MULTISEGMENT INTEGRATION TECHNIQUE FOR POST-BUCKLING ANALYSIS OF A PINNED-FIXED SLENDER ELASTIC ROD

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ABSTRACT

This paper investigates the post-buckling behavior of a slender axially inextensible elastic rod with pinned-fixed end. The set of five first order nonlinear ordinary differential equations with boundary conditions specified at both ends constitutes a complex two point boundary value problem. By using multisegment integration technique, the highly nonlinear boundary value problems are numerically solved. Results are presented in non-dimensional graphs for a range of prescribed loading condition. The secondary equilibrium paths and the post-buckling configurations of the rod are presented.

Key words: Post-buckling**,** slender rods, Multisegment integration technique, Equilibrium path.

I. INTRODUCTION

The elastic buckling of rods has always been a fundamental topic in structural mechanics. An originally straight rod would buckle if the axial compressive load exceeded a certain critical value. Traditionally buckling is used as a failure criterion. However, nowadays mechanical systems such as satellite tethers, marine cables, robotic arms, linkages, large antennas and so on, employ slender elements for transmission of forces, signals and power. They are usually designed to accept large displacements but deformations are kept within the elastic regime. Hence, the studies of post-buckling of elastic rods have wide engineering and applying backgrounds in recent days.

Based on the assumption that the axial line of the rod is inextensible, Timoshenko et al. [1] examined the post-buckling of compressed rod with both ends simply supported and presented a solution in elliptical integral form. Wang [2] dealt with the buckling of the axial compressive rod with pinned-fixed ends by using a shooting methods as well as a perturbation method, respectively. More recent studies on axially inextensible rods can be found in the literatures by Plaut et al. [3] and Lee [4]. Love's [5] seminal textbook on theory of mathematical elasticity has been extensively used in many fields of applied mechanics, establishing the basic for most research on the equilibrium of elastic rods. The asymptotic approach of Koiter [6]

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was a key development in the analysis of the initial post-buckling of a structure. With regard to extensibility (the stretching and contraction of the central axis), Stemple [7] and Antman [8] presented a consistent theory for extensional beam-column and derived an exact post-buckling solution for an extensible beam subjected to an applied compressible axial load. Nagai et al. [9] presented experimental results on chaotic oscillations of a post buckled reinforced beam subjected to lateral excitations. The beam was clamped at both ends; however, the motion at once end was arranged to be axially restricted by an elastic spring.

Analytical and experimental procedures were used by Tauchert and Lu [10] to investigate the large deformation of an initially imperfect elastic rod subjected to longitudinal and gravitational loads, where the ends of the rods rested on simple supports and were connected by a linear spring. Yin and Wang [11] studied the subcritical behavior of elastic clamped-free rods with constant original curvature under force load acting at the free end, where, in particular, the load displacement characteristics were examined. YuFeng and DeChao [12] developed the analytical solutions of rigid impact problems for two typical rod structures with elastic supports and thoroughly analyzed the boundary conditions effect on wave propagation and impact response. Li [13] presented a computational solution of elastica for a simply supported rod based on the theory of an extensible rod and gave a quantitative evaluation of the post-buckling deformation and the buckling rod. The post-buckling of extensible elastic rod was studied by Filipich and Rosales [14] using nonlinear geometric models. The classical strength-of-materials approach is compared and discussed with Lagrangian and Eulerian descriptions. Vaz and Silva [15] presented formulation and solution for the elastica of slender rods subjected to axial terminals forces and boundary conditions assumed hinged and elastically restrained with a rotational spring. Solutions for buckling, initial post-buckling (perturbation), large loads (asymptotic), and numerical integration were developed.

A relatively new numerical technique is the Multisegment integration method. Kalnins and Lestingi [16] introduced this method in the late sixties for solving linear and nonlinear system of ordinary differential equations. Multisegment integration technique is used to solve those boundary value problems of nonlinear ordinary

differential equations, which cannot be solved by direct integration, because the latter loses its accuracy in the process of subtraction of almost equal numbers in evaluating the unknown boundary values. Using this technique, the present paper solves the post-buckling problem of an elastic slender rod with one end fixed and the other end pinned. Finally, the equilibrium paths of the buckled rod are numerically obtained and the effects of load parameters on the buckling response are discussed.
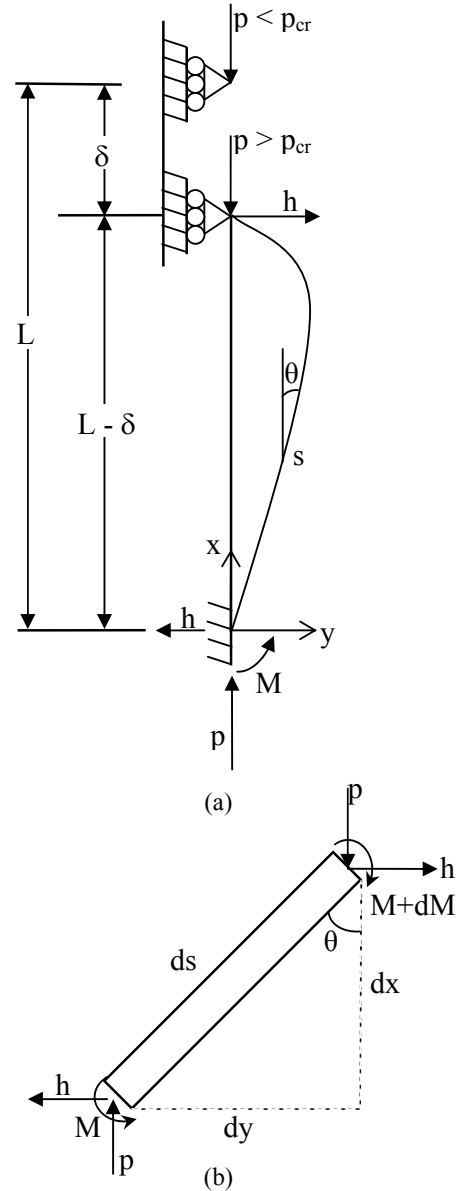

Figure 1. (a) The buckled rod, (b) elemental segment

II. GOVERNING EQUATIONS

Let us consider a long, slender, straight rod of initial undeformed length L, made of physically linear isotropic elastic material with the reference axis (x-axis) through the centroid of the rod cross-section. Transverse deformations (bending) are permitted, but shearing deformations are neglected. Figure 1(a) shows the deflected rod subjected to an axial load, p at the pinned end. There exists a horizontal reactive force h at the pin. Let a Cartesian coordinate system (x, y) be located at fixed end. Let s be the arc length from that end and θ be the local angle of inclination. The governing equations are derived from the geometrical compatibility, equilibrium of forces and moments and constitutive relations.

By analyzing the geometric relationship of the deformation of element dx to element ds as shown in Fig 1(b), it is easy to derive the geometric relations,

$$\frac{dx}{ds} = \cos\theta \tag{1}$$

$$\frac{dy}{ds} = \sin\theta \tag{2}$$

The general definition of the curvature κ is

$$\frac{d\theta}{ds} = \kappa \tag{3}$$

Equilibrium of forces in the x- and y- direction ($\Sigma F_x = 0$ and $\Sigma F_y = 0$) results in a constant reaction force component h and p acting on the rod end. Therefore

$$\frac{dh}{ds} = 0 \text{ and } \frac{dp}{ds} = 0 \tag{4}$$

Equilibrium of moments on the infinitesimal element of Fig. 1 gives

$$\left(M + dM\right) - M + h\,dx + p\,dy = 0 \tag{5}$$

where M is the bending moment. Dividing Eq. (5) by ds and employing Eq. (1) and (2) yields

$$\frac{dM}{ds} + p\sin\theta + h\cos\theta = 0 \tag{6}$$

Hooke's law applies for linear elastic materials and $M = EI\kappa$, where E is the Young modulus of elasticity and I is the cross-sectional area of inertia. For the sake of generality, the following non-dimensional terms are introduced:

$$X = \frac{x}{L} \tag{7a}$$

$$Y = \frac{y}{L} \tag{7b}$$

$$S = \frac{s}{L} \tag{7c}$$

$$K = \kappa L \tag{7d}$$

$$P = \frac{pL^2}{EI} \tag{7e}$$

$$H = \frac{hL^2}{EI} \tag{7f}$$

The governing equations (1), (2), (3), (4) and (6) can be transformed into the non-dimensional forms as follows:

$$\frac{dX}{dS} = \cos\theta \tag{8a}$$

$$\frac{dY}{dS} = \sin\theta \tag{8b}$$

$$\frac{d\theta}{dS} = K \tag{8c}$$

$$\frac{dK}{dS} = -P\sin\theta - H\cos\theta \tag{8d}$$

$$\frac{dH}{dS} = 0 \tag{8e}$$

A set of the boundary conditions must be specified:

$$X(0) = 0 \tag{9a}$$
$$Y(0) = 0 \tag{9b}$$
$$\theta(0) = 0 \tag{9c}$$
$$Y(1) = 0 \tag{9d}$$
$$K(1) = 0 \tag{9e}$$

Eqs. (9a)-(9c) represents non movable boundary conditions for fixed end whereas Eqs. (9d) and (9e) refer to a pinned condition at the upper end allowing movement in the x axis.

III. BUCKLING LOADS

Calculation of buckling loads follows straightforward approximation of moment equilibrium equation by assuming small displacement, i.e. $\sin\theta \approx \theta$, $\cos\theta \approx 1$. If small displacements are assumed, then the governing equation (8d) reduces to

$$\frac{d^4Y}{dX^4} + P\frac{d^2Y}{dX^2} = 0 \tag{10}$$

Four boundary conditions must be applied:

$$Y(0) = 0 \text{ and } \frac{dY}{dX}(0) = 0 \tag{11a}$$

$$Y(1) = 0 \text{ and } \frac{d^2Y}{dX^2}(1) = 0 \tag{11b}$$

Solution of Eq. (10) with boundary conditions (11a, b) are found in the published literature (Saha

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and Banu [17]). Hence the critical load is expressed by the following relation, $P_{cr} = 20.19$.

IV. NUMERICAL PROCEDURE

It is very cumbersome to obtain any analytical solutions of the complicated problem Eqs. (8) about the initial parameter vectors due to the inclusion of strong non-linearity and coupling in it. The associated boundary conditions are given at both ends, which characterize a two point boundary value problem. Several techniques can be employed for this problem (e.g. finite difference schemes, finite element methods, and energy methods). The finite difference method of solution or finite-element method of formulation of the nonlinear buckling problem governed by Eqs. (8) would convert it to a set of nonlinear algebraic equations, which are always solved by iteration as a sequence of solutions of a system of linear algebraic equations derived as approximations to the original equations, thus has often met the problem of non-convergence. Solutions via the shooting method with direct integration are conveniently employed in linear or non-linear problems when only one parameter is required for interpolation but they become rather complex if two conditions are sought in non-linear systems. Therefore, the multisegment integration method is employed to find numerical solutions to the problem. The idea behind the multisegment integration method is to replace the two point boundary value problem by a sequence of initial value problems. Thus, unknown values of the unknown functions at the initial point and unknown parameters are initially estimated to start the computing procedure (Kalnins and Lestingi, [16]) and these estimates are modified until specified boundary conditions at the terminal point are satisfied. The Runge-Kutta method is used to integrate the initial problem. Thus, the solution of the boundary value problem is obtained.

V. MULTISEGMENT INTEGRATION METHOD OF SOLUTION

The fundamental set of non-linear equations (8) together with the boundary conditions (9) has to be integrated over a finite range of the independent variable S. But the numerical integration of these equations is not possible beyond a very limited range of S due to the loss of accuracy in solving for the unknown initial values, as pointed out by

Sepetoski *et al.* [18]. Thus the multisegment method of integration developed by Kalnins and Lestingi [16] has been used in this analysis.

If the fundamental variables X, Y, θ, K and H of Eqs. (8) are represented in matrix notation by [w] in a standard form as follows,

$$\frac{dw}{dS} = f(S, w; P) \qquad (12)$$

in which,

$$w = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 \end{bmatrix}^T$$
$$= \begin{bmatrix} X & Y & \theta & K & H \end{bmatrix}^T$$

and

$$f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \\ K \\ -P\sin\theta - H\cos\theta \\ 0 \end{bmatrix} = \begin{bmatrix} \cos w_3 \\ \sin w_3 \\ w_4 \\ -P\sin w_3 - w_5 \cos w_3 \\ 0 \end{bmatrix}$$

The boundary conditions Eqs. (9) can be rearranged in the following form as follows:

$$Aw(0) + Bw(1) = C \qquad (13)$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let us consider the initial value problem corresponding to boundary value problem,

$$\frac{dW}{dS} = F(S, w, W; P) \qquad (14)$$

with

$$W(0) = I \qquad (15)$$

where,

$$W = \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} & W_{15} \\ W_{21} & W_{22} & W_{23} & W_{24} & W_{25} \\ W_{31} & W_{32} & W_{33} & W_{34} & W_{35} \\ W_{41} & W_{42} & W_{43} & W_{44} & W_{45} \\ W_{51} & W_{52} & W_{53} & W_{54} & W_{55} \end{bmatrix} \text{ and}$$

$$F = \begin{bmatrix} \dfrac{df_1}{dw_1(0)} & \dfrac{df_1}{dw_2(0)} & \dfrac{df_1}{dw_3(0)} & \dfrac{df_1}{dw_4(0)} & \dfrac{df_1}{dw_5(0)} \\[2mm] \dfrac{df_2}{dw_1(0)} & \dfrac{df_{21}}{dw_2(0)} & \dfrac{df_2}{dw_3(0)} & \dfrac{df_2}{dw_4(0)} & \dfrac{df_2}{dw_5(0)} \\[2mm] \dfrac{df_3}{dw_1(0)} & \dfrac{df_3}{dw_2(0)} & \dfrac{df_3}{dw_3(0)} & \dfrac{df_3}{dw_4(0)} & \dfrac{df_3}{dw_5(0)} \\[2mm] \dfrac{df_4}{dw_1(0)} & \dfrac{df_4}{dw_2(0)} & \dfrac{df_4}{dw_3(0)} & \dfrac{df_4}{dw_4(0)} & \dfrac{df_4}{dw_5(0)} \\[2mm] \dfrac{df_5}{dw_1(0)} & \dfrac{df_5}{dw_2(0)} & \dfrac{df_5}{dw_3(0)} & \dfrac{df_5}{dw_4(0)} & \dfrac{df_5}{dw_4(0)} \end{bmatrix}$$

In which,

$$\begin{bmatrix} F_{1j} \\ F_{2j} \\ F_{3j} \\ F_{4j} \\ F_{5j} \end{bmatrix}_{j=1,2,...5} = \begin{bmatrix} -W_{3j}\sin w_3 \\ W_{3j}\cos w_3 \\ W_{4j} \\ -PW_{3j}\cos w_3 - W_{5j}\cos w_3 + W_{3j}w_5\sin w_3 \\ 0 \end{bmatrix}$$

For a pinned-fixed rod, P is specified. Then K(0) and H(0) are varied until Eqs. (9d and 9e) are satisfied. The Runge-Kutta method is used to integrate Eqs. (12). In numerical computations, the relative error is 10^{-5} in the successive correction and numerical integration.

In short, the multisegment method of solving the boundary-value problem of Eq. (12) contains the following steps:

(1) Divisions of the given interval of S into R sufficiently small segments so that the length of each segment is less than the critical meridional length as defined by Sepetoski *et al.* [18].

(2) Arrange the boundary conditions in the form of Eq. (13).

(3) Deduce the governing ordinary differential equation for $W_{ij}(S)$. This is done by differentiating Eq. (12) partially with respect to w(0).

(4) An initial value integration of Eqs. (14) with the initial values of Eq. (15) is performed from 0 to 1 in every segment and only the elements of W(1) are stored.

(5) Solution of a system of R matrix equation for w(0), which ensures continuity of the variables at the points of the segments.

Eq. (12) are integrated from 0 to 1 in every segment with the initial values w(0) and the integration results at 1 are compared with w(1) as obtained from Eqs.(9d-9e). If the corresponding variables at the ends of consecutive segments match up to a desired number of significant figures, w(0) is accepted as the desired solution. If not, w(0) is taken as the next trial solution $w^t(0)$ and the process is repeated by returning to step (1).

VI. RESULTS AND DISCUSSIONS

The most important result is the force-displacement curve. Figure 2 shows our numerical result of downward force *P* versus the downward displacement δ of the pinned end. Our numerical
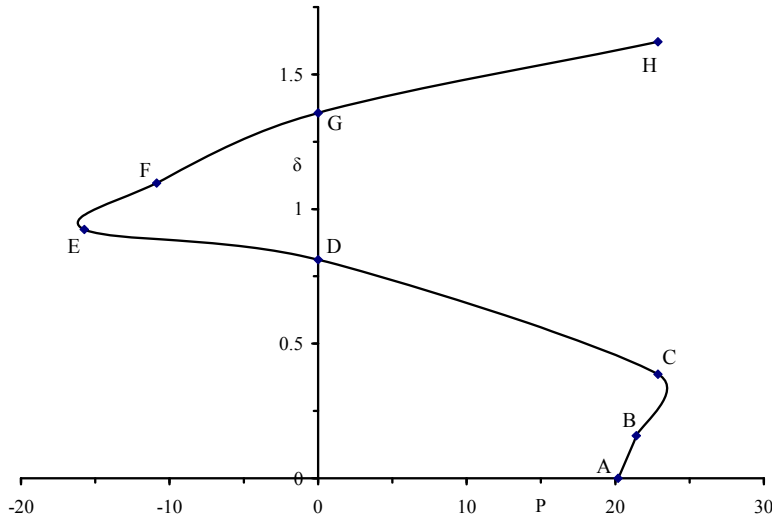

Figure 2. The force P-displacement δ curve.

solutions, in their respective regions of validity, compare quite well with the published results of Wang [2]. Notice the bifurcation curve is not monotonic. δ increases with *P* for the segments *ABC* and *EFGH*, but decreases with *P* for the segment *CDE*. Since the area under the *P*-δ curve represents the work done on the elastic, a negative slope signifies negative work for a positive displacement increment. Thus the equilibrium states on the segment *CDE* are statically unstable for a given constant load. For *P* < -15.74 (large upward force) the only solution is the stable

(trivial) straight rod; for -15.74 < *P* < 20.191 there are three solutions, two stable (one trivial) and one unstable; for 20.191 < *P* < 22.87, there are four solutions, two unstable (one trivial) and two stable; for *P* > 22.87 there are two solutions, one stable and one unstable trivial solution. The deformed shapes of the corresponding states in Fig. 2 are depicted in Fig. 3. For given end displacement δ the solution is unique. For given load *P*, the configuration may jump between stable solutions, given a suitable disturbance.
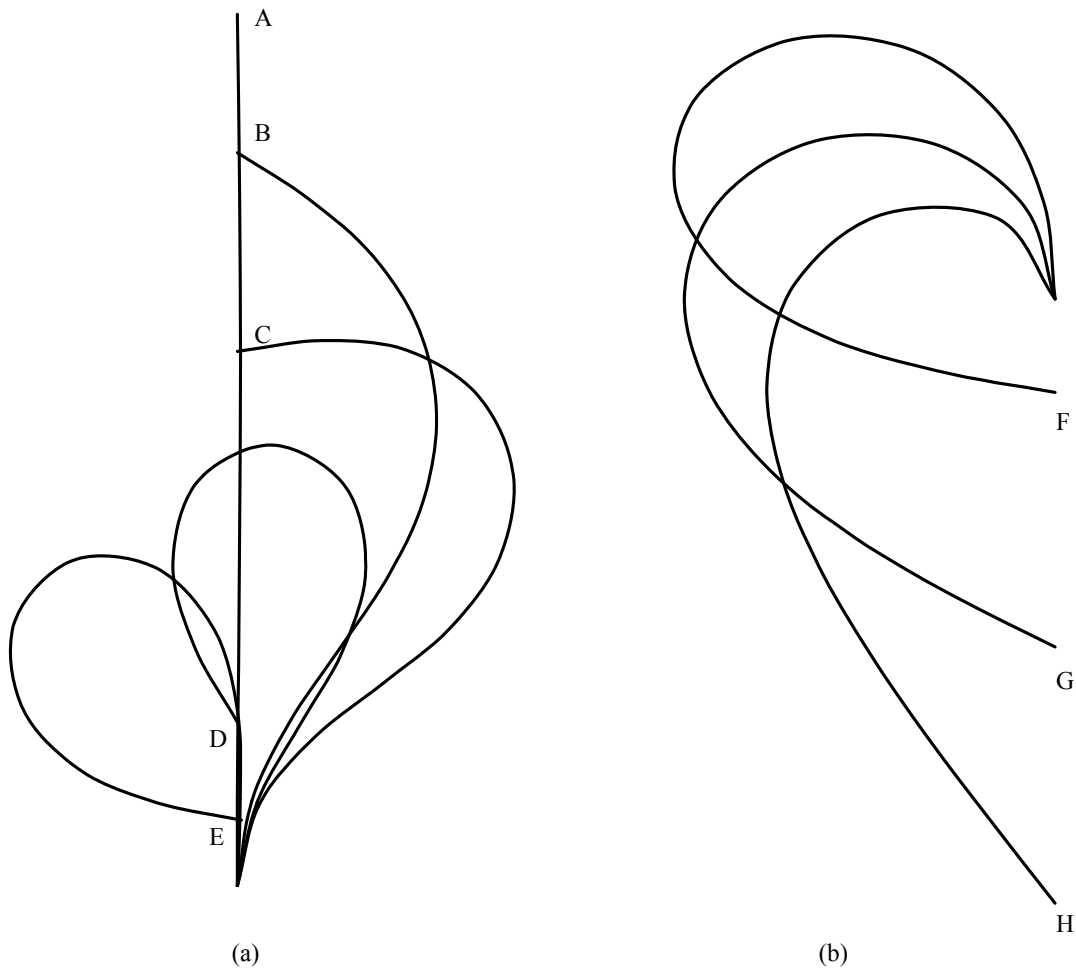

Figure 3. Rod configurations. States correspond to those indicated in Fig. 2. (a) A: δ = 0, P ≤ 20.191; B: δ = 0.158, P = 21.41; C: δ = 0.386, P = 22.87; D: δ = 0.812, P = 0; E: δ = 0.924, P = -15.74, (b) F: δ = 1.096, P = -10.87; G: δ = 1.357, P = 0; H: δ = 1.621, P = 22.87

Consider the straight rod with a gradual increase of *P* from zero. The rod remains straight until the

buckling load is reached, then it deforms through the States *A, B, C*. A further increase in *P* results in

a violent jump from State *C* to State *H*, and then follows the new branch forwarded from *H*. The unloading path is different. Gradual decrease of *P* causes the rod to go through the States *H, G, F, E*. Now the force is large and negative. Any further decrease in *P* causes the rod to snap back to State *J* which is the straight rod. A hysteresis loop thus exists. Notice the trivial state and States *D, G* all have zero *P*, i.e. the horizontal force *H* alone is sufficient to maintain the shapes.

VII. CONCLUSION

The numerical technique presented in this paper has been successfully employed in a two-point boundary value problem governed by a set of five first-order non-linear ordinary differential equations. The post-buckling configuration of slender elastic rods subjected to axial force is highly dependent on the prescribed boundary conditions. Analysis is carried out for pinned-fixed end condition by controlling the end axial force at the pinned end. Results, presented in non-dimensional format, reveal several interesting features such as limit load, jump, hysteresis, bifurcation and non-uniqueness.

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