charged particle worldlines cannot end. That is why the 0-form symmetries act on particles and local operators. But that is not the case for 1-form symmetries. They are topological objects that are 1 dimension lower, i.e., in Σ_{d-2}. So, if we have a particle sitting somewhere in Σ_{d-1}, the corresponding flux integral does not sit anywhere in Σ_{d-1}, rather it sits at infinity. That flux integral does not ever "cut" the location of the particle. This means that the flux integral and the local operator that created the particle commute, i.e. $[Q_{d-2}, \phi] = 0$, and the particle cannot carry 1-form charge. In the next section, we will see that the charged operator under 1-form symmetries are not point-like; rather, they are extended.

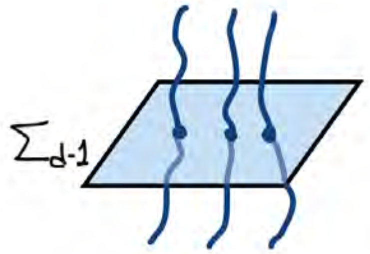

Figure 4.3. In 0-form symmetries the charge is integrated over a time slice Σ_{d-1}. All the particle worldlines (blue curves) must pass through this hypersurface.

We learned in 0-form symmetries that the charged objects (the point operators) are supposed to be linked with topological Σ_{d-1} surfaces. There is a natural analogue to this in higher form symmetries. For 1-form symmetries the symmetry defect operators live in Σ_{d-2}. So we need to think of objects that naturally link with 2-manifolds. Recall from section **??** that the dimensions of the two linked submanifolds add up to $d-1$. So in 4 dimensions the object that links with a 2-sphere is a line. So we should think about 1-dimensional defects in Maxwell theory.

4.7 Wilson Line and 't Hooft Line

The most familiar class of such defects are Wilson loops(or lines depending on the curve γ we are going to define them on). In abelian gauge theory we can define a family of Wilson loops by the curve γ on which they are supported with charge q. Let γ be a curve in Euclidean 4-manifold. Then

$$W_q(\gamma) \equiv e^{iq\int_\gamma A_\mu \mathrm{d}x^\mu} = e^{iq\int_\gamma A}. \tag{4.7.1}$$

In a theory without charged matter fields, Wilson loops are gauge invariant observables in the theory provided the curve γ is either closed or ends at infinity.

$$\begin{aligned} W_q(\gamma) \to \exp\left(iq\oint_\gamma A + d\lambda\right) \\ &= \exp\left(iq(\oint_\gamma A + \oint_\gamma d\lambda)\right) \\ &= \exp\left(iq(\oint_\gamma A + \oint_{\partial\gamma} \lambda)\right) \\ &= \exp\left(iq\oint_\gamma A\right) \end{aligned} \tag{4.7.2}$$

Here we have used Stokes' theorem the fact that γ is a closed loop without boundary or infinite.

As we shall see shortly, if the curve extends along the time direction, we can interpret the Wilson line as the worldline of an infinitely massive probe, which carries electric charge. Alternatively, we can localize the curve γ on a constant time slice, where it can be viewed as a loop of flux.

Before that, let us quickly see that for $W_q(\gamma)$ the charge $q \in \mathbb{Z}$.

For $U(1)$, $g(x) = e^{i\lambda(x)} \in U(1)$ and $A \to A + d\lambda$ with λ defined mod 2π.

$$W_q(\gamma) \to W'_q(\gamma) = \exp\left(iq\oint_\gamma (A + d\lambda)\right) = W_q(\gamma)\exp\left(iq\oint_\gamma d\lambda\right) \tag{4.7.3}$$

So gauge invariance requires $\exp(iq\oint_\gamma d\lambda) = 1$. We will parameterize the closed loop by $x(\theta)$ where $\theta \in [0, 2\pi]$. We will restrict $g|_\gamma : S^1 \to U(1)$.These maps are classified by winding number $n \in \mathbb{Z}$. If we pick a continuous branch $\lambda(\theta)$ with $g(\theta) = e^{i\lambda(\theta)}$, then the single-valuedness of g implies $\lambda(2\pi) = \lambda(0) + 2\pi n$ where $n \in \mathbb{Z}$. Hence,

$$\oint_\gamma d\lambda = \int_0^{2\pi} d\theta \frac{d\lambda}{d\theta} = \lambda(2\pi) - \lambda(0) = 2\pi n, \tag{4.7.4}$$

and so

$$\exp\left(iq\oint_\gamma d\lambda\right) = \exp(iq \cdot 2\pi n) = 1 \quad \forall n \in \mathbb{Z}, \tag{4.7.5}$$

which holds if and only if

$$q \in \mathbb{Z} \tag{4.7.6}$$

4.7.1 SDOs Acting on Wilson Line

Recall from section **??**, specifically from equation (4.3.9), the process of SDOs acting on line operators. By the same mechanism, SDO will act on Wilson line,

$$\langle U_g^e(\Sigma_2) W_q(\gamma)\rangle = e^{iq\lambda \mathrm{Link}(\Sigma_2,\gamma)}\langle W_q(\gamma) U_g^e(\Sigma_2')\rangle, \tag{4.7.7}$$

where Σ_2' is homotopic to Σ_2. We can see that after the deformation process Σ_2' does not link with γ. Also, the phase $e^{iq\lambda}$ can be interpreted as the following: deforming $\Sigma_2 \to \Sigma_2'$, the electric charge enclosed by the symmetry defect operator U_g^e goes from $q \to 0$ and we can see the corresponding effect in the symmetry group $U_g^e(\Sigma_2) \longmapsto e^{iq\lambda} U_g^e(\Sigma_2')$.

We will derive (4.7.7) explicitly. We mentioned before that the Wilson line can be thought of as the worldline of an infinitely massive probe which carries electric charge. The presence of which will source an electric flux and modify Maxwell's equations. To see this clearly, let us insert a Wilson line in the path integral as

$$\int [dA] e^{iq\int_\gamma A} e^{\frac{1}{2e^2}\int_{M_4} F\wedge * F}, \tag{4.7.8}$$

where $\gamma \subset M^4$ is an oriented curve. Let us define the distributional 3-form which is the Poincare dual of γ as

$$\delta^3(\gamma)(x) = \frac{1}{3!}\int_\gamma ds\ \dot{x}^\mu(s)\varepsilon_{\mu\nu\rho\sigma}\ \delta^{(4)}(x - x(s))\ dx^\nu \wedge dx^\rho \wedge dx^\sigma, \tag{4.7.9}$$

where we have chosen a parameterization $x^\mu(s)$ of γ and

$$\int_{M_3^T} \delta^3(\gamma) = \sum_{p\in M_3^T\cap\gamma} \mathrm{sign}(p) = 1 \tag{4.7.10}$$

meaning $M_3^T \subset M_4$ is a 3-manifold that transversely intersects γ once and positively.

We write $A = A_\alpha dx^\alpha$. Then

$$\begin{aligned}\delta^3(\gamma)\wedge A &= \int_\gamma ds\ \frac{\dot{x}^\mu}{3!}\varepsilon_{\mu\nu\rho\sigma}\delta^{(4)}(x - x(s))\ dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge A_\alpha dx^\alpha \\ &= \int_\gamma ds\ \dot{x}^\mu A_\mu(x)\delta^{(4)}(x - x(s))\ d^4x.\end{aligned} \tag{4.7.11}$$

where we have used $dx^\nu \wedge dx^\rho \wedge dx^\sigma \wedge dx^\alpha = \varepsilon^{\nu\rho\sigma\alpha} d^4x$ and $\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\nu\rho\sigma\alpha} = 3!\ \delta_\mu^\alpha$. Integrating over the whole M_4 gives us,

$$\int_{M_4} \delta^{(3)}(\gamma)\wedge A = \int_\gamma ds \dot{x}^\mu(s) A_\mu(x(s)) = \int_\gamma A \tag{4.7.12}$$

Let us identify the current $J = q\delta^{(3)}(\gamma)$. We shall see in a bit why this holds true. Now we write (4.7.8) as

$$\int [dA] e^{iq\int_\gamma A} e^{\frac{1}{2e^2}\int_{M_4} F\wedge * F} = \int [dA] e^{iq\int_{M_4}\delta^3(\gamma)\wedge A + \frac{1}{2e^2} F\wedge * F} \tag{4.7.13}$$

This makes it explicit that insertion of the Wilson line is equivalent to adding a source term to the action. So, the Wilson line acts as an electric source. To obtain the equation of motion for A, we vary the action with respect to A

$$\begin{aligned}\delta S &= \delta\left(\frac{1}{2e^2}\int_{M_4} F\wedge\star F + q\int_{M_4}\delta^3(\gamma)\wedge A\right)\\ &= \frac{1}{2e^2}\int_{M_4} d(\delta A)\wedge\star F + F\wedge\star d(\delta A) + q\int_{M_4}\delta^3(\gamma)\wedge\delta A.\end{aligned} \tag{4.7.14}$$

We can now use the fact that for 2-forms $\alpha,\beta\in\Omega^p(X)$ obey $\alpha\wedge\star\beta = \beta\wedge\star\alpha$ since the scalar product on the space $\Omega^p(X)$ can be written as $\langle\alpha,\beta\rangle = \alpha\wedge\star\beta$. Then $d(\delta A)\wedge\star F = F\wedge\star d(\delta A)$ and we get

$$\delta S = \frac{1}{e^2}\int_{M_4} d(\delta A)\wedge\star F + q\int_{M_4}\delta^3(\gamma)\wedge\delta A. \tag{4.7.15}$$

Next we can use the product rule $d(\alpha_p\wedge\beta_r) = d\alpha_p\wedge\beta_r + (-1)^p\alpha_p\wedge d\beta_q$ to get

$$\delta S = \frac{1}{e^2}\int_{M_4} d(\delta A\wedge\star F) - \frac{1}{e^2}\int_{M_4}\delta A\wedge d\star F + q\int_{M_4}\delta^3(\gamma)\wedge\delta A. \tag{4.7.16}$$

Using Stokes theorem $\int_{M_4} d(\delta A\wedge\star F) = \int_{\partial M_4}\delta A\wedge\star F$. We assume that all fields fall of sufficiently fast at infinity /at the boundary. Therefore,

$$\delta S = -\frac{1}{e^2}\int_{M_4}\delta A\wedge d\star F + q\int_{M_4}\delta^3(\gamma)\wedge\delta A = 0. \tag{4.7.17}$$

Since this is true for any variation δA we have the equation of motion

$$d\star F = qe^2\delta^3(x\in\gamma). \tag{4.7.18}$$

If we integrate (4.7.18) over a 3-manifold with boundary, the left side, by Stokes theorem, would be $\oint_{\Sigma_2}\star F$ which will the electric flux coming out of the boundary $\partial\Sigma_3$, and on the right side there would be a term $\oint_{\Sigma_{2+1}}\delta^3(x\in\gamma)$, which is just the linking number between Σ_2 and γ. Therefore,

$$\oint_{\Sigma_2}\star F = qe^2\mathrm{Link}(\Sigma_2,\gamma). \tag{4.7.19}$$

Exponentiation on (4.7.19) will give us

$$U_g^e(\Sigma_2)W_q(\gamma) = \exp\left(\frac{i\lambda}{e^2}\oint_{\Sigma_2}\star F\right) = e^{iq\lambda\mathrm{Link}(\Sigma_2,\gamma)}W_q(\gamma), \tag{4.7.20}$$

which reproduces the action in (4.7.7).

Another way to represent the action of SDO on the Wilson line is that the SDO acts on the gauge field by shifting it by a closed but not exact 1-form.

$$A \to A + \Lambda \ , \qquad d\Lambda = 0 \ , \qquad \oint \Lambda \in U(1). \tag{4.7.21}$$

4.7.2 SDOs Acting on 't Hooft line

Finally, we turn our attention to the 't Hooft lines. These are gauge invariant line operators that are magnetic dual to Wilson lines. Recall our discussions on S-duality in section **??**. Let us switch to that dual presentation. Here $\tilde{A}$ is the dual gauge field defined as

$$\star F_{(2)} = \frac{e^2}{2\pi} d\tilde{A}_{(1)} = \frac{e^2}{2\pi} \tilde{F}_{(2)}. \tag{4.7.22}$$

In this dual presentation in terms of $\tilde{A}$ the 't Hooft line operators can be defined just like Wilson lines

$$T_m(\gamma) = e^{im \int_\gamma \tilde{A}}. \tag{4.7.23}$$

The action of $U_g^m(\Sigma_2)$ on 't Hooft lines is identical to the action of $U_g^e(\Sigma_2)$ on Wilson lines:

$$\langle U_g^m(\Sigma_2) T(m,\gamma) \rangle = e^{im\lambda \,\mathrm{Link}(\Sigma_2,\gamma)} \langle T(m,\gamma) U_g^m(\Sigma_2') \rangle. \tag{4.7.24}$$

But we can also stay in the original presentation and define the 't Hooft line operator in terms of the gauge field A. To see this, let us first use the example of the Wilson line.

Suppose that a curve γ extended in time direction is a particle worldline. If we consider a particular time slice and zoom in, we see that the time slice is actually $\mathbb{R}^3$ and the particle sits somewhere in $\mathbb{R}^3$, taken to be the origin. So we have an electrically charged object at the origin, and then it can link with a 2-dimensional manifold (sphere) where the electric flux integral $Q_e(\Sigma_2) = \frac{1}{e^2} \int_{\Sigma_2} \star F$ is defined, and so we have electrically charged operators. See figure 4.4.

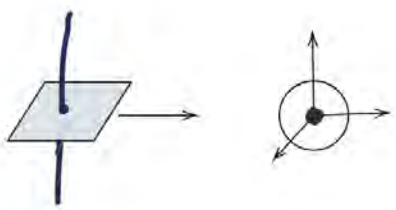

Figure 4.4. Wilson line in a particular time slice.

Now in this presentation, we are going to define 't Hooft line operators. Like before, fix a curve γ with charge m, in which we would like to insert our line operator. Now consider a small tubular neighborhood of γ and take a constant time slice and zoom in as before. Concretely, after zooming in, the space would look like $\mathbb{R}_\gamma \times \mathbb{R}^3_T$ where $\mathbb{R}^3_T$ is the transverse space, and in this transverse space the drilling out would locally look like an infinitesimal 3-ball surrounding a point. The boundary of this 3-ball is a transverse 2-sphere, and there we can impose the boundary condition

$$\frac{1}{2\pi}\int_{S^2} F_{(2)} = m. \tag{4.7.25}$$

This is a valid choice of boundary condition. Then we take a limit in which the size of this tubular neighborhood shrinks to zero. Because of the way in which we have defined the boundary condition, it is most certainly true that this line is magnetically charged and has magnetic charge m. We have a 't Hooft line operator. See figure 4.5

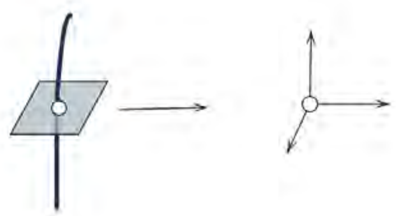

Figure 4.5. 't Hooft line in a particle time slice.

4.7.3 Charged Lines Cannot End

In these examples, 1-form SDOs and charges live on S^2. They are time independent, so they can smoothly move through time slices, and by linking they measure the electric and magnetic charge of these lines. One key consequence of this is that the electric and magnetic fluxes carried by these lines can never terminate. They would have to continue for ever. Because if they were to terminate, then we could take an allegedly topological surface and slide it off the end of the line. Imagine we could terminate the electric charged line at a certain t, (See figure 4.6) and measure the electric flux, which is topological. If we keep moving the flux operator, it will slide off the line and become trivial. If this were to be the case, then this SDO could not be topological or, equivalently, the line could not be charged. The only way to have charged lines and topological Σ_2 surfaces is if these lines never end.

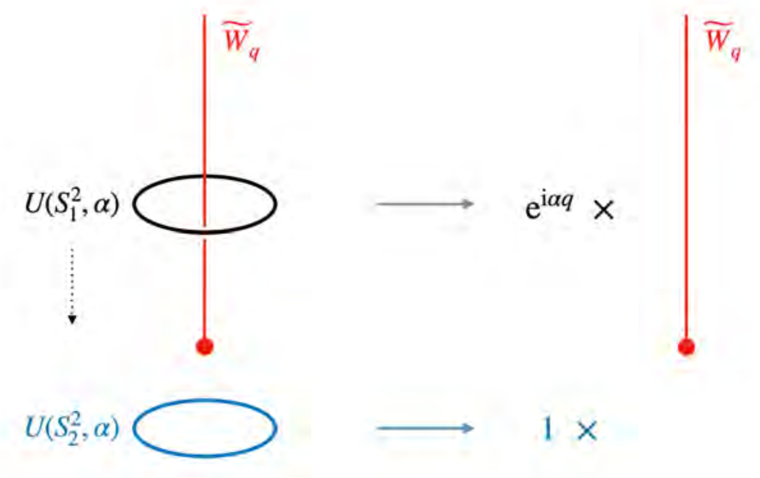

Figure 4.6. The figure illustrates what would happen if Wilson line could be endable; the SDO would cease to be topological, and act trivially. The figure is adapted from [17].

4.8 Coulomb Phase and Spontaneous 1-Form Symmetry Breaking

In this section, we will make an attempt to see whether the 1-form symmetries $U(1)^{(1)}_{e,m}$ of the Maxwell theory are spontaneously broken or not.

We learned in previous sections that in Maxwell's theory we have probe charges, i.e. Wilson lines and t'Hooft lines, which represent static, non-dynamic charges. Consider the static potential energy $V_{Q\overline{Q}}(r)$ between a source Q and its conjugate $\overline{Q}$ separated by a distance r. So, if Q has an electric charge, then $\overline{Q}$ would have a negative electric charge. In Maxwell's theory this is nothing but the Coulomb potential

$$V_{q_e\bar{q}_e}(r) \sim -\frac{e^2 q_e^2}{r}\,, \quad V_{q_m\bar{q}_m}(r) \sim -\frac{\widetilde{e}^2 q_m^2}{r} \sim -\frac{q_m^2}{e^2 r}. \tag{4.8.1}$$

Now recall that to determine whether a symmetry is spontaneously broken in a given vacuum, we examine whether a charged operator has a non-zero vacuum expectation value. A more convenient way is to consider the correlator $\langle\phi^\dagger\phi\rangle$ for large separations. If $\langle\phi^\dagger\phi\rangle \neq 0$ then by cluster decomposition

$$\langle\phi^\dagger(x)\phi(0)\rangle \longrightarrow |\langle\phi\rangle|^2 + \ldots\,, \qquad \text{as } |x| \to \infty \tag{4.8.2}$$

we see that $\langle\phi\rangle \neq 0$.

We need an analogue of $\langle\phi^\dagger\phi\rangle$ for the line operators $L(\gamma)$ along a large loop γ so that if we take the loop to be very large, then its asymptotic behavior will indicate whether $L(\gamma)$ has a non-zero vev, and that is

$$\langle L(\gamma)\rangle\,, \qquad \gamma\to\infty. \tag{4.8.3}$$

This makes sense because in the Euclidean picture, a closed, contractable loop is not charged under the 1-form symmetry. But as $\gamma\to\infty$, the loop goes from one point in spatial infinity to another point at spatial infinity and the loop clusters into products. Then it is charged under the 1-form symmetry.

Another subtlety to consider is loop counterterms. This is because the way we have defined (4.8.3) namely the expectation value of the the operator as the loop becomes large and ask whether it is zero or not, makes it a little ambiguous and scheme dependent. The counterterms that can modify the vev of the loop are not just constants. They can depend on the shape of the loop. It is reasonable to argue that the allowed counterterms that depend on the loop should reflect the geometric invariance of the loop. The most standard and important counterterm is just the length (perimeter) of the loop. Therefore, we can modify the loop by

$$L(\gamma)\to L(\gamma)\; e^{-mP(\gamma)}, \tag{4.8.4}$$

where m has the dimension of mass. It might be positive, negative, finite, or infinite. We can also think about higher derivative counterterms that involve not just the length of the loop but something like the extrinsic curvature. But these higher derivative counterterms are all suppressed in the cut-off.

Due to our redefinition in (4.8.4), we can affect its long distance behavior. This is because as we scale up the size of the loop, $P(\gamma)$ goes to infinity. So, if m is positive (negative), the counter term will decay (increase) exponentially with the length of the loop. Therefore, we need to modify our definition of what we mean by the loop having a constant vev.

This is what we mean: since m is non-universal, The universality class of loops where we allow adding such perimeter terms as (4.8.4), there is one representative that has non-zero vev, and all the other could decay or increase according to this perimeter law.

Perimeter Law: If the large loop behaves like

$$\langle L(\gamma\to\infty)\rangle\sim(\text{const.})\; e^{-mP(\gamma)}\,,\text{for some m} \tag{4.8.5}$$

then it is equivalent to saying that there exists a scheme in which the loop has a non-zero vev, and because of that, the 1-form symmetry acting on the loop is *spontaneously broken*. On the other hand, if $\langle L(\gamma\to\infty)\rangle$ decays faster than the perimeter law, then $L(\gamma)$ has zero vev and the 1-form symmetry is *unbroken*.

Now let us combine what we have discussed about the perimeter law and the effective potential $V_{Q\overline{Q}}(r)$, which for Maxwell's theory is just the Coulomb potential.

Consider in Euclidean space a large, rectangular Wilson loop extended in Euclidean time T direction. The other side is the spatial distance r. See figure 4.7. We interpret it as the creation of a static particle-antiparticle pair, which means that the particles can propagate only in time and have no kinetic term. They are separated by distance r. So, they are created; they propagate forward in time for time T and finally annihilate back to the vacuum. We know that for long times, the path integral projects the system onto the lowest energy state. Before the particles appear and after they disappear, this is the ground state of the system, which we can take to have zero energy. However, in the presence of two sources, the ground state of the system has energy $V_{Q\overline{Q}}(r)$.

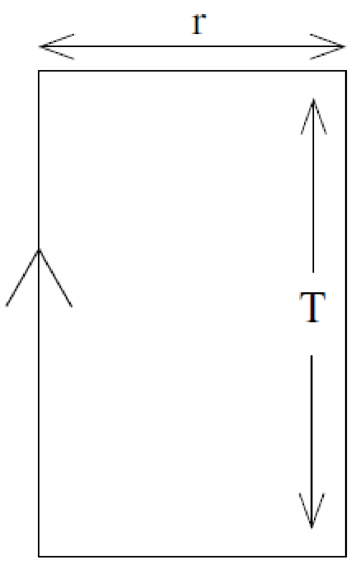

Figure 4.7. Rectangular Wilson loop. The original figure is from [11]

Therefore,

$$\lim_{T\to\infty} \langle W[\gamma]\rangle \sim e^{-V_{Q\overline{Q}}(r)T}. \tag{4.8.6}$$

If we want to account for a perimeter term like (4.8.4), then we can write

$$\langle W[\gamma]\rangle \sim e^{-V_{Q\overline{Q}}(r)\ T}\ e^{-m\ P(\gamma)}. \tag{4.8.7}$$

All the static potentials in Maxwell's theory (4.8.1) have Coulomb scalling. Therefore, at long distances $V_{Q\overline{Q}}(r)$ decays and we have

$$\langle W[\gamma]\rangle \sim (\text{const.})\ e^{-m\ P(\gamma)} \tag{4.8.8}$$

Hence, every single loop in Maxwell's theory abides by the perimeter law, which then means that both electric and magnetic 1-form symmetries are completely spontaneously broken. The photon is the Goldstone boson for both of them in the sense that the photon has two polarization states and since we have two symmetries, we get two independent propagating degrees of freedom that represent the Goldstone boson.

Chapter 5

p-Form Symmetries

5.1 Construction of p-Form symmetries

So far we have discussed about 0-form and 1-form symmetries. They are part of a more generalized form of symmetries called p-form symmetries. That is the topic that we will discuss here.

A *p-form global symmetry* is denoted by a symmetry group $G^{(p)}$, and it is a symmetry that acts on p-dimensional charged operators. Generalizing the notion of currents from previous sections, we could have guessed that a p-form symmetry has a corresponding $(p+1)$-form conserved current, which then would imply that the $(d-p-1)$-form dual current $*J_{p+1}$ is closed:

$$\partial_{\mu_1} J^{\left[\mu_1 \cdots \mu_{p+1}\right]}(x) = 0 \quad \Longrightarrow \quad d \star J_{p+1}(x) = 0 \tag{5.1.1}$$

Now that we have a closed dual current, $*J_{p+1}$ we can discuss coupling to background gauge fields. A theory with p-form global symmetry can be coupled to an abelian (p+1)-form background gauge field B_{p+1} by the term

$$S_p = i \int B_{p+1} \wedge \star J_{p+1}\,, \qquad B_{p+1} \to B_{p+1} + d\lambda_p \tag{5.1.2}$$

where λ_p is a p-form background gauge transformation parameter.

The variation in the action is given by

$$\delta S_p = i \int_{M_d} d\lambda_p \wedge \star J_{p+1} = i\ (-1)^{p+1} \int_{M_d} \lambda_p\ \wedge d \star J_{p+1} = 0 \tag{5.1.3}$$

Here we have used $d \star J_{p+1} = 0$. So $\star J_{p+1}$ being closed means that the action is invariant under background gauge transformation.

Now we can define a $(d-p-1)$-dimensional symmetry defect operator labeled by the group elements g,

$$U_g(\Sigma_{d-p-1}) = \exp\left\{ i\lambda \int_\Sigma \star J_{p+1} \right\} \quad , \quad g = e^{i\lambda} \tag{5.1.4}$$

We had noted when we were discussing 1-form symmetries that $G^{(1)}$ must be abelian. See section **??**. Similarly for p-form symmetries, the SDOs being defined on a co-dimension $(p+1)$-manifold does not allow a well defined notion of ordering of $U_g(\Sigma)$, and so $G^{(p)}$ must be abelian for $p \geq 1$. See figure 5.1.

As before we check that the symmetry defect operators obey a group law. Since $G^{(p)}$ is abelian, the currents $\star J_{p+1}$ satisfy

$$d \star J_{p+1}(x) \star J_{p+1}(y) = 0 \tag{5.1.5}$$

Then for two operators $U_{g_1}(\Sigma_{d-p-1})$ and $U_{g_2}(\Sigma_{d-p-1})$ we have

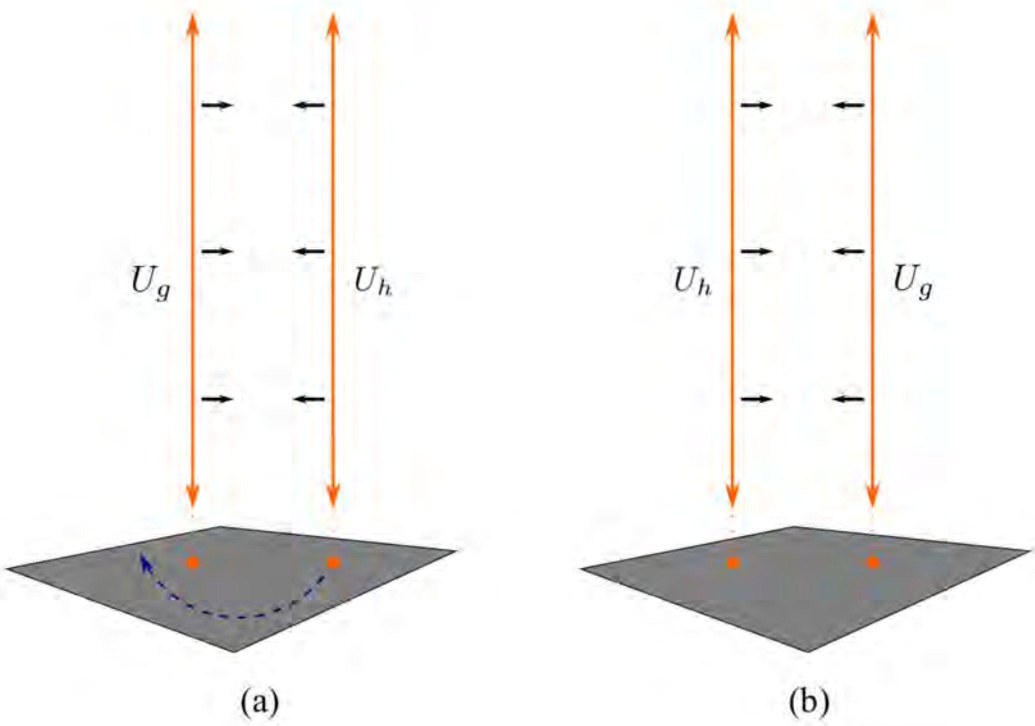

Figure 5.1. The figure above illustrates why $G^{(p)}$ must be abelian for $p \geq 1$. (a) Here to avoid any ill-defined intersection problems, we consider two parallel symmetry defect operators U_g and U_h. We assume that they have a natural ordering. Then they are projected onto the transverse 2-plane below. Since U_g and U_h are topological, they can be smoothly deformed as long as they do not cross a charged operator. Then they can be deformed such that they exchange their positions in the transverse 2-plane and now they have the ordering that is opposite to their natural ordering we assumed they had before. That means (a) and (b) describe the same physics. The figure is adapted from [2]

$$U_{g_1}(\Sigma_{d-p-1}) \cdot U_{g_2}(\Sigma_{d-p-1}) = \exp\left(i\lambda_1 \int_\Sigma \star J_{p+1}\right) \exp\left(i\lambda_2 \int_\Sigma \star J_{p+1}\right) \tag{5.1.6}$$

Now since the currents commute, the two exponentiated operators commute and simply combine:

$$U_{g_1}(\Sigma_{d-p-1}) \cdot U_{g_2}(\Sigma_{d-p-1}) = \exp\left(i(\lambda_1+\lambda_2)\int_\Sigma *J_{p+1}\right) = U_{g_1g_2}(\Sigma_{d-p-1}) \tag{5.1.7}$$

Here because of the commutativity of the integrands, the Baker-Campbell-Hausdorff series reduces to addition of parameters.

To show that the symmetry defect operators are topological we take $U_g(\Sigma_{d-p-1})$ and $U_{g^{-1}}(\Sigma'_{d-p-1})$ where $\Sigma_{d-p-1} \sim \Sigma'_{d-p-1}$ are homotopic.

$$\begin{aligned} U_g(\Sigma_{d-p-1}) \cdot U_{g^{-1}}(\Sigma'_{d-p-1}) &= \exp\left\{i\lambda\left(\int_\Sigma \star J_{p+1} - \int_{\Sigma'} \star J_{p+1}\right)\right\} \\ &= \exp\left\{i\lambda \int_M d\star J_{p+1}\right\} = \mathbb{1} \end{aligned} \tag{5.1.8}$$

where M is the $(d-p)$-dimensional surface swept out by the deformation of Σ_{d-p-1} to Σ'_{d-p-1} such that $\partial M = \Sigma_{d-p-1} \cup \overline{\Sigma'_{d-p-1}}$.

5.2 Ward Identity for p-Form symmetries

Recall that in (5.1.2) we saw that we can couple the theory to an $(p+1)$-form background gauge field B_{p+1}

$$S_p = i\int B_{p+1} \wedge \star J_{p+1} \tag{5.2.1}$$

with background gauge transformation:

$$B_{p+1} \to B_{p+1} + d\lambda_p \tag{5.2.2}$$

Now let $W_q(\Gamma_p)$ which is supported on Γ_p, be the charged operator under p-form symmetry with charge q. Then under the background gauge transformation $W_q(\Gamma_p)$ transforms as

$$W_q(\Gamma_p) \to \exp\left(iq\int_{\Gamma_p} \lambda_p\right) W_q(\Gamma_p) \tag{5.2.3}$$

Expanding to first order in λ_p

$$\delta W_q(\Gamma_p) = iq\left(\int_{\Gamma_p} \lambda_p\right) W_q(\Gamma_p) \tag{5.2.4}$$

Let δ_{Γ_p} be the Poincare dual to Γ_p in M_d such that

$$\int_M \lambda_p \ \wedge \ \delta_{\Gamma_p} = \int_{\Gamma_p} \lambda_p \tag{5.2.5}$$

Then we can write (5.2.4) as

$$\delta W_q(\Gamma_p) = iq \left(\int_M \lambda_p \ \wedge \ \delta_{\Gamma_p} \right) W_q(\Gamma_p) \tag{5.2.6}$$

Also from (5.1.3) we have

$$\delta S_p = i \ (-1)^{p+1} \int_{M_d} \lambda_p \ \wedge d \star J_{p+1} \tag{5.2.7}$$

Now consider the partition function

$$Z[B] = \int \mathcal{D}\phi \exp \left(-S[\phi] + i \int_M B_{p+1} \wedge \star J_{p+1} \right) \tag{5.2.8}$$

And so

$$\langle W_q(\Gamma_p) \rangle_B = \frac{1}{Z[B]} \int \mathcal{D}\phi \ \exp \left(-S + i \int B_{p+1} \wedge \star J_{p+1} \right) \ W_q(\Gamma_p) \tag{5.2.9}$$

Now consider the background gauge transformation on (5.2.6) and (5.2.7) Then the correlator becomes

$$\langle W'_q(\Gamma_p) \rangle_B = \frac{1}{Z[B]} \int \mathcal{D}\phi \ \exp \left(-S + i \int B_{p+1} \wedge \star J_{p+1} + \delta S_p \right) \ W'_q(\Gamma_p) \tag{5.2.10}$$

Now if we expand $\exp(\delta S_p)$ to first order in λ_p and write

$$\begin{aligned} e^{\delta S_p} \ W'_q(\Gamma_p) &\approx (1 + \delta S_p)(W_q(\Gamma_p) + \delta W_q(\Gamma_p)) \\ &\approx W_q(\Gamma_p) + \delta W_q(\Gamma_p) + \delta S_p \ W_q(\Gamma_p) \end{aligned} \tag{5.2.11}$$

So the total variation is

$$\delta_\lambda \langle W_q(\Gamma_p) \rangle_B = \int_M \lambda_p \wedge \left[i(-1)^{p+1} \langle d \star J_{p+1}(x) W_q(\Gamma_p) \rangle_B + iq \ \delta_{\Gamma_p}(x) \langle W_q(\Gamma_p) \rangle_B \right] \tag{5.2.12}$$

Finally, we make the argument that gauge transformations are redundancies and not physical symmetries. So the full correlator must be invariant under $B_{p+1} \to B_{p+1} + d\lambda_p$:

$$\delta_\lambda \langle W_q(\Gamma_p) \rangle_B = 0 \,, \qquad \forall \lambda_p \tag{5.2.13}$$

Now since λ_p is arbitrary we have

$$\langle d \star J_{p+1}(x)\ W_q(\Gamma_p)\rangle_B = q\ \delta_{\Gamma_p}(x)\langle W_q(\Gamma_p)\rangle_B \tag{5.2.14}$$

Where we have absorbed a factor of $(-1)^p$ into the defintion of the poincare dual. Now setting $B = 0$ gives us the Ward identity for p-form symmetries

$$d \star J_{p+1}(x)\ W_q(\Gamma_p) = q\ \delta_{\Gamma_p}(x \in \Gamma_p)\ W_q(\Gamma_p) \tag{5.2.15}$$

If we integrate (5.2.15) over a small $(d-p)$-ball B^{d-p} and use Stokes theorem

$$\left(\int_{B^{d-p}} d \star J_{p+1}\right)\ W_q(\Gamma_p) = \left(\int_{S^{d-p-1}} \star J_{p+1}\right)\ W_q(\Gamma_p) = q\left(\int_{B^{d-p}} \delta_{\Gamma_p}(x \in \Gamma_p)\right)\ W_q(\Gamma_p) \tag{5.2.16}$$

But from **??** we know

$$\int_{B^{d-p}} \delta^{d-p}_{\Gamma_p}(x \in \Gamma_p) = \#(B^{d-p}, \Gamma_p) = \mathrm{Link}(S^{d-p-1}, \Gamma_p) \tag{5.2.17}$$

where $\#(B^{d-p}, \Gamma_p)$ is the intersection number in M_d.

If Γ_p tranversaly interesects B^{d-p} at exactly one point, then upto a sign determined by the orientation,

$$\#(B^{d-p}, \Gamma_p) = \mathrm{Link}(S^{d-p-1}, \Gamma_p) = 1 \tag{5.2.18}$$

Therefore

$$\left(\int_{S^{d-p-1}} \star J_{p+1}\right)\ W_q(\Gamma_p) = Q_{S^{d-p-1}}\ W_q(\Gamma_p) = q\ W_q(\Gamma_p) \tag{5.2.19}$$

So the flux of $\star J_{p+1}$ through the linking sphere measures the p-form charge.

5.3 Action of SDO on Charged Operators

Now that we have the Ward identity, we can derive the action of $U_g(\Sigma_{d-p-1})$ on $W_q(\Gamma_p)$. Let us fix a particular Γ_p and pick a manifold $\Sigma_{d-p-1})$ that links with Γ_p and a manifold $\Sigma'_{d-p-1})$ that is homotopic to $\Sigma_{d-p-1})$ but does not link Γ_p. From (5.1.8) we get

$$U_g(\Sigma_{d-p-1}) \cdot U_{g^{-1}}(\Sigma'_{d-p-1}) = \exp\left\{i\lambda \int_M d \star J_{p+1}\right\} \tag{5.3.1}$$

Where $g = e^{i\lambda}$. Now we insert a charged operator $W_q(\Gamma_p)$ and take the expectation value:

$$\langle U_g(\Sigma_{d-p-1})U_{g^{-1}}(\Sigma'_{d-p-1})W_q(\Gamma_p)\rangle = \left\langle \exp\left(i\lambda\int_M d\star J\right)W_q(\Gamma_p)\right\rangle \tag{5.3.2}$$

Now if we expand the exponential and use the Ward identity, $d\star J_{p+1}(x)\ W_q(\Gamma_p) = q\ \delta_{\Gamma_p}(x\in\Gamma_p)\ W_q(\Gamma_p)$ the exponent becomes

$$\begin{aligned}\left(i\lambda\int_M d\star J\right)W_q(\Gamma_p) &= \left(i\lambda\int_M q\ \delta_{\Gamma_p}\right)W_q(\Gamma_p)\\ &= iq\lambda\ \#(M,\Gamma_p)\ W_q(\Gamma_p)\\ &= iq\lambda\ \ \mathrm{Link}(\Sigma_{d-p-1},\Gamma_p)\ W_q(\Gamma_p)\end{aligned} \tag{5.3.3}$$

Where $\#(M,\Gamma_p)$ is the intersection number of M with Γ_p. Equivalently, that integer is also the linking number between Σ_{d-p-1} and Γ_p. Now with (5.3.3) we can write

$$\exp\left(i\lambda\int_M d\star J\right)W_q(\Gamma_p) = \exp\left(iq\lambda\,\mathrm{Link}(\Sigma_{d-p-1},\Gamma_p)\right)\ W_q(\Gamma_p) \tag{5.3.4}$$

Substituting this expression in (5.3.2) would get us:

$$\langle U_g(\Sigma_{d-p-1})U_{g^{-1}}(\Sigma'_{d-p-1})W_q(\Gamma_p)\rangle = \exp\left(iq\lambda\,\mathrm{Link}(\Sigma_{d-p-1},\Gamma_p)\right)\ \langle W_q(\Gamma_p)\rangle \tag{5.3.5}$$

Now since this is a time-ordered correlator, we can move the insertions past one another and it would not change anything:

$$\langle U_g(\Sigma_{d-p-1})\ W_q(\Gamma_p)\ U_{g^{-1}}(\Sigma'_{d-p-1})\rangle = \exp\left(iq\lambda\,\mathrm{Link}(\Sigma_{d-p-1},\Gamma_p)\right)\ \langle W_q(\Gamma_p)\rangle \tag{5.3.6}$$

Finally if we insert another symmetry defect operator along Σ'_{d-p-1} then inserting both $U_{g^{-1}}$ and U_g along Σ'_{d-p-1} is equivalent to inserting no operator along Σ_{d-p-1} which means

$$U_{g^{-1}}(\Sigma'_{d-p-1})\cdot U_g(\Sigma'_{d-p-1}) = \mathbb{1} \tag{5.3.7}$$

And with that we have the action of the SDO on $W_q(\Gamma_p)$:

$$\langle U_g(\Sigma_{d-p-1})\ W_q(\Gamma_p)\rangle = \exp\left(iq\lambda\,\mathrm{Link}(\Sigma_{d-p-1},\Gamma_p)\right)\ \langle W_q(\Gamma_p)\ U_g(\Sigma'_{d-p-1})\rangle \tag{5.3.8}$$

Chapter 6

Spontaneous Symmetry Breaking

6.1 Goldstone's Theorem for 0-Form Symmetries

There are multiple ways to approach this theorem. We will consider a Euclidean path-integral proof which originally appeared in [23]. For a continuous 0-form global symmetry with conserved current j^μ and local operator $\mathcal{O}(x)$ at the origin of charge q the Ward identity is given by

$$\partial_\mu j^\mu(x)\mathcal{O}(0) = iq\delta^{(d)}(x)\mathcal{O}(0) \tag{6.1.1}$$

Now let us integrate both sides over the d-ball of radius R centered at the origin and use the divergence theorem. The left-hand side becomes

$$\int_{B^d(R)} d^dx\partial_\mu j^\mu(x)\mathcal{O}(0) = \oint_{S^{d-1}(R)} dS_i \ j^i(x) \ \mathcal{O}(0) \tag{6.1.2}$$

Here $S^{d-1}(R)$ is the sphere of radius R centered at the origin. B^d is the interior of the sphere. Also $dS_i = n_i \ R^{d-1} \ d\Omega_{d-1}$ is the oriented surface element where $d\Omega_{d-1}$ is the unit sphere measure and n_i is the outward unit normal to the surface.

The integral on the right hand side of (6.1.1) is trivial. it just picks up the delta function at the origin.

$$i \ q \int_{B^d(R)} d^dx \ \delta^{(d)}(x)\mathcal{O}(0) = iq \ \mathcal{O}(0) \tag{6.1.3}$$

And so we get

$$\oint_{S^{d-1}(R)} dS_i \ j^i(x) \ \mathcal{O}(0) = iq \ \mathcal{O}(0) \tag{6.1.4}$$

Now taking the expectation value of both side gives us

$$\begin{aligned}
&\left\langle \oint_{S^{d-1}(R)} dS_i \; j^i(x) \; \mathcal{O}(0) \right\rangle = iq \; \langle \mathcal{O} \rangle \\
&\Rightarrow \oint_{S^{d-1}(R)} dS_i \; \langle j^i(x) \; \mathcal{O}(0) \rangle = iq \; \langle \mathcal{O} \rangle
\end{aligned} \tag{6.1.5}$$

Now let x be a point on the sphere $S^{d-1}(R)$. Let n^i be the unit outward normal at that point so $x^i = R \; n^i$. And consider the correlator

$$V^i(x) = \langle j^i(x) \; \mathcal{O}(0) \rangle, \qquad x \in S^{d-1}(R) \tag{6.1.6}$$

Fix the point x. The subgroup of rotations that leaves x fixed is $SO(d-1)$ and so $V^i(x)$ is invariant under $SO(d-1)$. Now the only vectors in $\mathbb{R}^d$ that are invariant under all of $SO(d-1)$ are precisely those that are proportional to the radial vector n^i. Therefore,

$$V^i = \langle j^i(x) \; \mathcal{O}(0) \rangle = F(R) \; n^i, \qquad n^i = \frac{x^i}{R} \tag{6.1.7}$$

where $F(R)$ is a scalar function.
Substituting (6.1.7) into (6.1.5) we get

$$\oint_{S^{d-1}(R)} dS_i V^i = \int_{S^{d-1}} R^{d-1} d\Omega_{d-1} n_i n^i F(R) = R^{d-1} \Omega_{d-1} F(R) = iq \langle \mathcal{O} \rangle \tag{6.1.8}$$

Here $\Omega_{d-1} \equiv \mathrm{Vol}(S^{d-1})$ is the area of the unit (d-1)-sphere. Then

$$F(R) = \frac{iq \; \langle \mathcal{O} \rangle}{\Omega_{d-1} R^{d-1}} \tag{6.1.9}$$

So the correlator on the sphere is explicitly

$$\langle j^i(x) \mathcal{O}(0) \rangle = \frac{iq \langle \mathcal{O} \rangle}{\Omega_{d-1}} \frac{x^i}{|x|^d}. \tag{6.1.10}$$

There are now two possibilities to consider here:

1. This is the case where $\langle \mathcal{O} \rangle = 0$. The left-hand side also vanishes. The symmetry is *unbroken* here.
2. The $\langle \mathcal{O} \rangle \neq 0$ case corresponds to *spontaneous symmetry breaking.*

$$\langle j^i(x) \mathcal{O}(0) \rangle \sim \frac{iq \langle \mathcal{O} \rangle n^i}{R^{d-1}} \tag{6.1.11}$$

We can see explicitly that there is a power-law in distance correlation in the theory, and so there must be at least one gapless mode to generate this power-law correlation. This gapless mode is the Goldstone mode.

6.2 Golstone's Theorem for p-Form Symmetries

We will begin by consedering the Ward identity that we derived in 4.2

$$d \star J_{p+1}(x)\ W_q(\Gamma_p) = iq\ \delta_{\Gamma_p}(x \in \Gamma_p)\ W_q(\Gamma_p) \tag{6.2.1}$$

We just need to show that the perimeter law corresponds to a correlation that implies a gapless mode. So let us just assume that we are in a spontaneously broken phase with the vev obeying a perimeter law.

$$\langle W(\Gamma_p)\rangle \sim \exp(-m\ \mathrm{P}[\Gamma_p]) \tag{6.2.2}$$

where the perimeter $P[\Gamma_p]$ denotes the p-volume of the submanifold Γ_p. Recall in **??** we discussed at length that we can have a perimeter term as a loop counter term.

$$\overline{W}(\Gamma_p) \equiv \exp(+m\ P[\Gamma_p])\ W(\Gamma_p) \tag{6.2.3}$$

$\overline{W}_q(\Gamma_p)$ satisfies the same Ward identity as $W(\Gamma_p)$,

$$d \star J_{p+1}(x)\ \overline{W}_q(\Gamma_p) = iq\ \delta_{\Gamma_p}(x \in \Gamma_p)\ \overline{W}_q(\Gamma_p) \tag{6.2.4}$$

Now consider a $(d-p)$ ball B^{d-p} that is transversal to Γ_p,i.e, $B^{d-p} \pitchfork \Gamma_p$ such that they transversely intersect at only one point p with sign(p) $= +1$. The ball B^{d-p} has as its boundary the sphere, S^{d-p-1} of radius R. So

$$\#(B^{d-p},\ \Gamma_p) = \mathrm{Link}(S^{d-p-1},\ \Gamma_p) = 1 \tag{6.2.5}$$

Now integrate x on both side of (6.2.4) over the ball B^{d-p}. The left hand side gives us

$$\int_{B^{d-p}} d \star J_{p+1}(x)\ \overline{W}_q(\Gamma_p) = \oint_{S^{d-p-1}(R)} \star J_{p+1}(\theta)\ \overline{W}(\Gamma_p). \tag{6.2.6}$$

where θ is a point on the sphere S^{d-p-1}. Now since

$$\int_{B^{d-p}} \delta_{\Gamma_p}(x \in \Gamma_p) = \#(B^{d-p},\ \Gamma_p) = \mathrm{Link}(S^{d-p-1},\ \Gamma_p) = 1 \tag{6.2.7}$$

we have

$$\oint_{S^{d-p-1}(R)} \star J_{p+1}(\theta)\overline{W}(\Gamma_p) = iq\overline{W}(\Gamma_p) \tag{6.2.8}$$

Taking the expectation value we get:

$$\langle \oint_{S^{d-p-1}(R)} \star J_{p+1}(\theta)\overline{W}(\Gamma_p)\rangle = iq\langle\overline{W}(\Gamma_p)\rangle \tag{6.2.9}$$

We had already assumed that we are in a spontaneously broken phase,i.e., $\langle \overline{W}(\Gamma_p) \rangle \neq 0$. And so the left hand side is both non-zero and independent of the radius R. This implies that the correlation function must be of the form,

$$\langle \star J_{p+1}(\theta) \overline{W}(\Gamma_p) \rangle \sim \frac{iq}{R^{d-p-1}} \tag{6.2.10}$$

Chapter 7

The Strong CP Problem and Axions

7.1 The $U(1)_A$ Problem and Its Resolution

Consider QCD Lagrangian for N flavors

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \overline{q}\,(i\gamma^\mu D_\mu - M)\, q\,, \qquad q(x) = \begin{pmatrix} q_1(x) \\ \vdots \\ q_N(x) \end{pmatrix}, \tag{7.1.1}$$

where $F^a_{\mu\nu}$ is the gluon field strength, $D_\mu = \partial_\mu - ig_s A^a_\mu T^a$ and $\overline{q} = q^\dagger \gamma^0$. Define the chiral projectors to get the left-handed and right-handed Weyl spinors

$$\begin{aligned} P_L &= \frac{1-\gamma_5}{2}, \quad P_R = \frac{1+\gamma_5}{2}\,, \\ q_L &= P_L\, q\,, \quad q_R = P_R\, q. \end{aligned} \tag{7.1.2}$$

Then the quark part of the Lagrangian(with $M = 0$) is

$$\mathcal{L}_{\text{quark},\ M=0} = \overline{q}\ i\gamma^\mu D_\mu q = \overline{q}_L\ i\gamma^\mu D_\mu q_L + \overline{q}_R\ i\gamma^\mu D_\mu q_R. \tag{7.1.3}$$

In the limit of vanishing quark masses $m_f = 0$ the Lagrangian has a classical global symmetry: $U(N)_L \times U(N)_R$ acting as

$$q_L \to U_L\ q_L\,, \quad q_R \to U_R\ q_R, \tag{7.1.4}$$

with $U_L, U_R \in U(N)$. We can rewrite this using vector and axial rotations

$$U(N)_V = \{U_L = U_R\}\,, \qquad U(N)_A = \{U_L = U_R^\dagger\}. \tag{7.1.5}$$

Then

$$U(N)_L \times U(N)_R \simeq U(N)_V \times U(N)_A. \tag{7.1.6}$$

Now we would work with $N = 2$ since $m_u, m_d << \Lambda_{QCD}$, and so, for at least these quarks, it would be reasonable to consider the limit of sending quark masses to zero. Therefore, we should expect the strong interactions to be approximately invariant under $U(2)_V \times U(2)_A$. Let us focus on $U(2)_A$ as we have experiments showing vector symmetry to be a good approximate symmetry of nature. But things change when we consider axial symmetries $U(2)_A \simeq SU(2)_A \times U(1)_A$. Here, quark condensates $< \overline{u}u >=< \overline{d}d >\neq 0$ form, breaking the axial symmetry spontaneously. There are 3 broken generators in $SU(2)_A$, so we expect 3 Goldstone bosons, and indeed corresponding to these we have the pions. But the η' meson associated with spontaneous breaking of $U(1)_A$, as it turns out, is not light and much heavier than other mesons. Weinberg called it the $U(1)_A$ problem, suggesting that, somehow, $U(1)_A$ is not a true symmetry of QCD.

A resolution of this problem would seem to be provided by the chiral anomaly for axial currents. We call classical symmetries anomalous if they are broken in the quantum theory. Fujikawa in 1979 showed that in the path integral, assuming that the classical action $S[\Phi_a, \partial\Phi_a]$ is invariant under $\Phi_a \to \Phi'_a$, if there is an anomaly then the measure $\mathcal{D}\Phi_a$ is not invariant and in our case the divergence of the axial current gets an anomalous contribution

$$\partial_\mu J^\mu_A = \frac{g^2 N}{32\pi^2} F^{\mu\nu}_a \tilde{F}_{a\mu\nu}, \tag{7.1.7}$$

where $\tilde{F}_{a\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}_a$. Therefore, the chiral anomaly affects the action

$$\delta S = \alpha \int d^4x \partial_\mu J^\mu_A = \alpha \frac{g^2 N}{32\pi^2} \int d^4x F^{\mu\nu}_a \tilde{F}_{a\mu\nu}, \tag{7.1.8}$$

even when classically, in the massless quark limit, QCD is invariant under a $U(1)_A$ transformation

$$q_f \to e^{i\alpha\gamma_5} q_f. \tag{7.1.9}$$

So, it would seem that the $U(1)_A$ problem is resolved, but it turns out that there are some subtleties to it. One is that this term is CP violating term, and the second is that the anomalous contribution that we did get is a total derivative.

$$F^{\mu\nu}_a \tilde{F}_{a\mu\nu} = \partial_\mu K^\mu, \tag{7.1.10}$$

where $K^\mu = \epsilon^{\mu\alpha\beta\gamma} A_{a\alpha}[F_{a\beta\gamma} - \frac{g}{3} f_{abc} A_{b\beta} A_{c\gamma}]$.
Therefore, δS is a pure surface integral

$$\delta S = \alpha \frac{g^2 N}{32\pi^2} \int d^4x \partial_\mu K^\mu = \alpha \frac{g^2 N}{32\pi^2} \int d\sigma_\mu K^\mu. \tag{7.1.11}$$

If we just use the boundary condition $A^a_\mu = 0$ at spatial infinity, the contribution to the action will be zero, and it would appear that $U(1)_A$ is again a symmetry. It was not until t'Hoofts' realization that the $U(1)_A$ problem would finally be resolved. He understood that the QCD vacuum has a much more complicated structure than anyone initially thought. It is because of this nature of the QCD vacuum that, in effect, makes $U(1)_A$ not a true symmetry of QCD. t'Hooft showed that the appropriate choice of boundary condition is to take A^a_μ as a pure gauge field. That means A^a_μ at spatial infinity should either be 0 or a gauge transformation of 0. With this boundary condition, apparently, there are gauge configurations for which $\int d\sigma_\mu K^\mu \neq 0$, and we can conclude that $U(1)_A$ is not a symmetry of QCD.

Let us work on the temporal gauge $A^a_0 = 0$ and consider $SU(2)$ QCD for simplicity. Then, under a gauge transformation, these fields transform as

$$A_i \to A'_i = \Omega A_i \Omega^{-1} + \frac{i}{g}(\partial_i \Omega)\Omega^{-1}. \tag{7.1.12}$$

So, the vacuum configurations either vanish or have the form of pure gauge $\frac{i}{g}(\partial_i \Omega)\Omega^{-1}$. In addition, preserving $A^a_0 = 0$ requires that the residual gauge transformations be time independent. It is useful to further restrict the residual gauge transformations by letting them approach a constant, which we choose to be unity, at spatial infinity

$$\Omega(\vec{x}) \to 1 \qquad \text{as} \qquad |\vec{x}| \to \infty. \tag{7.1.13}$$

From a gauge transformation point of view, this restriction above effectively leads to the space $\mathbb{R}^3$ being compactified to S^3. Therefore, any residual gauge transformation $\Omega(\vec{x})$ defines a map

$$S^3 \to SU(2) \simeq S^3, \tag{7.1.14}$$

which fall into homotopy classes labeled by

$$\pi_3(S^3) \cong \mathbb{Z}, \tag{7.1.15}$$

where $n \in \mathbb{Z}$ is the integer winding number. This means that $\Omega_n : S^3 \to S^3$ has a degree (winding number) $n \in \pi_3(S^3) \cong \mathbb{Z}$, so each pure gauge configuration is labeled by an integer. Two different integers correspond to two topologically distinct vacua.

It turns out that the integer winding number is given by [3]

$$n = \frac{ig^3}{24\pi^2} \int d^3x \ \mathrm{Tr}(\epsilon_{ijk} \ A^i_n A^j_n A^k_n). \tag{7.1.16}$$

Now recall that in (**??**) we identified the current K^μ. In the temporal gauge $A^0_a = 0$ the only nonzero component is K^0, and one can find [3]

$$K^0 = -\frac{g}{3}\epsilon_{ijk}\ \epsilon_{abc}A^i_a A^j_b A^k_c = \frac{4}{3}ig\ \epsilon_{ijk}\mathrm{Tr}.(A^i A^j A^k) \tag{7.1.17}$$

So far for a fixed time t, we have looked at gauge field on that spatial slice. That $3d$ configuration has some winding number corresponding to how many times the map from S^3 to the gauge group wraps around. We can see that

$$n(t) = \frac{g^2}{32\pi^2}\int d^3x\ K^0. \tag{7.1.18}$$

At early and late times, the field relaxes to pure-gauge vacua. At $t \to -\infty$, the configuration is gauge-equivalent to a vacuum of winding $n(-\infty)$. At $t \to +\infty$, it approaches another vacuum of winding $n(+\infty)$. So, the difference in winding number or topological charge, as it is called ν, is given by

$$\nu = n|_{t=+\infty} - n|_{t=-\infty} = \int dt \frac{d}{dt}\ n(t) = \frac{g^2}{32\pi^2}\int_{-\infty}^{+\infty} dt \int d^3x \partial_0 K^0 = \frac{g^2}{32\pi^2}\int d^4x \partial_0 K^0. \tag{7.1.19}$$

Now $\int d^4x \partial_0 K^0 = \int d^4x \partial_\mu K^\mu - \int d^4x \partial_i K^i$. By the $3d$ divergence theorem $\int d^4x \partial_i K^i = \int_{-\infty}^{+\infty} dt \int d^3x \partial_i K^i(t, \mathbf{x}) = \int_{-\infty}^{+\infty} dt \oint_{S^2_\infty} dS_i K^i = 0$(by the boundary conditions at spatial infinity). Then we get

$$\nu = n|_{t=+\infty} - n|_{t=-\infty} = \frac{g^2}{32\pi^2}\int d^4x \partial_\mu K^\mu = \frac{g^2}{32\pi^2}\int d\sigma_\mu K^\mu \Big|_{t=-\infty}^{t=+\infty}\quad , \tag{7.1.20}$$

where we have used the $4d$ Gauss theorem.

The pure gauges $A_i^{(n)} = \frac{i}{g}(\partial_i \Omega_n(x))\Omega_n^{-1}(x)$, with $\Omega_n : S^3 \to S^3$ of winding number n, correspond to different classical vacuum $|n>$. This is due to the set of all continuous maps $S^3 \to S^3$ splitting into disconnected sectors. Each such configuration has $S_E = 0$. They are all degenerate classical vacuums. These are the troughs shown in figure 7.1.

The red path is an *instanton* trajectory that interpolates between $A^{(1)}$ and $A^{(2)}$. An instanton is a particular classical solution of the Euclidean equation of motion with finite action that at $\tau \to -\infty$ tends to a pure gauge of winding n, and at $\tau \to +\infty$ tends to a pure gauge of winding $n+1$. In the configuration space, it is a trajectory that starts in a vacuum trough n, climbs over the barrier, where $F \neq 0$, so $S_E \neq 0$, and ends in a trough $n+1$.

The n-vacuum in itself is not an eigenstate of the Hamiltonian; rather, it is the superposition of these, called the θ-vacuum, which is the eigenstate of the Hamiltonian. This is the true vacuum defined as

$$|\theta\rangle\ =\ \sum_n e^{-in\theta}\,|n\rangle\,. \tag{7.1.21}$$

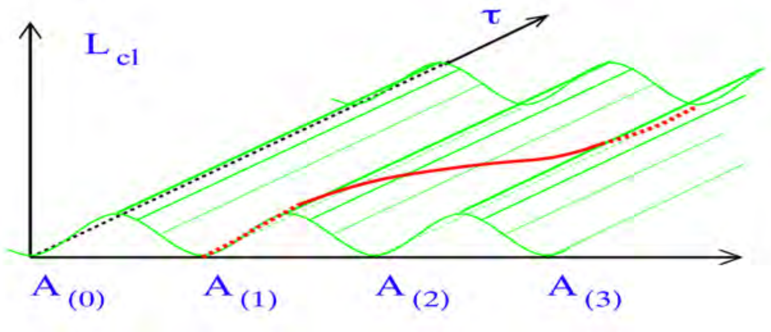

Figure 7.1. The classical QCD Lagrangian for different classes of gauge field configurations. The original figure is from [4]

Let us calculate the vacuum to vacuum transition amplitude using (7.1.20)

$$\begin{aligned}\langle\theta|\theta\rangle = \sum_{m,n} e^{im\theta}\, e^{-in\theta}\, \langle m|n\rangle &= \sum_{m,n} e^{i(m-n)\theta}\, \langle m|n\rangle \\ &= \sum_{\nu} e^{i\nu\theta}\, \langle n+\nu|m\rangle \\ &= \sum_{\nu,n} e^{i\theta\frac{g^2}{32\pi^2}\int d^4x F_a^{\mu\nu}\tilde{F}_{a\mu\nu}}\, \langle n+\nu|m\rangle\,,\end{aligned} \tag{7.1.22}$$

where we have defined $\nu = m - n$. So, the amplitude is the sum over all the cases that change the winding number by ν. If we rewrite the amplitude as a path integral over gauge fields, we get

$$\langle\theta|\theta\rangle = \sum_{\nu}\int \mathcal{D}A\; e^{iS_{\text{eff}}[A]}\;\delta\left[\nu - \frac{g^2}{32\pi^2}\int d^4x F_a^{\mu\nu}\tilde{F}_{a\mu\nu}\right], \tag{7.1.23}$$

where

$$S_{\text{eff}}[A] = S_{\text{QCD}}[A] + \theta\frac{g^2}{32\pi^2}\int d^4x F_a^{\mu\nu}\tilde{F}_{a\;\mu\nu}. \tag{7.1.24}$$

The delta function

$$\delta\left[\nu - \frac{g^2}{32\pi^2}\int d^4x F_a^{\mu\nu}\tilde{F}_{a\;\mu\nu}\right] \tag{7.1.25}$$

restricts the integral to the topological sector with integer charge ν.

Thus, we arrive at the consequence of acknowledging the complex structure of the QCD vacuum. It provides a resolution of the $U(1)_A$ problem, but there is a caveat: it adds an extra term to the QCD Lagrangian

$$\mathcal{L}_\theta = \theta \frac{g^2}{32\pi^2} F_a^{\mu\nu} \tilde{F}_{a\ \mu\nu}. \tag{7.1.26}$$

This term is a CP violating term (it violates Parity and Time reversal, but not charge conjugation, so it violates CP).

Now, here is the real problem: the existing bound on the neutron electric dipole moment requires θ to be extremely small ($\theta \leq 10^{-10}$). This is concerning since there is nothing in our discussions that would make us expect θ to be this small. Equivalently, it indicates that strong interactions preserve CP to an surprisingly exact degree. The θ term in the Lagrangian in (7.1.26) does not, in any way, necessitate that CP should be preserved to such a high and exact degree. This is known as the *strong CP problem.*

7.2 The Peccei-Quinn Mechanism and Axion

The most promising theoretical solution to the strong CP problem is the Peccei-Quinn mechanism [20]. They introduce a new global chiral $U(1)$ symmetry, most commonly referred to as $U(1)_{PQ}$, which provides a dynamical solution to the strong CP problem by effectively replacing θ with a dynamical CP-preserving field-*the axion*. The $U(1)_{PQ}$ symmetry is spontaneously broken, and the axion is the corresponding pseudo-Nambu-Goldstone boson. As a result, the axion field $a(x)$ enjoys a shift symmetry

$$a(x) \to a(x) + \alpha f_a \ , \tag{7.2.1}$$

where f_a is the order parameter associated with the breaking of $U(1)_{PQ}$.

The way it works is that the $U(1)_{PQ}$ current has a chiral anomaly and the anomalous term also represents an effective potential for the axion field. At its minimum, the θ term is canceled, which effectively restores CP for strong interactions.

But, as always, there is a problem, a *quality problem* to be more precise.

7.3 The Axion Quality Problem

We have seen that the Peccei-Quinn mechanism relies on the global $U(1)_{PQ}$ symmetry. An obvious problem that was even mentioned in the previous section is that $U(1)_{PQ}$ has an ABJ anomaly, which means that this symmetry is not, in any way, a symmetry of the quantum theory. There is one more problem, which is that even if we ignore the anomaly, the $U(1)_{PQ}$ is a global symmetry, and in a theory of quantum gravity, there is no place for a global symmetry. This is due to the *no global symmetries conjecture* [17, 21], which states there are no global symmetries in quantum gravity (i.e. *any symmetry is either broken or gauged*).

This is a serious problem, since the experimental bound on θ requires an abnormally intact PQ symmetry, which in turn means that the global axion shift symmetry must be well preserved. Any small explicit breaking of that symmetry will give rise to potential terms, which would hamper the PQ mechanism and its ability to cancel the θ term in QCD. This is the so-called *axion quality problem*.

One elegant solution to this stands out because of its ability to produce axions of *exponentially* good quality (see next chapter). This requires us to abandon this whole idea of the $U(1)_{PQ}$ and its spontaneous breaking and instead work in a higher-dimensional gauge theory, where the axion arises as the zero mode of the higher dimensional gauge field. The next chapter will explore this.

Chapter 8

Higher-Form Symmetry As a Probable Solution To The Axion Quality Problem

8.1 Introduction

The quality problem being exponentially less severe indicates that it favors axions, which naturally arise from the compactification of extra spacetime dimensions. In this chapter, we will consider a $U(1)$ gauge theory in $5d$, where the gauge field C is compactified to $4d$ on S^1. We will find out why we claim that constructions like these produce axions of *exponentially good quality*, and how non-local effects play a role in it. The discussion that follows will mostly follow [17, 22]

8.2 The Axion As a Kaluza-Klein Zero Mode

For our purposes, let us consider a minimal extra-dimensional axion model consisting of a one-form $U(1)$ gauge field and a one-form $SU(3)$ gauge field A_G on a $\mathbb{R}^{1,3} \times S^1$ spacetime. The field strength of the $U(1)$ and $SU(3)$ gauge fields will be denoted by dC and G,respectively. Let us consider the following $5D$ action

$$\begin{aligned} S^{5\mathrm{D}} = \int \bigg(& -\frac{1}{2g_5^2}\mathrm{d}C \wedge \star\mathrm{d}C - \frac{1}{2e_5^2}\mathrm{tr}\left[G \wedge \star G\right] \\ & + \frac{N}{8\pi^2} C \wedge \mathrm{tr}\left[G \wedge G\right] \bigg), \end{aligned} \tag{8.2.1}$$

where g_5 and e_5 are the 5D gauge couplings of C and A_G respectively, and $N \in \mathbb{Z}$. We define the $4D$ axion θ as a Kaluza-Klein zero mode of C

$$\theta(x) = \int_{S^1} C = \int_0^{2\pi R} C_5(x,y)\ dy \tag{8.2.2}$$

The axion has a periodicity $\theta \mapsto \theta + 2\pi n$ that comes from the $5D$ large gauge transformation $C \mapsto C + d\lambda$ where the gauge parameter λ is not single-valued on the circle. λ can wind the circle. Consider λ whose values after one loop around S^1 shifts by $2\pi n$:

$$\lambda(x, y + 2\pi R) = \lambda(x, y) + 2\pi n\,, \qquad n \in \mathbb{Z} \tag{8.2.3}$$

and so if we recall our discussion on Wilson loops in (4.7.4) we can write

$$\oint_{S^1} d\lambda = \lambda(y + 2\pi R) - \lambda(y) = 2\pi n \tag{8.2.4}$$

Therefore under $C \mapsto C + d\lambda$

$$\theta(x) \mapsto \int_{S^1} (C + d\lambda) = \theta(x) + \int_{S^1} d\lambda = \theta(x) + 2\pi n \tag{8.2.5}$$

So the 2π shift symmetry of the axion comes from the invariance of the theory under large gauge transformations.

Now let us see what happens to each of the terms in the action (8.2.1) after dimensional reduction. Let us use coordinates $x^M = (x^\mu, y)$ where $y \in [0, 2\pi R)$. The field strength $F_{MN} = \partial_M C_N - \partial_N C_M$. First of all, if we keep to zero mode, the axion becomes

$$\theta(x) = \int_{S^1} C = \int_0^{2\pi R} dy\; C_5(x, y) = 2\pi R\; C_5^{(0)}(x) \tag{8.2.6}$$

We can write the $5D$ kinetic term for the $U(1)$ gauge field as

$$-\frac{1}{2g_5^2}\int dC \wedge \star dC = -\frac{1}{4g_5^2}\int d^5x\; F_{MN}\; F^{MN} \tag{8.2.7}$$

For zero modes the only nonzero components relevant for the axion are $F_{\mu 5} = \partial_\mu C_5$, $F^{\mu 5} = \partial^\mu C_5$ and since

$$F_{MN}F^{MN} = 2F_{\mu 5}F^{\mu 5} + F_{\mu\nu}F^{\mu\nu}. \tag{8.2.8}$$

Dropping $F_{\mu\nu}$ zero mode piece gives us

$$\begin{aligned} -\frac{1}{4g_5^2}\int d^5x\; F_{MN}\; F^{MN} &\sim -\frac{1}{4g_5^2}\int d^5x\; 2\partial_\mu C_5\; \partial^\mu C_5 = -\frac{1}{2g_5^2}\int d^5x\; \partial_\mu C_5\; \partial^\mu C_5 \\ &= -\frac{2\pi R}{2g_5^2}\int d^4x\; \partial_\mu C_5^{(0)}\; \partial^\mu C_5^{(0)} \end{aligned} \tag{8.2.9}$$

Now using (8.2.6) we can write

$$
\begin{aligned}
-\frac{2\pi R}{2g_5^2}\int d^4x\ \partial_\mu C_5^{(0)}\ \partial^\mu C_5^{(0)} &= -\frac{2\pi R}{2g_5^2}\int d^4x \frac{1}{(2\pi R)^2}\partial_\mu\theta\ \partial^\mu\theta \\
&= -\frac{1}{2}\left(\frac{1}{g_5^2\ 2\pi R}\right)\int d^4x\ \partial_\mu\theta\ \partial^\mu\theta.
\end{aligned}
\tag{8.2.10}
$$

And so after dimensional reduction the $5D$ kinetic term for the $U(1)$ gauge field becomes

$$
\int -\frac{1}{2}f^2 \mathrm{d}\theta\wedge\star\mathrm{d}\theta, \quad f^2 = \frac{1}{2\pi R\ g_5^2}. \tag{8.2.11}
$$

8.3 Associated Higher-Form Symmetries

Let us ignore the gluon coupling in (8.2.1), as those are only relevant when we want to generate the canonical QCD axion potential, and just look at the free abelian gauge field C in $5d$

$$
S_{5D} = \int\left(-\frac{1}{2g_5^2}dC\wedge\star dC\right). \tag{8.3.1}
$$

In the free $5d$ theory there is a global *electric 1-form symmetry* $U(1)_e^{(1)}$ and a *magnetic 2-form symmetry* $U(1)_m^{(2)}$. They have currents J_e and J_m, respectively

$$
J_e = \frac{1}{g_5^2}\star\mathrm{d}C \quad , \quad J_m = \frac{1}{2\pi}\mathrm{d}C, \tag{8.3.2}
$$

where J_e is conserved

$$
dJ_e = \frac{1}{g_5^2}d(\star dC) = 0 \tag{8.3.3}
$$

because of the equation of motion $d\star dC = 0$ and conservation of J_m follows from the Bianchi identity $ddC = 0$.

Recall from chapter 4 that an $1-$form symmetry shifts the gauge field by a closed 1-form. Similarly in $5d$, the electric 1-form symmetry transformation shifts the gauge field

$$
C \to C + \Lambda, \tag{8.3.4}
$$

where Λ is a closed 1-form but not exact.

Now, here is the important part. We find that the axion shift symmetry is embedded in this electric one-form symmetry. If we focus on the S^1 part of the symmetry and consider the axion $\theta = \oint_{S^1} C$, then under the one-form symmetry

$$
\theta \to \theta + \oint_{S^1}\Lambda = \theta + \alpha \quad , \qquad \alpha\in[0,2\pi). \tag{8.3.5}
$$

We can also see this in another way if we consider the dimensional reduction of the current J_e.

Recall that

$$C_5^{(0)}(x) = \frac{\theta(x)}{2\pi R}, \tag{8.3.6}$$

then we adopt the following ansatz

$$C = C_5^{(0)}(x)\ dy = \frac{\theta(x)}{2\pi R}\ dy \tag{8.3.7}$$

to get

$$dC = \frac{1}{2\pi R}\ d\theta \wedge dy. \tag{8.3.8}$$

Since we are working with a metric that is a direct product, we can use the Hodge star property to write

$$\star_5\ dC = \frac{1}{2\pi R} \star_5 (d\theta \wedge dy) = \frac{1}{2\pi R} \star_4 d\theta \quad , \tag{8.3.9}$$

where $\star_4 d\theta$ is a 3-form in $\mathbb{R}^{1,3}$. Then after dimensional reduction the current becomes

$$J_e^{4d} = \frac{1}{g_5^2\ 2\pi R} \star d\theta = f^2 \star d\theta \quad , \tag{8.3.10}$$

which is exactly the 0-form axion shift symmetry current $J_s = f^2 \star d\theta$.

On the other hand, the $\mathbb{R}^{1,3}$ part of the shift provides the familiar $4d$ one-form symmetry.

Therefore, in this higher-dimensional model the 0-form axion shift symmetry naturally arises from the $U(1)_e$ electric 1-form symmetry.

The reason why this is a crucial observation and also why we are bothering to work our way through these kinds of models is this: the axion shift symmetry is what protects the $4d$ axion potential [22]. We can see this if we consider the $4d$ action after dimensional reduction

$$S_{4D} \supset \int \left(-\frac{1}{2} f^2 d\theta \wedge \star d\theta \right). \tag{8.3.11}$$

Here the $4d$ action is invariant under the global shift

$$\theta(x) \to \theta(x) + \alpha, \tag{8.3.12}$$

with constant α. The associated Noether current is exactly what we got above, i.e. $J_s = f^2 \star d\theta$. Now, if we have an exact symmetry $\theta \to \theta + \alpha$, then any local operator in the $4d$ field theory must be invariant under it. That forbids a mass term and any potential function $V(\theta)$. The only local terms allowed are terms depending on $d\theta$ [17]. However, these cannot generate an axion potential.

Returning to our original observation, since this axion shift symmetry, as we have seen, is embedded in the $U(1)_e$ electric 1-form symmetry, to assess the quality of these axions, we need to look at the ways in which this 1-form symmetry can be broken or gauged. The embedding of the shift symmetry then requires that if the axion were to acquire a potential, then this shift symmetry, and in which it is embedded, i.e. the electric 1-form symmetry must be broken.

Fortunately, 1-form symmetries have their advantages which will help us. First of all, there are, in general, less ways that a 1-form symmetry can break, and second, even if it breaks explicitly, it would have very little effect in the IR physics compared to 0-form symmetries. This robustness of higher-form symmetries is discussed in [14]. We will learn more about this in the next section, where we discuss electrically charged matter.

8.4 Modifying the Electric One Form Symmetry

8.4.1 Electrically Charged Matter

The first case we will consider is when we add electrically charged matter to the free theory. It is the most straightforward way to explicitly break the electric $U(1)_e^{(1)}$ one-form symmetry. Consider that a charged particle has field ϕ with electric charge $q \in \mathbb{Z}$ under C. Then the action becomes

$$S[C,\phi] = \int \left[-\frac{1}{2g_5^2} dC \wedge \star dC + (D\phi) \wedge \star(D\bar{\phi}) + V(|\phi|) \right] \tag{8.4.1}$$

where $D\phi = d\phi - iq\, C\phi$. If we vary C then the free part becomes

$$\begin{aligned} \delta S_{\text{free}} &= -\frac{1}{g_5^2} \delta(dC) \wedge \star dC \\ &= -\frac{1}{g_5^2} d(\delta C) \wedge \star dC \\ &= -\frac{1}{g_5^2} (\delta C) \wedge d \star dC \end{aligned} \tag{8.4.2}$$

where in the last line we did integration by parts and discarded the boundary.

And if $S_m = \int (D\phi) \wedge \star(D\bar{\phi})$ then $\delta(D\phi) = -iq(\delta C)\phi$ and $\delta(D\bar{\phi}) = iq(\delta C)\bar{\phi}$ then variation of the matter term is

$$
\begin{aligned}
\delta S_m &= \int [\delta(D\phi) \wedge \star(D\bar{\phi}) + (D\phi) \wedge \star\delta(D\bar{\phi})] \\
&= \int [-iq(\delta C)\phi \wedge \star(D\bar{\phi}) + (D\phi) \wedge \star\, iq(\delta C)\bar{\phi}] \\
&= \int \delta C \wedge [iq\bar{\phi} \wedge \star(D\phi) - iq\phi \wedge \star(D\bar{\phi})]
\end{aligned}
\tag{8.4.3}
$$

Define the 4-form current $\star j^{(1)} = iq\bar{\phi}\wedge\star(D\phi) - iq\phi\wedge\star(D\bar{\phi})$ and let $j_{\text{matter}} = \star j^{(1)}$ be the electric flux source, then the $U(1)_e^{(1)}$ current equation gets modified and has the form

$$
dJ_e = j_{\text{matter}} \tag{8.4.4}
$$

Therefore the electric current is no longer conserved. If we define the 1-form charge on a closed 3-manifold Σ_3

$$
Q_e(\Sigma_3) = \int_{\Sigma_3} J_e \tag{8.4.5}
$$

then deforming Σ_3 to Σ_3' through a 4-manifold M_4

$$
Q_e(\Sigma_3) - Q_e(\Sigma_3') = \int_{M_4} dJ_e = \int_{M_4} j_{\text{matter}} \tag{8.4.6}
$$

we see that Q_e is no longer topological. The operator on the right hand side on (8.4.6) measures the total electric charge enclosed by M_4. Now since all dynamical charges come in integer multiples of q then the violation Q_e is also a integer multiple of q.

$$
\int_{M_4} j_{\text{matter}} \in q\mathbb{Z} \tag{8.4.7}
$$

And so Q_e is no longer invariant under deformations of Σ_3 that sweep across matter worldlines. It now jumps by multiples of q. Consequently $U_e(\Sigma_3)$ ceases to be topological unless

$$
e^{i\alpha(\Delta Q_e)} = e^{i\alpha q m} = 1 \quad \forall m \in \mathbb{Z} \Longleftrightarrow \alpha \in \frac{2\pi}{q}\mathbb{Z}. \tag{8.4.8}
$$

Therefore the continuous $U(1)_e^{(1)}$ is explicitly broken down to a discrete subgroup

$$
U(1)_e^{(1)} \quad \longrightarrow \quad \mathbb{Z}_q^{(1)} \tag{8.4.9}
$$

And so only the exponentiated generator

$$
U_{e,k}(\Sigma_3) := \exp\left(\frac{2\pi i k}{q} Q_e(\Sigma_3)\right), \quad k \in \mathbb{Z} \tag{8.4.10}
$$

is topological. These generate the discrete $\mathbb{Z}_q^{(1)}$ symmetry. One way to see this is to note that now we can have endable Wilson lines. A field ϕ would have to be attached to a Wilson line of charge q to make a gauge invariant operator out of it. Then such a Wilson line can end or, as occasionally seen in literature, be screened. So Wilson lines of charge nq, where $n \in \mathbb{Z}$, can end on ϕ operator insertions. Therefore, the continuous $U(1)_e^{(1)}$ breaks down to a discrete $\mathbb{Z}_q^{(1)}$ 1-form symmetry.

We can now discuss how this explicit breaking of the electric 1-form symmetry gives rise to an effective potential for the axion. Local $5d$ terms that depend on the field strength $F = dC$, after dimensional reduction, become $4d$ terms depending on $d\theta$. These terms cannot generate an axion potential. To get $V(\theta)$ we can make a heuristic argument: instead of integrating over the charged field ϕ in the path integral, we can sum over the 1-dimensional worldlines γ traced out by $5d$ particles, which we will treat as heavy objects. So we are summing over closed loops γ in the $5d$ Euclidean spacetime, including the loops formed by the particle wrapping the $5d$ circle S^1 with winding ω. The worldline action for the particle is,

$$S(\gamma) = \int_\gamma m_{\mathrm{5d}}\ \mathrm{d}\tau + \mathrm{i}q \int_\gamma C \tag{8.4.11}$$

where τ is the proper length along the worldline. The coupling of the particle to the gauge field is $iq \int_\gamma C = iq \int_\gamma (C_\mu\ dx^\mu + C_5\ dy)$. We will focus on the term involving C_5 because that is what will generate $V(\theta)$.

If we integrate over y for a loop that does not wind the circle ($\omega = 0$), then $\int_\gamma C_5\ dy = 0$, which has no dependence on θ. On the other hand, for a loop that winds ω times around the compact dimension:

$$\int_\gamma C_5\ dy = \omega \int_{S^1} C_5\ dy = \omega\theta \tag{8.4.12}$$

So for the potential $V(\theta)$ we will only have to consider loops with $\omega \neq 0$. This integer $\omega \in \mathbb{Z}$ labels topologically distinct sectors of the path integral. A deformation that stays within a given w cannot change ω homotopically. Therefore sectors with different ω are disconnected. Now there are saddle point contributions to the path integral from each of these topological sectors, which means within each sector, the classical configuration that minimises the Euclidean action is a circular loop wrapping the S^1 exactly ω times. Such wrapped worldlines are sometimes called Euclidean worldline instantons. Then the proper length of the loop is just $2\pi R\ |\omega|$. And the action becomes

$$\begin{aligned} S_E &= m_{\mathrm{5d}} \int_\gamma \mathrm{d}\tau + iq \int_\gamma C \\ &\simeq 2\pi R\omega m_{5d} + iq\omega\theta \end{aligned} \tag{8.4.13}$$

We can now estimate the sum of contributions from winding numbers $\pm\omega$. The effective potential as a sum over all worldline instantons

$$\begin{aligned} V(\theta) &\propto \sum_{\omega\in\mathbb{N}} e^{-S_E(\gamma_\omega)} + e^{-S_E(\gamma_{-\omega})} \\ &\propto e^{-2\pi R m_{5d}\omega}(e^{iq\omega\theta} + e^{-iq\omega\theta}) \\ &\propto e^{-2\pi R m_{5d}\omega}\cos(\omega q\theta) \end{aligned} \tag{8.4.14}$$

This potential is *exponentially small* when the extra dimension is large [17]. This exponential suppression of the axion potential is the exponentially good quality that we have been mentioning in the previous sections.

8.4.2 Gauging the Magnetic Two-Form Symmetry

The next way through which the one-form symmetry can be broken is by gauging the magnetic two-form symmetry. We can do this by introducing a three-form gauge field K, coupling C. Consider the action

$$S^{5d} = \int -\frac{1}{2g_5^2}\, dC\wedge\star dC - \frac{1}{2e_K^2}\, dK\wedge\star dK + \frac{M}{2\pi}K\wedge dC \tag{8.4.15}$$

Here, e_K is the three-form gauge coupling and $M\in\mathbb{Z}$.

Let us try to see the conservation equation for J_e using the equation of motion of C

$$\delta S^{5d} = \int\left[-\frac{1}{g_5^2}\delta(dC)\wedge\star dC + \frac{M}{2\pi}K\wedge\delta(dC)\right] = 0 \tag{8.4.16}$$

Performing an integration by parts and identifying the current $J_e = \frac{1}{g_5^2}\star dC$, we arrive at the equation of motion

$$dJ_e = \frac{M}{2\pi}dK, \tag{8.4.17}$$

which clearly indicates that the electric 1-form symmetry is explicitly broken. If one tries to make it work by redefining the current,

$$J_e' = \frac{1}{g_5^2}\star dC - \frac{M}{2\pi}K, \tag{8.4.18}$$

which would satisfy $dJ_e' = 0$ but if we observe the SDO built from the current,

$$U(\Sigma_3) = \exp\left(i\alpha\int_{\Sigma_3} J_e'\right) \tag{8.4.19}$$

we can see that it is not gauge invariant under large gauge transformations of K unless $\alpha = 2\pi\mathbb{Z}/M$. A large gauge transformation of K is $K\to K + d\Lambda'_{\text{large}}$, where $d\Lambda'_{\text{large}}$ is a closed 3-form whose periods are quantized $\int_{\Sigma_3} d\Lambda_{\text{large}} = 2\pi\mathbb{Z}$ for any closed 3-cycle Σ_3. Under $K\to K + d\Lambda'_{\text{large}}$ the integral over Σ_3 shifts by

$$\int_{\Sigma_3} J'_e \longrightarrow \int_{\Sigma_3} J'_e - \frac{M}{2\pi}\int_{\Sigma_3} d\Lambda'_{\text{large}}, \tag{8.4.20}$$

and the SDO under this transformation picksupa phase

$$U(\Sigma_3) = \exp\left(i\alpha \int_{\Sigma_3} J'_e\right) \cdot \exp\left(-i\alpha M n\right). \tag{8.4.21}$$

Then for SDO to be gauge invariant,

$$\alpha M n \in 2\pi\mathbb{Z} \ , \qquad \forall n \in \mathbb{Z} \implies \alpha M \in 2\pi\mathbb{Z} \tag{8.4.22}$$

Thus

$$\alpha = \frac{2\pi k}{M} \quad , \qquad k \in \mathbb{Z} \tag{8.4.23}$$

What this means is that the electric $U(1)_e$ 1-form symmetry is broken down to a $\mathbb{Z}_M^{(1)}$ symmetry.

Now we vary the action with respect to K

$$\delta S^{5d} = \int \left[-\frac{1}{e_K^2}\delta K \wedge d\star dK + \frac{M}{2\pi}\delta K \wedge dC\right] = 0. \tag{8.4.24}$$

So, the equation of motion for K is

$$d\star dK = e_K^2 \frac{M}{2\pi} dC. \tag{8.4.25}$$

If we integrate this equation, we get

$$\star dK = e_K^2 \frac{M}{2\pi}(C - \beta), \tag{8.4.26}$$

where β is an arbitrary closed 1-form, i.e. $d\beta = 0$. Then substituting the expression for dK into the equation of motion of C would give us

$$d\star dC = g_5^2 e_K^2 \frac{M^2}{4\pi^2} \star (C - \beta). \tag{8.4.27}$$

This is the equation of motion of a massive vector field. So, C becomes massive with mass $m_C = g_5 e_K M/2\pi$.

Now, if we dimensionally reduce this equation, then it becomes a mass equation for the axion

$$d\star d\theta = \frac{e_{K,4}^2}{f^2}\frac{M^2}{4\pi^2} \star (\theta - \kappa), \tag{8.4.28}$$

where

$$m_\theta = \frac{e_{K,4}\ M}{2\pi f}, \tag{8.4.29}$$

and $\kappa$is an arbitrary closed 0-form.

8.4.3 Gauging the Electric One-Form Symmetry

An interesting case to study is when we gauge the electric $U(1)^{(1)}_e$ one-form symmetry. As we will see, gauging makes symmetries exact, which means that the electric one-form symmetry cannot be broken. Therefore it is natural to expect that the axion would not acquire a mass; however, via a Stueckelberg mechanism we can think of the axion getting a mass as it gets "eaten" by a gauge field.

In the ungauged theory, the electric 1-form global symmetry acts as a shift by a flat 1-form Λ

$$C \mapsto C + \Lambda \tag{8.4.30}$$

To gauge this 1-form symmetry, we promote Λ to an arbitrary 1-form $\Lambda(x)$ without needing any flatness condition. We introduce a 2-form gauge field B so that the action,

$$S^{5D} = \int \left(-\frac{1}{2e_B^2} \mathrm{d}B \wedge \star \mathrm{d}B - \frac{1}{2g_5^2} (\mathrm{d}C - kB) \wedge \star (\mathrm{d}C - kB) \right), \qquad k \in \mathbb{Z} \tag{8.4.31}$$

where e_B is the $5d$ gauge coupling of the 2-form B, is invariant under the combined gauge transformation

$$C \mapsto C + k\Lambda\,, \qquad B \mapsto B + d\Lambda \tag{8.4.32}$$

Therefore, the electric 1-form global symmetry has now been promoted to a local symmetry. Requiring Λ to be flat would recover the global symmetry. If we vary the action with respect to C we get

$$\delta S^{5D}|_{\delta C} = -\frac{1}{g_5^2} \int d(\delta C) \wedge \star (dC - kB) = +\frac{1}{g_5^2} \int \delta C \wedge d(\star (dC - kB)), \tag{8.4.33}$$

where we have done integration by parts and dropped the boundary term. Then the equation of motion is

$$d \left[\frac{1}{g_5^2} \star (dC - kB) \right] = 0. \tag{8.4.34}$$

This motivates us to define the gauge invariant refined electric current

$$J'_e \equiv \frac{1}{g_5^2} \star (dC - kB) = \underbrace{\frac{1}{g_5^2} \star dC}_{J_e} - \frac{k}{g_5^2} \star B \tag{8.4.35}$$

This current is exact. To see it, we need to look at the equation of motion for B. Let $X \equiv dC - kB$ then, $\delta X = -k\delta B$ and

$$\begin{aligned} \delta S^{5d} &= \int \left[-\frac{1}{e_B^2} d(\delta B) \wedge \star dB - \frac{1}{g_5^2} \delta X \wedge \star X \right] \\ &= \int \left[\frac{1}{e_B^2} \delta B \wedge d(\star dB) + \frac{1}{g_5^2} (k\delta B) \wedge \star X \right] \\ &= \int \delta B \wedge \left[\frac{1}{e_B^2} d(\star dB) + \frac{k}{g_5^2} \star X \right] \end{aligned} \tag{8.4.36}$$

Recall that we had defined the refined electric current as

$$J'_e \equiv \frac{1}{g_5^2} \star X = \frac{1}{g_5^2} \star dC - \frac{k}{g_5^2} \star B \tag{8.4.37}$$

Then the equation of motion for B comes out to be

$$kJ'_e = -\frac{1}{e_B^2} d(\star dB) \tag{8.4.38}$$

One immediate consequence of this exactness is that the current is conserved $dJ'_e = 0$.

After dimensional reduction the action in (8.4.31) becomes

$$S^{4d} = \int \left(-\frac{1}{2e_{B,4}^2} \mathrm{d}\widetilde{A} \wedge \star \mathrm{d}\widetilde{A} - \frac{1}{2} f^2 (\mathrm{d}\theta - k\widetilde{A}) \wedge \star (\mathrm{d}\theta - k\widetilde{A}) + \dots \right), \tag{8.4.39}$$

Where $\tilde{A}$ is the one-form zero mode of B and $e_{B,4}^2 \equiv e_B^2(2\pi R)$. In components that is

$$S^{4d} = \int \left[-\frac{1}{4e_{B,4}^2} \widetilde{F}_{\mu\nu} \widetilde{F}^{\mu\nu} - \frac{f^2}{2} (\partial_\mu \theta - k\widetilde{A}_\mu)(\partial^\mu \theta - k\widetilde{A}^\mu) + \dots \right] \tag{8.4.40}$$

The local symmetry(descended form $C \to C + k\Lambda,\ B \to B + d\Lambda$ in $5d$) is

$$\theta \to \theta + k\lambda(x), \quad \widetilde{A}_\mu \to \widetilde{A}_\mu + \partial_\mu \lambda(x) \tag{8.4.41}$$

Since θ shifts by $k\lambda(x)$ we can always choose $\lambda(x) = -\theta(x)/k$. In this gauge

$$\theta' = 0, \quad \widetilde{A}'_\mu = \widetilde{A}_\mu - \frac{1}{k} \partial_\mu \theta \tag{8.4.42}$$

Then the action

$$S^{4d} = \int -\frac{1}{4e_{B,4}^2}\widetilde{F}_{\mu\nu}\widetilde{F}^{\mu\nu} - \frac{f^2}{2}k^2\widetilde{A}_\mu\widetilde{A}^\mu \tag{8.4.43}$$

We can see that θ has completely disappeared from the theory. and the vector is massive with mass $m_{\tilde{A}} = k\ e_{B,4}f$. The shift in θ was a redundancy in the theory.

Chapter 9

Conclusion

In this thesis, we developed a systematic framework for higher-form symmetries and demonstrated how they provide a natural resolution of the axion quality problem. The central conceptual shift underlying this work is the transition from viewing symmetries as acting solely on local point operators, to recognizing that symmetries can act on extended objects such as lines, surfaces, and higher-dimensional defects. This generalized perspective reorganizes the structure of gauge theories and imposes powerful non-perturbative constraints invisible in the traditional symmetry framework.

We began by formulating ordinary global symmetries using topological symmetry defect operators, showing that their action is encoded through linking with charged operators. This naturally generalizes to 1-form and p-form symmetries, whose charged objects are extended rather than point-like. For 1-form symmetries, we constructed the corresponding defect operators, proved their abelian nature, and analyzed their action on Wilson and 't Hooft lines using intersection theory. In this framework, the Coulomb phase of electrodynamics emerges as the spontaneously broken phase of a continuous electric 1-form symmetry, with the photon playing the role of the associated Goldstone mode.

With this symmetry structure established, we turned to the strong CP problem and axions. While the Peccei–Quinn mechanism solves the strong CP problem by promoting the QCD θ-angle to a dynamical field, the exceptional quality required of the axion shift symmetry poses a deep theoretical puzzle, especially in light of expectations that global symmetries are forbidden by quantum gravity. The central result of this thesis is that higher-form symmetries provide a precise and technically natural explanation of axion quality when the axion arises from a higher-dimensional gauge field.

We showed that wrapped worldlines of heavy charged particles generate the allowed axion potential with exponential suppression. In this way, higher-form symmetry dictates the structure of the axion potential, while dynamics dictates its small magnitude.

While this thesis has focused on a five dimensional gauge theory compactified on a circle, the underlying mechanism could be extended to broader and much more interesting theories. Axions may arise from general p-form gauge fields integrated over non-trivial p-cycles Σ_p in higher-dimensional spaces [17], with exponential suppression controlled by the cycle volumes suggesting that axion quality can be further improved in large-volume compactifications.

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