Warped Compactifications in Supergravity: A p-Brane Model

by

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Declaration

It is hereby declared that

1. The thesis submitted is my own original work while completing degree at BRAC University.
2. The thesis does not contain material previously published or written by a third party, except where this is appropriately cited through full and accurate referencing.
3. The thesis does not contain material which has been accepted, or submitted, for any other degree or diploma at a university or other institution.
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Approval

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Abstract

In 4D particle phenomenology beyond the Standard Model, warped compactification of higher dimensional gravity provides a geometric mechanism for generating large hierarchies of 4D physical scales. However, realizing such backgrounds in supergravity is constrained by the Maldacena–Nuñez no-go theorem. Under its assumptions, two-derivative supergravity with positive-energy ingredients obstructs nontrivially warped compactifications to de Sitter or Minkowski space. In string theory, we can evade such a restriction by adding negative tension sources, such as orientifold planes, to the theory. We study the role of negative tension sources for a p-brane model, which can be generalised to a type II supergravity. We explicitly construct the localized source action for this model using Dirac–Born–Infeld and Wess–Zumino terms and match it to the p-brane background with compact internal space. We find that the compactness of the internal space requires an effective negative-tension source. Hence, we get an analogous realization of the mechanism by which orientifold-like negative-tension sources allow warped backgrounds in string theory.

Keywords: Warped compactification; Supergravity; Localized sources; p-brane; Orientifold planes; Maldacena–Nuñez no-go theorem.

Dedication

In loving memory of my *dadi* and *nani*.

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Chapter 1

Introduction

One of the central motivations for physics beyond the Standard Model is the hierarchy problem. The Standard Model successfully describes particle physics at presently accessible energies, but it contains a scalar Higgs field whose mass is sensitive to high-energy quantum corrections. The observed electroweak scale is of order

$$m_{\mathrm{EW}} \sim 10^2 \text{ GeV},$$

whereas the gravitational scale is set by the Planck mass (calculated from the measured 4D Newton's constant),

$$M_{\mathrm{Pl}} \sim 10^{19} \text{ GeV}.$$

Thus there is an enormous separation,

$$\frac{M_{\mathrm{Pl}}}{m_{\mathrm{EW}}} \sim 10^{17}.$$

The hierarchy problem asks why the electroweak scale is so much smaller than the Planck scale and why this separation remains stable against quantum corrections [24].

Extra-dimensional models provide a geometric way to think about this problem. In ordinary Kaluza–Klein compactification, a higher-dimensional theory appears four-dimensional at low energies because the extra dimensions are compact [1]. The four-dimensional Planck scale is then related to the higher-dimensional gravitational scale and the volume of the compact internal space. This suggests that the observed gravitational scale may be an effective quantity determined by the geometry of extra dimensions.

The Randall–Sundrum model gives a different and particularly striking geometric mechanism. Instead of relying on a large compactification volume, it uses a warped five-dimensional geometry. In the Randall–Sundrum setup, physical mass scales depend on position in the extra dimension [7]. A mass parameter m_0 localized on the visible brane is redshifted according to

$$m_{\mathrm{phys}} = e^{-kL} m_0.$$

For $e^{-kL} \sim 10^{-16}$, a Planck-scale mass parameter can appear as a TeV-scale physical mass. Thus the hierarchy is generated by the warp factor rather than by a large extra-dimensional volume. This makes warped compactification an attractive idea from the point of view of particle phenomenology.

However, the Randall–Sundrum model is a five-dimensional effective model. If one wants to embed warped geometries into a more fundamental framework, it is natural to turn to string theory and its low-energy limit, supergravity. Type II supergravity contains gravity, the dilaton, higher-form gauge fields, and localized extended objects such as D-branes and orientifold planes. These are the ingredients used in string-theoretic compactifications and in the study of warped backgrounds.

A major difficulty is that warped compactification to 4D solutions is strongly constrained by supergravity no-go theorems. In particular, the Maldacena–Nuñez no-go theorem shows that, under broad assumptions, ordinary two-derivative supergravity with positive-energy ingredients does not allow nontrivially warped compactifications to de Sitter space or to Minkowski space [8]. Thus the phenomenologically attractive idea of warping becomes highly nontrivial in supergravity.

String theory provides controlled ways to evade the assumptions of the no-go theorem. String theory has orientifold planes ubiquitously, which carry negative tension and negative Ramond–Ramond charge relative to D-branes [12] (when we consider theory of the unoriented strings). These negative contributions are important in warped compactification. Negative-tension localized objects are required by the global consistency of many warped compactified solutions. This mechanism has been illustrated by Giddings, Kachru, and Polchinski (GKP), where type IIB string theory is compactified with fluxes and localized sources [10]. In GKP compactifications, NS–NS and Ramond–Ramond three-form fluxes contribute to the warp factor, while orientifold planes help satisfy the integrated D3-brane tadpole condition. The warp factor is determined by a Poisson-type equation on the compact internal space, which requires the total source to cancel. This is the global reason why negative charge and negative tension play a central role.

The purpose of this thesis is to study this mechanism in a simplified supergravity setting. We focus on an Einstein–dilaton–form p-brane model, where space-filling p-branes are charged under $(p+1)$-form potential. This model keeps the metric, the dilaton, and a single $(p+2)$-form field strength. The main calculation of the thesis is the construction and matching of the localized source action. The source action contains a Dirac–Born–Infeld term, which contributes the localized tension and stress tensor, and a Wess–Zumino term, which contributes the charge coupling to the $(p+1)$-form gauge potential. By inserting the p-brane background from [20] into the gauge-field, dilaton, and Einstein equations, we determine the source parameters.

The thesis is organized as follows. Chapter 2 reviews the basic idea of Kaluza–Klein compactification and the emergence of lower-dimensional fields. Chapter 3 discusses the Randall–Sundrum model and its warped resolution of the hierarchy problem. Chapter 4 introduces the relevant ingredients of type II supergravity, including the NS–NS and Ramond–Ramond sectors, D-branes, DBI and Wess–Zumino actions, and orientifold planes. Chapter 5 reviews the Maldacena–Nuñez no-go theorem. Chapter 6 explains how GKP compactifications evade the obstruction using fluxes and orientifolds. Chapter 7 studies the simplified p-brane model and performs the localized source matching. Finally, Chapter 8 summarizes the results and discusses future directions.

Chapter 2

Klauza-Klein Theory and Extra Dimensions

This chapter is based on [19] and [25].

We begin with an idea that the spacetime observed at low energies might arise from a higher-dimensional spacetime in which the extra dimensions are compact. For a five-dimensional spacetime of the form

$$M_5 = M_4 \times S^1, \tag{2.1}$$

where M_4 is the four-dimensional spacetime and S^1 is a circle of radius r. If y denotes the coordinate on the circle, then

$$y \sim y + 2\pi r. \tag{2.2}$$

Because the extra direction is compact, fields must be periodic in y. This periodicity forces a Fourier expansion along the compact direction.

For example, consider a massless scalar field $\Phi(x^\mu, y)$ in five dimensions. Since y is periodic, we may write

$$\Phi(x^\mu, y) = \sum_{n\in\mathbb{Z}} \phi_n(x^\mu) e^{iny/r}. \tag{2.3}$$

The five-dimensional kinetic term contains a derivative with respect to the compact coordinate,

$$\partial_y \Phi = \sum_{n\in\mathbb{Z}} \frac{in}{r} \phi_n(x^\mu) e^{iny/r}. \tag{2.4}$$

Therefore, the momentum along the compact direction appears as a mass term from the four-dimensional point of view. The n-th Fourier mode has

$$m_n^2 = \frac{n^2}{r^2}. \tag{2.5}$$

Thus a single massless field in five dimensions appears in four dimensions as an infinite tower of fields with masses

$$m_0 = 0, \qquad m_n = \frac{|n|}{r}, \qquad n \neq 0.$$

We call these the Kaluza–Klein modes.

The physical implication of this is that if the compactification radius r is small, then the first excited Kaluza–Klein mass scale

$$m_{\mathrm{KK}} \sim \frac{1}{r} \tag{2.6}$$

is very large. At energies

$$E \ll m_{\mathrm{KK}}, \tag{2.7}$$

only the zero mode $n = 0$ is accessible, and the higher-dimensional theory appears effectively four-dimensional. Keeping only the zero modes is called dimensional reduction, while keeping the full tower is called compactification.

Moreover, Kaluza–Klein reduction explains how different lower-dimensional fields can arise from a single higher-dimensional field. For example, a five-dimensional vector field decomposes as

$$A_M = (A_\mu, A_5). \tag{2.8}$$

From the four-dimensional point of view, A_μ is a vector field, while A_5 is a scalar field. Similarly, a higher-dimensional metric G_{MN} decomposes into

$$G_{MN} \quad \longrightarrow \quad \begin{cases} G_{\mu\nu}, & \text{four-dimensional graviton,} \\ G_{\mu m}, & \text{four-dimensional vector fields,} \\ G_{mn}, & \text{four-dimensional scalar fields.} \end{cases} \tag{2.9}$$

Thus gauge fields and scalar moduli can arise geometrically from higher-dimensional gravity.

To conclude, a higher-dimensional theory can appear four-dimensional at low energies using the Kaluza–Klein mechanism. However, ordinary Kaluza–Klein compactification hides the extra dimensions by making the compactification radius small. In the next chapter, we discuss another mechanism called the Randall–Sundrum model, where physical scales can be redshifted by introducing a warp factor even when the hierarchy is not produced by a large Kaluza–Klein mass gap.

Chapter 3

Randall-Sundrum Model

This chapter is based on [14] and [7].

The Randall–Sundrum model provides an important example of a warped compactification. In contrast with ordinary Kaluza–Klein compactification, where the extra dimension is usually taken to be small, the Randall–Sundrum construction uses a non-factorizable five-dimensional geometry. The four-dimensional metric is multiplied by a warp factor depending on the extra coordinate. This warp factor can generate an exponential hierarchy between physical mass scales on different branes.

We consider a five-dimensional spacetime with coordinates

$$x^M = (x^\mu, y), \qquad \mu, \nu = 0, 1, 2, 3, \qquad M, N = 0, 1, 2, 3, 5.$$

The extra dimension is compactified on an orbifold

$$S^1/\mathbb{Z}_2.$$

Thus y is identified with $-y$, and the physical interval may be taken to be

$$0 \leq y \leq L.$$

The two fixed points of the orbifold are located at

$$y = 0, \qquad y = L.$$

At these fixed points we place two three-branes. The brane at $y = 0$ is usually called the Planck brane or hidden brane, while the brane at $y = L$ is called the TeV brane or visible brane.

3.1 Bulk and brane action

The five-dimensional action is taken to be the sum of a bulk Einstein–Hilbert term, a bulk cosmological constant, and brane-localized tension terms:

$$S = \int d^4x \int_{-L}^{L} dy \, \sqrt{-g} \, \left(M^3 R - \Lambda\right) - \int d^4x \, \sqrt{-g_0} \, \lambda_0 - \int d^4x \, \sqrt{-g_L} \, \lambda_L. \tag{3.1}$$

Here M is the fundamental five-dimensional Planck scale, R is the five-dimensional Ricci scalar, Λ is the bulk cosmological constant, and λ_0, λ_L are the brane tensions at $y = 0$ and $y = L$, respectively. The induced metrics on the two branes are denoted by

$$(g_0)_{\mu\nu} = g_{\mu\nu}(x, y = 0), \qquad (g_L)_{\mu\nu} = g_{\mu\nu}(x, y = L).$$

Varying the bulk part of the action with respect to the inverse metric gives

$$\delta S_{\text{bulk}} = \int d^5x \sqrt{-g} \left[M^3 \left(R_{MN} - \frac{1}{2} g_{MN} R \right) + \frac{1}{2} \Lambda g_{MN} \right] \delta g^{MN}. \tag{3.2}$$

The brane tensions contribute only to the four-dimensional components of the energy-momentum tensor. Therefore the Einstein equation can be written as

$$M^3 \left(R_{MN} - \frac{1}{2} g_{MN} R \right) = -\frac{1}{2} \Lambda g_{MN} - \frac{1}{2} \left[\lambda_0 \, g_{\mu\nu} \delta^\mu_M \delta^\nu_N \delta(y) + \lambda_L \, g_{\mu\nu} \delta^\mu_M \delta^\nu_N \delta(y - L) \right]. \tag{3.3}$$

Equivalently, if

$$\mathcal{G}_{MN} \equiv R_{MN} - \frac{1}{2} g_{MN} R$$

denotes the five-dimensional Einstein tensor, then

$$\mathcal{G}_{MN} = -\frac{\Lambda}{2M^3} g_{MN} - \frac{1}{2M^3} \left[\lambda_0 \, g_{\mu\nu} \delta^\mu_M \delta^\nu_N \delta(y) + \lambda_L \, g_{\mu\nu} \delta^\mu_M \delta^\nu_N \delta(y - L) \right]. \tag{3.4}$$

3.2 Warped metric ansatz

The Randall–Sundrum ansatz assumes four-dimensional Poincare invariance along the brane directions. Therefore the metric is taken to be

$$ds^2 = e^{-2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \tag{3.5}$$

where

$$\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1).$$

The function $A(y)$ is called the warp function, and $e^{-2A(y)}$ is the warp factor. From (3.5), the nonzero metric components are

$$g_{\mu\nu}(y) = e^{-2A(y)} \eta_{\mu\nu}, \qquad g_{55} = 1, \tag{3.6}$$

and the inverse metric is

$$g^{\mu\nu}(y) = e^{2A(y)} \eta^{\mu\nu}, \qquad g^{55} = 1. \tag{3.7}$$

Since the metric depends only on y, the calculation of the Christoffel symbols is simple. The only nonzero Christoffel symbols are

$$\Gamma^\mu_{\nu 5} = \Gamma^\mu_{5\nu} = -A'(y) \delta^\mu_\nu, \tag{3.8}$$

$$\Gamma^5_{\mu\nu} = A'(y) e^{-2A(y)} \eta_{\mu\nu} = A'(y) g_{\mu\nu}. \tag{3.9}$$

Here a prime denotes differentiation with respect to y.

Using these Christoffel symbols, we compute the Ricci tensor components

$$R_{\mu\nu} = \left(A'' - 4(A')^2\right) g_{\mu\nu}, \tag{3.10}$$
$$R_{55} = 4A'' - 4(A')^2. \tag{3.11}$$

The Ricci scalar is therefore

$$R = 8A'' - 20(A')^2. \tag{3.12}$$

Hence the Einstein tensor components are

$$\mathcal{G}_{\mu\nu} = \left(-3A'' + 6(A')^2\right) g_{\mu\nu}, \tag{3.13}$$
$$\mathcal{G}_{55} = 6(A')^2. \tag{3.14}$$

3.3 Solving the bulk equations

Away from the branes, the delta-function sources vanish. Therefore, in the bulk, the Einstein equation becomes

$$\mathcal{G}_{MN} = -\frac{\Lambda}{2M^3} g_{MN}. \tag{3.15}$$

The 55-component gives

$$6(A')^2 = -\frac{\Lambda}{2M^3}. \tag{3.16}$$

Thus

$$(A')^2 = -\frac{\Lambda}{12M^3}. \tag{3.17}$$

For $A(y)$ to be real, we must have

$$\Lambda < 0. \tag{3.18}$$

Therefore the five-dimensional bulk is anti-de Sitter space. We define

$$k^2 \equiv -\frac{\Lambda}{12M^3}. \tag{3.19}$$

Then the bulk equation becomes

$$(A')^2 = k^2. \tag{3.20}$$

Therefore

$$A'(y) = \pm k. \tag{3.21}$$

Because the orbifold identifies y with $-y$, the warp function must be even:

$$A(y) = A(-y).$$

The solution consistent with this $\mathbb{Z}_2$ symmetry is

$$A(y) = k|y|. \tag{3.22}$$

Thus the Randall–Sundrum metric is

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \qquad -L \leq y \leq L. \tag{3.23}$$

This is a slice of five-dimensional anti-de Sitter space.

3.4 Brane tensions and fine-tuning conditions

The function $A(y) = k|y|$ is not smooth at the orbifold fixed points. Its first derivative is

$$A'(y) = k\,\mathrm{sgn}(y), \tag{3.24}$$

and the second derivative contains delta functions. On the compact orbifold interval, this is written as

$$A''(y) = 2k\left[\delta(y) - \delta(y-L)\right]. \tag{3.25}$$

The delta functions must be matched with the brane sources in the Einstein equation.

Using

$$\mathcal{G}_{\mu\nu} = \left(-3A'' + 6(A')^2\right) g_{\mu\nu},$$

the delta-function part of the $\mu\nu$-component is

$$(\mathcal{G}_{\mu\nu})_{\text{delta}} = -3A'' g_{\mu\nu} = -6k\left[\delta(y) - \delta(y-L)\right] g_{\mu\nu}. \tag{3.26}$$

On the other hand, the brane part of the Einstein equation gives

$$(\mathcal{G}_{\mu\nu})_{\text{brane}} = -\frac{1}{2M^3}\left[\lambda_0\delta(y) + \lambda_L\delta(y-L)\right] g_{\mu\nu}. \tag{3.27}$$

Matching the coefficients of the delta functions gives

$$\lambda_0 = +12M^3 k, \tag{3.28}$$

$$\lambda_L = -12M^3 k. \tag{3.29}$$

Together with (3.19), the fine-tuning conditions are

$$\Lambda = -12M^3k^2, \qquad \lambda_0 = +12M^3k, \qquad \lambda_L = -12M^3k. \tag{3.30}$$

Thus the brane at $y = 0$ has positive tension, while the brane at $y = L$ has negative tension. This tuning is required in order to obtain a flat four-dimensional metric on the branes.

3.5 Comments

Four-dimensional Plank scale. The effective four-dimensional Planck scale is obtained by inserting the warped metric into the five-dimensional Einstein–Hilbert action and integrating over the extra dimension. The coefficient of the four-dimensional Ricci scalar is

$$M_{\text{Pl}}^2 = M^3 \int_{-L}^{L} dy\, e^{-2k|y|}. \tag{3.31}$$

Evaluating the integral gives

$$M_{\text{Pl}}^2 = M^3 \frac{1 - e^{-2kL}}{k}. \tag{3.32}$$

For $kL \gg 1$, the exponential term is negligible, and therefore

$$M_{\text{Pl}}^2 \simeq \frac{M^3}{k}. \tag{3.33}$$

Thus if M and k are both of order the fundamental Planck scale, the observed four-dimensional Planck scale is naturally reproduced.

Warping and the hierarchy problem. The main physical consequence of the Randall–Sundrum metric is that mass scales depend on position in the extra dimension. If we suppose the Standard Model fields are localized on the visible brane at $y = L$. The induced metric on this brane is

$$(g_L)_{\mu\nu} = e^{-2kL}\eta_{\mu\nu}. \tag{3.34}$$

For example, a scalar field H localized on the visible brane. Its action contains terms of the form

$$S_H = \int d^4x\,\sqrt{-g_L}\left[g_L^{\mu\nu}(D_\mu H)^\dagger(D_\nu H) - \lambda\left(|H|^2 - v_0^2\right)^2\right]. \tag{3.35}$$

Since

$$\sqrt{-g_L} = e^{-4kL}, \qquad g_L^{\mu\nu} = e^{2kL}\eta^{\mu\nu},$$

the kinetic term is not canonically normalized. Defining the canonically normalized field

$$H_{\text{phys}} = e^{-kL}H, \tag{3.36}$$

one finds that all physical masses on the visible brane are redshifted:

$$m_{\text{phys}} = e^{-kL}m_0. \tag{3.37}$$

Therefore a fundamental mass parameter m_0 of order the Planck scale can appear on the visible brane as a TeV-scale mass if

$$e^{-kL} \sim 10^{-16}. \tag{3.38}$$

Equivalently,

$$kL \sim 37. \tag{3.39}$$

This exponential suppression is the key mechanism by which the Randall–Sundrum model addresses the hierarchy problem. Unlike ordinary compactifications, the hierarchy does not arise from a large volume of the extra dimension. Instead, it is generated by the exponential warp factor.

Chapter 4

Type II Supergravity

Type II supergravity is the low-energy effective description of type II superstring theory. There are two ten-dimensional type II theories: type IIA and type IIB. They have the same Neveu–Schwarz–Neveu–Schwarz sector, usually called the NS–NS sector, but differ in their Ramond–Ramond sector, usually called the RR sector. These theories provide the natural spacetime framework for studying flux compactifications, D-branes, orientifold planes, and warped geometries. The discussion of this chapter is based on [6], [11], [16], [22], [29].

4.1 The NS–NS sector

The NS–NS sector is common to both type IIA and type IIB supergravity. It contains three bosonic fields:

$$g_{MN}, \qquad B_{MN}, \qquad \phi.$$

Here g_{MN} is the ten-dimensional spacetime metric, B_{MN} is an antisymmetric two-form field, and ϕ is the dilaton. The metric describes gravity. The dilaton controls the string coupling,

$$g_s = e^{\phi}.$$

The two-form B_2 has field strength

$$H_3 = dB_2.$$

This field is called the NS–NS three-form flux.

The terminology NS–NS comes from the worldsheet description of the string. Closed string states are built from a left-moving and a right-moving sector. The NS–NS fields arise from states in which both the left-moving and right-moving parts are in the Neveu–Schwarz sector. The massless NS–NS fields appear in both type IIA and type IIB string theory. In particular, every perturbative closed string theory contains a graviton, and therefore the metric belongs to this universal NS–NS sector.

In the string frame, the NS–NS part of the type II supergravity action is

$$S_{\mathrm{NS-NS}} = \frac{1}{2\kappa_{10}^2} \int d^{10}X \sqrt{-g^{(s)}}\, e^{-2\phi} \left[R^{(s)} + 4(\partial\phi)^2 - \frac{1}{2}|H_3|^2 \right]. \tag{4.1}$$

Here $g^{(s)}$ is the string-frame metric and

$$|H_3|^2 = \frac{1}{3!} H_{MNP} H^{MNP}.$$

The overall factor $e^{-2\phi}$ is characteristic of the closed-string tree-level action.

4.2 The RR sector

The RR sector contains antisymmetric gauge potentials. Unlike the NS–NS fields, the RR fields differ between type IIA and type IIB theory. These fields arise from closed string states in which both the left-moving and right-moving parts belong to the Ramond sector.

In type IIA supergravity, the RR gauge potentials have odd degree:

$$C_1, \qquad C_3,$$

together with their magnetic dual potentials in the democratic formulation. Their corresponding field strengths are schematically

$$F_2 = dC_1, \qquad F_4 = dC_3 + \cdots .$$

Thus type IIA contains RR field strengths of even degree.

In type IIB supergravity, the RR gauge potentials have even degree:

$$C_0, \qquad C_2, \qquad C_4,$$

again together with their magnetic duals in the democratic formulation. Their field strengths are schematically

$$F_1 = dC_0, \qquad F_3 = dC_2 + \cdots , \qquad F_5 = dC_4 + \cdots .$$

Thus type IIB contains RR field strengths of odd degree. The five-form field strength of type IIB satisfies the self-duality condition

$$F_5 = \star_{10} F_5. \tag{4.2}$$

Because of this condition, the covariant type IIB action is usually written as a pseudo-action. We vary the action first and impose (4.2) afterward.

The RR fields are important because D-branes are charged under them. A Dp-brane couples electrically to a $(p+1)$-form RR potential C_{p+1}. Therefore RR fields are the natural gauge fields for D-brane charge.

4.3 Type IIA action

The bosonic string-frame action of massless type IIA supergravity may be written as

$$\begin{aligned} S_{\mathrm{IIA}} = {} & \frac{1}{2\kappa_{10}^2} \int d^{10}X \, \sqrt{-g^{(s)}} \, e^{-2\phi} \left[R^{(s)} + 4(\partial\phi)^2 - \frac{1}{2}|H_3|^2 \right] \\ & - \frac{1}{4\kappa_{10}^2} \int d^{10}X \, \sqrt{-g^{(s)}} \left[|F_2|^2 + |\widetilde{F}_4|^2 \right] - \frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4. \end{aligned} \tag{4.3}$$

Here

$$F_2 = dC_1, \qquad F_4 = dC_3, \qquad \widetilde{F}_4 = F_4 - C_1 \wedge H_3.$$

The last term is a Chern–Simons coupling. It implies that the RR field strengths are not independent of the NS–NS two-form.

4.4 Type IIB action

The bosonic string-frame pseudo-action of type IIB supergravity may be written as

$$\begin{aligned} S_{\mathrm{IIB}} = &\frac{1}{2\kappa_{10}^2} \int d^{10}X \, \sqrt{-g^{(s)}} \, e^{-2\phi} \left[R^{(s)} + 4(\partial\phi)^2 - \frac{1}{2}|H_3|^2 \right] \\ &- \frac{1}{4\kappa_{10}^2} \int d^{10}X \, \sqrt{-g^{(s)}} \left[|F_1|^2 + |\widetilde{F}_3|^2 + \frac{1}{2}|\widetilde{F}_5|^2 \right] - \frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3. \end{aligned} \tag{4.4}$$

Here

$$F_1 = dC_0, \qquad F_3 = dC_2, \qquad F_5 = dC_4,$$

and

$$\begin{aligned} \widetilde{F}_3 &= F_3 - C_0 H_3, \\ \widetilde{F}_5 &= F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3 \end{aligned}$$

with the self-duality condition

$$\widetilde{F}_5 = \star_{10} \widetilde{F}_5.$$

4.5 Einstein frame

The string-frame action is natural from the worldsheet point of view, but many supergravity calculations are cleaner in Einstein frame. In ten dimensions, the Einstein-frame metric is related to the string-frame metric by

$$g^{(E)}_{MN} = e^{-\phi/2} g^{(s)}_{MN}. \tag{4.5}$$

In Einstein frame, the gravitational action has the standard Einstein–Hilbert form and the dilaton has a canonical kinetic term.

For a RR n-form field strength F_n, the Einstein-frame kinetic term has the schematic form

$$S^{(E)}_{RR,n} = -\frac{1}{2\kappa_{10}^2} \int d^{10}X \, \sqrt{-g_E} \, \frac{1}{2n!} e^{\frac{5-n}{2}\phi} F_n^2. \tag{4.6}$$

This formula will be useful later because a Dp-brane couples electrically to a RR potential C_{p+1}, whose field strength is

$$F_{p+2} = dC_{p+1}.$$

Setting $n = p + 2$, the RR kinetic term becomes

$$S^{(E)}_{RR,p} = -\frac{1}{2\kappa_{10}^2} \int d^{10}X \, \sqrt{-g_E} \, \frac{1}{2(p+2)!} e^{\frac{3-p}{2}\phi} F_{p+2}^2. \tag{4.7}$$

In the simplified p-brane model studied later, we replace the specific type II dilaton coupling by a general coupling parameter λ.

4.6 D-branes and their worldvolume actions

D-branes are nonperturbative extended objects in string theory. A Dp-brane has p spatial dimensions and therefore a $(p+1)$-dimensional worldvolume. Open strings can end on D-branes, while closed strings propagate in the full ten-dimensional spacetime. This is why D-branes are both dynamical objects and sources for spacetime fields.

At low energies, the dynamics of a single Dp-brane is described by the Dirac–Born–Infeld action and the Wess–Zumino action [5], [6], [12].

The Dirac–Born–Infeld action is

$$S_{\text{DBI}} = -T_p \int_{\Sigma_{p+1}} d^{p+1}\xi \, e^{-\phi} \sqrt{-\det\left(P[g^{(s)} + B]_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right)}. \tag{4.8}$$

Here Σ_{p+1} is the brane worldvolume, T_p is the D-brane tension, $P[\cdots]$ denotes pullback to the brane worldvolume, B is the NS–NS two-form, and $F_{\alpha\beta}$ is the field strength of the gauge field living on the brane. This action describes how the brane tension couples to the spacetime metric, the dilaton, the NS–NS two-form, and the worldvolume gauge field.

The Wess–Zumino action is

$$S_{\text{WZ}} = \mu_p \int_{\Sigma_{p+1}} P\left[\sum_q C_q\right] \wedge e^{P[B]+2\pi\alpha' F}. \tag{4.9}$$

Here μ_p is the RR charge of the D-brane and $\sum_q C_q$ denotes the formal sum of RR potentials. The Wess–Zumino term describes the coupling of the brane to RR gauge fields. In the simplest case, when the B-field and the worldvolume gauge field are set to zero, the leading coupling reduces to

$$S_{\text{WZ}} = \mu_p \int_{\Sigma_{p+1}} P[C_{p+1}]. \tag{4.10}$$

The two actions have different physical meanings. The DBI action contains the brane tension and therefore contributes to the stress tensor. It tells us how the brane gravitates. The Wess–Zumino action contains the brane charge and therefore tells us how the brane sources RR gauge fields. For BPS D-branes, the tension and charge are related in magnitude, reflecting the balance between gravitational/dilatonic attraction and RR repulsion.

The distinction between NS–NS and RR sectors is also visible here. The DBI action couples naturally to NS–NS fields such as the metric, dilaton, and B_2-field. The Wess–Zumino action couples naturally to RR potentials. Thus D-branes are objects that carry tension with respect to the NS–NS sector and charge with respect to the RR sector.

4.7 Orientifold planes

In addition to D-branes, type II string theory also contains another important class of localized objects called orientifold planes. Orientifold planes arise when one quotients string theory by a symmetry that combines worldsheet parity with a spacetime involution. The worldsheet parity operation is usually denoted by Ω_{ws}, and it reverses the orientation of the string worldsheet [6], [12], [21], [27]. An orientifold projection has the schematic form

$$\Omega_{\mathrm{ws}}\mathcal{R},$$

where $\mathcal{R}$ is a geometric involution of the spacetime or internal manifold. The fixed-point locus of $\mathcal{R}$ is an orientifold plane.

An orientifold p-plane, denoted O_p, extends along p spatial directions and has a $(p+1)$-dimensional worldvolume. Like a Dp-brane, it couples to the spacetime metric, the dilaton, and Ramond–Ramond gauge potentials. However, an orientifold plane is not a dynamical brane in the same sense as a D-brane. Rather, it is a fixed locus of the orientifold quotient. In the low-energy supergravity description, it appears as a localized source with negative tension and negative Ramond–Ramond charge relative to a D-brane.

The negative tension and charge of orientifold planes are their most important features in compactification problems. A Dp-brane has positive tension and positive RR charge, whereas an O_p-plane contributes with the opposite sign. Schematically,

$$T_{O_p} < 0, \qquad \mu_{O_p} < 0,$$

in units where a Dp-brane has

$$T_{D_p} > 0, \qquad \mu_{D_p} > 0.$$

More precisely, the charge and tension of a standard O_p^--plane are proportional to those of a Dp-brane by

$$\mu_{O_p^-} = -2^{p-5}\mu_{D_p}, \qquad T_{O_p^-} = -2^{p-5}T_{D_p},$$

up to convention-dependent normalizations. We note that orientifold planes provide localized negative-tension and negative-charge sources.

In the supergravity approximation, an orientifold plane is often represented by localized source terms analogous to the DBI and Wess–Zumino terms of a D-brane, but with negative tension and charge. We can write schematically

$$S_{O_p} = -|T_{O_p}| \int_{\Sigma_{p+1}} d^{p+1}\xi\, e^{-\phi}\sqrt{-\det P[g^{(s)}]_{\alpha\beta}} + \mu_{O_p}\int_{\Sigma_{p+1}} P[C_{p+1}], \qquad (4.11)$$

where μ_{O_p} is negative for a standard O_p^--plane. We note that this is not a fundamental worldvolume theory with open-string degrees of freedom in the same way as for a D-brane. This effective description is only reliable away from the orientifold fixed locus. The full string-theoretic quotient description is required near the locus.

The negative tension and negative Ramond–Ramond charge of orientifold planes are essential in many warped compactifications with compact internal spaces. On a compact internal space, the warp-factor equation is typically subject to an integrated source constraint. Positive-tension and positive-charge sources alone cannot satisfy this constraint in the relevant cases. Orientifold planes provide the negative localized contributions needed in many type II compactifications.

Chapter 5

Maldacena-Nuñez No Go Theorem

In this chapter, we review the no-go theorem by Maldacena and Nuñez [8]. The aim is to understand why we cannot obtain a non-singular warped compactification to $\mathbb{R}^d$ or to de Sitter space dS_d with finite d-dimensional Newton constant under a set of assumptions. The argument is especially useful in the context of Randall–Sundrum-type compactifications, because it shows that the desired warped compactifications are impossible in ordinary two-derivative supergravity unless one violates at least one of the assumptions.

5.1 Setup and assumptions

We consider a D-dimensional gravitational theory compactified to d external dimensions. The full D-dimensional coordinates are denoted by

$$x^M = (x^\mu, y^m),$$

where

$$M, N, L = 0, 1, \ldots, D-1, \qquad \mu, \nu, \rho = 0, 1, \ldots, d-1,$$

and

$$m, n, l = d, \ldots, D-1.$$

Thus x^μ are coordinates on the d-dimensional external spacetime, while y^m are coordinates on the internal $(D-d)$-dimensional space.

The working assumptions are the following.

1. The gravitational action is the ordinary two-derivative Einstein action. In particular, there are no higher-curvature corrections.
2. The scalar potential is non-positive,

$$V \leq 0. \tag{5.1}$$

   The potential may be a negative cosmological constant or a scalar-field potential, but it is assumed not to become positive in the region of field space explored by the solution.

3. The theory contains massless fields with positive kinetic terms. These are described by n-form field strengths

$$F_{M_1 \cdots M_n}, \qquad n < D.$$

   The case $n = D$ is excluded from this discussion because a top-form field strength contributes as an effective potential term.

4. The lower-dimensional Newton constant is finite. Equivalently, after dimensional reduction, the coefficient of the d-dimensional Einstein term must be finite.

5. The external spacetime is maximally symmetric and is taken to be either $\mathbb{R}^d$ or dS_d. Therefore its scalar curvature satisfies

$$R(\eta) \geq 0,$$

   with $R(\eta) = 0$ for $\mathbb{R}^d$ and $R(\eta) > 0$ for de Sitter space.

6. The fields respect the external d-dimensional isometries. Hence flux components are allowed only if they are either purely internal or proportional to the full d-dimensional external volume form.

The trace-reversed form of the D-dimensional Einstein equation is

$$R_{MN} = T_{MN} - \frac{1}{D-2} g_{MN} T^L{}_L. \tag{5.2}$$

This is the form used in the original argument.

5.2 Warped metric ansatz

We take the D-dimensional metric to be

$$ds_D^2 = \Omega^2(y) \left(ds_d^2 + \widehat{g}_{mn}(y)\, dy^m dy^n \right), \tag{5.3}$$

where

$$ds_d^2 = \eta_{\mu\nu}(x)\, dx^\mu dx^\nu. \tag{5.4}$$

Here $\Omega(y)$ is the warp factor and $\widehat{g}_{mn}$ is the internal metric before the overall conformal rescaling. Hatted covariant derivatives and contractions are always taken with respect to $\widehat{g}_{mn}$.

Equivalently, the metric components are

$$g_{\mu\nu} = \Omega^2(y) \eta_{\mu\nu}, \tag{5.5}$$
$$g_{mn} = \Omega^2(y) \widehat{g}_{mn}, \tag{5.6}$$

and the inverse metric is

$$g^{\mu\nu} = \Omega^{-2}(y) \eta^{\mu\nu}, \tag{5.7}$$
$$g^{mn} = \Omega^{-2}(y) \widehat{g}^{mn}. \tag{5.8}$$

Let

$$u(y) \equiv \log \Omega(y).$$

Since the warp factor depends only on the internal coordinates, the conformal transformation formula for the Ricci tensor gives the external components

$$R_{\mu\nu} = R_{\mu\nu}(\eta) - \eta_{\mu\nu} \left[\widehat{\nabla}^2 u + (D-2)(\widehat{\nabla} u)^2 \right]. \tag{5.9}$$

In terms of Ω, this is

$$R_{\mu\nu} = R_{\mu\nu}(\eta) - \eta_{\mu\nu} \left[\widehat{\nabla}^2 \log \Omega + (D-2) \left(\widehat{\nabla} \log \Omega \right)^2 \right]. \tag{5.10}$$

This is the Ricci component obtained from the warped metric ansatz. The derivation is shown in Appendix A.2.

Substituting (5.10) into the external components of the trace-reversed Einstein equation (5.2), we get

$$R_{\mu\nu}(\eta) - \eta_{\mu\nu} \left[\widehat{\nabla}^2 \log \Omega + (D-2)(\widehat{\nabla} \log \Omega)^2 \right] = T_{\mu\nu} - \frac{1}{D-2} \Omega^2 \eta_{\mu\nu} T^L{}_L. \tag{5.11}$$

5.3 Tracing the external Einstein equation

Now take the trace of (5.11) with $\eta^{\mu\nu}$. Since

$$\eta^{\mu\nu} R_{\mu\nu}(\eta) = R(\eta), \qquad \eta^{\mu\nu}\eta_{\mu\nu} = d,$$

we obtain

$$R(\eta) - d \left[\widehat{\nabla}^2 \log \Omega + (D-2)(\widehat{\nabla} \log \Omega)^2 \right] = \eta^{\mu\nu} T_{\mu\nu} - \frac{d}{D-2} \Omega^2 T^L{}_L. \tag{5.12}$$

But the external trace with the full D-dimensional metric is

$$T^\mu{}_\mu = g^{\mu\nu} T_{\mu\nu} = \Omega^{-2} \eta^{\mu\nu} T_{\mu\nu}. \tag{5.13}$$

Therefore

$$\eta^{\mu\nu} T_{\mu\nu} = \Omega^2 T^\mu{}_\mu. \tag{5.14}$$

Substituting this into (5.12) gives

$$R(\eta) - d \left[\widehat{\nabla}^2 \log \Omega + (D-2)(\widehat{\nabla} \log \Omega)^2 \right] = \Omega^2 T^\mu{}_\mu - \frac{d}{D-2} \Omega^2 T^L{}_L. \tag{5.15}$$

Rearranging, we obtain

$$\widehat{\nabla}^2 \log \Omega + (D-2)(\widehat{\nabla} \log \Omega)^2 = \frac{1}{d} R(\eta) + \frac{\Omega^2}{d} \left(-T^\mu{}_\mu + \frac{d}{D-2} T^L{}_L \right). \tag{5.16}$$

We absorb the factor of $1/d$ into the normalization of $R(\eta)$. Thus we can write

$$\widehat{\nabla}^2 \log \Omega + (D-2)(\widehat{\nabla} \log \Omega)^2 = R(\eta) + \Omega^2 \left(-T^\mu{}_\mu + \frac{d}{D-2} T^L{}_L \right). \tag{5.17}$$

Since

$$\Omega^{D-2} = e^{(D-2)\log\Omega},$$

we have

$$\widehat{\nabla}_m \Omega^{D-2} = (D-2)\Omega^{D-2}\widehat{\nabla}_m \log\Omega, \tag{5.18}$$

$$\begin{aligned} \widehat{\nabla}^2 \Omega^{D-2} &= \widehat{\nabla}_m \left[(D-2)\Omega^{D-2}\widehat{\nabla}^m \log\Omega\right] \\ &= (D-2)\left[\Omega^{D-2}\widehat{\nabla}^2 \log\Omega + \widehat{\nabla}_m \Omega^{D-2}\widehat{\nabla}^m \log\Omega\right] \\ &= (D-2)\Omega^{D-2}\left[\widehat{\nabla}^2 \log\Omega + (D-2)(\widehat{\nabla}\log\Omega)^2\right]. \end{aligned} \tag{5.19}$$

Therefore

$$\widehat{\nabla}^2 \log\Omega + (D-2)(\widehat{\nabla}\log\Omega)^2 = \frac{1}{(D-2)\Omega^{D-2}}\widehat{\nabla}^2\Omega^{D-2}. \tag{5.20}$$

Using (5.20), the warp-factor equation becomes

$$\frac{1}{(D-2)\Omega^{D-2}}\widehat{\nabla}^2\Omega^{D-2} = R(\eta) + \Omega^2\widetilde{T}, \tag{5.21}$$

where we have defined

$$\widetilde{T} \equiv -T^\mu{}_\mu + \frac{d}{D-2}T^L{}_L. \tag{5.22}$$

5.4 Positivity of $\widetilde{T}$

The main argument of the no-go theorem is that $\widetilde{T}$ is non-negative under the assumptions stated above. We now check this explicitly by following the analysis done in [30].

Potential contribution

Suppose the matter Lagrangian contains a scalar potential term

$$\mathcal{L}_V \sim -V.$$

Then its stress tensor is proportional to

$$T^{(V)}_{MN} \sim -V g_{MN}. \tag{5.23}$$

Hence

$$T^{(V)\mu}{}_\mu = g^{\mu\nu}T^{(V)}_{\mu\nu} \sim -dV, \tag{5.24}$$

$$T^{(V)L}{}_L = g^{MN}T^{(V)}_{MN} \sim -DV. \tag{5.25}$$

Therefore

$$\begin{aligned} \widetilde{T}^{(V)} &= -T^{(V)\mu}{}_\mu + \frac{d}{D-2}T^{(V)L}{}_L \\ &\sim dV - \frac{dD}{D-2}V \\ &= -\frac{2d}{D-2}V. \end{aligned} \tag{5.26}$$

Since $V \leq 0$, it follows that

$$\widetilde{T}^{(V)} \geq 0. \tag{5.27}$$

n-form field strengths

Now consider an n-form field strength $F_{M_1\cdots M_n}$. Up to irrelevant positive numerical constants, its stress tensor is

$$T^{(F)}_{MN} = F_{ML_1\cdots L_{n-1}} F_N{}^{L_1\cdots L_{n-1}} - \frac{1}{2n} g_{MN} F^2, \tag{5.28}$$

where

$$F^2 := F_{L_1\cdots L_n} F^{L_1\cdots L_n}. \tag{5.29}$$

First compute the external trace:

$$\begin{aligned} T^{(F)\mu}{}_\mu &= g^{\mu\nu} T^{(F)}_{\mu\nu} \\ &= F_{\mu L_1\cdots L_{n-1}} F^{\mu L_1\cdots L_{n-1}} - \frac{d}{2n} F^2. \end{aligned} \tag{5.30}$$

The full D-dimensional trace is

$$\begin{aligned} T^{(F)L}{}_L &= g^{MN} T^{(F)}_{MN} \\ &= F_{ML_1\cdots L_{n-1}} F^{ML_1\cdots L_{n-1}} - \frac{D}{2n} F^2. \end{aligned} \tag{5.31}$$

Since every component of F^2 is counted n times in $F_{ML_1\cdots L_{n-1}} F^{ML_1\cdots L_{n-1}}$, we have

$$F_{ML_1\cdots L_{n-1}} F^{ML_1\cdots L_{n-1}} = F^2. \tag{5.32}$$

Thus

$$T^{(F)L}{}_L = \left(1 - \frac{D}{2n}\right) F^2. \tag{5.33}$$

Substituting (5.30) and (5.33) into the definition of $\widetilde{T}$, we find

$$\begin{aligned} \widetilde{T}^{(F)} &= -T^{(F)\mu}{}_\mu + \frac{d}{D-2} T^{(F)L}{}_L \\ &= -F_{\mu L_1\cdots L_{n-1}} F^{\mu L_1\cdots L_{n-1}} + \frac{d}{2n} F^2 + \frac{d}{D-2}\left(1 - \frac{D}{2n}\right) F^2 \\ &= -F_{\mu L_1\cdots L_{n-1}} F^{\mu L_1\cdots L_{n-1}} + \frac{d}{D-2}\left(1 - \frac{1}{n}\right) F^2. \end{aligned} \tag{5.34}$$

This is the expression used to analyze the possible flux components.

There are two cases compatible with the external spacetime isometries.

Case 1: purely internal flux. If F is purely internal, then

$$F_{\mu L_1\cdots L_{n-1}} = 0.$$

Therefore (5.34) becomes

$$\widetilde{T}^{(F)} = \frac{d}{D-2}\left(1 - \frac{1}{n}\right) F^2. \tag{5.35}$$

For a purely internal flux,

$$F^2 \geq 0.$$

Hence

$$\widetilde{T}^{(F)} \geq 0. \tag{5.36}$$

For $n > 1$, the coefficient is strictly positive if the flux is nonzero. For $n = 1$, the coefficient vanishes, so a purely internal one-form gives $\widetilde{T}^{(F)} = 0$ even when the field strength is nonzero.

Case 2: flux with d external legs. Now suppose F has components along all d external directions and the remaining $n - d$ indices internal. This is the only possible external component compatible with the full external Lorentz or de Sitter symmetry.

In this case, when we contract over one external index μ, we are choosing one of the d external slots out of the total n slots of F. Therefore

$$F_{\mu L_1 \cdots L_{n-1}} F^{\mu L_1 \cdots L_{n-1}} = \frac{d}{n} F^2. \tag{5.37}$$

Substituting this into (5.34), we get

$$\begin{aligned}
\widetilde{T}^{(F)} &= -\frac{d}{n} F^2 + \frac{d}{D-2}\left(1 - \frac{1}{n}\right) F^2 \\
&= dF^2 \left[-\frac{1}{n} + \frac{n-1}{n(D-2)} \right] \\
&= dF^2 \left[\frac{-(D-2) + (n-1)}{n(D-2)} \right] \\
&= -F^2 \frac{d(D-1-n)}{n(D-2)}.
\end{aligned} \tag{5.38}$$

For a form with external time components,

$$F^2 < 0.$$

Moreover, since $n \leq D - 1$, we have

$$D - 1 - n \geq 0.$$

Therefore

$$\widetilde{T}^{(F)} \geq 0. \tag{5.39}$$

Combining the potential contribution and the allowed n-form contributions, we conclude that

$$\widetilde{T} \geq 0. \tag{5.40}$$

5.5 Condition on the warp factor

We now derive the central warp-factor constraint explicitly.

From (5.21), we have

$$\frac{1}{(D-2)\Omega^{D-2}}\widehat{\nabla}^2\Omega^{D-2} = R(\eta) + \Omega^2\widetilde{T}. \tag{5.41}$$

By assumption, the external spacetime is either Minkowski or de Sitter. Hence

$$R(\eta) \geq 0.$$

We have also shown that

$$\widetilde{T} \geq 0.$$

Therefore the right-hand side of (5.41) is non-negative:

$$R(\eta) + \Omega^2\widetilde{T} \geq 0. \tag{5.42}$$

Since $\Omega > 0$ in the regular region of the compactification, it follows that

$$\widehat{\nabla}^2\Omega^{D-2} \geq 0. \tag{5.43}$$

Multiplying by Ω^{D-2}, we obtain

$$\Omega^{D-2}\widehat{\nabla}^2\Omega^{D-2} \geq 0. \tag{5.44}$$

This is the basic inequality controlling the warp factor.

The d-dimensional Newton constant is determined by the warped internal volume:

$$\frac{1}{G_N^{(d)}} \sim \int d^{D-d}y\,\sqrt{\widehat{g}}\,\Omega^{D-2}. \tag{5.45}$$

By assumption this integral is finite.

Compact internal space with bounded warp factor

First suppose that the internal space is compact and that the warp factor is bounded above and below:

$$0 < \Omega_{\min} \leq \Omega(y) \leq \Omega_{\max} < \infty.$$

Define

$$f(y) := \Omega^{D-2}(y). \tag{5.46}$$

Then the inequality (5.44) becomes

$$f\,\widehat{\nabla}^2 f \geq 0. \tag{5.47}$$

Integrating over the compact internal space gives

$$\int d^{D-d}y\,\sqrt{\widehat{g}}\,f\,\widehat{\nabla}^2 f \geq 0. \tag{5.48}$$

On the other hand, integration by parts on a compact space without boundary gives

$$\begin{aligned}\int d^{D-d}y\,\sqrt{\hat{g}}\,f\,\widehat{\nabla}^2 f &= -\int d^{D-d}y\,\sqrt{\hat{g}}\,\widehat{\nabla}_m f\,\widehat{\nabla}^m f\\ &= -\int d^{D-d}y\,\sqrt{\hat{g}}\,\left(\widehat{\nabla} f\right)^2 \leq 0.\end{aligned} \tag{5.49}$$

Equations (5.48) and (5.49) imply

$$\int d^{D-d}y\,\sqrt{\hat{g}}\,\left(\widehat{\nabla} f\right)^2 = 0. \tag{5.50}$$

Therefore

$$\widehat{\nabla}_m f = 0, \tag{5.51}$$

and hence

$$f = \text{constant}. \tag{5.52}$$

Since $f = \Omega^{D-2}$, the warp factor must be constant:

$$\Omega = \text{constant}. \tag{5.53}$$

If Ω is constant, then

$$\widehat{\nabla}^2 \Omega^{D-2} = 0.$$

The warp-factor equation (5.41) then gives

$$R(\eta) + \Omega^2 \widetilde{T} = 0. \tag{5.54}$$

But both terms are non-negative. Hence each term must vanish:

$$R(\eta) = 0, \qquad \widetilde{T} = 0. \tag{5.55}$$

Therefore the external space cannot be de Sitter. It must be Minkowski.

Moreover, $\widetilde{T} = 0$ severely restricts the allowed fluxes. From the positivity analysis above, any flux that gives a strictly positive contribution to $\widetilde{T}$ must vanish. Thus the only possible nonzero fluxes are those whose contribution to $\widetilde{T}$ vanishes, such as the special $n = 1$ internal case and its dual $(D-1)$-form case.

Allowing singularities with $\Omega \to 0$

Now suppose the internal space may have singular regions where

$$\Omega \to 0.$$

The criterion used by Maldacena and Nuñez is that physically acceptable singularities should not have an increasing g_{00} as the singularity is approached. Since

$$g_{00} = \Omega^2 \eta_{00},$$

this means that the warp factor should not increase as one approaches the singularity. In particular, singularities with $\Omega \to \infty$ are excluded.

Let R_ϵ be the internal region obtained by removing small neighborhoods of the singularities, chosen so that

$$\Omega > \epsilon$$

inside R_ϵ. The boundary ∂R_ϵ lies near the excised singular regions. Because the warp factor does not increase as one moves toward the singularity, we can choose the outward-pointing unit normal ν^m on ∂R_ϵ so that

$$\nu^m \widehat{\nabla}_m \Omega^{D-2} \leq 0. \tag{5.56}$$

Again define

$$f = \Omega^{D-2}.$$

Integrating the pointwise inequality

$$f \widehat{\nabla}^2 f \geq 0$$

over R_ϵ, we get

$$0 \leq \int_{R_\epsilon} d^{D-d}y \sqrt{\widehat{g}}\, f\, \widehat{\nabla}^2 f. \tag{5.57}$$

Using integration by parts on the region with boundary,

$$\int_{R_\epsilon} d^{D-d}y \sqrt{\widehat{g}}\, f\, \widehat{\nabla}^2 f = \int_{\partial R_\epsilon} d\Sigma\, f\, \nu^m \widehat{\nabla}_m f - \int_{R_\epsilon} d^{D-d}y \sqrt{\widehat{g}} \left(\widehat{\nabla} f\right)^2. \tag{5.58}$$

By (5.56), the boundary term is non-positive:

$$\int_{\partial R_\epsilon} d\Sigma\, f\, \nu^m \widehat{\nabla}_m f \leq 0. \tag{5.59}$$

Therefore

$$\begin{aligned} 0 &\leq \int_{R_\epsilon} \sqrt{\widehat{g}}\, f\, \widehat{\nabla}^2 f \\ &= \int_{\partial R_\epsilon} d\Sigma\, f\, \nu^m \widehat{\nabla}_m f - \int_{R_\epsilon} \sqrt{\widehat{g}} \left(\widehat{\nabla} f\right)^2 \\ &\leq - \int_{R_\epsilon} \sqrt{\widehat{g}} \left(\widehat{\nabla} f\right)^2 \leq 0. \end{aligned} \tag{5.60}$$

Thus all inequalities must be equalities, and we conclude that

$$\widehat{\nabla}_m f = 0 \qquad \text{inside } R_\epsilon. \tag{5.61}$$

Since this holds for arbitrarily small ϵ, it follows that

$$\Omega = \text{constant} \tag{5.62}$$

throughout the allowed internal region.

As before, substituting a constant warp factor into the warp-factor equation gives

$$R(\eta) + \Omega^2 \widetilde{T} = 0. \tag{5.63}$$

Since both terms are non-negative, one must have

$$R(\eta) = 0, \qquad \widetilde{T} = 0. \tag{5.64}$$

Therefore de Sitter compactifications are excluded. Moreover, a Randall–Sundrum-type compactification with a nontrivial warp factor is also excluded under these assumptions.

Thus, the Maldacena–Nuñez argument shows that, for ordinary two-derivative supergravity theories with non-positive potential, positive kinetic terms, and finite lower-dimensional Newton constant, the warp factor must be constant if the only singularities allowed are those for which the warp factor does not increase toward the singularity. Consequently, there are no compactifications to de Sitter space and no smooth Randall–Sundrum-type compactifications in this class of theories.

The theorem can be evaded only by violating at least one assumption. For example, in string compactifications, this can happen by including localized negative-tension sources such as orientifold planes, higher-derivative corrections, or other ingredients outside the simple two-derivative supergravity framework.

Chapter 6

GKP Compactification

The Maldacena–Nuñez no-go theorem shows that ordinary two-derivative supergravity compactifications are highly constrained. Under the assumptions of compactness, finite lower-dimensional Newton constant, positive kinetic terms, non-positive potential, and acceptable singularities, one cannot obtain a de Sitter compactification or a nontrivially warped compactification to Minkowski space using only ordinary positive-energy ingredients [8]. Therefore, to realize warped compactifications with compact internal manifolds in string theory, at least one of the assumptions of the no-go theorem must be violated.

A central example is the construction of Giddings, Kachru, and Polchinski (GKP), which studies warped compactifications of type IIB string theory with background fluxes and localized sources [10]. The GKP construction provides a string-theoretic realization of warped compactification and explains how negative-tension objects, such as orientifold planes, enter the compact background. The goal of this chapter is to explain how the GKP construction evades the no-go obstruction.

6.1 An overview

The GKP setup is type IIB supergravity compactified from ten dimensions to four dimensions. The ten-dimensional spacetime is taken to be a warped product of a four-dimensional spacetime and a compact six-dimensional internal space. In Einstein frame, the metric is written as

$$ds_{10}^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)}\widetilde{g}_{mn}(y)dy^m dy^n. \tag{6.1}$$

Here $A(y)$ is the warp factor and $\widetilde{g}_{mn}$ is the unwarped internal metric. The essential point is that the warp factor depends on the internal coordinates. Therefore different regions of the compact internal space can correspond to different physical redshift scales in the four-dimensional effective theory.

This is analogous to the Randall–Sundrum model, where the warp factor depends on an extra coordinate and redshifts physical masses along that direction. In the GKP setting, the same basic idea is realized in a ten-dimensional string compactification: the warp factor is generated by fluxes and localized sources inside the compact internal space.

The relevant type IIB fields are the axio-dilaton

$$\tau = C_0 + ie^{-\phi},$$

the NS–NS three-form flux

$$H_3 = dB_2,$$

the RR three-form flux

$$F_3 = dC_2,$$

and the self-dual five-form flux $\widetilde{F}_5$. It is useful to combine the three-form fluxes into

$$G_3 = F_3 - \tau H_3. \tag{6.2}$$

In supersymmetric GKP backgrounds, this complex three-form flux satisfies an imaginary self-duality condition with respect to the unwarped internal metric,

$$\widetilde{\star}_6 G_3 = iG_3. \tag{6.3}$$

The three-form fluxes contribute to the warp-factor equation, but in a compact space their contribution must be balanced by localized sources.

6.2 Warp factor and tadpole condition

The five-form field strength in type IIB supergravity is self-dual:

$$\widetilde{F}_5 = \star_{10} \widetilde{F}_5. \tag{6.4}$$

In a warped four-dimensional compactification, it can be written in terms of a scalar function $\alpha(y)$ as

$$\widetilde{F}_5 = (1 + \star_{10}) \left[d\alpha(y) \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \right]. \tag{6.5}$$

In BPS GKP solutions, α is related to the warp factor, typically

$$\alpha = e^{4A}. \tag{6.6}$$

Thus the five-form flux and the warp factor are not independent. The Bianchi identity for $\widetilde{F}_5$ becomes the equation determining the warp factor.

The five-form Bianchi identity receives contributions from the three-form fluxes and from localized D3-brane charge:

$$d\widetilde{F}_5 = H_3 \wedge F_3 + 2\kappa_{10}^2 T_3 \, \rho_3^{\rm loc}. \tag{6.7}$$

Here $\rho_3^{\rm loc}$ denotes the localized D3-brane charge density, including contributions from D3-branes and from objects carrying D3-charge, such as O3-planes and wrapped D7/O7 configurations with induced D3-charge.

Schematically, this leads to a Poisson-type equation for the warp factor:

$$\widetilde{\nabla}^2 e^{-4A} = \text{flux contribution} + \text{localized D3-charge contribution}. \tag{6.8}$$

Thus the warp factor is sourced both by fluxes and by localized objects.

Because the internal space is compact, this Poisson equation is subject to an integrated consistency condition. On a compact manifold without boundary,

$$\int_{M_6} d^6y \sqrt{\widetilde{g}}\, \widetilde{\nabla}^2 f = 0. \tag{6.9}$$

Therefore a Poisson equation

$$\widetilde{\nabla}^2 f = \rho \tag{6.10}$$

can have a solution only if

$$\int_{M_6} d^6y \sqrt{\widetilde{g}}\, \rho = 0. \tag{6.11}$$

For the GKP compactification, this condition becomes the D3-brane tadpole cancellation condition. Schematically,

$$\frac{1}{2\kappa_{10}^2 T_3} \int_{M_6} H_3 \wedge F_3 + Q_3^{\rm loc} = 0. \tag{6.12}$$

Here $Q_3^{\rm loc}$ includes the signs of the localized D3-brane charges. On a compact space, the total charge must cancel because there is no boundary through which flux lines can escape. This is analogous to Gauss's law in compact space.

6.3 Evasion of the no-go obstruction

The tadpole condition explains why negative localized sources are important. In supersymmetric GKP backgrounds, the flux contribution to the D3-brane tadpole is positive, and ordinary D3-branes also contribute positive D3-brane charge. If only such positive sources were present, the integrated source constraint could not be satisfied on a compact internal space. Orientifold planes provide negative D3-brane charge, and in particular O3-planes also carry negative tension relative to D3-branes.

This is how the GKP construction evades the Maldacena–Nuñez obstruction. The no-go theorem assumes ordinary two-derivative supergravity with positive-energy ingredients and no negative-tension localized sources. Orientifold planes lie outside this class of assumptions. Their negative charge and negative tension allow the integrated field equations and the warp-factor equation to be satisfied on a compact internal space.

In the full string-theoretic setting, orientifold planes provide the required negative contribution. In the simplified p-brane model studied later, an analogous mechanism appears through the sign of the localized source tension determined by the equations of motion.

6.4 Toroidal orientifold

A standard example is the orientifold compactification

$$T^6/\mathbb{Z}_2.$$

This background contains O3-planes at the fixed points of the $\mathbb{Z}_2$ involution. These O3-planes carry negative D3-brane charge and negative D3-brane tension. In a common normalization, each O3-plane contributes

$$T_{O3} = -\frac{1}{4}T_3, \qquad \mu_{O3} = -\frac{1}{4}\mu_3.$$

Since $T^6/\mathbb{Z}_2$ has 64 O3-planes, the total O3-plane charge is

$$64\left(-\frac{1}{4}\right) = -16$$

in D3-brane units. This negative contribution can be balanced by positive D3-brane charge from D3-branes and fluxes. The example illustrates how negative localized sources are required in order to satisfy the global source constraint.

Chapter 7

A p-Brane Model

The GKP construction gives a full type IIB realization of warped compactification with fluxes and orientifold planes. However, we limit our focus to a simpler Einstein–dilaton–form model for a p-brane background. We only keep the metric, the dilaton, and a single $(p+2)$-form field strength. Then, we compute explicitly how localized DBI and Wess–Zumino source terms enter the gauge-field, dilaton, and Einstein equations. The goal is to see how warped backgrounds after compactification require a negative-tension source, which is similar to orientifolds in the GKP construction.

7.1 The effective action

From [20], we write a D-dimensional Einstein–dilaton–form model. This model should be understood as a truncation of the type II supergravity field content in which only the metric, the dilaton, and one $(p+1)$-form gauge potential are retained.

The effective action

$$S = \frac{1}{2\kappa_D^2}\int d^DX\,\sqrt{-g_D}\left[R_D - \frac{1}{2}(\partial\phi)^2 - \frac{e^{-\lambda\phi}}{2(p+2)!}\,F_{p+2}^2\right] + S_{\rm loc}, \qquad F_{p+2} = dC_{p+1}. \tag{7.1}$$

here $S_{\rm loc}$ is the action for localized sources that are coupled with gauge field.

7.2 Compact p-brane background

The p-brane solutions for our effective action (7.1) are given in [2], [3], [4], [20]. We have our metric, dialton, and gauge field as

$$ds^2 = H_0(y)^{2a}\,\widehat{\eta}_{\mu\nu}\,dx^\mu dx^\nu + H_0(y)^{2b}\,\widetilde{g}_{mn}(y)\,dy^m dy^n, \tag{7.2}$$

$$e^{-\phi} = H_0(y)^{\lambda/2}, \tag{7.3}$$

$$C_{p+1} = s\,H_0(y)^{-1}\,\widehat{\epsilon}_{p+1}, \qquad s = \pm 1, \tag{7.4}$$

Our internal space is Ricci flat $\tilde{R}_{mn}(\tilde{g}) = 0$. We also have our metric exponents defined as

$$a = -\frac{D-p-3}{2(D-2)}, \qquad b = \frac{p+1}{2(D-2)}, \qquad q = D-p-1. \tag{7.5}$$

and we require

$$\lambda^2 = 4 - \frac{2(p+1)(D-p-3)}{(D-2)} \tag{7.6}$$

We can compute

$$(p+1)a + b(q-2) = 0 \tag{7.7}$$

and

$$\frac{\lambda^2}{2} = 2 + 2a(p+1) = 2 - 2b(q-2) \tag{7.8}$$

$H_0(y)$ is a harmonic function on the internal space. It satisfies

$$\widetilde{\nabla}^2 H_0 = \sum_n Q_n \, \delta^{(q)}(y - y_n). \tag{7.9}$$

We also compute

$$\phi = -\frac{\lambda}{2} \ln H_0, \qquad e^{\gamma\phi} = H_0^{-\gamma\lambda/2}. \tag{7.10}$$

7.3 Localized source action

7.3.1 Background metric and setup

Our background metric is

$$ds^2 = H_0(y)^{2a} \, \hat{\eta}_{\mu\nu} \, dx^\mu dx^\nu + H_0(y)^{2b} \, \tilde{g}_{mn}(y) \, dy^m dy^n. \tag{7.11}$$

We have a D-dimensional spacetime with coordinates $X^M = (x^\mu, y^m)$ and a $(p+1)$ dimensional object called p-brane. We describe the brane by a map

$$X^M = X^M(\xi^\alpha) \tag{7.12}$$

here, $\xi^\alpha, \alpha = 0, \ldots, p$ are worldvolume coordinates and $X^M(\xi)$ is the position of the brane in spacetime. Here, $\mu = 0, 1, \ldots p$, $m = p+1, \ldots D-1$ and we define $q \equiv D - p - 1$. Then we have D-dimensional volume density

$$\sqrt{-g_D} = H_0^{a(p+1)+bq} \sqrt{-\hat{\eta}} \sqrt{\tilde{g}}. \tag{7.13}$$

We write the transverse volume form

$$\begin{aligned} \tilde{\epsilon}_q &= \sqrt{\tilde{g}} dy^{p+1} \wedge \cdots \wedge dy^{D-1} \\ &= \frac{1}{q!} \sqrt{\tilde{g}} \varepsilon_{m_1 \ldots m_q} dy^{m_1} \wedge \cdots \wedge dy^{m_q} \end{aligned} \tag{7.14}$$

We define the delta function normalized with respect to the metric

$$\int d^D X \sqrt{-g_D} \delta^{(D)}(X - X(\xi)) f(X) = f(X(\xi)). \tag{7.15}$$

We also compute an identity related to the Hodge star that will be used later.

$$\star_D\left(\widehat{\epsilon}_{p+1}\wedge\widetilde{d}(H_0^{-1})\right)=H_0^{-a(p+1)+b(q-2)}\widetilde{\star}\,\widetilde{d}(H_0^{-1}). \tag{7.16}$$

We can write

$$\widetilde{d}(H_0^{-1})=\partial_m(H_0^{-1})\,dy^m. \tag{7.17}$$

Then

$$\widehat{\epsilon}_{p+1}\wedge\widetilde{d}(H_0^{-1})=\sqrt{-\widehat{\eta}}\,\partial_m(H_0^{-1})\,dx^0\wedge\cdots\wedge dx^p\wedge dy^m. \tag{7.18}$$

Taking the D-dimensional Hodge dual gives a purely transverse $(q-1)$-form. The warp-factor dependence comes from the volume factor and from raising the indices of the form. The volume factor contributes

$$H_0^{a(p+1)+bq}. \tag{7.19}$$

The form has $p+1$ external indices and one transverse index. Therefore, raising its indices contributes

$$H_0^{-2a(p+1)}H_0^{-2b}. \tag{7.20}$$

Thus the total warp factor is

$$\begin{aligned}H_0^{a(p+1)+bq}H_0^{-2a(p+1)}H_0^{-2b}&=H_0^{a(p+1)+bq-2a(p+1)-2b}\\&=H_0^{-a(p+1)+b(q-2)}.\end{aligned} \tag{7.21}$$

The remaining contraction is exactly the transverse Hodge dual with respect to $\widetilde{g}_{mn}$. Therefore, we get

$$\star_D\left(\widehat{\epsilon}_{p+1}\wedge\widetilde{d}(H_0^{-1})\right)=H_0^{-a(p+1)+b(q-2)}\widetilde{\star}\,\widetilde{d}(H_0^{-1}). \tag{7.22}$$

7.3.2 Dirac-Born-Infeld action

The Dirac-Born-Infeld (DBI) action is the standard low-energy D-brane effective action, which is written in terms of the induced metric on the brane worldvolume [9]. We write (ignoring the gauge field and B-field)

$$S_{\text{DBI}}=-T_p\int d^{p+1}\xi e^{\gamma\phi}\sqrt{-\det P[g]_{\alpha\beta}} \tag{7.23}$$

Here,

$$P[g]_{\alpha\beta}=g_{MN}(X(\xi))\frac{\partial X^M}{\partial\xi^\alpha}\frac{\partial X^N}{\partial\xi^\beta} \tag{7.24}$$

is the pullback metric, which is induced on the brane. The DBI action describes the localized p-branes because it is defined on the worldvolume of the brane rather than full D-dimensional metric.

Now, we want to write the action as a D-dimensional integral. Using this inside (7.23) and taking $P = \det P[g]_{\alpha\beta}$, we get

$$
\begin{aligned}
S_{\text{DBI}} &= -T_p \int_{\Sigma_{p+1}} d^{p+1}\xi e^{\gamma\phi(X(\xi))}\sqrt{-P} \\
&= -T_p \int_{\Sigma_{p+1}} d^{p+1}\xi \sqrt{-P} \int d^D X \sqrt{-g_D}\delta^{(D)}(X - X(\xi))\, e^{\gamma\phi(X)} \\
&= -T_p \int d^D X \sqrt{-g_D}\, e^{\gamma\phi(X)} \int_{\Sigma_{p+1}} d^{p+1}\xi \sqrt{-P}\, \delta^{(D)}(X - X(\xi)) \\
&= -T_p \int d^D X \sqrt{-g_D}\, e^{\gamma\phi(X)} \left[\int_{\Sigma_{p+1}} d^{p+1}\xi \sqrt{-P}\, \delta^{(D)}(X - X(\xi)) \right].
\end{aligned} \tag{7.25}
$$

We define the DBI scalar density

$$
\rho(X) \equiv \int_{\Sigma_{p+1}} d^{p+1}\xi \sqrt{-P}\, \delta^{(D)}(X - X(\xi)) \,. \tag{7.26}
$$

Therefore, the DBI action in full D-dimensions is

$$
S_{\text{DBI}} = -T_p \int_{\mathcal{M}_D} d^D X \sqrt{-g_D}\, e^{\gamma\phi(X)} \rho(X). \tag{7.27}
$$

For a static brane fixed at $y = y_0$,

$$
X^M(\xi) = (x^\mu = \xi^\mu, y^m = y_0^m) \tag{7.28}
$$

We now calculate the pullback metric. Using

$$
\partial_\alpha X^M = \begin{cases} \delta^\mu_\alpha & M = \mu \\ 0 & M = m \end{cases} \tag{7.29}
$$

and

$$
\partial_\beta X^N = \begin{cases} \delta^\nu_\beta & N = \nu \\ 0 & N = n \end{cases} \tag{7.30}
$$

we get

$$
P[g]_{\alpha\beta} = H_0^{2a}(y_0)\hat{\eta}_{\mu\nu}\delta^\mu_\alpha\delta^\nu_\beta = H_0^{2a}(y_0)\hat{\eta}_{\alpha\beta} \tag{7.31}
$$

Then we take the determinant

$$
\det P[g]_{\alpha\beta} = H_0^{2a(p+1)} \det \hat{\eta}_{\alpha\beta}, \tag{7.32}
$$

$$
\sqrt{-\det P[g]_{\alpha\beta}} = H_0^{a(p+1)}\sqrt{-\hat{\eta}}. \tag{7.33}
$$

Then our DBI action becomes

$$
S_{DBI} = -T_p \int d^{p+1}\xi e^{\gamma\phi(y_0)} H_0^{a(p+1)}(y_0)\sqrt{-\hat{\eta}} \tag{7.34}
$$

Now the DBI scalar density in (7.26) becomes

$$\rho(X) = \int d^{p+1}\xi H_0^{a(p+1)} \sqrt{-\hat{\eta}}\, \delta^{(D)}\left(X - X(\xi)\right) \tag{7.35}$$

$$= \int d^{p+1}\xi H_0^{a(p+1)} \sqrt{-\hat{\eta}} \frac{\delta^{(p+1)}(x-\xi)\delta^{(q)}_{\text{coord}}(y-y_0)}{H_0^{a(p+1)+bq}\sqrt{-\hat{\eta}}\sqrt{\tilde{g}}} \tag{7.36}$$

$$= H_0^{-bq} \frac{1}{\sqrt{\tilde{g}}} \delta^{(q)}_{\text{coord}}(y-y_0) \tag{7.37}$$

We used

$$\delta^{(D)}(X - X(\xi)) = \frac{\delta^{(p+1)}(x-\xi)\delta^{(q)}_{\text{coord}}(y-y_0)}{\sqrt{-g_D}}. \tag{7.38}$$

The integration over the worldvolume coordinates removes the worldvolume delta function, leaving only the transverse delta function. We define metric-normalized transverse delta function as

$$\delta^{(q)}(y-y_0) \equiv \frac{1}{\sqrt{\tilde{g}}} \delta^{(q)}_{\text{coord}}(y-y_0) \tag{7.39}$$

Thus, the DBI scalar density for a collection of static branes becomes

$$\rho(y) = H_0^{-bq} \sum_n \delta^{(q)}(y-y_n). \tag{7.40}$$

Here, the DBI scalar density depends only on the transverse coordinates because the brane fills the worldvolume directions and is localized only in the transverse directions. Hence, the source is uniformly extended along the brane directions and localized in the transverse space.

7.3.3 Wess-Zumino action

A charged p-brane couples with the $(p+1)$-form gauge potential C_{p+1}. This localized coupling is described by the Wess-Zumino term

$$S_{WZ} = \mu_p \int_{\Sigma_{p+1}} P[C_{p+1}] \tag{7.41}$$

Here, Σ_{p+1} is the brane worldvolume, $P[C_{p+1}]$ is the pullback of the bulk $(p+1)$-form potential onto the brane and μ_p is the constant brane charge. The pullback

$$P[C_{p+1}] = \frac{1}{(p+1)!} C_{M_0 \ldots M_p}(X(\xi)) \frac{\partial X^{M_0}}{\partial \xi^{\alpha_0}} \cdots \frac{\partial X^{M_p}}{\partial \xi^{\alpha_p}} d\xi^{\alpha_0} \wedge \cdots \wedge d\xi^{\alpha_p} \tag{7.42}$$

In the static gauge, the pullback becomes

$$P[C_{p+1}] = \frac{1}{(p+1)!} C_{\mu_0 \ldots \mu_p}(x, y_0) d\xi^{\mu_0} \wedge \cdots \wedge d\xi^{\mu_p} \tag{7.43}$$

Using Poincaré-dual delta-form current of the brane worldvolume, we can write the WZ term as an integral over the full D-dimensional spacetime. We write

$$S_{WZ} = \mu_p \int_{\mathcal{M}_D} C_{p+1} \wedge j_q \tag{7.44}$$

Here, the Poincaré-dual current for our static brane is

$$
\begin{aligned}
j_q &= \delta^{(q)}_{\text{coord}}(y - y_0) dy^{p+1} \wedge \cdots \wedge dy^{D-1} \\
&= \frac{1}{\sqrt{\tilde{g}}} \delta^{(q)}_{\text{coord}}(y - y_0) \tilde{\epsilon}_q \\
&= \delta^{(q)}(y - y_0) \tilde{\epsilon}_q.
\end{aligned} \tag{7.45}
$$

Thus, for a collection of static branes, the current becomes

$$
j_q = \sum_n \delta^{(q)}(y - y_n) \tilde{\epsilon}_q. \tag{7.46}
$$

7.3.4 Local action

The localized action is taken to be

$$
S_{\text{loc}} = S_{\text{DBI}} + S_{\text{WZ}}, \tag{7.47}
$$

where

$$
S_{\text{DBI}} = -T_p \int_{\mathcal{M}_D} d^D X \sqrt{-g_D}\, e^{\gamma \phi(X)} \rho(X), \tag{7.48}
$$

and

$$
S_{\text{WZ}} = \mu_p \int_{\mathcal{M}_D} C_{p+1} \wedge j_q. \tag{7.49}
$$

Here, T_p, γ, μ_p are the initial unknowns.

7.3.5 Negative source contribution

Similar to the GKP compactification (6.12), we need to introduce negative source contributions to satisfy the Poisson-type equation. The localized positive and negative source contributions must cancel in the integrated equation. We can write

$$
\widetilde{\nabla}^2 H_0 = \sum_i Q_i^{(+)} \delta^{(q)}(y - y_i) + \sum_j Q_j^{(-)} \delta^{(q)}(y - y_j), \tag{7.50}
$$

then compactness requires

$$
\sum_i Q_i^{(+)} + \sum_j Q_j^{(-)} = 0. \tag{7.51}
$$

This is the p-brane analogue of tadpole cancellation in GKP compactifications. In the full string-theoretic setting, the negative terms are supplied by orientifold-like sources.

7.4 Equations of motion

7.4.1 Gauge-field equation of motion

We now vary the total action with respect to the gauge potential C_{p+1}. Since S_{DBI} does not depend on C_{p+1}, it does not contribute to the gauge-field variation.

The gauge-field dependent part of the action is therefore

$$S[C_{p+1}] = -\frac{1}{2\kappa_D^2}\int d^DX\,\sqrt{-g_D}\,\frac{e^{-\lambda\phi}}{2(p+2)!}\,F_{p+2}^2+\mu_p\int_{\mathcal{M}_D} C_{p+1}\wedge j_q, \qquad F_{p+2}=dC_{p+1}. \tag{7.52}$$

Using the convention

$$\int A_r\wedge\star_D B_r = (-1)^{r(D-r)}\frac{1}{r!}\int d^DX\sqrt{-g_D}\,B_{M_1\cdots M_r}A^{M_1\cdots M_r}, \tag{7.53}$$

with $r = p+2$, the component kinetic term can be written in differential-form notation as

$$-\frac{1}{2\kappa_D^2}\int d^DX\,\sqrt{-g_D}\,\frac{e^{-\lambda\phi}}{2(p+2)!}\,F_{p+2}^2 = -\frac{(-1)^{(p+2)(D-p-2)}}{4\kappa_D^2}\int e^{-\lambda\phi}F_{p+2}\wedge\star_D F_{p+2}. \tag{7.54}$$

Equivalently,

$$S[C_{p+1}] = -\frac{(-1)^{(p+2)(D-p-2)}}{4\kappa_D^2}\int e^{-\lambda\phi}F_{p+2}\wedge\star_D F_{p+2}+\mu_p\int C_{p+1}\wedge j_q. \tag{7.55}$$

We now vary C_{p+1}, keeping g_{MN}, ϕ, and j_q fixed. Since

$$\delta F_{p+2} = d(\delta C_{p+1}),$$

we get

$$\delta_C S = -\frac{(-1)^{(p+2)(D-p-2)}}{4\kappa_D^2}\int e^{-\lambda\phi}\delta(F_{p+2}\wedge\star_D F_{p+2})+\mu_p\int \delta C_{p+1}\wedge j_q. \tag{7.56}$$

Using the symmetry identity

$$A_r\wedge\star_D B_r = B_r\wedge\star_D A_r,$$

we have

$$\begin{aligned}\delta(F_{p+2}\wedge\star_D F_{p+2}) &= \delta F_{p+2}\wedge\star_D F_{p+2}+F_{p+2}\wedge\star_D\delta F_{p+2}\\ &= \delta F_{p+2}\wedge\star_D F_{p+2}+\delta F_{p+2}\wedge\star_D F_{p+2}\\ &= 2\,\delta F_{p+2}\wedge\star_D F_{p+2}.\end{aligned} \tag{7.57}$$

Therefore,

$$\begin{aligned}\delta_C S &= -\frac{(-1)^{(p+2)(D-p-2)}}{2\kappa_D^2}\int e^{-\lambda\phi}\delta F_{p+2}\wedge\star_D F_{p+2}+\mu_p\int\delta C_{p+1}\wedge j_q\\ &= -\frac{(-1)^{(p+2)(D-p-2)}}{2\kappa_D^2}\int e^{-\lambda\phi}d(\delta C_{p+1})\wedge\star_D F_{p+2}+\mu_p\int\delta C_{p+1}\wedge j_q\\ &= -\frac{(-1)^{(p+2)(D-p-2)}}{2\kappa_D^2}\int d(\delta C_{p+1})\wedge\big(e^{-\lambda\phi}\star_D F_{p+2}\big)+\mu_p\int\delta C_{p+1}\wedge j_q.\end{aligned} \tag{7.58}$$

Now we use the Leibniz rule. Since δC_{p+1} is a $(p+1)$-form,

$$\begin{aligned} d\left[\delta C_{p+1} \wedge e^{-\lambda\phi} \star_D F_{p+2}\right] = d(\delta C_{p+1}) \wedge e^{-\lambda\phi} \star_D F_{p+2} \\ + (-1)^{p+1} \delta C_{p+1} \wedge d\left(e^{-\lambda\phi} \star_D F_{p+2}\right). \end{aligned} \tag{7.59}$$

Hence

$$\begin{aligned} d(\delta C_{p+1}) \wedge e^{-\lambda\phi} \star_D F_{p+2} = d\left[\delta C_{p+1} \wedge e^{-\lambda\phi} \star_D F_{p+2}\right] \\ - (-1)^{p+1} \delta C_{p+1} \wedge d\left(e^{-\lambda\phi} \star_D F_{p+2}\right). \end{aligned} \tag{7.60}$$

Substituting (7.60) into (7.58), we obtain

$$\begin{aligned} \delta_C S = -\frac{(-1)^{(p+2)(D-p-2)}}{2\kappa_D^2} \int d\left[\delta C_{p+1} \wedge e^{-\lambda\phi} \star_D F_{p+2}\right] \\ + \frac{(-1)^{(p+2)(D-p-2)+p+1}}{2\kappa_D^2} \int \delta C_{p+1} \wedge d\left(e^{-\lambda\phi} \star_D F_{p+2}\right) \\ + \mu_p \int \delta C_{p+1} \wedge j_q. \end{aligned} \tag{7.61}$$

Dropping the total derivative term, this becomes

$$\delta_C S = \int \delta C_{p+1} \wedge \left[\frac{(-1)^{(p+2)(D-p-2)+p+1}}{2\kappa_D^2} d\left(e^{-\lambda\phi} \star_D F_{p+2}\right) + \mu_p j_q\right]. \tag{7.62}$$

Since δC_{p+1} is arbitrary, the gauge-field equation of motion is

$$\frac{(-1)^{(p+2)(D-p-2)+p+1}}{2\kappa_D^2} d\left(e^{-\lambda\phi} \star_D F_{p+2}\right) + \mu_p j_q = 0. \tag{7.63}$$

Equivalently,

$$d\left(e^{-\lambda\phi} \star_D F_{p+2}\right) = (-1)^{(p+2)(D-p-2)+p}\, 2\kappa_D^2 \mu_p\, j_q. \tag{7.64}$$

Together with $F_{p+2} = dC_{p+1}$, we also have the Bianchi identity

$$dF_{p+2} = 0. \tag{7.65}$$

Gauge invariance of the WZ coupling

We now check gauge invariance under

$$C_{p+1} \longrightarrow C_{p+1} + d\Lambda_p. \tag{7.66}$$

The bulk kinetic term is gauge invariant because it depends only on $F_{p+2} = dC_{p+1}$. The DBI term is also gauge invariant because it does not contain C_{p+1}. The WZ term changes as

$$\delta_\Lambda S_{\text{WZ}} = \mu_p \int d\Lambda_p \wedge j_q. \tag{7.67}$$

Using the Leibniz rule,

$$d(\Lambda_p \wedge j_q) = d\Lambda_p \wedge j_q + (-1)^p \Lambda_p \wedge dj_q, \tag{7.68}$$

so that

$$d\Lambda_p \wedge j_q = d(\Lambda_p \wedge j_q) - (-1)^p \Lambda_p \wedge dj_q. \tag{7.69}$$

Therefore

$$\delta_\Lambda S_{\mathrm{WZ}} = \mu_p \int d(\Lambda_p \wedge j_q) - \mu_p (-1)^p \int \Lambda_p \wedge dj_q. \tag{7.70}$$

Dropping the total derivative term, we get

$$\delta_\Lambda S_{\mathrm{WZ}} = (-1)^{p+1} \mu_p \int \Lambda_p \wedge dj_q. \tag{7.71}$$

Thus gauge invariance requires

$$dj_q = 0. \tag{7.72}$$

7.4.2 Dilaton equation of motion

The ϕ-dependent part of the action is

$$S[\phi] = \frac{1}{2\kappa_D^2} \int d^D X \sqrt{-g_D} \left[-\frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi - \frac{e^{-\lambda\phi}}{2(p+2)!} F_{p+2}^2 \right] - T_p \int d^D X \sqrt{-g_D}\, e^{\gamma\phi} \rho(X). \tag{7.73}$$

Varying with respect to the dilaton gives

$$\begin{aligned}
\delta_\phi S &= \delta_\phi \left[\frac{1}{2\kappa_D^2} \int d^D X \sqrt{-g_D} \left(-\frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi - \frac{e^{-\lambda\phi}}{2(p+2)!} F_{p+2}^2 \right) - T_p \int d^D X \sqrt{-g_D}\, e^{\gamma\phi} \rho(X) \right] \\
&= \frac{1}{2\kappa_D^2} \int d^D X \sqrt{-g_D} \Bigg[-\frac{1}{2} g^{MN} \delta_\phi (\partial_M \phi \partial_N \phi) - \frac{1}{2(p+2)!} \delta_\phi \left(e^{-\lambda\phi}\right) F_{p+2}^2 \Bigg] \\
&\qquad - T_p \int d^D X \sqrt{-g_D}\, \delta_\phi \left(e^{\gamma\phi}\right) \rho(X).
\end{aligned} \tag{7.74}$$

Using

$$\delta_\phi (\partial_M \phi \partial_N \phi) = \partial_M(\delta\phi) \partial_N \phi + \partial_M \phi \partial_N (\delta\phi), \tag{7.75}$$

we get

$$\begin{aligned}
-\frac{1}{2} g^{MN} \delta_\phi (\partial_M \phi \partial_N \phi) &= -\frac{1}{2} g^{MN} \left[\partial_M(\delta\phi) \partial_N \phi + \partial_M \phi \partial_N (\delta\phi) \right] \\
&= -g^{MN} \partial_M \phi \partial_N (\delta\phi) \\
&= -\partial^M \phi \, \partial_M (\delta\phi).
\end{aligned} \tag{7.76}$$

And

$$\delta_\phi \left(e^{-\lambda\phi}\right) = -\lambda e^{-\lambda\phi} \delta\phi, \qquad \delta_\phi \left(e^{\gamma\phi}\right) = \gamma e^{\gamma\phi} \delta\phi. \tag{7.77}$$

Substituting (7.76) and (7.77) into (7.74), we obtain

$$\begin{aligned}\delta_\phi S &= \frac{1}{2\kappa_D^2}\int d^DX\,\sqrt{-g_D}\left[-\partial^M\phi\,\partial_M(\delta\phi) + \frac{\lambda}{2(p+2)!}e^{-\lambda\phi}F_{p+2}^2\,\delta\phi\right] \\ &\qquad - \gamma T_p\int d^DX\sqrt{-g_D}\,e^{\gamma\phi}\rho(X)\,\delta\phi \\ &= \frac{1}{2\kappa_D^2}\int d^DX\,\sqrt{-g_D}\,\delta\phi\,\nabla^2\phi - \frac{1}{2\kappa_D^2}\int d^DX\,\partial_M\left[\sqrt{-g_D}\,\delta\phi\,\partial^M\phi\right] \\ &\qquad + \frac{1}{2\kappa_D^2}\int d^DX\,\sqrt{-g_D}\left[\frac{\lambda}{2(p+2)!}e^{-\lambda\phi}F_{p+2}^2\,\delta\phi\right] - \gamma T_p\int d^DX\sqrt{-g_D}\,e^{\gamma\phi}\rho(X)\,\delta\phi.\end{aligned} \tag{7.78}$$

Dropping the boundary term, (7.78) becomes

$$\begin{aligned}\delta_\phi S &= \frac{1}{2\kappa_D^2}\int d^DX\,\sqrt{-g_D}\,\delta\phi\left[\nabla^2\phi + \frac{\lambda}{2(p+2)!}e^{-\lambda\phi}F_{p+2}^2\right] - \gamma T_p\int d^DX\sqrt{-g_D}\,e^{\gamma\phi}\rho(X)\,\delta\phi \\ &= \int d^DX\,\sqrt{-g_D}\,\delta\phi\left[\frac{1}{2\kappa_D^2}\left(\nabla^2\phi + \frac{\lambda}{2(p+2)!}e^{-\lambda\phi}F_{p+2}^2\right) - \gamma T_p e^{\gamma\phi}\rho(X)\right].\end{aligned} \tag{7.79}$$

Since $\delta\phi$ is arbitrary, the dilaton equation of motion is

$$\frac{1}{2\kappa_D^2}\left(\nabla^2\phi + \frac{\lambda}{2(p+2)!}e^{-\lambda\phi}F_{p+2}^2\right) - \gamma T_p e^{\gamma\phi}\rho(X) = 0. \tag{7.80}$$

Equivalently,

$$\nabla^2\phi + \frac{\lambda}{2(p+2)!}e^{-\lambda\phi}F_{p+2}^2 = 2\kappa_D^2\gamma T_p e^{\gamma\phi}\rho(X). \tag{7.81}$$

In differential-form notation

$$(-1)^{D-1}d\star_D d\phi + \frac{\lambda}{2}(-1)^{(p+2)(D-p-2)}e^{-\lambda\phi}F_{p+2}\wedge\star_D F_{p+2} = 2\kappa_D^2\gamma T_p e^{\gamma\phi}\rho(X)\;\star_D 1. \tag{7.82}$$

7.4.3 Einstein equation of motion

We now vary the total action with respect to the inverse metric g^{MN}. The metric dependent part of the action is

$$\begin{aligned}S[g] = \frac{1}{2\kappa_D^2}\int d^DX\,\sqrt{-g_D}\left[R_D - \frac{1}{2}g^{AB}\partial_A\phi\partial_B\phi - \frac{e^{-\lambda\phi}}{2(p+2)!}F_{p+2}^2\right] \\ - T_p\int_{\Sigma_{p+1}} d^{p+1}\xi\,e^{\gamma\phi(X)}\sqrt{-P}\end{aligned} \tag{7.83}$$

We write the DBI action in its worldvolume form to vary the metric dependence in $P = \det P[g]_{\alpha\beta}$. The WZ term has no metric dependence as j_q is a fixed Poincaré-dual current.

Now varying with respect to the metric gives

$$\delta_g S = \delta_g \Bigg\{ \frac{1}{2\kappa_D^2} \int d^D X \sqrt{-g_D} \left[R_D - \frac{1}{2} g^{AB} \partial_A \phi \partial_B \phi - \frac{e^{-\lambda\phi}}{2(p+2)!} F_{p+2}^2 \right] \\ - T_p \int_{\Sigma_{p+1}} d^{p+1}\xi \, e^{\gamma\phi(X)} \sqrt{-P} \Bigg\}. \tag{7.84}$$

Using

$$\delta\sqrt{-g_D} = -\frac{1}{2}\sqrt{-g_D}\, g_{MN} \delta g^{MN}, \tag{7.85}$$

$$\delta\left(\sqrt{-g_D} R_D\right) = \sqrt{-g_D} \left(R_{MN} - \frac{1}{2} g_{MN} R_D \right) \delta g^{MN} + \text{boundary term}, \tag{7.86}$$

$$\delta\left(g^{AB} \partial_A \phi \partial_B \phi\right) = \partial_M \phi \partial_N \phi \, \delta g^{MN}, \tag{7.87}$$

and

$$\delta\left(F_{p+2}^2\right) = (p+2) F_{M A_1 \cdots A_{p+1}} F_N{}^{A_1 \cdots A_{p+1}} \delta g^{MN}, \tag{7.88}$$

we obtain, after dropping the boundary term,

$$\delta_g S = \frac{1}{2\kappa_D^2} \int d^D X \sqrt{-g_D} \Bigg[R_{MN} - \frac{1}{2} g_{MN} R_D - \frac{1}{2} \partial_M \phi \partial_N \phi + \frac{1}{4} g_{MN} (\partial\phi)^2 \\ - \frac{e^{-\lambda\phi}}{2(p+1)!} F_{M A_1 \cdots A_{p+1}} F_N{}^{A_1 \cdots A_{p+1}} + \frac{e^{-\lambda\phi}}{4(p+2)!} g_{MN} F_{p+2}^2 \Bigg] \delta g^{MN} \\ - T_p \int_{\Sigma_{p+1}} d^{p+1}\xi \, e^{\gamma\phi(X)} \delta\sqrt{-P}. \tag{7.89}$$

For the DBI variation, we use

$$\delta\sqrt{-P} = \frac{1}{2}\sqrt{-P}\, (P[g])^{\alpha\beta} \delta P[g]_{\alpha\beta}, \tag{7.90}$$

and

$$\begin{aligned} \delta P[g]_{\alpha\beta} &= \partial_\alpha X^M \partial_\beta X^N \delta g_{MN}(X(\xi)) \\ &= -\partial_\alpha X^M \partial_\beta X^N g_{MP} g_{NQ} \, \delta g^{PQ}(X(\xi)). \end{aligned} \tag{7.91}$$

Therefore

$$- T_p \int_{\Sigma_{p+1}} d^{p+1}\xi \, e^{\gamma\phi(X)} \delta\sqrt{-P} \\ = \frac{T_p}{2} \int_{\Sigma_{p+1}} d^{p+1}\xi \, e^{\gamma\phi(X)} \sqrt{-P} (P[g])^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N g_{MP} g_{NQ} \delta g^{PQ}(X(\xi)). \tag{7.92}$$

Now we want to convert into a D-dimensional integral. Using

$$\delta g^{PQ}(X(\xi)) = \int d^D X \sqrt{-g_D} \delta^{(D)}(X - X(\xi)) \, \delta g^{PQ}(X) \tag{7.93}$$

and

$$\rho(X) = \int_{\Sigma_{p+1}} d^{p+1}\xi \sqrt{-P}\,\delta^{(D)}(X - X(\xi))\,. \tag{7.94}$$

The DBI contribution becomes

$$\begin{aligned}
&-T_p \int_{\Sigma_{p+1}} d^{p+1}\xi\, e^{\gamma\phi(X)} \delta\sqrt{-P} \\
&= \frac{T_p}{2} \int d^D X \sqrt{-g_D}\, e^{\gamma\phi(X)} \\
&\quad\times \left[\int_{\Sigma_{p+1}} d^{p+1}\xi \sqrt{-P}\,(P[g])^{\alpha\beta} \partial_\alpha X^A \partial_\beta X^B g_{AM} g_{BN} \delta^{(D)}(X - X(\xi)) \right] \delta g^{MN}(X).
\end{aligned} \tag{7.95}$$

Substituting (7.95) into (7.89), we get

$$\begin{aligned}
\delta_g S = \int d^D X \sqrt{-g_D}\, \delta g^{MN} \Bigg\{ &\frac{1}{2\kappa_D^2} \Bigg[R_{MN} - \frac{1}{2} g_{MN} R_D - \frac{1}{2} \partial_M \phi \partial_N \phi + \frac{1}{4} g_{MN} (\partial\phi)^2 \\
&- \frac{e^{-\lambda\phi}}{2(p+1)!} F_{MA_1\cdots A_{p+1}} F_N{}^{A_1\cdots A_{p+1}} + \frac{e^{-\lambda\phi}}{4(p+2)!} g_{MN} F_{p+2}^2 \Bigg] \\
&+ \frac{T_p}{2} e^{\gamma\phi(X)} \int_{\Sigma_{p+1}} d^{p+1}\xi \sqrt{-P}(P[g])^{\alpha\beta} \partial_\alpha X^A \partial_\beta X^B g_{AM} g_{BN} \delta^{(D)}(X - X(\xi)) \Bigg\}.
\end{aligned} \tag{7.96}$$

Since δg^{MN} is arbitrary, the Einstein equation is

$$\begin{aligned}
0 = \frac{1}{2\kappa_D^2} &\Bigg[R_{MN} - \frac{1}{2} g_{MN} R_D - \frac{1}{2} \partial_M \phi \partial_N \phi + \frac{1}{4} g_{MN} (\partial\phi)^2 \\
&- \frac{e^{-\lambda\phi}}{2(p+1)!} F_{MA_1\cdots A_{p+1}} F_N{}^{A_1\cdots A_{p+1}} + \frac{e^{-\lambda\phi}}{4(p+2)!} g_{MN} F_{p+2}^2 \Bigg] \\
&+ \frac{T_p}{2} e^{\gamma\phi(X)} \int_{\Sigma_{p+1}} d^{p+1}\xi \sqrt{-P}(P[g])^{\alpha\beta} \partial_\alpha X^A \partial_\beta X^B g_{AM} g_{BN} \delta^{(D)}(X - X(\xi))\,.
\end{aligned} \tag{7.97}$$

Multiplying by $2\kappa_D^2$, we obtain

$$\begin{aligned}
R_{MN} - \frac{1}{2} g_{MN} R_D = \frac{1}{2} &\partial_M \phi \partial_N \phi - \frac{1}{4} g_{MN} (\partial\phi)^2 \\
&+ \frac{e^{-\lambda\phi}}{2(p+1)!} F_{MA_1\cdots A_{p+1}} F_N{}^{A_1\cdots A_{p+1}} - \frac{e^{-\lambda\phi}}{4(p+2)!} g_{MN} F_{p+2}^2 \\
&- \kappa_D^2 T_p e^{\gamma\phi(X)} \int_{\Sigma_{p+1}} d^{p+1}\xi \sqrt{-P}(P[g])^{\alpha\beta} \partial_\alpha X^A \partial_\beta X^B g_{AM} g_{BN} \delta^{(D)}(X - X(\xi))\,.
\end{aligned} \tag{7.98}$$

Trace-reversed form. The DBI source trace is obtained by contracting the last term in (7.98) with g^{MN}. Since

$$\begin{aligned} g^{MN}g_{AM}g_{BN}(P[g])^{\alpha\beta}\partial_\alpha X^A\partial_\beta X^B &= (P[g])^{\alpha\beta}g_{AB}\partial_\alpha X^A\partial_\beta X^B \\ &= (P[g])^{\alpha\beta}P[g]_{\alpha\beta} \\ &= p+1, \end{aligned} \tag{7.99}$$

The trace-reversed equation becomes

$$\begin{aligned} R_{MN} = &\frac{1}{2}\partial_M\phi\partial_N\phi \\ &+ \frac{e^{-\lambda\phi}}{2(p+1)!}\left[F_{MA_1\cdots A_{p+1}}F_N{}^{A_1\cdots A_{p+1}} - \frac{p+1}{(p+2)(D-2)}g_{MN}F^2_{p+2}\right] \\ &- \kappa_D^2 T_p e^{\gamma\phi(X)}\left[\int_{\Sigma_{p+1}} d^{p+1}\xi\,\sqrt{-P}\,(P[g])^{\alpha\beta}\partial_\alpha X^A\partial_\beta X^B g_{AM}g_{BN}\delta^{(D)}(X-X(\xi))\right] \\ &+ \kappa_D^2\frac{p+1}{D-2}T_p e^{\gamma\phi(X)}\rho(X)g_{MN}. \end{aligned} \tag{7.100}$$

Localized source term in static-gauge. After imposing a static gauge and metric background, the pullback metric is calculated in (7.31). Taking its inverse

$$P[g]^{\alpha\beta} = H_0^{-2a}\hat{\eta}^{\alpha\beta} \tag{7.101}$$

For external components, we have

$$(P[g])^{\alpha\beta}\partial_\alpha X^A\partial_\beta X^B g_{A\mu}g_{B\nu} = (P[g])^{\alpha\beta}H_0^{4a}(y_0)\delta^\rho_\alpha\delta^\sigma_\beta\hat{\eta}_{\rho\mu}\hat{\eta}_{\sigma\nu} = H_0^{2a}\hat{\eta}_{\mu\nu}. \tag{7.102}$$

Thus, we can write

$$\begin{aligned} &\int_{\Sigma_{p+1}} d^{p+1}\xi\,\sqrt{-P}\,(P[g])^{\alpha\beta}\partial_\alpha X^A\partial_\beta X^B g_{A\mu}g_{B\nu}\delta^{(D)}(X-X(\xi)) \\ &\qquad = H_0^{2a}\hat{\eta}_{\mu\nu}\int_{\Sigma_{p+1}} d^{p+1}\xi\sqrt{-P}\delta^{(D)}(X-X(\xi)) \\ &\qquad = H_0^{2a}\hat{\eta}_{\mu\nu}\rho(X) \end{aligned} \tag{7.103}$$

Therefore the direct DBI source in the $(\mu\nu)$ equation of (7.98) is proportional to

$$-T_p e^{\gamma\phi}\rho(X)H_0^{2a}\hat{\eta}_{\mu\nu}. \tag{7.104}$$

For internal components,

$$(P[g])^{\alpha\beta}\partial_\alpha X^A\partial_\beta X^B g_{Am}g_{Bn} = 0, \tag{7.105}$$

because $\partial_\alpha X^m = 0$. Hence the direct DBI source in the (mn) equation vanishes.

Now, in the trace-reversed equation, the localized contribution to the external Einstein equation becomes

$$\begin{aligned} R^{\text{loc}}_{\mu\nu} &= -\kappa_D^2 T_p e^{\gamma\phi}\rho(X)H_0^{2a}\hat{\eta}_{\mu\nu} + \kappa_D^2\frac{p+1}{D-2}T_p e^{\gamma\phi}\rho(X)H_0^{2a}\hat{\eta}_{\mu\nu} \\ &= -\kappa_D^2\frac{D-p-3}{D-2}T_p e^{\gamma\phi}\rho(X)H_0^{2a}\hat{\eta}_{\mu\nu}. \end{aligned} \tag{7.106}$$

The localized contribution to the internal Einstein equation is

$$\begin{aligned} R^{\text{loc}}_{mn} &= 0 + \kappa_D^2 \frac{p+1}{D-2} T_p e^{\gamma\phi} \rho(X) H_0^{2b} \tilde{g}_{mn} \\ &= +\kappa_D^2 \frac{p+1}{D-2} T_p e^{\gamma\phi} \rho(X) H_0^{2b} \tilde{g}_{mn}. \end{aligned} \tag{7.107}$$

In static gauge, the DBI source contributes only along the worldvolume directions. The internal contribution appears after the trace-reversal. Hence, the DBI action gives different coefficients in the external and internal Einstein equations.

7.5 Fixing the localized source

We now determine the unknown source parameters μ_p, T_p, and γ by inserting the compact p-brane background into the gauge-field, dilaton, and Einstein equations. The gauge equation fixes the RR charge coefficient, the dilaton equation fixes the dilaton coupling and tension coefficient, and the Einstein equation checks that the DBI stress tensor has the correct external and internal components.

7.5.1 From gauge field EOM

The gauge-field equation determines the charge coefficient μ_p. Since the DBI term does not depend on C_{p+1}, only the bulk kinetic term and the Wess–Zumino term contribute. The gauge-field equation is

$$d\left(e^{-\lambda\phi} \star_D F_{p+2}\right) = (-1)^{(p+2)(D-p-2)+p}\, 2\kappa_D^2 \mu_p\, j_q. \tag{7.108}$$

We write the field strength

$$F_{p+2} = s\, \widetilde{d}(H_0^{-1}) \wedge \widehat{\epsilon}_{p+1} = (-1)^{p+1} s\, \widehat{\epsilon}_{p+1} \wedge \widetilde{d}(H_0^{-1}). \tag{7.109}$$

For our metric, the relevant Hodge-star identity (see (7.16) for proof) is

$$\star_D \left(\widehat{\epsilon}_{p+1} \wedge \widetilde{d}(H_0^{-1})\right) = H_0^{-a(p+1)+b(q-2)} \widetilde{\star}\, \widetilde{d}(H_0^{-1}). \tag{7.110}$$

Using (7.109), we therefore get

$$\begin{aligned} e^{-\lambda\phi} \star_D F_{p+2} &= (-1)^{p+1} s\, e^{-\lambda\phi} H_0^{-a(p+1)+b(q-2)} \widetilde{\star}\, \widetilde{d}(H_0^{-1}) \\ &= (-1)^p s\, H_0^{\lambda^2/2-a(p+1)+b(q-2)-2} \widetilde{\star}\, \widetilde{d} H_0 \\ &= (-1)^p s\, \widetilde{\star}\, \widetilde{d} H_0. \end{aligned} \tag{7.111}$$

We used the relation among the exponents

$$\frac{\lambda^2}{2} - a(p+1) + b(q-2) - 2 = 0. \tag{7.112}$$

Hence

$$
\begin{aligned}
d\left(e^{-\lambda\phi} \star_D F_{p+2}\right) &= (-1)^p s\, \widetilde{d}\left(\widetilde{\star}\, \widetilde{d} H_0\right) \\
&= (-1)^p s\, (\widetilde{\nabla}^2 H_0)\widetilde{\epsilon}_q \\
&= (-1)^p s \sum_n Q_n \delta^{(q)}(y - y_n)\widetilde{\epsilon}_q. \qquad (7.113)
\end{aligned}
$$

Inserting j_q on the right-hand side of the gauge equation, we get

$$
(-1)^{(p+2)(D-p-2)+p} 2\kappa_D^2 \mu_p j_q = (-1)^{(p+2)(D-p-2)+p} 2\kappa_D^2 \mu_p \sum_n \delta^{(q)}(y - y_n)\widetilde{\epsilon}_q. \qquad (7.114)
$$

Therefore, matching the coefficients of the localized sources gives

$$
(-1)^p s\, Q_n = (-1)^{(p+2)(D-p-2)+p} 2\kappa_D^2 \mu_p. \qquad (7.115)
$$

Equivalently,

$$
2\kappa_D^2 \mu_p = (-1)^{(p+2)(D-p-2)} s\, Q_n. \qquad (7.116)
$$

7.5.2 From dilaton EOM

The dilaton equation determines the dilaton coupling γ and relates the brane tension to the harmonic source coefficient. The dilaton equation is

$$
\nabla^2 \phi + \frac{\lambda}{2(p+2)!} e^{-\lambda\phi} F_{p+2}^2 = 2\kappa_D^2 \gamma T_p e^{\gamma\phi} \rho(y). \qquad (7.117)
$$

We now evaluate each term on the background. We write the dilaton term, which only depends on the transverse coordinates

$$
\begin{aligned}
\phi &= -\frac{\lambda}{2} \ln H_0 \qquad (7.118) \\
\implies \partial_m \phi &= -\frac{\lambda}{2} H_0^{-1} \partial_m H_0. \qquad (7.119)
\end{aligned}
$$

Thus, the scalar Laplacian on the dilaton becomes

$$
\begin{aligned}
\nabla^2 \phi &= \frac{1}{\sqrt{-g_D}} \partial_m \left(\sqrt{-g_D}\, g^{mn} \partial_n \phi\right) \\
&= \frac{1}{H_0^{a(p+1)+bq} \sqrt{-\widehat{\eta}} \sqrt{\widetilde{g}}} \partial_m \left(H_0^{a(p+1)+bq} \sqrt{-\widehat{\eta}} \sqrt{\widetilde{g}}\, H_0^{-2b} \widetilde{g}^{mn} \partial_n \phi\right) \\
&= H_0^{-a(p+1)-bq} \frac{1}{\sqrt{\widetilde{g}}} \partial_m \left(H_0^{a(p+1)+bq-2b} \sqrt{\widetilde{g}}\, \widetilde{g}^{mn} \partial_n \phi\right) \\
&= -\frac{\lambda}{2} H_0^{-a(p+1)-bq} \frac{1}{\sqrt{\widetilde{g}}} \partial_m \left(H_0^{a(p+1)+bq-2b-1} \sqrt{\widetilde{g}}\, \widetilde{g}^{mn} \partial_n H_0\right) \\
&= -\frac{\lambda}{2} H_0^{-2b} \frac{1}{\sqrt{\widetilde{g}}} \partial_m \left(H_0^{-1} \sqrt{\widetilde{g}}\, \widetilde{g}^{mn} \partial_n H_0\right) \\
&= -\frac{\lambda}{2} \left[H_0^{-1-2b} \widetilde{\nabla}^2 H_0 - H_0^{-2-2b} (\widetilde{\nabla} H_0)^2\right]. \qquad (7.120)
\end{aligned}
$$

Here, we used the relation (7.7) in the exponents. Thus

$$\nabla^2\phi = -\frac{\lambda}{2}H_0^{-1-2b}\widetilde{\nabla}^2 H_0 + \frac{\lambda}{2}H_0^{-2-2b}(\widetilde{\nabla}H_0)^2. \tag{7.121}$$

Now we compute the F_{p+2}^2 term. Our field strength is

$$\begin{aligned} F_{p+2} &= s\,\widetilde{d}(H_0^{-1}) \wedge \widehat{\epsilon}_{p+1} \\ \implies F_{m\mu_1\ldots\mu_{p+1}} &= s\partial_m(H_0^{-1})\hat{\epsilon}_{\mu_1\ldots\mu_{p+1}} \end{aligned} \tag{7.122}$$

Then

$$\begin{aligned} F_{p+2}^2 &= F_{M_1\ldots M_{p+2}}F^{M_1\ldots M_{p+2}} \\ &= (p+2)H_0^{-2b}\tilde{g}^{mn}H_0^{-2a(p+1)}\hat{\eta}^{\mu_1\nu_1}\ldots\hat{\eta}^{\mu_{p+1}\nu_{p+1}}F_{m\mu_1\ldots\mu_{p+1}}F_{n\nu_1\ldots\nu_{p+1}} \\ &= (p+2)H_0^{-2b-2a(p+1)}\partial_m(H_0^{-1})\partial^m(H_0^{-1})\hat{\epsilon}_{\mu_1\ldots\mu_{p+1}}\hat{\epsilon}^{\mu_1\ldots\mu_{p+1}} \\ &= -(p+2)H_0^{-2b-2a(p+1)}H_0^{-4}(\tilde{\nabla}H_0)^2(p+1)! \\ &= -(p+2)!H_0^{-2b-2a(p+1)-4}(\tilde{\nabla}H_0)^2 \end{aligned} \tag{7.123}$$

Since we are taking a mostly plus signature, the volume form for our Lorentzian metric contracts

$$\frac{1}{(p+1)!}\hat{\epsilon}_{\mu_1\ldots\mu_{p+1}}\hat{\epsilon}^{\mu_1\ldots\mu_{p+1}} = -1 \tag{7.124}$$

Putting (7.8), we obtain

$$\begin{aligned} \frac{1}{(p+2)!}F_{p+2}^2 &= -H_0^{-2-2b-\lambda^2/2}(\widetilde{\nabla}H_0)^2 \\ \implies \frac{\lambda}{2(p+2)!}e^{-\lambda\phi}F_{p+2}^2 &= -\frac{\lambda}{2}H_0^{-2-2b}(\widetilde{\nabla}H_0)^2. \end{aligned} \tag{7.125}$$

Combining (7.121) and (7.125), the gradient terms cancel

$$\begin{aligned} \nabla^2\phi + \frac{\lambda}{2(p+2)!}e^{-\lambda\phi}F_{p+2}^2 &= -\frac{\lambda}{2}H_0^{-1-2b}\widetilde{\nabla}^2 H_0 \\ &= -\frac{\lambda}{2}H_0^{-1-2b}\sum_n Q_n\delta^{(q)}(y-y_n). \end{aligned} \tag{7.126}$$

The right-hand side is

$$\begin{aligned} 2\kappa_D^2\gamma T_p e^{\gamma\phi}\rho(y) &= 2\kappa_D^2\gamma T_p H_0^{-\gamma\lambda/2}\left[H_0^{-bq}\sum_n \delta^{(q)}(y-y_n)\right] \\ &= 2\kappa_D^2\gamma T_p H_0^{-\gamma\lambda/2-bq}\sum_n \delta^{(q)}(y-y_n). \end{aligned} \tag{7.127}$$

Matching the powers of H_0 between (7.126) and (7.127), we require

$$-\frac{\gamma\lambda}{2} - bq = -1 - 2b. \tag{7.128}$$

Equivalently,

$$\frac{\gamma\lambda}{2} = 1 + 2b - bq = 1 + b(2-q). \tag{7.129}$$

Using

$$b = \frac{p+1}{2(D-2)}, \qquad q = D - p - 1,$$

we find

$$1 + b(2-q) = 1 - \frac{(p+1)(D-p-3)}{2(D-2)}. \tag{7.130}$$

We also have

$$\lambda^2 = 4 - \frac{2(p+1)(D-p-3)}{D-2}. \tag{7.131}$$

Therefore

$$1 - \frac{(p+1)(D-p-3)}{2(D-2)} = \frac{\lambda^2}{4}. \tag{7.132}$$

Thus

$$\frac{\gamma\lambda}{2} = \frac{\lambda^2}{4}, \tag{7.133}$$

and hence

$$\gamma = \frac{\lambda}{2}. \tag{7.134}$$

Then the dilaton equation becomes

$$\begin{aligned} -\frac{\lambda}{2} H_0^{-1-2b} \sum_n Q_n \delta^{(q)}(y-y_n) &= 2\kappa_D^2 \left(\frac{\lambda}{2}\right) T_p H_0^{-1-2b} \sum_n \delta^{(q)}(y-y_n) \\ -Q_n &= 2\kappa_D^2 T_p \end{aligned} \tag{7.135}$$

Equivalently,

$$2\kappa_D^2 T_p = -Q_n \tag{7.136}$$

7.5.3 From Einstein EOM

The Einstein equation checks whether the stress tensor produced by the DBI term has precisely the external and internal components required by the warped p-brane geometry. We found that the localized contribution to the external Einstein equation is

$$R^{\rm loc}_{\mu\nu} = -\kappa_D^2 \frac{D-p-3}{D-2} T_p e^{\gamma\phi(y)} \rho(y) H_0^{2a} \hat{\eta}_{\mu\nu}. \tag{7.137}$$

And the localized contribution to the internal Einstein equation is

$$R^{\rm loc}_{mn} = +\kappa_D^2 \frac{p+1}{D-2} T_p e^{\gamma\phi(y)} \rho(y) H_0^{2b} \tilde{g}_{mn}. \tag{7.138}$$

Using $\gamma = \lambda/2$, we have

$$\begin{aligned} e^{\frac{\lambda}{2}\phi}\rho(y) &= H_0^{-\lambda^2/4} H_0^{-bq} \sum_n \delta^{(q)}(y-y_n) \\ &= H_0^{-1-2b} \sum_n \delta^{(q)}(y-y_n) \end{aligned} \tag{7.139}$$

Here, from (7.8), we used

$$-\frac{\lambda^2}{4} - bq = -1 - 2b. \tag{7.140}$$

Inserting T_p that we found earlier

$$\begin{aligned}\kappa_D^2 T_p e^{\frac{\lambda}{2}\phi}\rho(y) &= -\frac{1}{2}H_0^{-1-2b}\sum_n Q_n\delta^{(q)}(y-y_n)\\ &= -\frac{1}{2}H_0^{-1-2b}\widetilde{\nabla}^2 H_0.\end{aligned} \tag{7.141}$$

Substituting (7.141) into (7.137), we get

$$R^{\text{loc}}_{\mu\nu} = +\frac{1}{2}\frac{D-p-3}{D-2}H_0^{-1-2b+2a}\widetilde{\nabla}^2 H_0\,\hat{\eta}_{\mu\nu}. \tag{7.142}$$

Since

$$a = -\frac{D-p-3}{2(D-2)},$$

this becomes

$$R^{\text{loc}}_{\mu\nu} = -a\,H_0^{2a-2b-1}\widetilde{\nabla}^2 H_0\,\hat{\eta}_{\mu\nu}. \tag{7.143}$$

Similarly,

$$R^{\text{loc}}_{mn} = -\frac{1}{2}\frac{p+1}{D-2}H_0^{-1}\widetilde{\nabla}^2 H_0\,\tilde{g}_{mn}. \tag{7.144}$$

Since

$$b = \frac{p+1}{2(D-2)},$$

we obtain

$$R^{\text{loc}}_{mn} = -b\,H_0^{-1}\widetilde{\nabla}^2 H_0\,\tilde{g}_{mn}. \tag{7.145}$$

Coefficient of R^{loc}_{mn} and $R^{\text{loc}}_{\mu\nu}$ terms both match the coefficient from the Ricci tensor computed directly from our metric in Appendix A.3. Thus the DBI source gives precisely the different external and internal coefficients required by the compact p-brane geometry.

7.6 Results

Finally, we have fixed all three initial unknowns

$$\gamma = \frac{\lambda}{2}, \tag{7.146}$$

$$T_p = -\frac{Q_n}{2\kappa_D^2}, \tag{7.147}$$

$$\mu_p = \pm(-1)^{(p+2)(D-p-2)}\,\frac{Q_n}{2\kappa_D^2}. \tag{7.148}$$

And for a collection of static p-branes, the DBI scalar density and Poincaré-dual current are

$$\rho(y) = H_0^{-bq} \sum_n \delta^{(D-p-1)}(y - y_n), \tag{7.149}$$

$$j_{D-p-1} = \sum_n \delta^{(D-p-1)}(y - y_n) \tilde{\epsilon}_{D-p-1}. \tag{7.150}$$

Therefore the localized action becomes

$$S_{\text{loc}} = -T_p \int_{\mathcal{M}_D} d^D X \sqrt{-g_D}\, e^{\frac{\lambda}{2}\phi(X)} \rho(X) + \mu_p \int_{\mathcal{M}_D} C_{p+1} \wedge j_q. \tag{7.151}$$

In worldvolume form,

$$S_{\text{loc}} = -T_p \int_{\Sigma_{p+1}} d^{p+1}\xi\, e^{\frac{\lambda}{2}\phi(X)} \sqrt{-\det P[g]_{\alpha\beta}} + \mu_p \int_{\Sigma_{p+1}} P[C_{p+1}]. \tag{7.152}$$

μ_p and T_p give the BPS-type relation between magnitudes

$$|\mu_p| = |T_p|. \tag{7.153}$$

The magnitude of the charge carried by the Wess–Zumino coupling equals the magnitude of the tension carried by the DBI coupling. For $Q_n > 0$, the coefficient $T_p = -Q_n/(2\kappa_D^2)$ is negative. Thus the source required by the compact solution is not an ordinary positive-tension D-brane. It is orientifold-like in the effective supergravity sense.

Chapter 8

Discussion and Outlook

The purpose of this thesis was to understand the role of localized sources in warped supergravity compactifications. We started from the hierarchy problem. The Randall–Sundrum model showed that a warp factor can redshift physical mass scales and thereby generate a large hierarchy geometrically. However, embedding this mechanism into supergravity or string theory is constrained by the Maldacena–Nuñez no-go theorem. Under its assumptions, the theorem excludes de Sitter compactifications and nontrivially warped Randall–Sundrum-type compactifications to Minkowski space. The GKP construction provides a string-theoretic framework for obtaining warped compactified solutions using three-form fluxes and localized sources. In particular, orientifold planes contribute negative Ramond–Ramond charge and negative tension and help satisfy the integrated tadpole condition.

In this thesis, we studied an analogous mechanism in a simplified Einstein–dilaton–form p-brane model. Starting with the bulk action, we introduced localized source terms through DBI and Wess–Zumino actions. Using the p-brane background, we fixed the localized source parameters as

$$\gamma = \frac{\lambda}{2}, \qquad T_p = -\frac{Q_n}{2\kappa_D^2}, \qquad \mu_p = \pm(-1)^{(p+2)(D-p-2)}\frac{Q_n}{2\kappa_D^2}. \tag{8.1}$$

Their magnitudes, therefore, satisfy

$$|\mu_p| = |T_p|. \tag{8.2}$$

For $Q_n > 0$, the tension is negative:

$$T_p < 0. \tag{8.3}$$

Thus the localized source required by the compactness of the internal manifold of the background is analogous to stringy orientifold planes at the level of the effective supergravity description.

Clarification of these parameters in (8.1) supports the solutions discussed in [20]. Such solutions earlier proved useful in clarifying gauge-invariant fluctuations of the compactification. The simplest such fluctuation is the breathing mode, which describes an overall expansion or contraction of the internal space. In an unwarped compactification, it is associated with the internal volume modulus. However, in a warped background, the fluctuation generally mixes with fluctuations of the warp factor (which is volume mode) and the dilaton. The resulting gauge-invariant scalar degree of freedom is termed the warped breathing mode.

Our work is similar in spirit to the Randall–Sundrum construction. In the 5D RS model, localized source terms support the warped compactification and are interpreted as 3-branes. However, those branes do not have a stringy interpretation, which we have in our work.

Lastly, there are many possible directions to extend the current work. We list a few relevant future directions below:

1. **Higher-derivative corrections.** The toy model studied in this thesis is based on two-derivative supergravity. A natural extension is to include higher-derivative corrections in the gravitational and form-field sectors, as well as in the brane action.

2. **Other types of fluctuations.** Only the warped breathing mode was discussed in [20]. However, a Ricci-flat internal manifold may also possess other deformations. For example, Kähler and complex-structure deformations which preserves Ricci flatness. Studying these modes around the background is still an open problem. This could give understanding of the 4D dynamics of such modes via dimensional reduction, and could capture essential features of more realistic EFTs obtained from string compactifications.

3. **Time- and internal-coordinate-dependent backgrounds.** We only considered the background with a warp factor $H(y)$ depending only on the internal coordinates. Recently, [30] studied time-dependent type IIB backgrounds with an overall time-dependent scale factor for the internal space, but without localized sources. We can extend this to consider backgrounds with dependence on both time and the internal coordinates, $H = H(t, y)$. Such a framework should admit early time-dependent phase and GKP-type solutions in late time warped compactification. This has relevance in cosmology.

4. **Dimensional reduction and four-dimensional cosmology.** The breathing mode and the other internal moduli appear as scalar fields after dimensional reduction [15], [28]. Deriving the four-dimensional effective action would make it possible to identify their kinetic terms, interactions, and effective potential. This would provide the appropriate framework for studying the cosmological implications of the higher-dimensional background [26]. In particular, one could investigate whether the resulting theory admits physically relevant time-dependent solutions.

5. **The de Sitter problem.** An ongoing program in string cosmology is whether warped compactifications can produce controlled four-dimensional de Sitter or accelerated backgrounds. Constructions such as GKP and KKLT add fluxes, orientifolds, nonperturbative effects, and uplift terms, while time-dependent compactifications explore a different route beyond the assumptions of static no-go theorems [10], [13]. Studying the present toy model in the context of finding an accelerating background following the steps of KKLT (i.e., step by step adding more elements in the action) could capture the essential mechanism of the string theory de Sitter construction (if any).

Appendix A

Ricci Tensor Calculations

In this appendix, we present the Ricci tensor calculations used in the main text. The tensor algebra was checked using `Cadabra` [17], [18], [23]. The corresponding codes are available at

https://github.com/hasan-walid/p-brane-background.

A.1 Conventions for curvature tensors

For a metric g_{MN}, the Levi–Civita connection is defined by

$$\Gamma^P{}_{MN} = \frac{1}{2} g^{PQ} \left(\partial_M g_{NQ} + \partial_N g_{MQ} - \partial_Q g_{MN} \right). \tag{A.1}$$

This connection is torsion-free and metric-compatible:

$$\Gamma^P{}_{MN} = \Gamma^P{}_{NM}, \qquad \nabla_P g_{MN} = 0. \tag{A.2}$$

Our convention for the Riemann tensor is

$$R^P{}_{QMN} = \partial_M \Gamma^P{}_{NQ} - \partial_N \Gamma^P{}_{MQ} + \Gamma^P{}_{MR} \Gamma^R{}_{NQ} - \Gamma^P{}_{NR} \Gamma^R{}_{MQ}. \tag{A.3}$$

The Ricci tensor is obtained by contracting the first and third indices of the Riemann tensor:

$$R_{MN} = R^P{}_{MPN}. \tag{A.4}$$

In terms of Christoffel symbols, this gives

$$R_{MN} = \partial_P \Gamma^P{}_{MN} - \partial_N \Gamma^P{}_{MP} + \Gamma^P{}_{PQ} \Gamma^Q{}_{MN} - \Gamma^P{}_{NQ} \Gamma^Q{}_{MP}. \tag{A.5}$$

This is the convention used in the Cadabra computations and in the curvature components displayed below.

The Ricci scalar is

$$R = g^{MN} R_{MN}. \tag{A.6}$$

Greek indices $\mu, \nu, \rho, \ldots$ denote external spacetime directions, Latin indices $m, n, p, \ldots$ denote internal or transverse directions, and capital Latin indices $M, N, P, \ldots$ denote full D-dimensional spacetime directions.

A.2 Maldacena–Nuñez metric

We first consider the warped metric used in the Maldacena–Nuñez no-go theorem:

$$ds_D^2 = \Omega^2(y)\,(\eta_{\mu\nu}(x)\,dx^\mu dx^\nu + \hat{g}_{mn}(y)\,dy^m dy^n)\,. \tag{A.7}$$

The external space has dimension d, the internal space has dimension q, and hence

$$D = d + q. \tag{A.8}$$

The warp factor $\Omega(y)$ and the internal metric $\hat{g}_{mn}(y)$ depend only on the internal coordinates, while $\eta_{\mu\nu}(x)$ depends only on the external coordinates.

We define

$$u_m := \Omega^{-1}\partial_m\Omega = \partial_m \log\Omega, \qquad u^m := \hat{g}^{mn}u_n, \qquad u^2 := u_m u^m = (\hat{\nabla}\log\Omega)^2. \tag{A.9}$$

All hatted covariant derivatives are computed with respect to $\hat{g}_{mn}$.

A.2.1 Simplification of $R_{\mu\nu}$

The direct computation of $R_{\mu\nu}$ gives many terms. Following the Cadabra output, we organize the result into four parts:

$$R_{\mu\nu} = \mathcal{R}^{(\eta)}_{\mu\nu} + \mathcal{L}_{\mu\nu} + \mathcal{Q}_{\mu\nu} + \mathcal{M}_{\mu\nu}, \tag{A.10}$$

where:

- $\mathcal{R}^{(\eta)}_{\mu\nu}$ contains terms built purely from the external metric $\eta_{\mu\nu}$;
- $\mathcal{L}_{\mu\nu}$ contains the Laplacian-type terms involving second derivatives of Ω;
- $\mathcal{Q}_{\mu\nu}$ contains terms quadratic in first derivatives of Ω;
- $\mathcal{M}_{\mu\nu}$ contains mixed external-internal terms.

1. Pure external terms. The terms containing only $\eta_{\mu\nu}$ and its external derivatives combine to give the intrinsic Ricci tensor of the external metric:

$$\mathcal{R}^{(\eta)}_{\mu\nu} = R_{\mu\nu}(\eta). \tag{A.11}$$

2. Mixed terms. For the metric (A.7), both Ω and $\hat{g}_{mn}$ depend only on the internal coordinates. Therefore there is no external dependence in the internal block. In particular,

$$\Gamma^m{}_{\mu n} = 0, \qquad \Gamma^m{}_{\mu m} = 0, \qquad \Gamma^m{}_{m\mu} = 0. \tag{A.12}$$

Consequently, all mixed terms vanish:

$$\mathcal{M}_{\mu\nu} = 0. \tag{A.13}$$

3. Laplacian-type terms. The Laplacian-type terms are those linear in second derivatives of Ω or in derivatives of $u_m = \Omega^{-1}\partial_m\Omega$. They combine as

$$\mathcal{L}_{\mu\nu} = -\eta_{\mu\nu}\left[\partial_m(\hat{g}^{mn}u_n) + \hat{\Gamma}^m{}_{mp}\hat{g}^{pq}u_q\right]. \tag{A.14}$$

The expression in brackets is the covariant divergence of u^m with respect to the internal metric:

$$\hat{\nabla}_m u^m = \partial_m u^m + \hat{\Gamma}^m{}_{mp}u^p. \tag{A.15}$$

Therefore

$$\begin{aligned}\mathcal{L}_{\mu\nu} &= -\eta_{\mu\nu}\hat{\nabla}_m u^m \\ &= -\eta_{\mu\nu}\hat{\nabla}^2 \log\Omega.\end{aligned} \tag{A.16}$$

4. Quadratic gradient terms. The terms quadratic in first derivatives of Ω combine into

$$\begin{aligned}\mathcal{Q}_{\mu\nu} &= (-d - q + 1 + 1)\,\eta_{\mu\nu}u^2 \\ &= -(D-2)\eta_{\mu\nu}u^2 \\ &= -(D-2)\eta_{\mu\nu}(\hat{\nabla}\log\Omega)^2.\end{aligned} \tag{A.17}$$

Here we used $D = d + q$.

5. Final result. Combining (A.11), (A.13), (A.16), and (A.17), we obtain

$$R_{\mu\nu} = R_{\mu\nu}(\eta) - \eta_{\mu\nu}\hat{\nabla}^2\log\Omega - (D-2)\eta_{\mu\nu}(\hat{\nabla}\log\Omega)^2. \tag{A.18}$$

Thus

$$R_{\mu\nu} = R_{\mu\nu}(\eta) - \eta_{\mu\nu}\left[\hat{\nabla}^2\log\Omega + (D-2)(\hat{\nabla}\log\Omega)^2\right]. \tag{A.19}$$

This is the expression used in the Maldacena–Nuñez no-go theorem chapter.

If $\eta_{\mu\nu}$ is flat Minkowski, then $R_{\mu\nu}(\eta) = 0$, and

$$R_{\mu\nu} = -\eta_{\mu\nu}\left[\hat{\nabla}^2\log\Omega + (D-2)(\hat{\nabla}\log\Omega)^2\right]. \tag{A.20}$$

If $\eta_{\mu\nu}$ is de Sitter, then $R_{\mu\nu}(\eta)$ is the intrinsic Ricci tensor of the d-dimensional de Sitter metric.

A.3 Ricci tensor for the p-brane metric

We now compute the Ricci tensor for the warped metric used in the compact p-brane background:

$$ds_D^2 = H(y)^{2a}\hat{\eta}_{\mu\nu}dx^\mu dx^\nu + H(y)^{2b}\tilde{g}_{mn}(y)dy^m dy^n. \tag{A.21}$$

Here

$$n := p + 1, \qquad q := D - p - 1, \tag{A.22}$$

where q denotes the dimension of the transverse space. Thus

$$D = n + q. \tag{A.23}$$

The inverse metric is

$$g^{\mu\nu} = H^{-2a}\hat{\eta}^{\mu\nu}, \qquad g^{mn} = H^{-2b}\tilde{g}^{mn}, \qquad g^{\mu n} = 0. \tag{A.24}$$

All tilded covariant derivatives are computed using the transverse metric $\tilde{g}_{mn}$.

We use the definitions

$$C := a(p+1) + b(D-p-1), \tag{A.25}$$
$$(\tilde{\nabla}H)^2 := \tilde{g}^{mn}\partial_m H \partial_n H, \tag{A.26}$$
$$\tilde{\nabla}_m\tilde{\nabla}_n H := \partial_{mn}H - \tilde{\Gamma}^r{}_{mn}\partial_r H, \tag{A.27}$$
$$\tilde{\nabla}^2 H := \tilde{g}^{mn}\tilde{\nabla}_m\tilde{\nabla}_n H. \tag{A.28}$$

A.3.1 Christoffel symbols

The nonzero Christoffel symbols are

$$\Gamma^\rho{}_{\mu n} = aH^{-1}\partial_n H\,\delta^\rho{}_\mu, \tag{A.29}$$
$$\Gamma^m{}_{\mu\nu} = -aH^{2a-2b-1}\tilde{g}^{mr}\partial_r H\,\hat{\eta}_{\mu\nu}, \tag{A.30}$$
$$\Gamma^m{}_{nr} = \tilde{\Gamma}^m{}_{nr} + bH^{-1}\left(\delta^m{}_n\partial_r H + \delta^m{}_r\partial_n H - \tilde{g}_{nr}\tilde{g}^{ms}\partial_s H\right). \tag{A.31}$$

A.3.2 Simplification of $R_{\mu\nu}$

The Cadabra computation gives terms involving $\tilde{\nabla}^2 H$ and $(\tilde{\nabla}H)^2$. We simplify them in the same order as in the computation.

First, the last two product terms in the Ricci tensor combine as

$$-\Gamma^\rho{}_{\nu r}\Gamma^r{}_{\mu\rho} = a^2 H^{2a-2b-2}(\tilde{\nabla}H)^2\hat{\eta}_{\mu\nu}, \tag{A.32}$$
$$-\Gamma^r{}_{\nu\rho}\Gamma^\rho{}_{\mu r} = a^2 H^{2a-2b-2}(\tilde{\nabla}H)^2\hat{\eta}_{\mu\nu}. \tag{A.33}$$

Therefore they contribute

$$2a^2 H^{2a-2b-2}(\tilde{\nabla}H)^2\hat{\eta}_{\mu\nu}. \tag{A.34}$$

Next, the Laplacian-type terms combine into

$$\begin{aligned} &-aH^{2a-2b-1}\left[\partial_j(\tilde{g}^{jr})\partial_r H + \tilde{g}^{jr}\partial_{jr}H + \tilde{\Gamma}^i{}_{ij}\tilde{g}^{jr}\partial_r H\right]\hat{\eta}_{\mu\nu} \\ &\quad = -aH^{2a-2b-1}\tilde{\nabla}^2 H\,\hat{\eta}_{\mu\nu}. \end{aligned} \tag{A.35}$$

Collecting the gradient-square terms gives

$$\begin{aligned} R_{\mu\nu} = &-aH^{2a-2b-1}\tilde{\nabla}^2 H\,\hat{\eta}_{\mu\nu} \\ &+ \left[-a(2a-2b-1) - (p+1)a^2 - ab(D-p-1) + 2a^2\right] H^{2a-2b-2}(\tilde{\nabla}H)^2\hat{\eta}_{\mu\nu}. \end{aligned} \tag{A.36}$$

The coefficient simplifies as

$$\begin{aligned} &-a(2a-2b-1) - (p+1)a^2 - ab(D-p-1) + 2a^2 \\ &\quad = a - (p+1)a^2 - ab(D-p-3). \end{aligned} \tag{A.37}$$

Therefore

$$R_{\mu\nu} = -aH^{2a-2b-1}\tilde{\nabla}^2 H\,\hat{\eta}_{\mu\nu} + \left[a-(p+1)a^2 - ab(D-p-3)\right] H^{2a-2b-2}(\tilde{\nabla}H)^2\hat{\eta}_{\mu\nu}. \tag{A.38}$$

Equivalently,

$$R_{\mu\nu} = -aH^{2a-2b}\hat{\eta}_{\mu\nu}\left[H^{-1}\tilde{\nabla}^2 H + ((p+1)a + (D-p-3)b - 1)\,H^{-2}(\tilde{\nabla}H)^2\right]. \tag{A.39}$$

A.3.3 Simplification of R_{mn}

We now simplify the internal Ricci tensor in groups.

1. Intrinsic transverse Ricci tensor. The terms built only from the transverse Christoffel symbols combine into the intrinsic Ricci tensor of $\tilde{g}_{mn}$:

$$\partial_q(\tilde{\Gamma}^q{}_{mn}) - \partial_n(\tilde{\Gamma}^q{}_{mq}) + \tilde{\Gamma}^q{}_{qr}\tilde{\Gamma}^r{}_{mn} - \tilde{\Gamma}^r{}_{ms}\tilde{\Gamma}^s{}_{nr} = \tilde{R}_{mn}. \tag{A.40}$$

2. Hessian term. Collecting the terms containing $\partial_{mn}H$ and $\tilde{\Gamma}^r{}_{mn}\partial_r H$, one obtains

$$\begin{aligned}(2b-C)H^{-1}\partial_{mn}H &+ (C-2b)H^{-1}\tilde{\Gamma}^r{}_{mn}\partial_r H \\ &= -(C-2b)H^{-1}\left(\partial_{mn}H - \tilde{\Gamma}^r{}_{mn}\partial_r H\right) \\ &= -(C-2b)H^{-1}\tilde{\nabla}_m\tilde{\nabla}_n H.\end{aligned} \tag{A.41}$$

Since

$$C - 2b = a(p+1) + b(D-p-3), \tag{A.42}$$

this block becomes

$$-\left[a(p+1) + b(D-p-3)\right] H^{-1}\tilde{\nabla}_m\tilde{\nabla}_n H. \tag{A.43}$$

3. Traced Laplacian term. The terms proportional to $\tilde{g}_{mn}$ combine as follows. Using metric compatibility of $\tilde{g}_{mn}$, the derivative of the metric cancels the corresponding connection terms. The remaining terms give

$$\begin{aligned}&-bH^{-1}\tilde{g}_{mn}\left[\partial_q(\tilde{g}^{qs})\partial_s H + \tilde{g}^{qs}\partial_{qs}H + \tilde{\Gamma}^q{}_{qr}\tilde{g}^{rs}\partial_s H\right] \\ &\quad + bH^{-2}\tilde{g}_{mn}(\tilde{\nabla}H)^2 - bCH^{-2}\tilde{g}_{mn}(\tilde{\nabla}H)^2.\end{aligned} \tag{A.44}$$

Hence

$$-bH^{-1}\tilde{g}_{mn}\tilde{\nabla}^2 H + (b - bC)H^{-2}\tilde{g}_{mn}(\tilde{\nabla}H)^2. \tag{A.45}$$

4. $\partial_m H\,\partial_n H$ terms. The remaining terms proportional to $H^{-2}\partial_m H\partial_n H$ combine to

$$\left[-2b + C + 2bC - (p+1)a^2 - (q+2)b^2\right] H^{-2}\partial_m H\partial_n H. \tag{A.46}$$

Using $q = D-p-1$, this coefficient can be rewritten as

$$(p+1)a(1+2b-a) + (D-p-3)b(1+b). \tag{A.47}$$

Therefore the $\partial_m H\,\partial_n H$ block is

$$\left[(p+1)a(1+2b-a) + (D-p-3)b(1+b)\right] H^{-2}\partial_m H\partial_n H. \tag{A.48}$$

5. Remaining $\tilde{g}_{mn}(\tilde{\nabla}H)^2$ terms. The remaining $\tilde{g}_{mn}(\tilde{\nabla}H)^2$ terms combine with the contribution from the traced Laplacian block to give

$$b\,[1-(p+1)a-(D-p-3)b]\,H^{-2}\tilde{g}_{mn}(\tilde{\nabla}H)^2. \tag{A.49}$$

6. Final result for R_{mn}. Combining all pieces, we obtain

$$\begin{aligned}R_{mn} = \tilde{R}_{mn} &- [a(p+1)+b(D-p-3)]\,H^{-1}\tilde{\nabla}_m\tilde{\nabla}_n H - bH^{-1}\tilde{g}_{mn}\tilde{\nabla}^2 H\\ &+ [(p+1)a(1+2b-a)+(D-p-3)b(1+b)]\,H^{-2}\partial_m H\,\partial_n H\\ &+ b\,[1-(p+1)a-(D-p-3)b]\,H^{-2}\tilde{g}_{mn}(\tilde{\nabla}H)^2.\end{aligned} \tag{A.50}$$

A.4 Ricci scalar for the p-brane metric

The Ricci scalar is

$$R = g^{MN}R_{MN} = g^{\mu\nu}R_{\mu\nu} + g^{mn}R_{mn}, \tag{A.51}$$

since $R_{\mu n} = 0$.

First,

$$\begin{aligned}g^{\mu\nu}R_{\mu\nu} = &-na\,H^{-2b-1}\tilde{\nabla}^2 H\\ &+ n\left[a-na^2-(q-2)ab\right]H^{-2b-2}(\tilde{\nabla}H)^2.\end{aligned} \tag{A.52}$$

Here we used

$$\hat{\eta}^{\mu\nu}\hat{\eta}_{\mu\nu} = n. \tag{A.53}$$

Second,

$$\begin{aligned}g^{mn}R_{mn} = H^{-2b}\tilde{R} &- [na+2(q-1)b]\,H^{-2b-1}\tilde{\nabla}^2 H\\ &+ \Big[na(1+2b-a)+(q-2)b(1+b)+qb\big(1-na-(q-2)b\big)\Big]H^{-2b-2}(\tilde{\nabla}H)^2.\end{aligned} \tag{A.54}$$

Adding (A.52) and (A.54), we find

$$\begin{aligned}R = H^{-2b}\tilde{R} &- 2\,[na+(q-1)b]\,H^{-2b-1}\tilde{\nabla}^2 H\\ &+ \Big[n\big(a-na^2-(q-2)ab\big)+na(1+2b-a)+(q-2)b(1+b)\\ &\quad + qb\big(1-na-(q-2)b\big)\Big]H^{-2b-2}(\tilde{\nabla}H)^2.\end{aligned} \tag{A.55}$$

The coefficient of $(\tilde{\nabla}H)^2$ simplifies as

$$\begin{aligned}&n\big(a-na^2-(q-2)ab\big)+na(1+2b-a)+(q-2)b(1+b)+qb\big(1-na-(q-2)b\big)\\ &\quad = 2na-n(n+1)a^2-2n(q-2)ab+2(q-1)b-(q-1)(q-2)b^2.\end{aligned} \tag{A.56}$$

Therefore

$$\begin{aligned}R = H^{-2b}\tilde{R} &- 2\,[na+(q-1)b]\,H^{-2b-1}\tilde{\nabla}^2 H\\ &+ \Big[2na+2(q-1)b-n(n+1)a^2-2n(q-2)ab-(q-1)(q-2)b^2\Big]H^{-2b-2}(\tilde{\nabla}H)^2.\end{aligned} \tag{A.57}$$

Restoring $n = p + 1$, this becomes

$$\begin{aligned} R = {} & H^{-2b}\tilde{R} - 2\left[(p+1)a + (q-1)b\right] H^{-2b-1}\tilde{\nabla}^2 H \\ & + \Big[2(p+1)a + 2(q-1)b - (p+1)(p+2)a^2 \\ & \quad - 2(p+1)(q-2)ab - (q-1)(q-2)b^2\Big] H^{-2b-2}(\tilde{\nabla} H)^2. \end{aligned} \tag{A.58}$$

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