Modeling a differentiated goods Bertrand Duopoly under uncertain demand

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Declaration

It is hereby declared that

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2. The thesis does not contain material previously published or written by a third party, except where this is appropriately cited through full and accurate referencing.
3. The thesis does not contain material which has been accepted, or submitted, for any other degree or diploma at a university or other institution.
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Abstract

This thesis explores the design of information structures in a Bertrand duopoly with differentiated products under demand uncertainty. Specifically, we consider a setting in which two firms compete in prices while facing a common uncertain demand parameter, which we modeled as a random variable over the unit interval. Drawing inspiration from the framework of Bayesian persuasion, we examine how posterior beliefs, induced through noisy signals, affect equilibrium payoffs. By comparing the expected profits under prior and posterior beliefs, we show — using Jensen's inequality — that firms achieve strictly higher expected payoffs under the prior belief than under the posterior when the governing interaction terms lie in a certain interval. This result illustrates that more precise information can be strategically disadvantageous in a Bayesian game with continuous action and type spaces. Furthermore, we extend this finding to an n-player symmetric setting, where we discover that the aforementioned interval shrinks as the number of competing firms increases. These insights contribute to a deeper understanding regarding the role and value of private information in oligopoly pricing games.

Keywords: Information Design; Bertrand Duopoly; Bayesian Game; Industrial Organization, Bayesian Inference

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Nomenclature

The next list describes several symbols & abbreviation that will be later used within the body of the document

AC Asymmetric Case

BNE Bayesian Nash Equilibrium

BRF Best Response Function

CDF Cumulative Distribution Function

FOC First-Order Condition

FOM First-Order Maximization

GSC Game of Strategic Complements

GSS Game of Strategic Substitutes

PDF Probability Density Function

RV Random Variable

SC Symmetric Case

Chapter 1

Introduction

Markets with only two dominant firms—referred to as duopolies—are of enduring interest in economics, industrial organization, and game theory. These environments offer a tractable yet rich setting for analyzing how firms make strategic decisions under competition. From an economic standpoint, duopolies lie at the intersection between monopoly and perfect competition, making them crucial for understanding market efficiency, pricing behavior, and welfare implications. Mathematically, they allow the formulation of elegant yet nontrivial models involving optimization, equilibrium analysis, and strategic uncertainty. In particular, game-theoretic approaches to duopoly, such as Bertrand or Cournot competition, enable rigorous analysis of how firms respond to informational asymmetries, external signals, and beliefs about rivals. This makes the study of duopolies not only foundational in microeconomic theory but also a fertile ground for applying Bayesian methods and mechanism design. To that end, let us begin the discourse by examining the structure of a basic duopoly game:

1.1 Defintion of Duopoly

A **duopoly** can be defined as a strategic (normal-form) game with the following components[14]:

- **Players:** Two firms, denoted as Firm 1 and Firm 2.
- **Strategy Sets:**
  - Each firm's strategy is a choice of elements:

$$s_i \in S_i = [0, \infty), \quad \text{for } i = 1, 2$$

- **Payoff Functions:** Each firm has a payoff function:

$$\pi_i(s_1, s_2) = \text{Revenue}_i - \text{Cost}_i$$

  where π_i is the profit of Firm i, and it depends on both its own strategy and the strategy of the other firm.

- **Information:** The game typically assumes *complete information*, meaning each firm knows the structure of the game and the payoff functions.

- **Solution Concept:** The outcome is analyzed using the concept of *Nash Equilibrium.* A pair of strategies (s_1^*, s_2^*) is a Nash equilibrium if:

$$\pi_1(s_1^*, s_2^*) \geq \pi_1(s_1, s_2^*) \quad \text{for all } s_1 \in S_1$$

$$\pi_2(s_1^*, s_2^*) \geq \pi_2(s_1^*, s_2) \quad \text{for all } s_2 \in S_2$$

That is, no firm has an incentive to unilaterally deviate from its equilibrium strategy.

1.2 History of Bertrand Duopoly

Historically, inception of the duopoly model took place in 1838 when Antoine Augustin Cournot began studying general oligopoly (n-firm) competition [1]. Cournot laid the foundation of this theory by proposing a solution concept for oligopolistic interaction, examining both cases of **substitute** and **complementary** products, studying the stability of proposed solutions. The Cournot model distinguishes itself by modeling competition in quantities, i.e, each $s_i \in S_i = [0, \infty)$ is a specific quantity to be chosen by each firm. In his suggested solution concept, the quantities produced by each firm would fluctuate until the output of any firm is the best response to the output of all other firms. This occurs when each firm chooses an output level that maximizes profits given the outputs chosen by the rivals. As a result, a Nash equilibrium is established for both substitute and complementary goods and - as asserted by Cournot - at a price above the cost of production.

However, in 1883, Joseph Bertrand presented a duopoly model where the strategy space is composed of prices rather quantities. In this model Bertrand proved that despite Cournot's assertion, the equilibrium price **does not** settle above of the cost of production [24]. According to Bertrand's result, even in a 2-firm (duopoly) game, when firm 1 decreases its price, firm 2 must also decrease its price for consumers will always choose the firm offering the lower price. Thus the prices offered by the firms will sequentially decrease until it equals the cost of production. This is the state known as **perfect competition** where all firms accrue zero profit. As this solution is trivial, in this paper we shall consider the Bertrand Pricing Game with differentiated goods in order to avoid the state of perfect competition.

1.3 Supermodular and Submodular

In more modern treatments of oligopoly pricing games, we introduce the notion of a **smooth supermodular** game defined as:

$$G := \langle N, (S_i, \pi_i)_{i \in N} \rangle,$$

where $N := \{1, 2, \ldots, n\}$ is the set of players, S_i is player i's strategy space, a compact cube in $\mathbb{R}^m$, and $\pi_i : S \to \mathbb{R}$ is her twice continuously differentiable payoff function, where $S := \times_{i \in N} S_i$. Let $s_{-i} \in S_{-i} := \times_{j \in N \setminus \{i\}} S_j$ denote the strategy profile of all players except i, and s_{ih} denote the hth component of the strategy s_i of player i. Then, G is smooth supermodular if, for every player $i \in N$, $\pi_i(s_i, s_{-i})$

(i) is supermodular in s_i:

$$\frac{\partial^2 \pi_i(s_i, s_{-i})}{\partial s_{ih} \partial s_{ik}} \geq 0 \quad \text{for every } h, k \in \{1, \ldots, m\} \text{ with } h \neq k \text{ and every } s \in S, \text{ and}$$

(ii) has increasing differences in (s_i, s_{-i}):

$$\frac{\partial^2 \pi_i(s_i, s_{-i})}{\partial s_{ih} \partial s_j} \geq 0 \quad \text{for every } s \in S, \text{ every } i, j \in N \text{ with } i \neq j, \text{ and every } h, k \in \{1, \ldots, m\}.$$

Analogously, we define a **smooth submodular game** as follows:

$$G := \langle N, (S_i, u_i)_{i \in N} \rangle,$$

where $N := \{1, 2, \ldots, n\}$ is the set of players, S_i is player i's strategy space, a compact cube in $\mathbb{R}^m$, and $\pi_i : S \to \mathbb{R}$ is her twice continuously differentiable payoff function, where $S := \times_{i \in N} S_i$. Let $s_{-i} \in S_{-i} := \times_{j \in N \setminus \{i\}} S_j$ denote the strategy profile of all players except i, and s_{ih} denote the hth component of the strategy s_i of player i.

Then, G is a **smooth submodular game** if, for every player $i \in N$, $\pi_i(s_i, s_{-i})$

(i) is **submodular** in s_i:

$$\frac{\partial^2 \pi_i(s_i, s_{-i})}{\partial s_{ih} \partial s_{ik}} \leq 0 \quad \text{for every } h, k \in \{1, \ldots, m\} \text{ with } h \neq k \text{ and every } s \in S, \text{ and}$$

(ii) has **decreasing differences** in (s_i, s_{-i}):

$$\frac{\partial^2 \pi_i(s_i, s_{-i})}{\partial s_{ih} \partial s_j} \leq 0 \quad \text{for every } s \in S, \text{ every } i, j \in N \text{ with } i \neq j, \text{ and every } h, k \in \{1, \ldots, m\}.$$

As seen through the works of Topkis (1978, 1998) [4][11], Vives (2001)[13] and Tirole (1998), [6] classification of oligoploy games into supermodular games allowed the usage of Tarki's fixed-point method and related results (altogether termed the lattice-theoretic method) to efficiently find a Nash equilibrium. In oligopoly terminology, supermodular and submodular games are synonymously referred to as games of strategic complements and strategic substitutes, respectively. In the latter case, the lattice-theoretic approach is not viable and it is pertinent to use the Kakutani fixed-point theorem instead. Regardless, a slew of results on supermodular oligopoly games compiled by Vives (2018) [21] elaborate on the existence and uniqueness of equilibrium when regular calculus and convex analysis fails.

1.4 Incomplete Information

Since the 1990s, research on oligopoly games has shifted towards focusing on incomplete information scenarios[22], where each firm has does not have complete knowledge of the actions, strategies or payoffs of its competitors. Since our paper revolves around a game of this type, let us look at a concrete example of an incomplteme information oligoply, taken from Vives (2001) [13].

1.4.1 An Illustrative Example

Consider a Cournot oligopoly with n firms, indexed by $i \in N = \{1, 2, \dots, n\}$. The market inverse demand function is given by:

$$P(Q) = a - Q, \quad \text{where } Q = \sum_{j=1}^{n} q_j$$

Each firm simultaneously chooses a quantity $q_i \geq 0$ to maximize its expected profit. This game has the following major components:

- Each firm i has a **private** marginal cost $c_i \in \Theta_i = \Theta =\subseteq [\underline{c}, \overline{c}]$, independently drawn from a commonly known distribution $F(c_i)$. The **type** of firm i is $\theta_i = c_i$
- The profit function of firm i is

$$\pi_i(q_i, q_{-i}, c_i) = (P(Q) - c_i)q_i = \left(a - \sum_{j=1}^{n} q_j - c_i \right) q_i$$

- A pure strategy for firm i is a measurable function

$$s_i : \Theta \to \mathbb{R}_+, \quad \text{mapping type } c_i \text{ to output } q_i$$

In these types of games the goal is to maximize the profit conditional on the information structure produced by the uncertain marginal costs of each firm. By definition, these are **Bayesian** games which introduces more restrictions to the solution concept used to find a pure strategy Nash equilibrium. Formally, they are defined as:

Definition 1. *A Bayesian game consists of*

- *A set of players N;*
- *A set of strategies for each player i: S_i;*
- *A set of types for each player i: $\theta_i \in \Theta_i$;*
- *A payoff function for each player i: $\pi_i(s_1, \dots, s_I,\ \theta_1, \dots, \theta_I)$;*
- *A probability distribution $P(\theta_1, \dots, \theta_I)$ over types*

With the solution concept determined by a *Bayesian Nash Equilibrium* (BNE):

Definition 2 (Bayesian Nash Equilibrium)**.** *The strategy profile $s(\cdot)$ is a (pure strategy) Bayesian Nash equilibrium if for all $i \in N$ and for all $c_i \in \Theta_i$, we have that*

$$s_i(c_i) \in \arg\max_{s_i' \in S_i} \int \pi_i(s_i', s_{-i}(c_{-i}), c_i, c_{-i}) P(c_{-i}|c_i).$$

For Cournot Oligopoly under consideration, a BNE is a profile of strategies $(s_1^*, \ldots, s_n^*)$ such that, for each firm $i \in N$ and for all $c_i \in \Theta_i$,

$$s_i^*(c_i) \in \arg\max_{q_i \geq 0} \mathbb{E}_{c_{-i}} \left[\left(a - q_i - \sum_{j \neq i} s_j^*(c_j) - c_i \right) q_i \right]$$

Note that the $\mathbb{E}_{c_{-i}}$ written above is the **expectation operator** which represnts integrating a function with respect to the probability distribution of c_{-i}. Hence, this is the same expression given in **Definition 2**. Additionally, Vives (2001) [13] provides a set of sufficient conditions for guaranteeing the existence of a pure strategy Bayesian Nash equilibrium in both Bertrand and Cournot settings with incomplete information (e.g., uncertainty in marginal cost or demand intercepts).

1. **Compact and Convex Action Sets:** For all i, the strategy space $A_i \subset \mathbb{R}$ is non-empty, compact, and convex.

2. **Continuity:** The payoff function $\pi_i(a_i, a_{-i}; \theta_i)$ is continuous in (a_i, a_{-i}) for every fixed $\theta_i \in \Theta_i$.

3. **Measurability:** The function $\pi_i(a_i, a_{-i}; \theta_i)$ is measurable in θ_i for every fixed strategy profile (a_i, a_{-i}).

4. **Quasi-concavity:** The payoff function $\pi_i(a_i, a_{-i}; \theta_i)$ is quasi-concave in a_i for each a_{-i} and θ_i.

5. **Finite Number of Players:** The player set N is finite.

Keeping the above developments in mind, we shall look at one last important segment of modern game theoretic research before we explicitly present the duopoly model this paper concerns with.

1.5 Persuation Theory & Information Design

The study of Bayesian games underwent a remarkable evolution with the publication of the seminal paper Bayesian Persuasion by Kamenica and Gentzkow (2011)[15], which gave rise to the modern theory of **information design** in strategic settings. The central insight of the paper is that a sender—an agent who has access to private information—can influence the behavior of a receiver by strategically controlling the flow of information. Rather than fully revealing their private signal, the sender may commit to a signaling mechanism (or information structure) that shapes the receiver's beliefs and ultimately their actions.

This problem is formalized using tools from convex analysis, where the sender's optimal strategy is to induce a distribution over posterior beliefs that maximizes their expected utility. The key mathematical device is concavification—that is, the sender's value function is characterized as the concave envelope of the expected utility as a function of the receiver's beliefs. Importantly, the paper shows that more information is not always better: withholding or garbling information can yield more favorable outcomes for the sender, a result that has significant implications for strategic communication and market design.

The framework introduced by Kamenica and Gentzkow has since evolved into a broader theory of information design, applicable to a variety of economic environments, including auctions, regulation, and **industrial organization** - the latter of which concerns the oligopoly games we are currently studying. Within this context, researchers have extended persuasion theory to games with multiple agents and strategic interactions.

Our thesis draws inspiration from this literature by applying the tools of information design to a Bertrand duopoly with differentiated products and uncertainty. Specifically, we reinterpret the classic price competition model through the lens of Bayesian persuasion, where a *metaphorical* sender designs private signals about a underlying demand parameter. The goal is to understand how signal structures affect equilibrium pricing strategies and firm profits in settings where both players react strategically to the same uncertain economic fundamentals.

1.6 The Model

In this thesis we consider a simplified version of the influential model proposed by Vives & Singh (1984) [5]. Particularly, we look at a linear demand function governing the price competition between two firms where we introduce a random variable Y (realized by y) with support [0,1]. This RV acts as a common uncertain parameter. Looking at the explicit model, let

$$q_i = D_i(y, p_i, p_{-i}) = 1 + y - p_i + (\beta_i + \gamma_i \cdot y) \cdot p_{-i} \tag{1.1}$$

be the demand of the product produced by firm $i \in 1, 2$ where:

- β_i, γ_i are known parameters, $(\beta_i + \gamma_i \cdot y) \in (-1, 1)$, and $y \sim F_Y(\cdot)$ over $[0, 1]$,
- p_i and p_{-i} refer to the prices of each firm and its rival respectively and $p_i, p_{-i} \in [0, \overline{p}]$ where $\overline{p}$ is some theoretical maximum price

While we will look at the reasons for doing so in a later chapter, in order to maintain $(\beta_i + \gamma_i \cdot y) \in (-1, 1)\ \forall y \in [0, 1]$ we must let $\beta \in (-1, 1), \gamma \in (-1 - \beta, 1 - \beta)$. We shall split our work into two cases:

1. $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$; meaning symmetric parameters. This case aptly leads to a **symmetric game**.
2. $\beta_1 \neq \beta_2$ and $\gamma_1 \neq \gamma_2$; meaning asymmetric parameters. This case aptly leads to an **asymmetric game**

From the above demand function we can derive the revenue of each firm by considering the product:

$q_i \cdot p_i = p_i(1 + y - p_i + \beta_i p_{-i} + \gamma_i y p_{-i}) = (1 + y)p_i - p_i^2 + (\beta_i + \gamma_i \cdot y) \cdot p_i \cdot p_{-i}$

As our focus is on solely studying the information structures arising from the uncertain demand, we shall allow the cost of production to be constant. On that note, we can even let the cost function become zero ($C_i(q_i) = 0$) for our convenience [1]. As a result, the profit (payoff) function of firm i is given by:

$$\pi_i(y, p_i, p_{-i}) = q_i p_i - C_i(q_i) = (1 + y)p_i - p_i^2 + (\beta_i + \gamma_i y)p_i p_{-i} \quad (1.2)$$

Observe that

$$\frac{\partial^2 \pi_i(y, p_i, p_{-i})}{\partial p_i \partial p_{-i}} = \beta_i + \gamma_i \cdot y \quad (1.3)$$

which means, using the definition from **Section 1.3**, our model produces a supermodular (submodular) game whenever $\beta_i + \gamma_i \cdot y$ is non-negative (non-positive).

1.6.1 The Information Game

At this point we explicate the information design aspect of this game by elaborating on the role of the parameter y. As stated before, it represents the realization of the RV Y but what does this RV really entail? In simple language, the demand function q_i of firm i tells us, in terms of quantity, the demand for the specific goods produced by firm i. Naturally, such a function should depend on some parameter which captures the total consumer demand of the goods produced by both firm i and its competitor firm $-i$. This parameter is y. However, as happens in the real world, this value is uncertain - making this a game of incomplete information.

As a result, both firms assign *subjective* probabilities (called **beliefs**) to the true value of total consumer demand. These beliefs, which we take to be **common** between the two firms, are represented by the RV Y over $F_Y(y)$ with support [0,1]. It must be noted the Cumulative Distribution Function (CDF) $F_Y(y)$ does not govern the **true** distribution of the demand but rather their subjective understanding of it.

To better this subjective understanding, each firm tries to gain some private information about the y. We represent this firm specific information as independent **signals** x_i which shed some new light on actual value of y. In real world terms, this x_i can be the manifestation of some survey or market analysis. Unlike in Kamenica & Gentzkow (2011) [15], we consider the sender of these signals as part of "Nature" rather than a player in the game. We assume the signals x_i come from a common CDF $F_X(x)$ over [0,1].

With this above addition, it is quite clear that the our model represents a Bayesian game since:

1. **Players:** We have a player set $N = \{1, 2\}$
2. **Strategy Sets:** Each firm chooses a price $p_i \in \mathbb{R}_+$ from a compact set $[0, \bar{p}]$.
3. **Type Space:** Each player has a private type x_i taken from the type space $\Theta_i = \Theta = [0, 1]$
4. **Payoffs:** The profit function for player i is:

   $$\pi_i(y, p_i, p_{-i}) = (1 + y - p_i + (\beta_i + \gamma_i y)p_{-i})\, p_i,$$

   which is continuous in actions.

[1] To what extent these assumptions affect the generality of the model will be addressed in a later chapter

5. **Distribution** A probability distribution $F_X(x)$ on the types x_i

We may also mention that the derived payoff function it fulfills the criteria of Rosen's Existence Theorem (Rosen, 1965) [3]:

- **Compact and convex strategy sets:** Each firm chooses its price $p_i \in [0, \bar{p}] \subset \mathbb{R}$. This strategy space is compact and convex.

- **Continuity and differentiability:** The profit function $\pi_i(y, p_i, p_{-i})$ is continuous and continuously differentiable in both p_i and p_{-i}.

- **Concavity in own strategy:** For a fixed y and opponent price p_{-i}, the profit function of firm i is a quadratic function in p_i with a negative coefficient on p_i^2. Specifically,

$$\pi_i(p_i) = (1 + y)p_i - p_i^2 + (\beta_i + \gamma_i \cdot y) \cdot p_i p_{-i}.$$

  The second derivative with respect to own strategy is:

$$\frac{\partial^2 \pi_i}{\partial p_i^2} = -2 < 0,$$

  which confirms that π_i is strictly concave in p_i.

- **Best response is well-defined and continuous:** Since the payoff functions are strictly concave in own strategy and continuously differentiable, the best response correspondence is single-valued and continuous.

Therefore, by Rosen's Theorem [3], a pure strategy Nash equilibrium exists and is characterized by the solution to the system of first-order conditions (FOC):

$$\frac{\partial \pi_i}{\partial p_i} = 0, \quad \text{for each } i = 1, 2.$$

If the reader is concerned, the above conditions also satisfy the existence of a pure strategy BNE [7]:

Theorem 1. *Consider a Bayesian game with continuous strategy spaces and continuous types. If strategy sets and type sets are compact, payoff functions are continuous and concave in own strategies, then a pure strategy Bayesian Nash equilibrium exists.*

Although in this paper we will use **Definition 2** to explicitly derive the BNE as we saw in the Cournot example, our main target is not the equilibrium solution itself but rather its behavior.

1.7 The Objective

Our ultimate goal is to study effects of **Bayesian Updating** on the equilibrium payoff function, i.e, the **value** function. To elaborate, we shall explicitly construct a likelihood function $F_{X|Y}(x = x_i|y)$ using which we will derive the **posterior** belief $F_{Y|X}(y|x = x_i)$ of firm i regarding the uncertain demand parameter. Note that is in contrast to $F_Y(y)$ which we call the **prior** belief. After computing the BNE using this posterior distribution, we invoke the 2nd derivative test and Jensen's Inequality to demonstrate that the **expected** payoff π_i^* of with respect to posterior belief is less than the expected payoff with respect to the prior belief - ultimately showing that more information is detrimental to the competing firms.

Chapter 2

Literature Review

Before we proceed with direct calculations we shall provide the reader with a brief literature review of relevant publications which informed the scope and structure of this study.

Starting off with the paper that is the inspiration behind our model, "Price and Quantity Competition in a Differentiated Duopoly" by Singh and Vives (1984) [5] explores the strategic behavior of firms in a duopoly when they compete either by setting prices (Bertrand competition) or quantities (Cournot competition), particularly in the context of differentiated products. The objective is to analyze which type of competition is more beneficial for firms and society, especially depending on whether products are substitutes or complements.

To address this, the authors model a two-stage game. In the first stage, firms choose whether to commit to a price or quantity contract with consumers; in the second stage, they compete based on their chosen contracts. The analysis is conducted under both linear and nonlinear demand structures, beginning with a linear inverse demand system of the form:

$$p_i = \alpha_i - \beta_i q_i - \gamma q_j, \quad i \neq j, \quad i, j = 1, 2$$

where $\gamma > 0$ represents substitutability and $\gamma < 0$ represents complementarity between products. They demonstrate that Cournot competition with substitutes is the *strategic dual* of Bertrand competition with complements, and vice versa, implying that the structure of firms' best responses mirrors under these two forms of rivalry.
The equilibrium comparison reveals that, when goods are substitutes ($\gamma > 0$), Cournot competition leads to **higher prices** and **lower quantities** than Bertrand competition:

$$p^C > p^B, \quad q^C < q^B$$

Additionally, Singh and Vives derive that the price difference between Cournot and Bertrand equilibria is directly proportional to the degree of substitutability:

$$p_i^C - p_i^B = \frac{\gamma}{2\beta - \gamma}$$

thus, as products become more differentiated (i.e., lower γ/β), the distinction between Cournot and Bertrand outcomes diminishes.

They also show that profit outcomes vary systematically: Cournot profits (Π_i^C) are higher than Bertrand profits (Π_i^B) when goods are substitutes, equal when goods are independent, and lower when goods are complements:

$$\Pi_i^C > \Pi_i^B \quad \text{if } \gamma > 0, \quad \Pi_i^C = \Pi_i^B \quad \text{if } \gamma = 0, \quad \Pi_i^C < \Pi_i^B \quad \text{if } \gamma < 0$$

Thus, when goods are substitutes, it is a dominant strategy for firms to commit to quantity contracts (Cournot competition), while for complements, committing to price contracts (Bertrand competition) is optimal. However, while firms may prefer the Cournot outcome for profitability when goods are substitutes, Bertrand competition generally yields **higher consumer surplus and total welfare** due to lower equilibrium prices and higher output levels.

In conclusion, the paper successfully achieves its objective of unifying and explaining the strategic differences between price and quantity competition in differentiated markets. It also highlights that firms' strategic incentives may conflict with broader social welfare, raising important considerations for market regulation and antitrust policy.

In "A General Model of Information Sharing in Oligopoly", Michael Raith (1996) [8] develops a general framework to determine when firms in an oligopoly have incentives to share private information about uncertain market conditions. Each firm's profit is modeled as

$$\pi_i = \alpha_i(\tau_i) - \sum_{j \neq i} \varepsilon s_i s_j + (\beta + \gamma_s \tau_i - \delta s_i) s_i,$$

where τ_i represents a stochastic cost or demand parameter, and s_i is the firm's strategic choice (quantity or price). Firms privately observe noisy signals $y_i = \tau_i + \eta_i$ and can decide whether to share this information before competition. Raith classifies uncertainty into **common value (CV)** (where $\text{cov}(\tau_i, \tau_j) = t$), **independent values (IV)** (where $\text{cov}(\tau_i, \tau_j) = 0$), and **perfect signals (PS)** (noise variance $u = 0$). He shows that under IV and PS, complete information sharing is always profitable, whereas under CV, sharing is profitable only when firms' actions are strategic complements ($\varepsilon < 0$). In settings with strategic substitutes ($\varepsilon > 0$), sharing may reduce profits and create a Prisoner's Dilemma. Raith formalizes that the expected change in profits from information sharing, $\Delta\mathbb{E}[\pi_i]$, can be decomposed into two terms:

$$\Delta\mathbb{E}[\pi_i] = (\text{direct adjustment effect}) + (\text{strategic adjustment effect}),$$

where the direct adjustment reflects the benefit from rivals' improved understanding of their own payoffs, and the strategic adjustment captures how rivals' reactions to better information affect the firm's outcome. He derives that the direct adjustment is always positive under strategic complements but can be negative under substitutes. Raith also critiques the prior literature by showing that changes in the correlation of strategies alone cannot explain disclosure incentives — the precision of signals and the endogenous response of strategies are the decisive factors. Overall, the paper successfully generalizes previous fragmented results, concluding that incentives to share information hinge on the interplay between the structure of uncertainty, signal precision, and the strategic environment of competition.

This survey paper by Currarini and Feri (2018)[20] provides a comprehensive overview of theoretical research on firms' incentives to share private information in oligopolistic markets, especially under uncertainty about demand or cost conditions. The authors build on Raith's (1996) [8] general framework and extend it to both multilateral and bilateral sharing contexts. The central objective is to understand when and how information sharing emerges as a rational strategic choice, and whether it should be interpreted as a form of collusion or competitive efficiency. The paper distinguishes between two main models: the strategic model, where firms disclose unilaterally, and the contractual model, where sharing occurs on a reciprocal basis. Under multilateral sharing, the paper confirms that information sharing is beneficial when firms' actions are strategic complements (e.g., Bertrand competition) but may be harmful under strategic substitutes (e.g., Cournot competition with homogeneous goods), unless products are sufficiently differentiated. Currarini and Feri (2018)[20] also explore smaller-scale or bilateral agreements, where firms selectively exchange information, forming endogenous networks. They show that such networks can be pairwise stable, even under strategic substitutes and homogeneous goods, if firms' private signals are conditionally correlated. Furthermore, they demonstrate that core–periphery structures can emerge endogenously when firms are asymmetric in the quality (variance) of their private signals. The paper concludes that bilateral sharing arrangements can sustain positive levels of information exchange under broader conditions than previously recognized, challenging earlier assumptions that full industry-wide sharing is necessary for information pooling to be profitable or stable.

In this paper, Sasaki (2001)[12] explores how the value of information in duopoly markets with uncertain demand depends on the strategic structure of the game—specifically, whether it exhibits supermodularity or submodularity. Using a two-stage model where firms can choose to acquire costly information about demand before competing in either a Bertrand or Cournot framework, the paper shows that information acquisition is strategically complemental when the game is supermodular (i.e., actions are strategic complements, such as pricing in Bertrand competition), and strategically substitutional when the game is submodular (i.e., actions are strategic substitutes, like quantities in Cournot competition). In these cases, the value of information for one firm increases or decreases, respectively, when its rival is also informed. This superadditivity or subadditivity of information value hinges on the nature of strategic interactions. Moreover, Sasaki (2001) [12]investigates how beliefs and asymmetries affect information value in games where firms cannot observe whether rivals have acquired information (covert acquisition). When actions are either mutually complementary or substitutive, a firm benefits more from acquiring information if it is believed by its rival to be informed. However, in cases of asymmetric modularity—where one firm's action is a complement and the other's is a substitute—the opposite occurs. The paper concludes that strategic context and belief dynamics significantly shape incentives for acquiring or withholding information and that these insights can generalize to a broader class of economic games with smooth payoffs and strategic interdependence.

This paper by Wang and Ma (2013)[17] investigates the dynamics and stability of a Cournot-Bertrand mixed duopoly model where one firm competes on quantity and the other on price, under the realistic assumption of bounded rationality and limited market information. The model incorporates linear demand and cost functions, with firms adjusting their strategies based on marginal profit expectations. Unlike traditional models assuming perfect foresight, this approach reflects more realistic adaptive behavior. Wang and Ma (2013)[17] derive the system's nonlinear difference equations and analyze the existence and local stability of equilibrium points, particularly focusing on the Nash equilibrium. Through both analytical derivation and numerical simulations—including bifurcation diagrams and parameter basin plots—the paper reveals that system stability is sensitive to the firms' adjustment speeds and the degree of product differentiation. It finds that excessive adjustment speed, especially in price-setting behavior, can lead to instability and chaotic dynamics. Moreover, greater product differentiation generally increases the stability of the system. These results imply that firms' strategic response speeds and market structure significantly affect the predictability and robustness of competitive interactions. The study concludes that bounded rational Cournot-Bertrand competition can lead to complex and even chaotic dynamics, offering important insights for both theoretical modeling and practical market regulation.

The famous paper "Information Design: A Unified Perspective" by Bergemann and Morris (2019) [22] The paper *"Information Design: A Unified Perspective"* by Dirk Bergemann and Stephen Morris presents a comprehensive framework that unifies various strands of the literature on information design. The objective is to understand how the strategic provision of information influences players' decisions in games with incomplete information, whether through a literal "information designer" or as an analytical tool for bounding equilibrium outcomes.

Central to the framework is the concept of a *Bayes Correlated Equilibrium* (BCE). A BCE is a probability distribution $\psi(a, \theta)$ over actions a and states θ satisfying two conditions: (i) obedience (incentive compatibility), and (ii) Bayes plausibility. Formally, the obedience condition requires that for every player i, given any deviation $\tilde{a}_i$,

$$\sum_{a_{-i}} \sum_{\theta} \psi(a_i, a_{-i}, \theta) u_i(a_i, a_{-i}, \theta) \geq \sum_{a_{-i}} \sum_{\theta} \psi(a_i', a_{-i}, \theta) u_i(a_i', a_{-i}, \theta), \quad \forall a_i', \forall i$$

where u_i is the utility function of player i.
The Bayes plausibility condition ensures that the marginal distribution over states remains consistent with the players' beliefs:

$$\sum_{a} \psi(a, \theta) = \lambda(\theta), \quad \forall \theta$$

where $\lambda(\theta)$ is the prior belief over the states of the world.
An information designer selects an information structure $\varphi(s|\theta)$, a mapping from states θ to signals s, to induce a distribution over actions that maximizes an objective function $W(\psi)$. The designer's problem is:

$$\max_{\psi \in \text{BCE}} W(\psi)$$

subject to the obedience and Bayes plausibility constraints. This problem can be formulated as a *linear program*, with ψ as the decision variable.

In particular, the authors show that Bayesian persuasion (Kamenica and Gentzkow,

2011) is a special case where the set of available actions is fixed, and the designer's objective depends solely on the action chosen, not on the specific realization of the state. vspace2mm Through examples such as a two-player investment game, Bergemann and Morris demonstrate how the designer can optimally control the extent of information disclosure, shifting the set of achievable equilibrium payoffs. They also discuss extensions to dynamic settings and environments where the designer controls multiple signals sequentially.

In conclusion, the paper achieves its objective by unifying diverse models under the BCE framework and demonstrating that many economic environments—including auctions, contracts, regulation, and communication games—can be reinterpreted as information design problems. This perspective highlights the central role of information structures in shaping strategic interactions.

The paper "Dynamic Oligopoly with Incomplete Information" by Bonatti et al. (2017)[19] investigates how firms in an oligopoly compete over time when they have private information about their costs and can only observe noisy market prices. The main objective is to characterize equilibrium behavior in such a setting, where firms simultaneously learn about competitors' costs and signal their own through pricing strategies.

To tackle this, Bonatti et al. (2017)[19] construct a dynamic Cournot model in continuous time, assuming private cost information and a market price subject to stochastic shocks. They focus on symmetric linear Markov equilibria—strategies depending only on time and firms' beliefs—to make the model tractable. Using Kalman filtering, they show that each firm's belief can be expressed as a weighted average of its private cost and a public estimate of industry costs. They characterize these equilibria as solutions to a boundary value problem involving a set of differential equations and demonstrate existence under certain parameter conditions.

The paper achieves its goal by showing that even in the presence of incomplete information and noise, firms eventually learn enough to approximate static complete information outcomes. The conclusion emphasizes that learning and signaling create rich, nonmonotonic behavior in firm strategies and prices, with implications for both market dynamics and policy.

Despite the vast intersection between Bayesian games and industrial organization, so far no research has addressed the question of how valuable information can be in context of oligopoly games. In finite form games we are aware of explicit cases where a player possessing more information led to him accruing lower payoffs - but no such counterexamples exist in the context of Bayesian games with continuous type and action spaces nor do researches have produced any general results which eliminate the possibility of these counterexamples. As a result, our main objective in this research would solve a price competition between 2 firms and clearly demonstrate that, under certain parameters, equilibrium payoffs **can** decrease with respect to information. This is a highly non-intuitive result in the context of a Bertrand Duopoly and would serve as the gateway to rigorously studying the role of private information in games imbued with oligopoly structures.

Chapter 3

Posterior Probability

3.1 Bayesian Updating

Bayesian updating is a process in decision theory where prior beliefs (represented by a prior distribution) are revised based on new knowledge, resulting in a posterior distribution that reflects a more informed understanding [16]. The foundation of

Bayesian updating is **Bayes' Rule** (or Bayes' Theorem), which relates the conditional and marginal probabilities of two events. The rule is stated as:

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)} \tag{3.1}$$

where:

- $P(A \mid B)$ is the **posterior probability**: the probability of event A occurring given that B is true.
- $P(B \mid A)$ is the **likelihood**: the probability of observing B given A.
- $P(A)$ is the **prior probability**: the initial belief about the probability of A.
- $P(B)$ is the **marginal likelihood**: the total probability of observing B under all possible hypotheses.

The word **updating** refers to the iterative process of refining the posterior probability as new information is observed. The steps are as follows:

1. **Specify the Prior:** Before observing any information, we define a prior distribution $P(A)$ representing our initial beliefs about the parameter A.

2. **Constructing Likelihood:** Upon observing some signal B, we compute the likelihood $P(B \mid A)$, which quantifies how probable the observed signal is under different parameter values.

3. **Apply Bayes' Theorem:** The posterior distribution $P(A \mid B)$ is obtained by applying Bayes' theorem:

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)} \tag{3.2}$$

where $P(B)$, like before, is the total probability of observing B, often computed as:

$$P(B) = \int P(B \mid A)P(A)\,dA \quad \text{(for continuous variables)} \tag{3.3}$$

4. **Update Beliefs:** The posterior becomes the new prior when additional information is observed, allowing for sequential updating of beliefs.

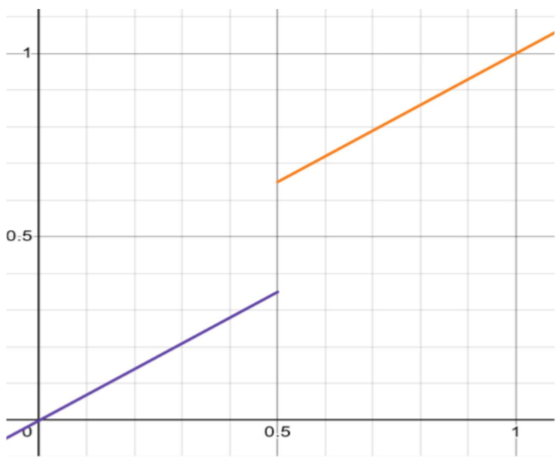
In our model the uncertain demand parameter y is distributed on some prior CDF $F_Y(y)$ over [0,1]. This is the initial belief held by both firms. On the other hand, $x_i \sim F_x$ is the distribution of signals (new knowledge) observed by firm i. If we are to apply Bayesian Updating in our model we must first explicitly construct the the likelihood probability distribution $F_{X|Y}(x|y)$ and then use Bayes' Rule to derive the posterior belief $F_{Y|X}(y|x)$ which serves as the required conditional distribution needed to compute BNE.

3.2 The Likelihood Function

We assume that the probability of the signal variable X accurately informing the market demand Y is denoted by a parameter $\theta \in [0, 1]$. Here θ governs the "noise" encumbering the signal x_i. In probabilistic terms, $P(X = Y, x_i = y) = \theta$. Hence, $P(X \neq Y, x_i \neq y) = 1 - \theta$. Explicitly, we assert:

$$F_{X|Y}(x = x_i|y) = \begin{cases} (1-\theta)x_i & x_i < y \\ \theta + (1-\theta)x_i & x_i \geq y \end{cases}$$

Note that θ is an **atom**; A point x in a probability space such that the probability of exactly x_i occurring is positive. For visual clarity we may compute the plot of $F_{X|Y}(x|y)$ against x_i using example values $y = 0.5$ & $\theta = 0.36$ where the magnitude of the latter value determines the size of the "jump".



Past researchers who have employed similar constructions include Kamenica & Gentzkow (2011) [15], Lyu & Suen (2024) [25] and Taneva (2019)[23].

3.3 Deriving the Posterior

We shall now use Bayes' Rule to derive the posterior $F_{Y|X}(y|x_i)$. However, to do this, we make a strong assumption: $y \sim F_Y(\cdot) = \mathcal{U}[0,1]$. That is, we let the demand parameter y be uniformly distributed over [0,1]. We remind the reader that F_y was always meant to be a **subjective** belief on the distribution of y held by both firms and is not the true distribution of y. This means we may as well assume an explicit (subjective) distribution of y without encountering a logical contradiction. However, to what extent this assumption undermines the generality of the forthcoming results will be addressed at a later section. For now, owing to ease of calculations, let $y \sim F_Y(\cdot) = \mathcal{U}[0,1]$.

3.3.1 Computing $F_x(x_i)$

Consider two random variables, X and Y with support $[0,1]$. Suppose $Y \sim F_Y = U[0,1]$ and the conditional CDF of X given Y for firm i is:

$$F_{X|Y}(x_i|y) = \mathbb{P}(X \leq x_i|Y = y) = \begin{cases} (1-\theta)x_i & \text{if } x_i < y, \\ \theta + (1-\theta)x_i & \text{if } x_i > y. \end{cases}$$

The marginal CDF of X is derived as:

$$F_X(x_i) = \int_0^1 F_{X|Y}(x_i|y) f_Y(y)\, dy,$$

$$\begin{aligned}
&\Rightarrow F_X(x_i) = \int_0^1 (\begin{cases} (1-\theta)x_i & x_i < y \\ \theta + (1-\theta)x_i & y \leq x_i \end{cases}) \cdot f_Y(y)dy \\
&\Rightarrow F_X(x_i) = \int_0^{x_i} \theta + (1-\theta)x_i \cdot f_Y(y)dy + \int_{x_i}^1 (1-\theta)x_i \cdot f_Y(y)dy \\
&\Rightarrow F_X(x_i) = \int_0^{x_i} \theta + (1-\theta)x_i \cdot 1dy + \int_{x_i}^1 (1-\theta)x_i \cdot 1dy \\
&\Rightarrow F_X(x_i) = x_i
\end{aligned}$$

using the uniform distribution for Y. Thus, $F_X(x_i)$ is also $\mathcal{U}[0,1]$.

3.3.2 Computing $F_{Y|X}(y|x_i)$

Let $F(x,y) := \mathbb{P}[X \leq x_i, Y \leq y]$. Note that $F(x,y)$ is non-decreasing in both x and y. The joint density is:

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial y \partial x}.$$

This derivative exists almost everywhere (via Lebesgue Differentiation Theorem) because $F(x,y)$ is non-decreasing.
The joint CDF can be expressed as:

$$F(x,y) = \int_0^y \int_0^x f(\hat{x},\hat{y})\, d\hat{x} d\hat{y}.$$

The conditional CDFs are:

$$F_{X|Y}(x|y) = \int_0^x f(\hat{x}|y)\, d\hat{x} = \int_0^x \frac{f(\hat{x}, y)}{f_Y(y)}\, d\hat{x} = \int_0^x f(\hat{x}, y)\, d\hat{x},$$

Where $f_Y(y) = 1$ since $F_Y(y) = \mathcal{U}[0,1]$. Thus:

$$F_{X|Y}(x|y) = \frac{\partial F(x,y)}{\partial y}.$$

Similarly:

$$F_{Y|X}(y|x) = \frac{\partial F(x,y)}{\partial x}.$$

Now given we have:

$$F_{X|Y}(x|y) = \mathbb{P}(X \leq x | Y = y) = \begin{cases} (1-\theta)x & \text{if } x < y \\ \theta + (1-\theta)x & \text{if } x > y \end{cases}$$

$$\begin{aligned}
\Rightarrow F(x,y) &= \int_0^y \frac{\partial F(x,\hat{y})}{\partial \hat{y}} d\hat{y} \\
\Rightarrow F(x,y) &= \int_0^y F_{X|Y}(x|\hat{y}) d\hat{y} \\
\Rightarrow F(x,y) &= \begin{cases} \int_0^x F_{X|Y}(x|\hat{y}) d\hat{y} + \int_x^y F_{X|Y}(x|\hat{y}) d\hat{y} & \text{if } x < y \\ \int_0^y F_{X|Y}(x|\hat{y}) d\hat{y} & \text{if } x > y \end{cases} \\
\Rightarrow F(x,y) &= \begin{cases} [\theta\hat{y} + (1-\theta)x\hat{y}]_{\hat{y}=0}^{\hat{y}=x} + [(1-\theta)x\hat{y}]_{\hat{y}=x}^{\hat{y}=y} & \text{if } x < y \\ [(1-\theta)x\hat{y}]_{\hat{y}=0}^{\hat{y}=y} & \text{if } x > y \end{cases} \\
\Rightarrow F(x,y) &= \begin{cases} \theta x + (1-\theta)x^2 + (1-\theta)x(y-x) & \text{if } x < y \\ (1-\theta)xy & \text{if } x > y \end{cases} \\
\Rightarrow F(x,y) &= \begin{cases} \theta x + (1-\theta)xy & \text{if } x < y \\ (1-\theta)xy & \text{if } x > y \end{cases}
\end{aligned}$$

$$\therefore F_{Y|X}(y|x) = \frac{\partial F(x,y)}{\partial x} = \begin{cases} (1-\theta)y & \text{if } y < x \\ \theta + (1-\theta)y & \text{if } y > x \end{cases}$$

where $x = x_i$ is implicit for convinience.

$F_{Y|X}(y|x = x_i)$ is the distribution of the subjective demand y with respect to the types x_i of firm i. The reader may note that it is a function from [0,1] to [0,1] due to the fact that $x_i, y, \theta \in [0,1]$. With this explicit form of $F_{Y|X}(y|x = x_i)$ we can now also derive the **expected** posterior (updated) demand $\mathbb{E}[Y|X = x_i]$ given by firm i's observed signal x_i

3.4 Conditional Expectation of Demand

$$\begin{aligned}
E[Y|X=x_i] &= \int_0^1 y \cdot \frac{d}{dy} F_{Y|X}(y|x_i)\, dy \\
&= \big[y \cdot F_{Y|X}(y|x_i)\big]_0^1 - \int_0^1 F_{Y|X}(y|x_i) \cdot \frac{d}{dy}(y)\, dy \\
&= 1 \cdot F(1) - 0 \cdot F(0) - \int_0^1 \begin{cases} (1-\theta)\, y & 0 \le y < x_i \\ \theta + (1-\theta)\, y & x_i \le y \le 1 \end{cases} dy \\
&= 1 - \int_0^{x_i} (1-\theta) \cdot y\, dy - \int_{x_i}^1 \theta + (1-\theta) \cdot y\, dy \\
&= 1 - \frac{(1-\theta)\, x_i^2}{2} - \theta\, (1-x_i) - \frac{(1-\theta)\,(-x_i^2+1)}{2} \\
&= \frac{1}{2} + \theta\, x_i - \frac{1}{2}\theta
\end{aligned}$$

With this construction in hand we will now move onto acquiring the expected payoff at equilibrium, otherwise known as the expected value function.

Chapter 4

Nash Equilibrium, Value Function & Concavity

In our setup we have two firms competing in a Bertrand Duopoly of differentiated goods with the cost of production assumed to be 0. Let the competing firms be referred to as firm 1 and firm 2. Then we get a pair of profit functions, π_1 and π_2, where π_1 is the payoff of firm 1 in terms of its own price and the price set by firm 2 while π_2 is the payoff received by firm 2 in terms of its own price and the price set by firm 1.

$\pi_1(y, p_1, p_2) = (1 + y)p_1 - (p_1)^2 + (\beta_1 + \gamma_1 \cdot y)p_1 \cdot p_2$ (firm 1 w.r.t firm 2)

$\pi_2(y, p_2, p_1) = (1 + y)p_2 - (p_2)^2 + (\beta_2 + \gamma_2 \cdot y)p_2 \cdot p_1$ (firm 2 w.rt to firm 1)

Here p_1 and p_2 refer to the prices set by firm 1 and 2 respectively. Using a concise notation, we can represent the two above functions as one:

$\pi_i(y, p_i, p_{-i}) = (1 + y)p_i - (p_i)^2 + (\beta_i + \gamma_i \cdot y)p_i \cdot p_{-i}$ where $i \in$ 1, 2.

At this point we delineate the role of β_i and γ_i. These parameters serve as "interaction terms" which govern the effect of firm $-i$'s price on firm i's profit. Intuitively, it is clear that in a real world setting firm i's profit should not be affected by firm $-i$'s price **more than its own price**. Hence we apply an affine transformation on y - as this term is the coefficient of p_{-i} - using $\beta \in (-1, 1), \gamma \in (-1 - \beta, 1 - \beta)$; forcing $(\beta_i + \gamma_i \cdot y) \in (-1, 1)$. There's also a mathematical reason for using this affine transformation; without it the payoff function "blows up" in equilibrium.

Additionally, in the classical Bertrand Duopoly we have seen the result that when the products sold by two firms are homogeneous (perfect substitutes), customers will always choose the product with lower price. This means - in equilibrium - profits eventually converge to 0. To avoid this trivial (and unrealistic) result, economists use the parameters β, γ to ensure **product differentiation** between the goods sold by each firm. The rules are [9]:

- When the p_{-i} coefficient $(\beta_i + \gamma_i \cdot y)$ is > 0 the goods are complements to each other. However, as shown by Bertrand, when two goods are perfect complements the firms begin to collude and the pure strategy equilibrium does not exist hence we must let $\beta_i + \gamma_i \cdot y < 1$. This leads to a Game of Strategic Complements (GSC).

- When the p_{-i} coefficient $(\beta_i + \gamma_i \cdot y)$ is < 0 the goods are substitutes to each other. However, as shown by Bertrand, when two goods are perfect substitutes the profit function converges to 0 hence we must let $\beta_i + \gamma_i \cdot y > -1$. This leads to a Game of Strategic Substitues (GSS).
- When the p_{-i} coefficient $(\beta_i + \gamma_i \cdot y)$ is $= 0$ the goods are fully independent of each other. Then p_{-i} does not affect π_i and the game reduces to a monopoly
- Thus $(\beta_i + \gamma_i \cdot y) \in$ (-1, 1). We allow negative values to fully explore the mathematical consequences of GSS.

With this preamble, we may now begin the process of First Order Maximization (FOM) of π_i with respect to p_i:

4.1 First Order Maximization

$$\pi_i(y, p_i, p_{-i}) = (1+y)p_i - (p_i)^2 + (\beta_i + \gamma_i \cdot y)p_{-i} \cdot p_i$$

We invoke the definition of BNE to compute the expected utility conditional on firm i's type x_i :

$$\begin{aligned}
&\Rightarrow \mathbb{E}(\pi_i(y, p_i, p_{-i})) = \int_0^1 \big((1+y)p_i - (p_i)^2 + (\beta_i + \gamma_i \cdot y)p_i \cdot p_{-i}\big) \cdot \frac{d}{dy}(F_{Y|X}(y|x_i))\, dy \\
&\Rightarrow \mathbb{E}(\pi_i(y, p_i, p_{-i})) = \int_0^1 p_i \cdot \frac{d}{dy}(F_{Y|X}(y|x_i)) + p_i \cdot y \cdot \frac{d}{dy}(F_{Y|X}(y|x_i)) - p_i^2 \frac{d}{dy}(F_{Y|X}(y|x_i) \\
&\quad + (\beta_i + \gamma_i \cdot y)p_i \cdot p_{-i} \cdot \frac{d}{dy}(F_{Y|X}(y|x_i))\, dy \\
&\Rightarrow \mathbb{E}(\pi_i(y, p_i, p_{-i})) = p_i \int_0^1 \frac{d}{dy}(F_{Y|X}(y|x_i))\, dy + p_i \int_0^1 y \cdot \frac{d}{dy}(F_{Y|X}(y|x_i))\, dy \\
&\quad - p_i^2 \int_0^1 \frac{d}{dy}(F_{Y|X}(y|x_i))\, dy + \beta_i \cdot p_i \cdot p_{-i} \int_0^1 \frac{d}{dy}(F_{Y|X}(y|x_i))\, dy \\
&\quad + \gamma_i \cdot p_i \cdot p_{-i} \cdot \int_0^1 y \cdot \frac{d}{dy}(F_{Y|X}(y|x_i))\, dy \\
&\Rightarrow \mathbb{E}(\pi_i(y, p_i, p_{-i})) = p_i \cdot [F_{Y|X}(1|x_i) - F_{Y|X}(0|x_i)] + p_i \cdot \mathbb{E}[Y|X] \\
&\quad - p_i^2 \cdot [F_{Y|X}(1|x_i) - F_{Y|X}(0|x)] + \beta_i \cdot p_i \cdot p_{-i} \cdot [F_{Y|X}(1|x_i) - F_{Y|X}(0|x_i)] \\
&\quad + \gamma_i \cdot p_i \cdot p_{-i} \cdot \mathbb{E}[Y|X] \\
&\Rightarrow \mathbb{E}(\pi_i(y, p_i, p_{-i})) = p_i + p_i \cdot \mathbb{E}[Y|X] - p_i^2 + \beta_i \cdot p_i \cdot p_{-i} + \gamma_i \cdot p_i \cdot p_{-i} \cdot \mathbb{E}[Y|X]
\end{aligned}$$

Where $\mathbb{E}[Y|X] = \mathbb{E}[Y|X = x_i]$ is the expected demand for the product sold by firm i based on his type x_i. Now taking $\frac{d}{dp_i}(\mathbb{E}(\pi_i(y, p_i, p_{-i}))) = 0$:

$$\begin{aligned}
&\frac{d}{dp_i}\big(p_i + p_i \cdot \mathbb{E}[Y|X] - p_i^2 + \beta_i \cdot p_i \cdot p_{-i} + \gamma_i \cdot p_i \cdot p_{-i} \cdot \mathbb{E}[Y|X]\big) = 0 \\
&\Rightarrow 1 + \mathbb{E}[Y|X] - 2p_i + \beta_i \cdot p_{-i} + \gamma_i \cdot p_{-i} \cdot \mathbb{E}[Y|X] = 0 \\
&\Rightarrow p_i = \frac{1}{2}\gamma_i \cdot p_{-i} \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta_i \cdot p_{-i} + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2}
\end{aligned}$$

We now have the best-response function (BRF) of firm i with respect to the price (action) set by firm $-i$. We will now use the the pair of best-response functions (one each for firm 1 and 2) in order to find the symmetric **Bayesian Nash Equilibrium**. In this demonstration, symmetric refers to the specific case where β and γ for both firm 1 and firm 2 are invariant, i.e, $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$.

4.2 Symmetric Equilibrium

$$p_1 = \frac{1}{2}\gamma \cdot p_2 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta \cdot p_2 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2} \tag{4.1}$$

$$p_2 = \frac{1}{2}\gamma \cdot p_1 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta \cdot p_1 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2} \tag{4.2}$$

Substituting Eq. 4.2 in Eq. 4.1,

$$\begin{aligned}
p_1 &= \frac{1}{2}\gamma \cdot p_2 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta \cdot p_2 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2} \\
\Rightarrow p_1 &= \frac{1}{2}\gamma \cdot (\frac{1}{2}\gamma \cdot p_1 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta \cdot p_1 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2}) \cdot \mathbb{E}[Y|X] \\
&+ \frac{1}{2}\beta \cdot (\frac{1}{2}\gamma \cdot p_1 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta \cdot p_1 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2}) + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2} \\
\Rightarrow p_1 &= \frac{(\gamma^2 p_1 + \gamma)\mathbb{E}[Y|X]^2}{4} + \frac{(2\beta\gamma p_1 + \beta + \gamma + 2)\mathbb{E}[Y|X]}{4} + \frac{\beta^2 p_1}{4} + \frac{\beta}{4} + \frac{1}{2} \\
\Rightarrow p_1 &- \frac{(\gamma^2 p_1 + \gamma)\mathbb{E}[Y|X]^2}{4} - \frac{(2\beta\gamma p_1 + \beta + \gamma + 2)\mathbb{E}[Y|X]}{4} - \frac{\beta^2 p_1}{4} = \frac{\beta}{4} + \frac{1}{2} \\
\Rightarrow p_1 &= \frac{\beta + 2 + \gamma\mathbb{E}[Y|X]^2 + \gamma\mathbb{E}[Y|X] + \beta\mathbb{E}[Y|X] + 2\mathbb{E}[Y|X]}{4} \div \\
&\frac{4 - \gamma^2\mathbb{E}[Y|X]^2 - 2\beta\gamma\mathbb{E}[Y|X] + \beta^2}{4} \\
\Rightarrow p_1^* &= \frac{1 + \mathbb{E}[Y|X]}{2 - (\beta + \gamma\mathbb{E}[Y|X])}
\end{aligned}$$

Substituting Eq. 4.1 in Eq. 4.2,

$$\Rightarrow p_2 = \frac{1}{2}\gamma \cdot p_1 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta \cdot p_1 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2}$$
$$\Rightarrow p_2 = \frac{1}{2}\gamma \cdot (\frac{1}{2}\gamma \cdot p_2 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta \cdot p_2 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2}) \cdot \mathbb{E}[Y|X]$$
$$+ \frac{1}{2}\beta \cdot (\frac{1}{2}\gamma \cdot p_2 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta \cdot p_2 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2}) + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2}$$
$$\Rightarrow p_2 = \frac{(\gamma^2 p_2 + \gamma)\mathbb{E}[Y|X]^2}{4} + \frac{(2\beta\gamma p_2 + \beta + \gamma + 2)\mathbb{E}[Y|X]}{4} + \frac{\beta^2 p_2}{4} + \frac{\beta}{4} + \frac{1}{2}$$
$$\Rightarrow p_2 - \frac{(\gamma^2 p_2 + \gamma)\mathbb{E}[Y|X]^2}{4} - \frac{(2\beta\gamma p_2 + \beta + \gamma + 2)\mathbb{E}[Y|X]}{4} - \frac{\beta^2 p_2}{4} = \frac{\beta}{4} + \frac{1}{2}$$
$$\Rightarrow p_2 = \frac{\beta + 2 + \gamma\mathbb{E}[Y|X]^2 + \gamma\mathbb{E}[Y|X] + \beta\mathbb{E}[Y|X] + 2\mathbb{E}[Y|X]}{4} \div$$
$$\frac{4 - \gamma^2\mathbb{E}[Y|X]^2 - 2\beta\gamma\mathbb{E}[Y|X] + \beta^2}{4}$$
$$\Rightarrow p_2^* = \frac{1 + \mathbb{E}[Y|X]}{2 - (\beta + \gamma\mathbb{E}[Y|X])}$$

Hence $p_1^* = p_2^* = \frac{1+\mathbb{E}[Y|X]}{2-(\beta+\gamma\cdot\mathbb{E}[Y|X])}$

Where p_1^* and p_2^* maximize the payoff functions of firm 1 and firm 2 respectively and are solutions to the symmetric equilibrium.

4.3 Symmetric Value Function

We will now plug in the solution p_1^*, p_2^* into the payoff function for each firm. However, the symmetry of the solution ($p_1^* = p_2^*$) assures us that both firms will have the same payoff given below:

$$\pi(y, p_1^*, p_2^*) = (1 + y)p_1^* - (p_1^*)^2 + (\beta + \gamma \cdot y)p_1^* \cdot p_2^*$$
$$\Rightarrow \pi(y, \mathbb{E}[Y|X]) = (1 + y) \cdot \left(\frac{1 + \mathbb{E}[Y|X]}{2 - (\beta + \gamma\mathbb{E}[Y|X])}\right) - \left(\frac{1 + \mathbb{E}[Y|X]}{2 - (\beta + \gamma\mathbb{E}[Y|X])}\right)^2$$
$$+ (\beta + \gamma \cdot y) \cdot \left(\frac{1 + \mathbb{E}[Y|X]}{2 - (\beta + \gamma\mathbb{E}[Y|X])}\right) \cdot \left(\frac{1 + \mathbb{E}[Y|X]}{2 - (\beta + \gamma\mathbb{E}[Y|X])}\right)$$
$$\Rightarrow \pi(y, \mathbb{E}[Y|X]) = -\frac{(1 + \mathbb{E}[Y|X])((-\beta + \gamma + 1)\mathbb{E}[Y|X] + \beta y - \gamma y - 2y - 1)}{(\gamma\mathbb{E}[Y|X] + \beta - 2)^2}$$
$$\Rightarrow \pi(y, \mathbb{E}[Y|X]) = -\frac{-\gamma\mathbb{E}[Y|X]^2 + \mathbb{E}[Y|X]^2\beta + \mathbb{E}[Y|X]\gamma y - \mathbb{E}[Y|X]\beta y - \mathbb{E}[Y|X]^2}{(\gamma\mathbb{E}[Y|X] + \beta - 2)^2}$$
$$- \frac{\gamma\mathbb{E}[Y|X] - \mathbb{E}[Y|X]\beta - 2\mathbb{E}[Y|X]y - \gamma y + \beta y - 2y - 1}{(\gamma\mathbb{E}[Y|X] + \beta - 2)^2}$$

The above expression is called the **value function**. We will now calculate its expectation w.r.t $F_{Y|X}(y|x)$ in order to acquire the **expected payoff at equilibrium**:

$$
\begin{aligned}
\Rightarrow \mathbb{E}(\pi(y, \mathbb{E}[Y|X])) = &- \frac{-\gamma\mathbb{E}[Y|X]^2 + \mathbb{E}[Y|X]^2\beta + \mathbb{E}[Y|X]^2\gamma}{(\gamma\mathbb{E}[Y|X] + \beta - 2)^2} \\
&- \frac{\mathbb{E}[Y|X]^2\beta - \mathbb{E}[Y|X]^2 - \gamma\mathbb{E}[Y|X]}{(\gamma\mathbb{E}[Y|X] + \beta - 2)^2} \\
&+ \frac{\mathbb{E}[Y|X]\beta + 2\mathbb{E}[Y|X]^2 + \gamma\mathbb{E}[Y|X] - \beta\mathbb{E}[Y|X] + 2\mathbb{E}[Y|X] + 1}{(\gamma\mathbb{E}[Y|X] + \beta - 2)^2} \\
\Rightarrow \mathbb{E}(\pi(y, \mathbb{E}[Y|X])) = &\ \pi^*(\mathbb{E}[Y|X]) = \frac{(1 + \mathbb{E}[Y|X])^2}{(\gamma\mathbb{E}[Y|X] + \beta - 2)^2}
\end{aligned}
$$

4.3.1 Concavity in the Symmetric Case

We will now use the second derivative test to check the concavity of π^* w.r.t $\mathbb{E}[Y|X]$. Since the sign of the 2nd derivative will depend on the values of β, γ we shall find an explicit range for which the 2nd derivative is ≤ 0.

Second Derivative,

$$
\begin{aligned}
&\frac{d^2}{d\mathbb{E}[Y|X]^2}\left(\frac{(1 + \mathbb{E}[Y|X])^2}{(\gamma\mathbb{E}[Y|X] + \beta - 2)^2}\right) \\
= &\frac{2}{(\gamma\mathbb{E}[Y|X] + \beta - 2)^2} - \frac{8(1 + \mathbb{E}[Y|X])\gamma}{(\gamma\mathbb{E}[Y|X] + \beta - 2)^3} + \frac{6(1 + \mathbb{E}[Y|X])^2\gamma^2}{(\gamma\mathbb{E}[Y|X] + \beta - 2)^4} \\
= &\frac{2(\beta - \gamma - 2)(-2\gamma\mathbb{E}[Y|X] + \beta - 3\gamma - 2)}{(\gamma\mathbb{E}[Y|X] + \beta - 2)^4}
\end{aligned}
$$

Since $\gamma\mathbb{E}[Y|X] + \beta \in (-1, 1)$, the denominator is always > 0. This means the second derivative is ≤ 0 as long as

$$(\beta - \gamma - 2)(-2\gamma\mathbb{E}[Y|X] + \beta - 3\gamma - 2) \leq 0$$

Case 1:

$$\beta - \gamma - 2 \geq 0 \tag{4.3}$$

$$-2\gamma\mathbb{E}[Y|X] + \beta - 3\gamma - 2 \leq 0 \tag{4.4}$$

From 4.4 we get $\beta - \gamma - 2 \leq 2\gamma + 2\gamma\mathbb{E}[Y|X]$. Combined with 4.3 we have $0 \leq \beta - \gamma - 2 \leq 2\gamma + 2\gamma\mathbb{E}[Y|X]$. But this inequality is impossible since $0 \leq \beta - \gamma - 2 \Rightarrow \gamma < 0$ which would make $2\gamma + 2\gamma\mathbb{E}[Y|X]$ negative while also being greater than zero. Hence this is a contradiction and we look at the other possibility of how the 2nd derivative can be non-positive.

Case 2:

$$\beta - \gamma - 2 \leq 0 \tag{4.5}$$

$$-2\gamma\mathbb{E}[Y|X] + \beta - 3\gamma - 2 \geq 0 \tag{4.6}$$

From 4.5 & 4.6 we get $0 \geq \beta - \gamma - 2 \geq 2\gamma + 2\gamma\mathbb{E}[Y|X]$. This inequality is true for $\gamma \geq \beta - 2$ and $\beta - \gamma - 2 \geq 2\gamma + 2\gamma\mathbb{E}[Y|X]$. From the latter inequality we can derive:

$\frac{\beta-2}{2\mathbb{E}[Y|X]+3} \geq \gamma \Rightarrow \frac{\beta-2}{3} \geq \gamma \forall \mathbb{E}[Y|X] \in [0,1]$

So for $\beta \in (-1,1) \wedge \gamma \in [\beta-2, \frac{\beta-2}{3}]$ the 2nd derivative $\frac{2(\beta-\gamma-2)(-2\gamma\mathbb{E}[Y|X]+\beta-3\gamma-2)}{(\gamma\mathbb{E}[Y|X]+\beta-2)^4} \leq 0$ $\forall \mathbb{E}[Y|X] \in [0,1]$. As a result, our equilibrium payoff is **concave** for these values of β, γ.

Why does it matter if it is concave? At this point we invoke Jensen's inequality:

Theorem 2. *Let ϕ be a* ***concave*** *function and X be a random variable such that $\mathbb{E}[X]$ and $\mathbb{E}[\phi(X)]$ exist. Then:*

$$\phi\left(\mathbb{E}[X]\right) \geq \mathbb{E}\left[\phi(X)\right].$$

If ϕ is ***strictly concave****, the inequality is strict unless X is deterministic (i.e., X is almost surely constant).*

This means that for our equilibrium payoff we have:

$$\mathbb{E}(\pi^*(\mathbb{E}[Y|x=x_i])) \leq \pi^*(\mathbb{E}(\mathbb{E}[Y|x=x_i]))$$

$$\Rightarrow \mathbb{E}(\pi^*(\mathbb{E}[Y|x=x_i])) \leq \pi^*(\mathbb{E}[Y])$$

Where $\pi^*(\mathbb{E}[Y]$ is the expected payoff of firm i **prior** to receiving the signal x_i while $\mathbb{E}(\pi^*(\mathbb{E}[Y|x=x_i]))$ is the **posterior** expected payoff of firm i . This demonstrates that receiving new information only served to reduce firm i's profit; showing the detrimental effect of information in a duopoly competition. Let us now repeat the whole process for the AC and test if the same results hold.

4.4 Asymmetric Equilibrium

Asymmetric payoff functions arise when the interaction terms β and γ are not invariant, i.e, $\beta_1 \neq \beta_2$ and $\gamma_1 \neq \gamma_2$. This leads to to the following pair of best response functions which we will use to solve, like before, to now find the asymmetric Bayesian Nash Equilibrium:

$$p_1 = \frac{1}{2}\gamma_1 \cdot p_2 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta_1 p_2 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2} \tag{4.7}$$

$$p_2 = \frac{1}{2}\gamma_2 \cdot p_1 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta_2 p_1 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2} \tag{4.8}$$

Substituting Eq. 4.8 in Eq. 4.7,

$$
\begin{aligned}
&p_1 = \frac{1}{2}\gamma_1 \cdot p_2 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta_1 p_2 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2} \\
\Rightarrow &p_1 = \frac{1}{2}\gamma_1 \cdot \left(\frac{1}{2}\gamma_2 \cdot p_1 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta_2 p_1 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2}\right) \cdot \mathbb{E}[Y|X] \\
&+ \frac{1}{2}\beta_1 \cdot \left(\frac{1}{2}\gamma_2 \cdot p_1 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta_2 p_1 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2}\right) + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2} \\
\Rightarrow &p_1 = \frac{\gamma_1(\gamma_2 p_1 + 1)\mathbb{E}[Y|X]^2}{4} + \frac{((\beta_2 p_1 + 1)\gamma_1 + p_1\beta_1\gamma_2 + \beta_1 + 2)\mathbb{E}[Y|X]}{4} \\
&+ \frac{\beta_1\beta_2 p_1}{4} + \frac{\beta_1}{4} + \frac{1}{2} \\
\Rightarrow &p_1 - \frac{\gamma_1(\gamma_2 p_1 + 1)\mathbb{E}[Y|X]^2}{4} + \frac{((\beta_2 p_1 + 1)\gamma_1 + p_1\beta_1\gamma_2 + \beta_1 + 2)\mathbb{E}[Y|X]}{4} \\
&+ \frac{\beta_1\beta_2 p_1}{4} = \frac{\beta_1}{4} + \frac{1}{2} \\
\Rightarrow &p_1 = \frac{\beta_1 + 2 + \gamma_1\mathbb{E}[Y|X]^2 + \gamma_1\mathbb{E}[Y|X] + \beta_1\mathbb{E}[Y|X] + 2\mathbb{E}[Y|X]}{4} \div \\
&\frac{4 - \gamma_1 \cdot \gamma_2\mathbb{E}[Y|X]^2 - \beta_1 \cdot \gamma_2\mathbb{E}[Y|X] - \beta_2\gamma_1\mathbb{E}[Y|X] + \beta_1 \cdot \beta_2}{4} \\
\Rightarrow &p_1^* = \frac{(1 + \mathbb{E}[Y|X])(\gamma_1\mathbb{E}[Y|X] + \beta_1 + 2)}{4 - (\gamma_1\mathbb{E}[Y|X] + \beta_1) \cdot (\gamma_2\mathbb{E}[Y|X] + \beta_2)}
\end{aligned}
$$

Substituting Eq. 4.7 in Eq. 4.8,

$$
\begin{aligned}
&p_2 = \frac{1}{2}\gamma_2 \cdot p_1 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta_2 p_1 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2} \\
\Rightarrow &p_2 = \frac{1}{2}\gamma_2 \cdot \left(\frac{1}{2}\gamma_1 \cdot p_2 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta_1 p_2 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2}\right) \cdot \mathbb{E}[Y|X] \\
&+ \frac{1}{2}\beta_2 \cdot \left(\frac{1}{2}\gamma_1 \cdot p_2 \cdot \mathbb{E}[Y|X] + \frac{1}{2}\beta_1 p_2 + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2}\right) + \frac{1}{2}\mathbb{E}[Y|X] + \frac{1}{2} \\
\Rightarrow &p_2 = \frac{\gamma_2(\gamma_1 p_2 + 1)\mathbb{E}[Y|X]^2}{4} + \frac{((\beta_1 p_2 + 1)\gamma_2 + p_2\beta_2\gamma_1 + \beta_2 + 2)\mathbb{E}[Y|X]}{4} \\
&+ \frac{\beta_1\beta_2 p_2}{4} + \frac{\beta_2}{4} + \frac{1}{2} \\
\Rightarrow &p_2 - \frac{\gamma_2(\gamma_1 p_2 + 1)\mathbb{E}[Y|X]^2}{4} + \frac{((\beta_1 p_2 + 1)\gamma_2 + p_2\beta_2\gamma_1 + \beta_2 + 2)\mathbb{E}[Y|X]}{4} \\
&+ \frac{\beta_1\beta_2 p_2}{4} = \frac{\beta_2}{4} + \frac{1}{2} \\
\Rightarrow &p_2 = \frac{\beta_2 + 2 + \gamma_2\mathbb{E}[Y|X]^2 + \gamma_2\mathbb{E}[Y|X] + \beta_2\mathbb{E}[Y|X] + 2\mathbb{E}[Y|X]}{4} \div \\
&\frac{4 - \gamma_1 \cdot \gamma_2\mathbb{E}[Y|X]^2 - \beta_1 \cdot \gamma_2\mathbb{E}[Y|X] - \beta_2\gamma_1\mathbb{E}[Y|X] + \beta_1 \cdot \beta_2}{4} \\
\Rightarrow &p_2^* = \frac{(1 + \mathbb{E}[Y|X])(\gamma_2\mathbb{E}[Y|X] + \beta_2 + 2)}{4 - (\gamma_1\mathbb{E}[Y|X] + \beta_1) \cdot (\gamma_2\mathbb{E}[Y|X] + \beta_2)}
\end{aligned}
$$

Hence

$$p_1^* = \frac{(1+\mathbb{E}[Y|X])(\gamma_1\mathbb{E}[Y|X]+\beta_1+2)}{4-(\gamma_1\mathbb{E}[Y|X]+\beta_1)\cdot(\gamma_2\mathbb{E}[Y|X]+\beta_2)}$$
$$p_2^* = \frac{(1+\mathbb{E}[Y|X])(\gamma_2\mathbb{E}[Y|X]+\beta_2+2)}{4-(\gamma_1\mathbb{E}[Y|X]+\beta_1)\cdot(\gamma_2\mathbb{E}[Y|X]+\beta_2)}$$

Unlike before, $p_1^* \neq p_2^*$, which means we will now have two separate value functions for each of the firms when we plug in p_1^*, p_2^* into $\pi_i(y,p_i,p_j) = (1+y)p_i - (p_i)^2 + (\beta_i + \gamma_i \cdot y)p_i \cdot p_j$.

4.5 Asymmetric Value Function

$$\begin{aligned}
&\pi_1(y,p_1^*,p_2^*) = (1+y)p_1^* - (p_1^*)^2 + (\beta_1+\gamma_1\cdot y)p_1^*\cdot p_2^* \\
&\Rightarrow \pi_1(y,\mathbb{E}[Y|X]) = (1+y)\cdot\left(\frac{(1+\mathbb{E}[Y|X])(\gamma_1\mathbb{E}[Y|X]+2)}{4-(\gamma_1\mathbb{E}[Y|X]+\beta_1)(\gamma_2\mathbb{E}[Y|X]+\beta_2)}\right) \\
&-\left(\frac{(1+\mathbb{E}[Y|X])(\gamma_1\mathbb{E}[Y|X]+2)}{4-(\gamma_1\mathbb{E}[Y|X]+\beta_1)(\gamma_2\mathbb{E}[Y|X]+\beta_2)}\right)^2 \\
&+(\beta_1+\gamma_1\cdot y)\cdot\left(\frac{(1+\mathbb{E}[Y|X])(\gamma_1\mathbb{E}[Y|X]+2)}{4-(\gamma_1\mathbb{E}[Y|X]+\beta_1)(\gamma_2\mathbb{E}[Y|X]+\beta_2)}\right) \\
&\cdot\left(\frac{(1+\mathbb{E}[Y|X])(\gamma_2\mathbb{E}[Y|X]+2)}{4-(\gamma_1\mathbb{E}[Y|X]+\beta_1)(\gamma_2\mathbb{E}[Y|X]+\beta_2)}\right)
\end{aligned}$$

We shall simplify the above expression and calculate its expectation w.r.t to $F_{Y|X}(y|x)$ as before since y is uncertain random variable and we cannot let the payoff depend on it.

$$\Rightarrow \mathbb{E}[\pi_1(y,\mathbb{E}[Y|X])] = \pi_1^*(\mathbb{E}[Y|X]) = \frac{(1+\mathbb{E}[Y|X])^2(\gamma_1\mathbb{E}[Y|X]+\beta_1+2)}{(4-(\gamma_1\mathbb{E}[Y|X]+\beta_1)\cdot(\gamma_2\mathbb{E}[Y|X]+\beta_2))^2}$$

The above is the value function of firm 1 with respect to firm 2. We will now compute the value function of firm 2 in terms of firm 1:

$$\begin{aligned}
&\pi_2(y,p_2^*,p_1^*) = (1+y)p_2^* - (p_2^*)^2 + (\beta_2+\gamma_2\cdot y)p_2^*\cdot p_1^* \\
&\Rightarrow \pi_2(y,\mathbb{E}[Y|X]) = (1+y)\cdot\left(\frac{(1+\mathbb{E}[Y|X])(\gamma_2\mathbb{E}[Y|X]+2)}{4-(\gamma_1\mathbb{E}[Y|X]+\beta_1)(\gamma_2\mathbb{E}[Y|X]+\beta_2)}\right) \\
&-\left(\frac{(1+\mathbb{E}[Y|X])(\gamma_2\mathbb{E}[Y|X]+2)}{4-(\gamma_1\mathbb{E}[Y|X]+\beta_1)(\gamma_2\mathbb{E}[Y|X]+\beta_2)}\right)^2 \\
&+(\beta_2+\gamma_2\cdot y)\cdot\left(\frac{(1+\mathbb{E}[Y|X])(\gamma_2\mathbb{E}[Y|X]+2)}{4-(\gamma_1\mathbb{E}[Y|X]+\beta_1)(\gamma_2\mathbb{E}[Y|X]+\beta_2)}\right) \\
&\cdot\left(\frac{(1+\mathbb{E}[Y|X])(\gamma_1\mathbb{E}[Y|X]+2)}{4-(\gamma_1\mathbb{E}[Y|X]+\beta_1)(\gamma_2\mathbb{E}[Y|X]+\beta_2)}\right)
\end{aligned}$$

Again we will simplify the expression and calculate its expectation:

$$\Rightarrow \mathbb{E}[\pi_2(y,\mathbb{E}[Y|X])] = \pi_2^*(\mathbb{E}[Y|X]) = \frac{(1+\mathbb{E}[Y|X])^2(\gamma_2\mathbb{E}[Y|X]+\beta_2+2)}{(4-(\gamma_1\mathbb{E}[Y|X]+\beta_1)\cdot(\gamma_2\mathbb{E}[Y|X]+\beta_2))^2}$$

4.5.1 Concavity in the Asymmetric Case

We will use the following concise notation

$$\pi_i^*(\mathbb{E}[Y|X]) = \frac{(1+\mathbb{E}[Y|X])^2(\gamma_i\mathbb{E}[Y|X]+\beta_i+2)^2}{(4-(\gamma_i\mathbb{E}[Y|X]+\beta_i)\cdot(\gamma_j\mathbb{E}[Y|X]+\beta_j))^2}$$

to check the concavity of firm i's profit in the asymmetric case via another computation of the 2nd derivative with respect to $\mathbb{E}[Y|X]$.

Second Derivative:

$$\begin{aligned}
&\frac{d^2}{d\mathbb{E}[Y|X]^2}\left(\frac{(1+\mathbb{E}[Y|X])^2(\gamma_i\mathbb{E}[Y|X]+\beta_i+2)}{4-(\gamma_i\mathbb{E}[Y|X]+\beta_i)\cdot(\gamma_j\mathbb{E}[Y|X]+\beta_j)}\right)\\
=&\frac{2(3(E+1)^2(\gamma_i(E\gamma_j+\beta_j)+\gamma_j(E\gamma_i+\beta_i))^2(E\gamma_i+\beta_i+2)^2}{((E\gamma_i+\beta_i)(E\gamma_j+\beta_j)-4)^4}\\
&+\frac{-2(E+1)((E\gamma_i+\beta_i)(E\gamma_j+\beta_j)-4)}{((E\gamma_i+\beta_i)(E\gamma_j+\beta_j)-4)^4}\frac{(E\gamma_i+\beta_i+2)(\gamma_i\gamma_j(E+1)(E\gamma_i+\beta_i+2)}{((E\gamma_i+\beta_i)(E\gamma_j+\beta_j)-4)^4}\\
&+\frac{2\gamma_i(E+1)(\gamma_i(E\gamma_j+\beta_j)+\gamma_j(E\gamma_i+\beta_i))}{((E\gamma_i+\beta_i)(E\gamma_j+\beta_j)-4)^4}\\
&+\frac{2(\gamma_i(E\gamma_j+\beta_j)+\gamma_j(E\gamma_i+\beta_i))(E\gamma_i+\beta_i+2))}{((E\gamma_i+\beta_i)(E\gamma_j+\beta_j)-4)^4}\\
&+\frac{((E\gamma_i+\beta_i)(E\gamma_j+\beta_j)-4)^2(\gamma_i^2(E+1)^2}{((E\gamma_i+\beta_i)(E\gamma_j+\beta_j)-4)^4}\\
&+\frac{4\gamma_i(E+1)(E\gamma_i+\beta_i+2)+(E\gamma_i+\beta_i+2)^2))}{((E\gamma_i+\beta_i)(E\gamma_j+\beta_j)-4)^4}
\end{aligned}$$

where $E \equiv \mathbb{E}[Y|X]$ for brevity.

Clearly this expression is too complex to check using the same method as the symmetric case. However, one may try using evaluating this expression at discreet points and check that it is indeed negative for certain values. Extending that idea, it can be verified that the asymmetric value function is highly concave whenever $\gamma_i \in (-2,-0.5] \wedge \gamma_j \in [0.5,2) \wedge \beta_i \in [0.5,1) \wedge \beta_j \in (-1,-0.5]$. Then, using Jensen's Inequality again, we can assert the posterior payoff at asymmetric equilibrium is less than the prior payoff. For a visual demonstration, we explicitly graph the asymmetric value function π_i^* against $\mathbb{E}[Y|X]$ for values in the aforementioned intervals.

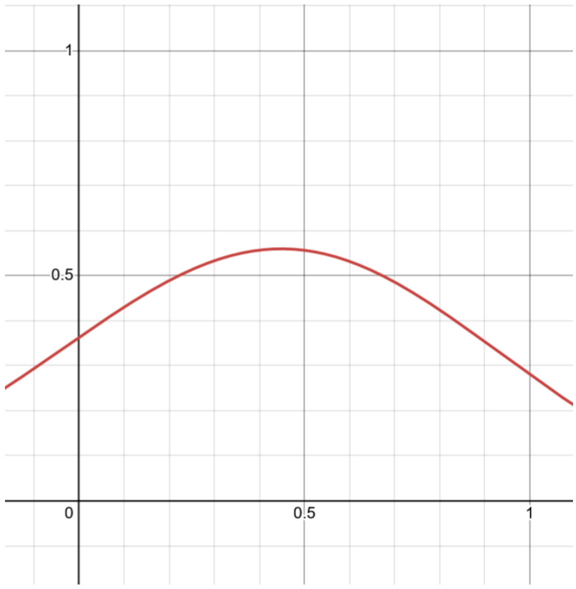


Figure 4.1: π_i^* vs. $\mathbb{E}[Y|X]$ for $\gamma_i = -1.54, \gamma_j = 1.54, \beta_i = 0.76, \beta_j = -0.76$

Observe that not only is the asymmetric value function concave, it is also decreasing in $\mathbb{E}[Y|X]$ after a certain value. So not only is this asymmetric posterior payoff less than the prior payoff, we also have that, after a certain point, higher quality information is explicitly detrimental.

Chapter 5

The Third Stage

Any Bayesian game has three stages (Myerson,1997) [10]:

1. **Ex-Ante stage game:** Players do not know their types or those of other players. A player recognizes payoffs as expected values based on a prior distribution of all possible types

2. **Interim stage game:** Players know their type but only a probability distribution of other players. When considering payoffs, a player studies the expected value of the other player's type

3. **Ex-post stage game:** Players know their types and those of other players. The payoffs are known to players

In chapter 4 we computed the interim stage equilibrium and proceeded to use Jensen's inequality in order to demonstrate that firm i acquires a higher payoff in the ex-ante stage. However, in order to show this result we did not need to use the explicit form of $\mathbb{E}[Y|X = x_i]$ which we derived in chapter 3. Yet, in order to reach the final stage where each firm knows their own payoffs and their opponent's payoffs, we must consider calculate the symmetric and asymmetric value functions over all possible values of $x_i \in [0, 1]$.

5.1 Symmetric Payoff

In chapter 4 we derived the symmetric value function as:

$$\pi^*(\mathbb{E}[Y|X = x_i]) = \frac{(1 + \mathbb{E}[Y|X = x_i])^2}{(\gamma\mathbb{E}[Y|X = x_i] + \beta - 2)^2} \tag{5.1}$$

and in chapter 3 we computed the conditional expectation as:

$$\mathbb{E}[Y|X = x_i] = \frac{1}{2} + \theta\, x_i - \frac{1}{2}\,\theta \tag{5.2}$$

Doing the relevant substitution we get:

$$\pi^* = \frac{(1+\frac{1}{2}+\theta\, x_i-\frac{1}{2}\,\theta)^2}{(\gamma(\frac{1}{2}+\theta\, x_i-\frac{1}{2}\,\theta)+\beta-2)^2} = \frac{(2\theta x_i-\theta+3)^2}{(2\gamma\theta x_i-\gamma\theta+2\beta+\gamma-4)^2}$$

For the sake of our convenience let us roll all x_i independent terms into constants. Let $-\theta+3=b$ and $-\gamma\theta+2\beta+\gamma-4=d$. Additionally, let us also suppress coefficients of x_i as $2\theta=a$ and $2\gamma\theta=c$. Thus our value function becomes:

$$\pi^* = \frac{(ax_i+b)^2}{(cx_i+d)^2} \tag{5.3}$$

In order to find the equilibrium payoff π^* over all possible types x_i, we must integrate with respect to the distribution $F_X(x)=\mathcal{U}[0,1]$.

$$\begin{aligned}
\mathbb{E}(\pi^*) &= \int_0^1 \frac{(a\cdot x+b)^2}{(c\cdot x+d)^2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(F_X(x)\right)\mathrm{d}x\\
\mathbb{E}(\pi^*) &= \int_0^1 \frac{(a\cdot x+b)^2}{(c\cdot x+d)^2}\cdot 1\mathrm{d}x\\
\mathbb{E}(\pi^*) &= \frac{(2a^2d^2-2abcd)\ln(|d|)+a^2d^2-2abcd+b^2c^2}{c^3d}\\
&- \frac{(2a^2d^2+(2a^2-2ab)\,cd-2abc^2)\ln(|c+d|)+a^2d^2+(-a^2-2ab)\,cd+(-a^2+b^2)\,c^2}{c^4+c^3d}.
\end{aligned}$$

Plugging back in the the pre-defined constants we get:

$$\begin{aligned}
\mathbb{E}(\pi^*) &= \frac{(8\theta^2(-\gamma\theta+2\beta+\gamma-4)^2)}{8\gamma^3\theta^3(-\gamma\theta+2\beta+\gamma-4)}\\
&- \frac{8\theta^2(-\theta+3)\gamma(-\gamma\theta+2\beta+\gamma-4)\ln|-\gamma\theta+2\beta+\gamma-4|}{8\gamma^3\theta^3(-\gamma\theta+2\beta+\gamma-4)}\\
&+ \frac{4\theta^2(-\gamma\theta+2\beta+\gamma-4)^2}{8\gamma^3\theta^3(-\gamma\theta+2\beta+\gamma-4)}\\
&- \frac{8\theta^2(-\theta+3)\gamma(-\gamma\theta+2\beta+\gamma-4)}{8\gamma^3\theta^3(-\gamma\theta+2\beta+\gamma-4)} + \frac{4(-\theta+3)^2\gamma^2\theta^2)}{8\gamma^3\theta^3(-\gamma\theta+2\beta+\gamma-4)}\\
\Rightarrow \mathbb{E}(\pi^*) &= -\frac{(8\theta^2(-\gamma\theta+2\beta+\gamma-4)^2}{16\gamma^4\theta^4+8\gamma^3\theta^3(-\gamma\theta+2\beta+\gamma-4)}\\
&+ \frac{2\left(8\theta^2-4\theta(-\theta+3)\right)\gamma\theta(-\gamma\theta+2\beta+\gamma-4)}{16\gamma^4\theta^4+8\gamma^3\theta^3(-\gamma\theta+2\beta+\gamma-4)}\\
&- \frac{16\theta^3(-\theta+3)\gamma^2)\ln\left(|\gamma\theta+2\beta+\gamma-4|\right)}{16\gamma^4\theta^4+8\gamma^3\theta^3(-\gamma\theta+2\beta+\gamma-4)}\\
&+ \frac{4\theta^2(-\gamma\theta+2\beta+\gamma-4)^2}{16\gamma^4\theta^4+8\gamma^3\theta^3(-\gamma\theta+2\beta+\gamma-4)} + \frac{2(-4\theta^2+(-\theta+3)^2)\gamma^2\theta^2}{16\gamma^4\theta^4+8\gamma^3\theta^3(-\gamma\theta+2\beta+\gamma-4)}
\end{aligned}$$

Clearly this expression is too intractable to provide any useful insights. Yet it represents the utility at equilibrium for the firms without dependence on any random variables. In other words, this is the **full information** payoff function of both firm. Hence we must analyze its behavior for different values of β, γ and θ. First let us look at the graph of $\mathbb{E}(\pi^*)$ against θ

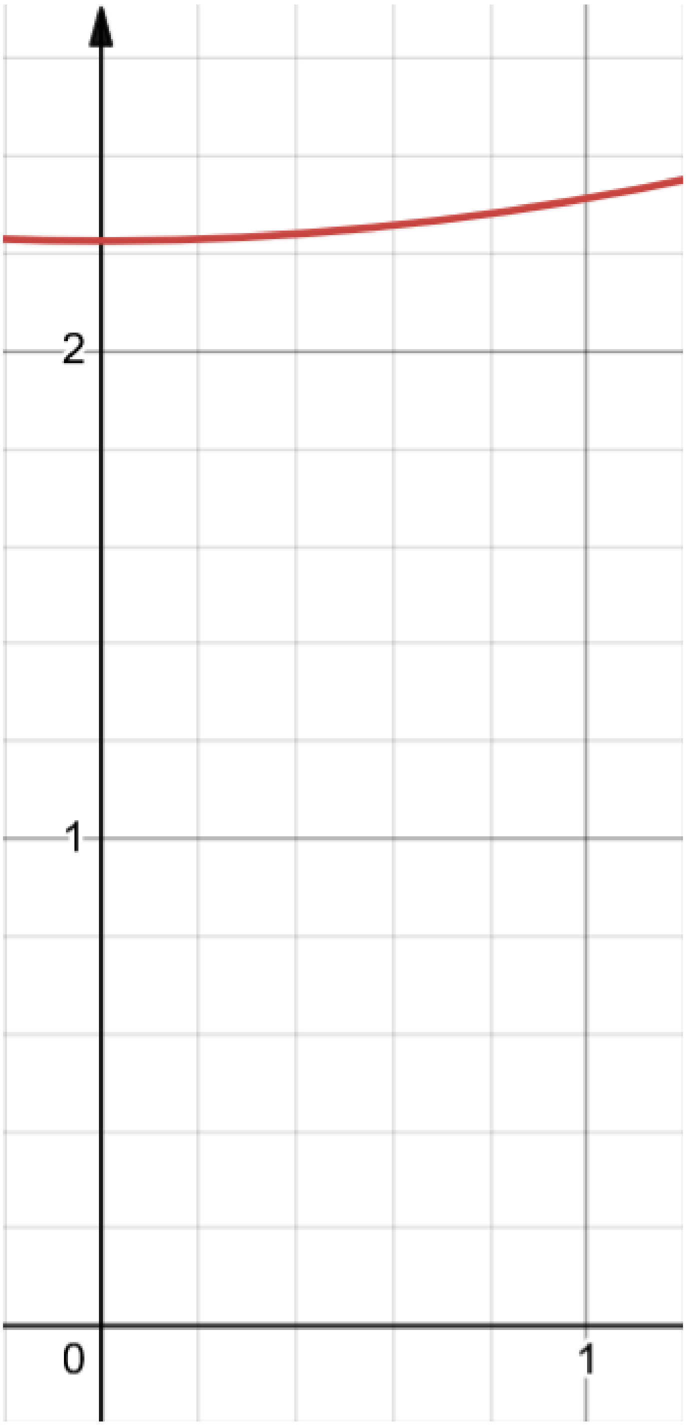


Figure 5.1: $\mathbb{E}(\pi^*)$ vs. θ for $\beta = 0.99, \gamma = 0.01$

As can be seen from Figure 5.1, the maximum value of $\mathbb{E}(\pi^*)$ is less than 2.4 while the function is increasing in θ, demonstrating that higher quality information leads to greater payoffs. However, if plot the graph for $\beta \in (-1, 1), \gamma \in (\beta - 2, \frac{\beta-2}{3})$ as in Figure 5.2, then it is easy to see that the $\mathbb{E}(\pi^*)$ is decreasing in θ. This is a visual confirmation of our previous result where we showed that greater information leads to lower payoffs for certain values of β, γ.

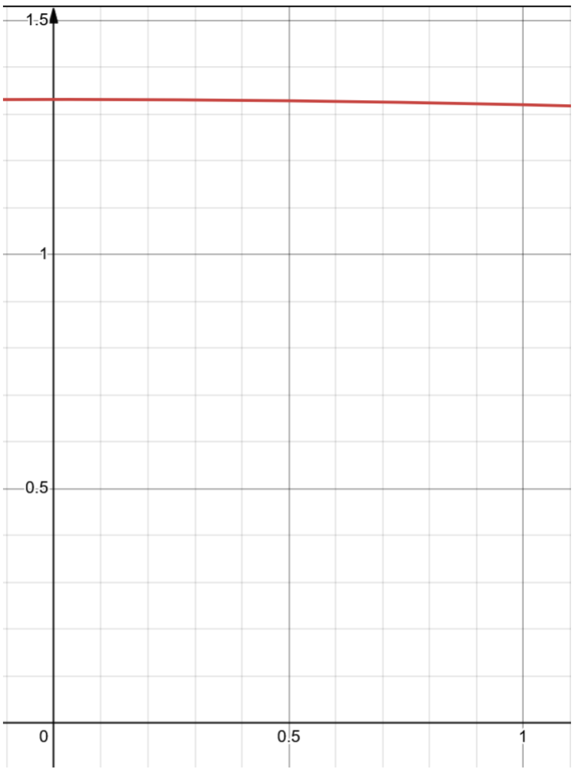


Figure 5.2: $\mathbb{E}(\pi^*)$ vs. θ for $\beta = 0.91, \gamma = -0.42$

5.2 Asymmetric Payoff

Previously we derived the asymmetric value function as:

$$\pi_i^*(\mathbb{E}[Y|X]) = \frac{(1+\mathbb{E}[Y|X])^2(\gamma_i\mathbb{E}[Y|X]+\beta_i+2)^2}{(4-(\gamma_i\mathbb{E}[Y|X]+\beta_i)\cdot(\gamma_j\mathbb{E}[Y|X]+\beta_j))^2} \tag{5.4}$$

and in chapter 3 we computed the conditional expectation as:

$$\mathbb{E}[Y|X = x_i] = \frac{1}{2} + \theta\, x_i - \frac{1}{2}\theta \tag{5.5}$$

Doing the relevant substitution we get:

$$\pi_i^* = \frac{\left(1+\left(\frac{1}{2}+\theta x-\frac{1}{2}\theta\right)\right)^2\left(\gamma_i\left(\frac{1}{2}+\theta x-\frac{1}{2}\theta\right)+\beta_i+2\right)^2}{\left(4-\left(\gamma_i\left(\frac{1}{2}+\theta x-\frac{1}{2}\theta\right)+\beta_i\right)\left(\gamma_j\left(\frac{1}{2}+\theta x-\frac{1}{2}\theta\right)+\beta_j\right)\right)^2}$$

Unfortunately, the definite integral of the above expression with respect to $F_X(x)$ is too large and sophisticated for any meaningful analysis. As a result, at this point we must consider some specific values of β, γ and then proceed with the integration.

Let us consider $\beta_i = 0.5, \beta_j = -0.8, \gamma_i = -0.03, \gamma_j = -0.15$. Then value function simplifies to:

$$\pi_i^* = \frac{0.25(2\theta x+\theta+3)^2(-2.485+0.03\theta x-0.015\theta)^2}{(-4.424375-0.0465\theta x+0.02325\theta+0.0045\theta^2x^2-0.0045\theta^2x+0.001125\theta^2)^2}$$

Integrating x over [0,1]:

$$\mathbb{E}(\pi_i^*) = \int_0^1 \frac{0.25(2\theta x+\theta+3)^2(-2.485+0.03\theta x-0.015\theta)^2}{(-4.424375-0.0465\theta x+0.02325\theta+0.0045\theta^2x^2-0.0045\theta^2x+0.001125\theta^2)^2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(F_X(x))\,\mathrm{d}x$$

$$\mathbb{E}(\pi_i^*) = \int_0^1 \frac{0.25(2\theta x+\theta+3)^2(-2.485+0.03\theta x-0.015\theta)^2}{(-4.424375-0.0465\theta x+0.02325\theta+0.0045\theta^2x^2-0.0045\theta^2x+0.001125\theta^2)^2} \cdot 1\mathrm{d}x$$

Which gives us:

$$\mathbb{E}(\pi_i^*) = \frac{(255600.0\theta^4 - 2.119605600\times 10^9\theta^2 + 3.953299013\times 10^{12})\ln(-9.0\theta^2 - 186.0\theta + 35395.0)}{81.0\theta^5 - 671706.0\theta^3 + 1.252806025\times 10^9\theta}$$
$$+ \frac{(-255600.0\theta^4 + 2.119605600\times 10^9\theta^2 - 3.953299013\times 10^{12})\ln(-9.0\theta^2 + 186.0\theta + 35395.0)}{81.0\theta^5 - 671706.0\theta^3 + 1.252806025\times 10^9\theta}$$
$$+ \frac{(-383443.5537\theta^4 + 3.179769577\times 10^9\theta^2 - 5.930622152\times 10^{12})\operatorname{arctanh}(0.015733784 86\theta - 0.1625}{81.0\theta^5 - 671706.0\theta^3 + 1.252806025\times 10^9\theta}$$
$$+ \frac{(-383443.5537\theta^4 + 3.179769577\times 10^9\theta^2 - 5.930622152\times 10^{12})\operatorname{arctanh}(0.015733784 86\theta + 0.1625}{81.0\theta^5 - 671706.0\theta^3 + 1.252806025\times 10^9\theta}$$
$$+ \frac{3600.0\theta^5 - 6.78753742\times 10^7\theta^3 + 2.341274395\times 10^{11}\theta}{81.0\theta^5 - 671706.0\theta^3 + 1.252806025\times 10^9\theta}$$

If we plot $\mathbb{E}(\pi_i^*)$ against θ we can see that this it is an increasing function; i.e for these values of β, γ possessing higher quality signals is better for the firms.

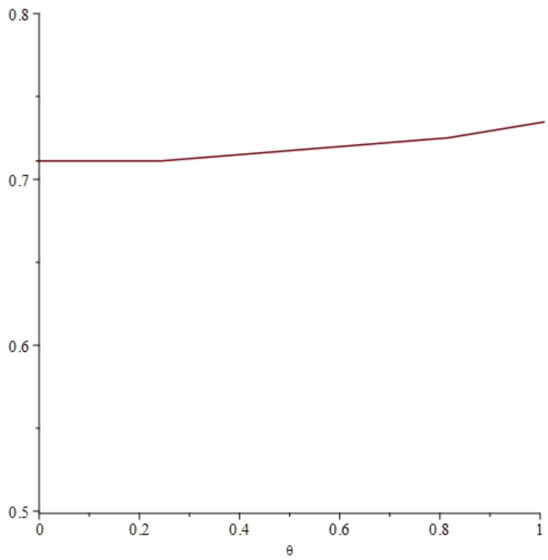


Figure 5.3: $\mathbb{E}(\pi_i^*)$ vs. θ for $\beta_i = 0.5, \beta_j = -0.8, \gamma_i = -0.03, \gamma_j = -0.15$

On the other hand if do the repeat the all of the above processes with π_i^* for $\beta_i = 0.76, \beta_j = -0.76, \gamma_i = -1.54, \gamma_j = 1.54$ then we get the following plot where clearly $\mathbb{E}(\pi_i^*)$ is decreasing θ; showing again that even in the asymmetric game there exists values of β, γ for which more information is bad for the firms.

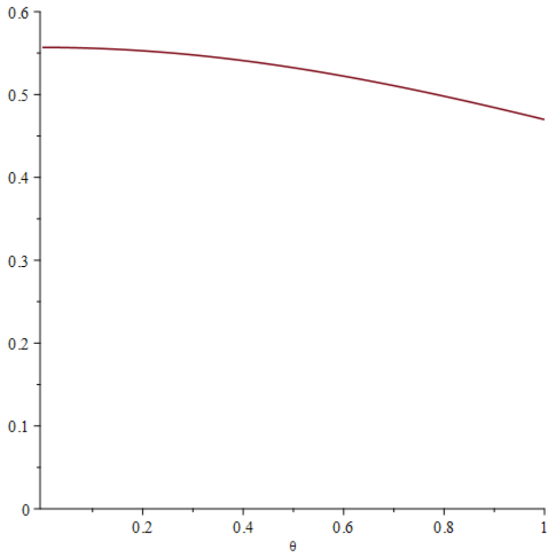


Figure 5.4: $\mathbb{E}(\pi_i^*)$ vs. θ for $\beta_i = 0.76, \beta_j = -0.76, \gamma_i = -1.54, \gamma_j = 1.54$

Chapter 6

Assumptions and Generalizations

Throughout the results demonstrated in the previous sections we made a number of assumptions, not all of which were WLOG. In this chapter we shall attempt to address these preconceptions and evaluate the extent to which our work can be generalized.

6.1 Universality of the Uniform

A strong assumption was made when we let the prior distribution $F_y(y) = \mathcal{U}[0, 1]$. However, invoking the results below [2] [18]

Theorem 3 (Probability Integral Transform)**.** *Let Y be a continuous random variable with strictly increasing cumulative distribution function (CDF) $F_Y : \mathbb{R} \to [0, 1]$. Define the random variable:*

$$U = F_Y(Y)$$

Then U follows a standard uniform distribution:

$$U \sim \text{Uniform}(0, 1)$$

Theorem 4 (Inverse Transform Sampling)**.** *Let $U \sim \text{Uniform}(0, 1)$ and let F_Y be a CDF with quantile function $F_Y^{-1} : [0, 1] \to \mathbb{R}$ defined as:*

$$F_Y^{-1}(u) = \inf\{y \in \mathbb{R} : F_Y(y) \geq u\}$$

Then the random variable $Y = F_Y^{-1}(U)$ has CDF F_Y.

It is quite clear that for any continuous CDF there exists a process to transform it into the Standard Uniform Distribution and vice versa. This ensures the assumption on $F_y(y)$ does not stagger generality as one may simply transform their general prior distribution into $\mathcal{U}[0, 1]$ and continue with the process of deriving $F_{Y|X}(y|x)$ via Bayes Rule.

We may also mention the assumed support [0,1] considered for random variables X & Y is WLOG since for any $[a, b] \subset \mathbb{R}$ one can always normalize it to $[0, 1]$ using $U = \frac{Y-a}{b-a}$.

6.2 Cost function is 0

In our derivation of the profit function $\pi_i = p_i \cdot q_i - C(q_i)$ we assumed $C(q_i) = 0$. The question is, how strong was this assumption in the context of this paper?

As stated before, the focus of this research is on studying information structures rather than the explicit equilibrium strategies or other numerical quantification. In that regard, it is not necessary to consider complex, heterogeneous cost functions. Rather, we may **can** consider constant marginal cost c for each firm without disturbing the results. In that case, the profit function for firm i with constant marginal cost c is:

$$\pi_i = (p_i - c)q_i, \tag{6.1}$$

where the demand function is:

$$q_i = 1 + y - p_i + (\beta_i + \gamma_i \cdot y)p_{-i}. \tag{6.2}$$

From this we get our new payoff function as:

$$\pi_i = (1 + y - p_i + (\beta_i + \gamma_i \cdot y)p_{-i}) \cdot (p_i - c) \tag{6.3}$$

Which, using the symmetric BRF, produces the following BNE:

$$p^* = \frac{1 + \mathbb{E}[Y|X] + c}{\gamma \cdot \mathbb{E}[Y|X] + \beta - 2} \tag{6.4}$$

Plugging (6.4) back into (6.3) and taking its expectation we get the following expected value function:

$$\pi^*(\mathbb{E}[Y|X]) = \frac{(1 + \mathbb{E}[Y|X] + c \cdot (\gamma\mathbb{E}[Y|X] + \beta - 1))^2}{(\gamma\mathbb{E}[Y|X] + \beta - 2)^2} \tag{6.5}$$

Which has a 2nd derivative given by:

$$\frac{4(c\,\gamma^2\mathbb{E}[Y|X] + \left(\beta c - \frac{1}{2}c + \mathbb{E}[Y|X] + \frac{3}{2}\right)\gamma - \frac{\beta}{2} + 1)((c+1)\gamma - \beta + 2)}{(\gamma\mathbb{E}[Y|X] + \beta - 2)^4} \tag{6.6}$$

The above expression is nonpositive $\forall \mathbb{E}[Y|X] \in [0,1]$ given $\beta \in (-1, 0], \gamma \in (-2, \frac{\beta-2}{c+1})$. Hence using Jensen's inequality we can again claim that the disadvantage stemming from more information remains for certain values of β, γ

6.3 Generalization to n-Player Oligopoly

Consider extending the original duopoly model to n firms competing in prices. The demand function for firm i becomes:

$$q_i = 1 + y - p_i + \sum_{j \neq i}^{n} (\beta_{ij} + \gamma_{ij} y) p_j \tag{6.7}$$

Like before we define the profit function:

$$\pi_i = p_i \cdot q_i - 0 = p_i(1 + y) - p_i^2 + p_i \sum_{j \neq i}^{n} (\beta_{ij} + \gamma_{ij} y) p_j \tag{6.8}$$

6.3.1 First-Order Conditions

Each firm maximizes expected profit $\mathbb{E}[\pi_i|x_i]$. The FOC for firm i is:

$$\frac{\partial \mathbb{E}[\pi_i|x_i]}{\partial p_i} = \frac{\partial \mathbb{E}\left[p_i(1+y) - p_i^2 + p_i \sum_{j\neq i}^{n}(\beta_{ij} + \gamma_{ij} y)p_j \middle| x_i\right]}{\partial p_i} = 0 \tag{6.9}$$

In symmetric equilibrium with $\beta_{ij} = \beta$, $\gamma_{ij} = \gamma$, and common signal structure:

$$p^* = \frac{1 + \mathbb{E}[Y|X = x_i]}{2 - (n-1)(\beta + \gamma \mathbb{E}[Y|X = x_i])} \tag{6.10}$$

To ensure the above equilibrium price is contained in the previously defined interval $[0, \overline{p}]$, we must implement new constraints on β, γ. In particular, we let $\beta \in (-\frac{1}{n-1}, \frac{1}{n-1})$ and $\gamma \in (-\frac{1}{n-1} - \beta, \frac{1}{n-1} - \beta)$. This forces $(n-1)(\beta + \gamma \mathbb{E}[Y|X] \in (-1, 1)$, extending the previous economic restriction to the n-player case, where we stated *the competitor's price cannot have a greater effect on firm i's price than its own price.*

6.4 Information Effects in the n-Player Game

In the n-player oligopoly the symmetric value function becomes

$$\frac{(1 + \mathbb{E}[Y|X])^2}{((n-1)(\gamma \mathbb{E}[Y|X] + \beta) - 2)^2} \tag{6.11}$$

Computing the 2nd derivative we get:

$$\frac{2((n-1)(\beta - \gamma) - 2)(-2(n-1)\gamma \mathbb{E}[Y|X] + \beta(n-1) - 3\gamma(n-1) - 2)}{((n-1)(\gamma \mathbb{E}[Y|X] + \beta) - 2)^4} \tag{6.12}$$

Once again, we may break it down into cases to find the range of values for which the 2nd derivative is non-positive $\forall \mathbb{E}[Y|X] \in [0, 1]$. This time the equations are:
Case 1:

$$(n-1)(\beta - \gamma) - 2 \geq 0 \tag{6.13}$$

$$-2(n-1)\gamma \mathbb{E}[Y|X] + \beta(n-1) - 3\gamma(n-1) - 2) \leq 0 \tag{6.14}$$

Case 2:

$$(n-1)(\beta - \gamma) - 2 \leq 0 \tag{6.15}$$

$$-2(n-1)\gamma \mathbb{E}[Y|X] + \beta(n-1) - 3\gamma(n-1) - 2) \geq 0 \tag{6.16}$$

Like before, case 1 leads to a contradiction while using algebra on case 2 leads to the discovery that the n-player symmetric value function is concave as long as $\beta \in (-\frac{1}{n-1}, \frac{1}{n-1})$ and $\gamma \in (\beta - \frac{2}{n-1}, \frac{\beta}{3} - \frac{2}{3(n-1)})$. Clearly, the range of values that allow for concavity becomes smaller as n becomes larger. In other words, as the number of firms increase the chances of the posterior payoff being less than the prior payoff decrease - illustrating the that the value of posterior knowledge (new information) is proportional to the number of players.

Chapter 7

Concluding Remarks

This study examined a specific linear demand function taken from the duopoly constructed by Singh & Vives (1984) and modified it by adding an uncertain demand parameter; transforming their full information game into a Bayesian game. Inspired by previous literature, we included the concept of private signals to delineate player types and explicitly constructed a likelihood function to deduce the compatibility of the signals with respect to the true demand parameter. Using Bayesian inference to update the beliefs of each firm we showed that, at equilibrium, the payoff after receiving the private signal is less than that of the payoff received before observing any information for both the symmetric and the asymmetric case for certain values of β, γ. While such detrimental effects of information have been displayed in finite normal form games and sender-receiver type games, this result is highly non-intuitive in a duopoly setting.

In the Ex-post stage of the game we graphed the payoffs against the noise parameter θ and explicitly showed how, for the aforementioned values of β, γ the payoff received by each firm decreases as the quality of signals increase. We were able to generalize these results to the n-player symmetric game while also addressing questions regarding loss of generality with respect to some of assumptions made throughout this paper.

Clearly, most of the difficulties that stem from this kind of approach manifest in the form of intractable algebraic expressions, particularly arising in the asymmetric case. In n-player oligopoly, the solutions are so intractable that we are unable to derive an explicit form of the value function. Even when evaluating the 2-player asymmetric value function in the Ex-Post stage, we were unable to directly plot its curve due to the complexity of the resulting function. As a result, one must experiment with numerical techniques and procedures to solve such quandries.

With that being said, this still paper succeeded in demonstrating the detreminental effect of information for at least one class of functions - thus opening avenues for further research on optimal information design in markets and the implications of signal structure under various competitive and cooperative regimes.

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