System of Linear Equations in Computed Tomography (CT)

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By

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Declaration

I do hereby declare that the thesis titled "System of Linear Equations in Computed Tomography (CT)" is submitted to the Department of Mathematics and Natural Sciences of BRAC University in partial fulfillment of the Bachelor of Science in Mathematics. This is my original work and has not been submitted elsewhere for the award of any other degree or any other publication. Every work that has been used as reference for this work has been cited properly.

Date: 27th September, 2018

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Abstract

In X-rays the internal structures often overlap, thereby it reduces the chances of detecting any anomalies (if any) and hence the patient may be given an improper diagnosis or they will have to suffer due to participating in multiple scans, which is inevitable if their illness continues and the radiologists are unable to find a cause. In that aspect Computed Tomography (CT), uses multiple X-rays at variable angles directed at a section of the patient's body, to acquire images (single X-ray shot gives image from a specific angle) of the entire cross section. These images are then taken by the scanner and the data (all the images) is then processed to create a detailed image of the cross section. This image is very detailed and so chances of detecting anomaly (if any) goes up. This thesis focuses on representing the scans of CT using a field of linear algebra, namely the System of Linear Equations to be more specific. That is to say, in this thesis we acquire a system of linear equations using the arbitrary data collected after sending and detecting X-ray beams at a patient's cross section (slice), then solving it using Gaussian Elimination and the Matrix Inversion method to acquire roots or solution values which represent the substances present in the slice. These values are then cross referenced with a table that shows the range of values (linearly attenuated for this arbitrary scan) that represent various substances like bones, tissues (healthy or tumorous) and metallic objects that may or may not be present in the slice. Thus by using this table we can identify as to what the roots actually are within the slice.

Executive Summary

Chapter 1: Here we define and describe the two core topics that make up this thesis, which are the System of Linear Equations and CT and how these two are connected. Then we end the chapter by showing how using the scan data we can form a system of linear equations.

Chapter 2: In this chapter we show the methods of solutions that is to be used in this thesis to solve the acquired system of linear equations. The methods being Gaussian Elimination (REF or RREF) method and the Matrix Inversion method (which uses either the adjoint of matrix or the row canonical form of a matrix to find the inverse of a matrix)

Chapter 3: This chapter gives a general understanding of X-ray and CT scanners to non-medical personnel who may need it to understand how the scan takes place, as understanding how the scans work in general may help to comprehend the tomography parts of this thesis.

Chapter 4: The last chapter considers two arbitrary cases of CT scans of a patient's slice where we form systems of linear equations and solve them using the solution methods mentioned in **Chapter 2** to acquire sets of values (from within the slice) that are then cross referenced with a table in **Chapter 1** to identify whether the sets of values are bone structures, tissues (healthy or tumorous) or metallic structures.

Contents

Chapter 1

System of Linear Equations in CT

1.1 Introduction

This chapter contains the definitions and descriptions of the core topics used to make this thesis. They are:

- System of Linear Equations
- Computed Tomography(CT)

Anyone well versed in Linear Algebra know the what, how and why behind the system of linear equations. So we've briefly explained this here and mentioned the types of solutions acquired when a system is solved.

In this thesis we use CT instead of X-ray scans as X-ray scanners scan a body from one directions and so some internal structures may overlap [1] giving inaccurate diagnosis. But this does not happen in CT as the X-rays in CT are sent through a section of a body from multiple directions in large numbers. Hence a 3 dimensional image can be acquired after the scan, rendering the overlapping of internal structures redundant. Now most people are also familiar with CT scans, as it is frequently prescribed by physicians when they deem it required, nowadays. But most may not be familiar as to how a CT machine scans a patient, as most machines have the procedure built in so that the acquired data by the scan is sent to a graphics processing machine, which then gives the image of the cross section being scanned. In this chapter we'll show how a CT machine scans a patient's cross section, acquires the data and then forms a system of linear equations using the data.

1.2 Systems of Linear Equations

1.2.1 Linear Equation

A **linear equation [**2**]** in the variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b \tag{1}$$

where b and the coefficients $a_1, a_2, \ldots, a_n$ are real or complex numbers, usually known in advance. The subscript n may be any positive integer. In text book examples and exercises, n is usually between 2 and 5. In real-life problems, n might be 100 or 10000 or even larger. The equations,

$$3x_1 - 3x_2 + 2 = x_1$$

$$x_2 = 5(\sqrt{9} + x_1) - x_3$$

are both linear because they can be rearranged algebraically as in equation (1):

$$\begin{aligned} 2x_1 - 3x_2 \qquad &= -2 \\ -5x_1 + x_2 + x_3 &= 5\sqrt{9} \end{aligned}$$

The equations,

$$7x_1 - 9x_2 = x_1x_2 \;\;\&\;\; x_2 = 5\sqrt{x_1} - 9$$

are not linear because of the presence of x_1x_2 in the first equation and $\sqrt{x_1}$ in the second.

1.2.2 Linear System

System of linear equations (or a **linear system**) [2] is a collection of one or more linear equations containing the same variables such as, $x_1, x_2, \ldots, x_n$.

An example is:

$$\begin{aligned} 3x_1 \quad\quad + \ 5x_3 &= -4 \\ x_1 + \ 2x_2 - \ 4x_3 &= \ 10 \end{aligned} \tag{2}$$

A solution of the system is a list $s_1, s_2, \ldots, s_n$ of numbers that makes each equation a true statement when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively. For instance, $(0, 3.4, -0.8)$ is a solution of system (2) because, when these values are substituted in system (2) for x_1 ; x_2 ; x_3 respectively, the equations simplify to $-4 = -4$ and $10 = 10$.

The set of all possible solutions is called the solution set of the linear system. Two linear systems are called equivalent if they have the same solution set. That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.

Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines: $p1$ and $p2$. A typical problem is:

$$-2x_1 + x_2 = 5$$

$$3x_1 + x_2 = 1$$

The graphs of these equations are lines, which we denote by p_1 and p_2 . A pair of numbers (x_1, x_2) satisfies both equations in the system if and only if the point (x_1, x_2) lies on both p_1 and p_2. In the system above, the solution is the single point $(-0.8, 3.4)$, as you can easily verify. ***In this thesis we will deal with systems having unique solutions only.*** See **Fig 1.**

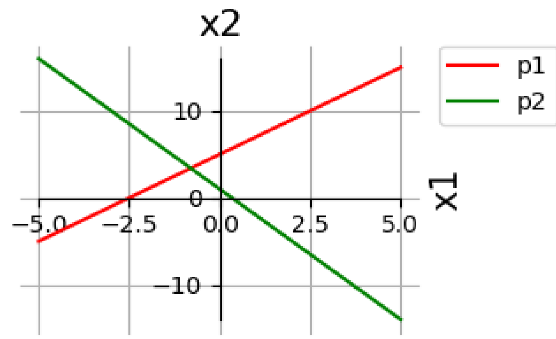
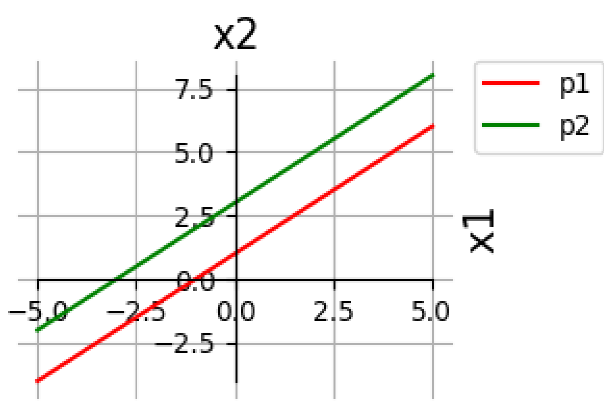


Fig 1. Unique solution

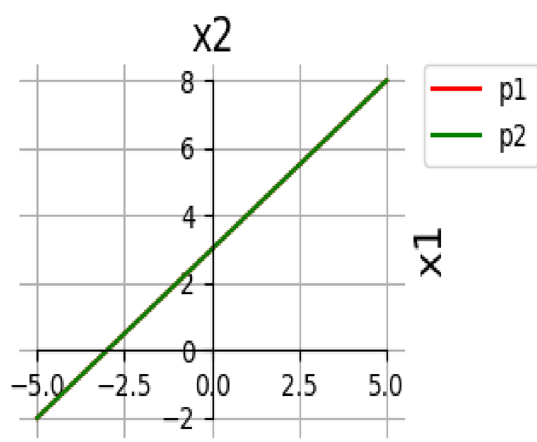
Of course, two lines need not intersect in a single point—they could be parallel, or they could coincide and hence "intersect" at every point on the line. **Fig 2.** shows the graphs that correspond to the following systems:

a) $$-x_1 + x_2 = 1$$ $$-x_1 + x_2 = 3$$

b) $$-x_1 + x_2 = 3$$ $$-x_1 + x_2 = 3$$



(a)



(b)

Fig 2. (a) No solution; (b) Infinitely many solutions

1.3 Computed Tomography (CT) and how it works

1.3.1 What is Computed Tomography (CT)?

CT is based on the absorption of X-rays(which hits a slice from various angles to cover the whole slice) by different tissues, computed tomography (CT) imaging, also known as "CAT scanning" (Computerized Axial Tomography), provides a different form of imaging known as cross-sectional imaging. The origin of the word "tomography" is from the Greek word "tomos" meaning "slice" or "section" and "graphe" meaning "drawing." A CT imaging system produces cross-sectional images or "slices" of anatomy, like the slices in a loaf of bread, as shown in the **Fig 3.** below:

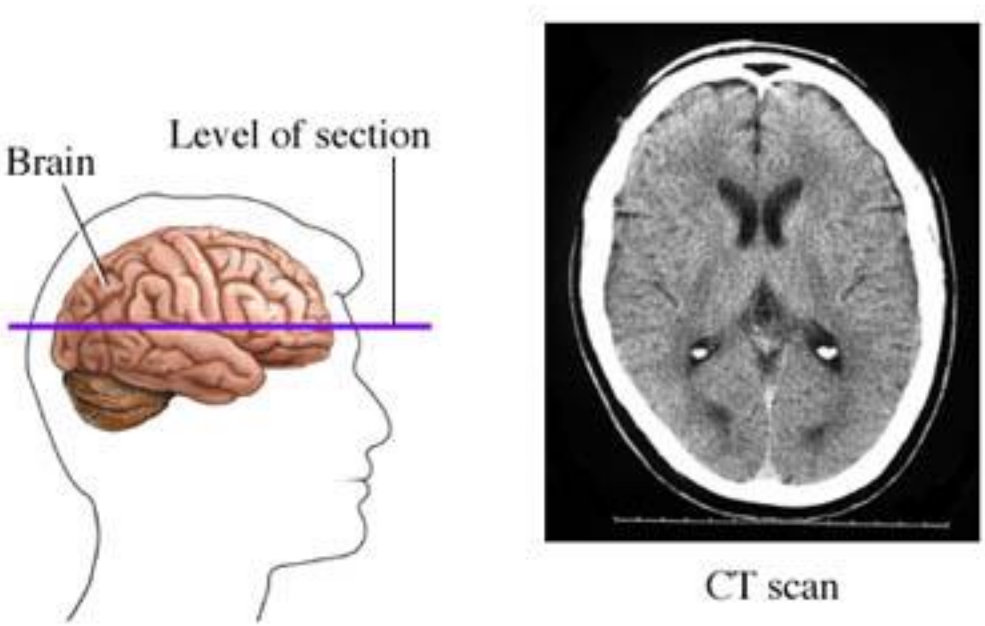


Fig 3. CT slice

1.3.2 How it works and acquires data

Now that we are familiar with how the CT works to acquire the necessary data from a specific cross section that is being scanned, we will now look at it with a mathematical perspective. That is, we will see how the data is being collected, analyzed and identified using the prowess of only mathematics, more specifically the field of linear algebra.

Say we want to scan the brain of a patient namely 'L'. CT scans in cross sections, hence we must consider a single cross section of the brain and divide it in grids as shown below and they are labeled as well for convenience:

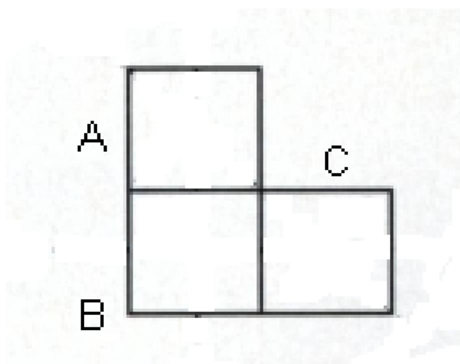


Fig 4.

When the scanner starts up, the gantry starts rotating around the patient. The X-ray source in the gantry emits beams of set intensity at various angles through the entire rotation, which are then detected by the x-ray detector set up in the gantry opposite to the source, as shown below:

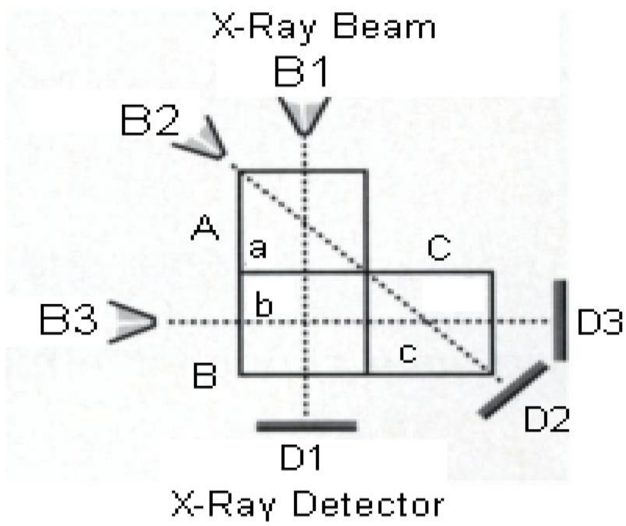


Fig 5.

Note.

- *All beams have same dimension due to the one and only source.*
- *For this thesis the dimension is of* 1 *units*

As mentioned above, a slice of the brain has been superimposed by a grid for simplicity and to allocate objects grid wise. That is, each and every grid contains an object to be scanned, analyzed and identified. So when a beam of x-ray passed through a grid cell, an amount of energy or intensity is lost. Since we don't have a scanner, we cannot acquire specific intensity values.

So we assume the lost intensity as percent attenuation 'x' which we then convert to linear attenuation units (LAUs) [3] using the following formula:

$$L = -ln(1 - x)$$

where, $x =$ *percent attenuation [* x_1 *,* x_2 *,* x_3 *, ... ;* a *,* b *,* c *, ...]*

$L =$ *Linear attenuation [*L_1 *,* L_2 *,* L_3 *, ...]*

So after a beam passes through more than one grid cell the percentage absorbed is determined, which is otherwise known as percent attenuation, that is then converted to LAUs.

Consider the scan in **Fig 5.** the whole rotation can be generalized in the system of linear equations that we seek, as it is shown below:

- Beam (B) 1 passes through grid cells A and B where the beam, intensity reduces by a & b units respectively. So from this information we have the following equation:

$$a + b = L_1$$

- B2 passes through grid cells A and C where the beam intensity is reduced by a & c units respectively. So we have:

$$a + c = L_2$$

- B3 passes through grid cells B and C where the beam intensity is reduced by b & c units respectively. So we have:

$$b + c = L_3$$

Hence arranging the above three equations in the proper order, we finally have acquired a system of linear equations:

$$\begin{array}{ccccc} a & + & b & = & L_1 \\ a & + & c & = & L_2 \\ b & + & c & = & L_3 \end{array}$$

Or,

$$\begin{array}{ccccccc} a & + & b & + & 0 & = & L_1 \\ a & + & 0 & + & c & = & L_2 \\ 0 & + & b & + & c & = & L_3 \end{array}$$

Now using the methods of Gaussian elimination, Gauss-Jordan elimination, Matrix inversion methods, etc., we can solve the system for a, b, c whose values, we will then cross reference with the table [3] given below to identify the contents of each grid cells.

TABLE 3 X-Ray Absorption		
Substance	Percent of X-rays Absorbed	Linear-Attenuation Values
Healthy tissues	15–25 percent	0.1625–0.2877
Tumorous tissues	23.5–32.5 percent	0.2679–0.3930
Bone	32–40 percent	0.3857–0.5108
Metal	78.6–100 percent	1.54–∞

Table 1. CAT/CT scanner ranges *(collected information)*

Thus we will have fulfilled our purpose of using the grid cells and conversion to LAUs to acquire a system and solve it to identify the content of the grids. This concludes a CAT scan where, after solving for the variables and cross referencing them with a standard data table, the structures such as bones, tissues, etc. in the gridded slice are identified and passed on for further diagnosis to the appropriate personnel (radiologists).

Chapter 2

Mathematical Methodologies

2.1 Introduction

Previously we showed how a system of linear equations could be formed from the data acquired through the CT scan of a patient's cross section. Now like all system of equations, we must solve this as well. There are many methods of solutions, of which we will be using two direct methods and its variances in this thesis. The two methods are:

- Gaussian Elimination method
- Matrix Inversion method

We'll explain in details what the methods are and how they are to be used to solve any system of linear equations.

2.2 Gaussian Elimination

Gaussian elimination is the most reliable and comprehensible method of solution for a given system of linear equations using our very hands and not any computational tools. But of course, it does not necessarily mean that using a tool like a computer program is prohibited. On contrary, a program would make things simpler and less time consuming. Also when the matrices get larger like a 4 x 4 system or a 10 x 10 the row operations, as well as the calculations per row become more tedious and chaotic for a person. In such cases using a computer program based on, in this case, Gauss Elimination is preferable over using pen and paper.

The goals of Gaussian elimination [4] are to make the upper-left corner element a 1, use **elementary row operations(ERO)** to get $0s$ in all positions underneath that first 1, get $1s$ for leading coefficients in every row diagonally from the upper-left to lower-right corner and get $0s$ beneath all leading coefficients. Basically, we eliminate all variables in the last row except for one, all variables except for two in the equation above that one, and so on and so forth to the top equation, which has all the variables. And hence we have acquired the **Row Echelon Form (REF)** of the augmented matrix, which is one of the two optimized forms an augmented matrix can be reduced to.

Then we can use **back substitution** to solve for one variable at a time by placing the values we know into the equations from the bottom up. We accomplish this elimination by eliminating the x(or whichever variable comes first) in all equations except for the first one. Then eliminate the second variable in all equations except for the first two. This process continues, eliminating one more variable per line, until only one variable is left in the last line. Then solve for that variable.

Definition 1. Consider a system of the form $AX = B$ with n linear equations,

where $A = a_{ij} \longrightarrow$ Coefficient matrix; $X = x_j \longrightarrow$ Variable matrix ; $B = b_j \longrightarrow$ Constant matrix.

An **augmented matrix** is formed when only the coefficient and the constant matrix adjoint into a single matrix separated (optional, it is done mainly for visual ease) by either a colon or a vertical bar ***(in this thesis we'll be using a colon)*** as shown in the example below:

$$[A:B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & b_2 \\ \vdots & \vdots & \ddots & \vdots & : & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & : & b_n \end{bmatrix}$$

Every column on the left side of the colon represents the coefficient of a single variable. That is, column 1 contains the coefficients of x_1 , column 2 represents the coefficients of x_2 , and so on till x_n . The right-most column represents the constants or the constant matrix which is the values we get on the right side of the equal sign in the form: $AX = B$.

2.2.1 Elementary Row Operations

The basic method for solving a system of linear equations is to replace the given system by a new system that has the same solution set but is easier to solve. This new system is generally obtained in a series of steps by applying the following three types of operations to eliminate unknowns systematically:

1. Multiply an equation through by a non-zero constant.
2. **Interchange** two equations.
3. Add a multiple of one equation to another.

Since the rows (horizontal lines) of an augmented matrix correspond to the equations in the associated system, these three operations correspond to the following operations on the rows ($R[p]$; $p \in [1,2,3, \dots , n]$) of the augmented matrix:

1. Multiply a row through by a non-zero constant.

   For example: $R1' = 2 * R1$,

   where row 1 is multiplied by a constant, 2 to give a new row 1.

2. Interchange two rows.
   For example: $R1 \Leftrightarrow R2$,
   where row 2 interchanges with row 1 to become the new row 1 and vice versa.

3. Add a multiple of one row to another row.
   For example: $R3' = 3 * R1 + R3$,
   where row 1 is multiplied by 3 and added to row 3 and the resulting row becomes the new row 3

2.2.2 Properties of matrix in REF and RREF

Earlier we mentioned that an augmented matrix, using the ERO, can be reduced to one of two forms called REF and reduced REF (RREF). For the matrix to be reduced to either forms, the reduced matrix, that is the new matrix that forms after ERO has been applied to the augmented matrix, it must have the following properties:

- If a row does not consist entirely of zeroes, then the first non-zero number in the row is a **1**. We call this a leading 1.
- If there are any rows that consist entirely of zeroes, then they are grouped together at the bottom of the matrix.
- In any two successive rows that do not consist entirely of zeroes, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
- Each column that contains a leading 1 has zeroes everywhere else in that column.

A matrix that has the first three properties is said to be in REF. An example of a matrix in REF is given below:

$$\begin{bmatrix} 1 & * & * & : & * \\ 0 & 1 & * & : & * \\ 0 & 0 & 1 & : & * \end{bmatrix}$$

where $\quad\quad * \rightarrow$ refers to the altered values acquired during the EROs

A matrix that has all four properties is said to be in RREF and the method that we achieve this by, is often named as **Gauss-Jordan Elimination**. An example is given below:

$$\begin{bmatrix} 1 & 0 & 0 & : & * \\ 0 & 1 & 0 & : & * \\ 0 & 0 & 1 & : & * \end{bmatrix}$$

Thus, a matrix in RREF is of necessity in REF, but not conversely. More often than not, assuming the coefficient matrix is a square matrix (rows=columns), the reduced matrix on the left side of the constants (constant matrix) often takes the form of an **Identity matrix** or mostly referred to as I. That is,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.3 Matrix Inversion Method

Let us consider the system of n linear equations in n unknowns as shown below:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array} \tag{i}$$

The system (i) can be written in the matrix form as shown below:

$$AX = B \tag{ii}$$

where,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Let A be non-singular so that A^{-1} exists. Then, multiplying both sides of (ii) by A^{-1} we obtain:

$$A^{-1}AX = A^{-1}B$$

which gives us,

$$X = A^{-1}B \tag{iii}$$

since,

$$A^{-1}A = I \text{ and } IX = X$$

Hence if A^{-1} is known, then the solution vector X can be found out from the above matrix relation in (iii). That is, acquiring the inverse of A is paramount for this method. Now to achieve this inverse there are many procedures but for this thesis we will be showing two such methods to acquire the inverse in the next two sections. But it should be mentioned that all the matrices must be square and have the determinant of zero ($det = 0$), that is non-singular matrices, as only such matrices can be inverted [5].

2.3.1 Inverse using adjoint of a matrix

Consider a system $[a_{i,j}x_j = b_j]$, where $A = [a_{i,j}]$; $X = x_j$, $B = b_j$ & $i = j$. **Definition 1.** The **adjoint** of a matrix A is the transpose of the cofactor matrix of A and it is denoted by $adj(A)$. An adjoint matrix is also called an adjugate matrix sometimes.

Definition 2. The **minor** of $a_{i,j}$ is denoted by $M_{i,j}$ and is defined to be the determinant of the sub matrix that remains after the i^{th} row and j^{th} column are deleted from. The number $(-1)^{i+j}M_{ij}$ is denoted by C_{ij} and is called the **cofactor** of entry $a_{i,j}$.

Definition 3. The **transpose** of A, denoted by A^T is defined to be the $j \times i$ matrix that results from interchanging the rows and columns of A. That is, the first column of A^T is the first row of A, the second column of A^T is the second row of A and so on.

The definitions above should give a clear idea as to how the adjoint matrix is created or derived from the given system and we all know how to acquire the determinant of A, denoted by $det(A)$. Now to achieve the inverse of A, denoted by A^{-1} we use the following formula:

$$A^{-1} = \frac{1}{det(A)} * adj(A) \qquad \text{(iv)}$$

Or

$$A^{-1} = \frac{1}{|A|} * adj(A)$$

Finally using the inverse of A acquired from formula (iv) in equation (iii) we can solve for the constant matrix X.

An example that utilizes (iv) and (iii) is shown below:

Let us consider the system that retains the form (ii), where

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 6 & 5 & 1 \\ 3 & 4 & 2 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

To acquire the $adj(A)$ we must first find the cofactor of A, that is, C_{ij}. The process is as follows:

$$C_{11} = (-1)^{1+1} * \begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} = (-1)^2 * (5 * 2 - 1 * 4) = 6$$

$$C_{12} = (-1)^3 * \begin{vmatrix} 6 & 1 \\ 3 & 2 \end{vmatrix} = -9$$

$$C_{13} = (-1)^4 * \begin{vmatrix} 6 & 5 \\ 4 & 3 \end{vmatrix} = 9$$

$$C_{21} = (-1)^3 * \begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} = 2$$

$$C_{22} = (-1)^4 * \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4$$

$$C_{23} = (-1)^5 * \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = 5$$

$$C_{31} = (-1)^4 * \begin{vmatrix} 3 & 2 \\ 5 & 1 \end{vmatrix} = -7$$

$$C_{32} = (-1)^5 * \begin{vmatrix} 1 & 2 \\ 6 & 1 \end{vmatrix} = 11$$

$$C_{33} = (-1)^6 * \begin{vmatrix} 1 & 3 \\ 6 & 5 \end{vmatrix} = -13$$

The cofactor of A:

$$C_{ij} = \begin{bmatrix} 6 & -9 & 9 \\ 2 & -4 & 5 \\ -7 & 11 & -13 \end{bmatrix}$$

Finally we can acquire the adjoint of A, that is:

$$adj(A) = (C_{ij})^T = \begin{bmatrix} 6 & 2 & -7 \\ -9 & -4 & 11 \\ 9 & 5 & -13 \end{bmatrix}$$

According to (iv) we will need the $det(A)$ to acquire the inverse, hence

$$det(A) = |A| = 1 * \begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} - 3 * \begin{vmatrix} 6 & 1 \\ 3 & 2 \end{vmatrix} + 2 * \begin{vmatrix} 6 & 3 \\ 5 & 4 \end{vmatrix} = -3$$

Using the $adj(A)$ and $det(A)$ in (iv) we have,

$$A^{-1} = \frac{1}{|A|} * adj(A) = \frac{1}{-3} * \begin{bmatrix} 6 & 2 & -7 \\ -9 & -4 & 11 \\ 9 & 5 & -13 \end{bmatrix} = \begin{bmatrix} -2 & -0.67 & 2.33 \\ 3 & 1.33 & -3.67 \\ -3 & -1.67 & 4.33 \end{bmatrix}$$

Using the acquired A^{-1} in (iii) we can solve for X, that is:

$$X = A^{-1}B = \begin{bmatrix} -2 & -0.67 & 2.33 \\ 3 & 1.33 & -3.67 \\ -3 & -1.67 & 4.33 \end{bmatrix} * \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5.64 \\ -8.36 \\ 10.64 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

[Solved]

2.3.2 Row canonical form

The row canonical form is the same as the form named RREF achieved through Gauss-Jordan elimination shown in **2.2**. Except in this case we are using a modified augmented matrix, that is $[A{:}\,I]$ instead of $[A{:}\,B]$, where $A -$ Coefficient matrix ; $B -$ Constant matrix ; $I -$Identity matrix and the system considered here is the same as that we used in **2.3** . For example:

$$[A{:}\,I] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & : & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & : & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & : & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & : & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Also, unlike what we discussed in **2.2** where the augmented matrices are reduced to either REF or RREF forms to solve a given system say (i), here we reduce the modified augmented matrix using Gauss-Jordan to acquire the inverse of A in the form $[I: A^{-1}]$. Hence using the acquired inverse of A in (iii) we can solve for X.

An example that uses the row canonical form and Gauss-Jordan is given below:

Let us consider the system that retains the form (ii), where

$$A = \begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

To acquire the inverse of A we must first reduce it to the row canonical form and how it is done is shown below:

$$5x_1 + 3x_2 = 3$$

$$2x_1 + 6x_2 = 5$$

Or,

$$\begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Which is in the form $AX = B$ and now we can acquire the modified augmented matrix that we talked about a bit earlier in this that is required to be reduced by the Gauss-Jordan method, after which we will achieve our inverse. The process of achieving the inverse is shown below:

The augmented matrix in the form $[A|I]$, that is:

$$\begin{bmatrix} 5 & 3 & : & 1 & 0 \\ 2 & 6 & : & 0 & 1 \end{bmatrix} \begin{matrix} \rightarrow R1 \\ \rightarrow R2 \end{matrix} \qquad (3)$$

Next we apply appropriate ERO to acquire our inverse.

In (3): firstly we multiply row 1(R1) by $\left(1/_5\right)$ and simultaneously multiply R2 by $\left(1/_2\right)$. So the new rows are:

$$\begin{bmatrix} 1 & 0.6 & : & 0.2 & 0 \\ 1 & 3 & : & 0 & 0.5 \end{bmatrix} \tag{4}$$

In (4): we subtract the constants of R1 from R2. The new R2 is:

$$\begin{bmatrix} 1 & 0.6 & : & 0.2 & 0 \\ 0 & 2.4 & : & -0.2 & 0.5 \end{bmatrix} \tag{5}$$

In (5): next we subtract the multiplication of R2 and $\left(1/_4\right)$ from R1. The new row is:

$$\begin{bmatrix} 1 & 0 & : & 0.25 & -0.125 \\ 0 & 2.4 & : & -0.2 & 0.5 \end{bmatrix} \tag{6}$$

In (6): next we multiply R2 with $\left(1/_{2.4}\right)$. The new row is:

$$\begin{bmatrix} 1 & 0 & : & 0.25 & -0.125 \\ 0 & 1 & : & -0.083 & 0.208 \end{bmatrix} \tag{7}$$

The matrix in (7) cannot be reduced any further and we do not need to go any further either since we have acquired the form $[I : A^{-1}]$. That is we have found the inverse of A which we can now use in (iii) to solve for X. The solution is shown below:

$$X = A^{-1}B = \begin{bmatrix} 0.25 & -0.125 \\ -0.083 & 0.208 \end{bmatrix} * \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -0.25 \\ 1.082 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

[Solved]

Chapter 3

Medical Methodologies

3.1 Introduction

Chapter 1 showed the core topics of my thesis and how I combined the two. Now my thesis involves a medical technology called CT (Computed Tomography). Before I could use this for my thesis I had to learn about the what-s, how-s and why-s of it. This chapter is entirely dedicated to the understanding of CT, so if any non-medical person(s) views my thesis, they can go through this chapter to get the general idea of CT and how I've adopted my idea from this.

3.2 X-rays in the medical field

3.2.1 Description

X-rays are a form of radiation similar to light rays, but they are more energetic than light rays and are invisible to the human eye. They are created when an electric current is passed through a vacuum tube. X-rays were accidentally discovered in 1895 by German physicist Wilhem Roentgen (1845-1923), who was later awarded the first Nobel Prize in physics for his discovery. Roentgen was also a photographer and almost immediately realized that the shadows created when x-rays passed through the body could be permanently recorded on photographic plates. His first x-ray picture was of his wife's hand. Within a few years, x-rays became a valued diagnostic tool of physicians all over the world.

3.2.2 Purpose

Diagnostic x-rays are useful in detecting abnormalities within the body. They are a painless, non-invasive way to help diagnose problems such as broken bones, tumors, dental decay and the presence of foreign bodies.

3.2.3 How X-rays work

X-rays pass easily through air and soft tissue of the body. When they encounter more dense material, such as a tumor, bone or a metal fragment, they are stopped. Diagnostic x-rays are performed by positioning the part of the body to be examined between a focused beam of x-rays and a plate containing film. This process is painless. The greater the density of the material that the x-rays pass through, the more rays are absorbed. Thus bone absorbs more x-rays than muscle or fat, and tumors may absorb more x-rays than surrounding tissue. The x-rays that pass through the body hit the photographic plate and interact with silver molecules on the surface of the film.

After the processing of the film plates are complete, dense material such as bone shows up as white, while softer tissue shows up as shades of gray, and airspaces look black. A radiologist, who is a physician trained to comprehend diagnostic x-rays, examines the pictures and reports to the doctor who ordered the tests. Plain film x-rays normally take only a few minutes to perform and can be done in a hospital, radiological center, clinic, doctor's or dentist's office, or at bedside with a portable x-ray machine.

3.2.4 Special types of x-ray procedures

Mammograms are fixed plate x-rays that are designed to locate tumors within the breasts. Dental x-rays are designed to locate decay within the tooth. Sometimes a liquid called contrast material (for example, barium) is used to help outline internal organs such as the intestines. The contrast material absorbs x-rays, helping to make soft tissue more easily visible on the x-ray films. Contrast material is commonly used in making x-rays of the digestive system. The contrast liquid can be swallowed or injected, depending on the part of the body being x-rayed. This may cause some minor discomfort.

Computed tomography or CT scan works on the same principles as fixed plate x-rays, only with a CT scan, an x-ray tube rotates around the individual, taking hundreds of images that are then compiled by a computer to produce a two-dimensional cross section of the body. Although many images are taken to produce a CT scan, the total dose of radiation the individual is exposed to is low. Other common imaging techniques such as **magnetic resonance imaging** (MRI) and ultrasound do not use x-rays.

3.2.5 How x-rays are performed

Fixed plate x-rays are extremely common diagnostic tests. A trained x-ray technologist takes the x-ray. The individual is first asked to remove clothing and jewelry and to wear a hospital gown. The x-ray technologist positions the patient appropriately, so that the part of the body to be x-rayed will be between the x-ray beam and the film plate. Usually the individual either lies on an adjustable table or stands. Parts of the body that are especially sensitive to damage by x-rays (for example, the reproductive organs, and the thyroid) are shielded with a lead apron. Lead is very dense and effectively protects the body by stopping all x-rays.

It is essential to remain motionless during the x-ray, since movement causes the resulting picture to be blurry. Sometimes patients are asked to hold their breath briefly during the procedure. Children who are not old enough follow directions or who cannot stay still may need to be restrained or given medication to sedate them in order to keep them still enough to obtain useful results. Sometimes parents can stay with children during an x-ray, unless the mother is pregnant, in which case she must protect the fetus from x-ray exposure.

If a contrast material is to be used, the individual will be given special instructions to prepare for the procedure and may be asked to remain afterwards until recovery is complete.

3.2.6 Precautions

Although unnecessary exposure to radiation should be avoided, the low levels of radiation one is exposed to during an x-ray does not cause harm with a few exceptions. Pregnant women should not have x-rays unless in emergencies the benefits highly outweigh the risks. Exposure of the fetus to x-rays, especially during early pregnancy can increase the risk of the child later developing leukemia. Body parts not being x-rayed should be shielded with a lead apron, especially the testes, ovaries, and thyroid [6].

3.3 Computed Tomography (CT) in the medical field

3.3.1 Definition

Computerized axial tomography (CAT), currently known as Computed tomography (CT) is a common diagnostic imaging procedure that uses x-rays to generate images (slices) of the anatomy.

3.3.2 Purpose

Computed tomography (CT) is a type of x-ray imaging procedure used for a variety of clinical applications. CT is used for spine and head imaging, gastrointestinal imaging, vascular imaging (e.g., detection of blood clots), cancer staging and radiotherapy treatment planning, screening for cancers and heart disease, rapid imaging of trauma, imaging of musculoskeletal disorders, detection of signs of infectious disease, and guidance of certain interventional procedures (e.g., biopsies). CT is the preferred imaging exam for diagnosing several types of cancers, and along with the chest x-ray, CT is the most commonly performed procedure for imaging the chest. CT is also used to perform non-invasive angiographic imaging to assess the large blood vessels.

CT may be performed on newborns, infants, children, and adolescents. In children, CT is most frequently used in the hospital emergency department to evaluate the effects of trauma, especially to the head, face, brain, and spine, and to diagnose or rule out appendicitis and other abdominal disorders because a scan can be completed in less than 20 seconds. Chest CT examinations are used to assess complications from infectious diseases, such as pneumonia and tuberculosis, inflammation of the airways, and birth defects. CT scans of the pelvic area are used to image ovarian cysts and tumors, bladder abnormalities, urinary tract stones, kidney disease, and bone disease. Head CT scans are used to examine the brain and sinuses. For children with cancer, CT is used to assist in treatment planning and to monitor cancer progression and response to treatment. For children requiring complex surgeries (e.g., brain, spine), CT is often used to produce images of the anatomy that help surgeons plan the surgery. Newer CT scanners, called multi-slice or multi-detector CT, are used to rapidly image newborns to assess congenital heart defects.

3.3.3 Description

CT is performed using a specialized scanner, an x-ray system, a patient table, and a computer workstation. The CT scanner is shaped like a large square with a hole in the center or round like a doughnut. X-rays are produced in the form of a beam that rotates around the patient. During a CT scan, the patient table is moved through the center hole as x-ray beams pass through the patient's body. The x-rays are converted into a series of black-and-white images, each of which represents a "slice" of the anatomy.

CT scans are conducted by a technologist with specialized training in x-rays and CT imaging. During scanning, the technologist operates the CT scanner using a computer located in an adjacent room. Because movement during the scan can cause inaccurate images, the technologist instructs the patient via an intercom system to hold their breath and not move during the x-rays. The scan itself may only take five to 15 minutes, but total examination time may be up to 30 minutes, since the patient must be prepped and positioned. Abdominal CT scans usually require that the child drink a solution that contains a dye, called oral contrast that shows up on the CT images to help better define internal organs (**Fig 6.**). For pelvic scans, contrast material may be delivered via the rectum. Some CT scans also require the injection of contrast material into the vein to help define the blood vessels and surrounding tissue.

The images from CT examination are called slices because they are acquired in very small (millimeter-size) sections of the body. The image slices are displayed on a computer monitor for viewing or printed as a film. A radiologist interprets the x-ray images produced during the CT examination. For emergency CT scans, images are interpreted immediately so that the child can be treated as soon as possible. For non-urgent outpatient CT scans, the radiologist interprets the images and sends a report to the referring physician within a few days.

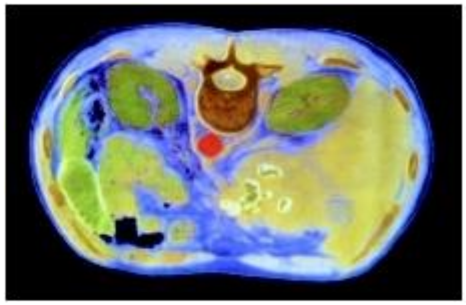


Fig 6. False color computed tomography scan through the abdomen, showing the liver (larger yellow organ) and spleen (smaller yellow organ). The abdominal aorta is colored red and located above the spine and between the kidneys. (Photo Researchers, Inc.)

3.3.4 Precautions

CT scans expose the child to radiation, and overuse of CT scanning has received attention from organizations that regulate medical radiation exposure. Although no side effects have been linked to radiation exposure from CT imaging, the Food and Drug Administration has issued guidance to physicians regarding levels of radiation during pediatric CT examinations. New CT scanners have preset imaging features that allow scanning at the lowest radiation dose for the child's weight and age.

Oral contrast may be unpleasant tasting, although chocolate, vanilla, and fruit flavors may be available. Injected contrast can cause sensations of heat or cold through the body. Some children may have allergic reactions to the contrast material, although severe reactions are rare. Parents should inform CT staff if their child has ever had a reaction to any medication, contrast material, or anesthesia. Because contrast material may contain iodine, sensitivity to contrast material may occur if the child has other allergies to iodine or seafood, and CT staff should be informed if the child has such allergies. Also, because CT contrast material can affect kidney function, parents should notify CT staff if their child has kidney disease [7].

Chapter 4

Mathematical Applications

4.1 Introduction

In the previous chapters we described, discussed and showed how the linear algebra can be related to CT and hence acquire sets of data from scanning a slice of a patient's brain or skull and use it to make a systems of linear equations by utilizing the grids that form around a slice according to our need (more the grids, the better the results). We also showed and described methods to solve this acquired system of linear equations by first acquiring their matrix forms and then applying the solution methods to solve them as shown in **Chapter 2**. The solution are mainly the LAUs of objects or substances that inhabit the slice or the grid cells. Then we compare these values with a given table (**Table 1.** in **Chapter 1**) to identify what each value represents, that is whether a value represents a bone structure or healthy or tumorous tissues.

The methods of solutions in **Chapter 2** can be coded in programs like Python, Java, etc. where the user only need to input the data into the code and the program will compute accordingly to solve the case (which is in its linear equations or matrix form) and give back the solution values (LAUs) of objects contained within the grid cells, which like above, we compare with the given table in **Chapter 1** to identify these objects. ***In this thesis we will be using Python only.***

Here, in this chapter we are going to consider arbitrary cases of CT scan of a patient's brain using different number of grids and hence x-ray beams. And using this arbitrary data we'll acquire the system of linear equations for the cases and solve them with few or all the methods of solution described in chapter 2. Thus solving the cases with these methods we can show that the field of linear algebra can be used to acquire the necessary data to identify the objects being scanned by CT.

4.2 Case 1

(Arbitrary CT report)

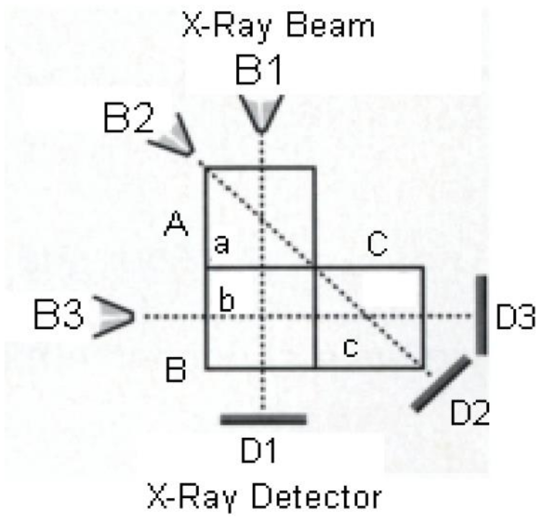


Fig 5.

Given that, grid cell A weakens beam by a units, grid cell B weakens beam by b units and grid cell C weakens the beam by c units as shown in **1.3.2; Fig 5.**

Suppose the CAT scanner reports that for some patient:

a) B1 is weakened by 0.70 units as it passes through grid cells A and B

b) B2 is weakened by 0.55 units as it passes through grid cells A and C

c) B3 is weakened by 0.75 units as it passes through grid cells B and C

Using the above data, a system of linear equations can be obtained as follows:

$$a + b = 0.70 \tag{8}$$

$$a + c = 0.55 \tag{9}$$

$$b + c = 0.75 \tag{10}$$

4.2.1 Solution using Gauss Elimination (REF)

Matrix form of equations (8), (9), (10) in a combined system:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} . \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.70 \\ 0.55 \\ 0.75 \end{bmatrix}$$

Which is in the form $AX = B$, where:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad B = \begin{bmatrix} 0.70 \\ 0.55 \\ 0.75 \end{bmatrix}$$

First we acquire the augmented matrix that we require to apply the method of solution to the sum, which is as follows:

$$\begin{bmatrix} 1 & 1 & 0 & : & 0.70 \\ 1 & 0 & 1 & : & 0.55 \\ 0 & 1 & 1 & : & 0.75 \end{bmatrix} \begin{matrix} \longrightarrow R1 \\ \longrightarrow R2 \\ \longrightarrow R3 \end{matrix} \qquad (11)$$

Now we must acquire the REF, using Gauss Elimination as follows:

Firstly we interchange row 2(R2) and row 3(R3) of the matrix in (11), then we have:

$$\begin{bmatrix} 1 & 1 & 0 & : & 0.70 \\ 0 & 1 & 1 & : & 0.75 \\ 1 & 0 & 1 & : & 0.55 \end{bmatrix} \qquad (12)$$

Next we subtract R1 from R3 for new R3 of the matrix in (12):

$$\begin{bmatrix} 1 & 1 & 0 & : & 0.70 \\ 0 & 1 & 1 & : & 0.75 \\ 0 & -1 & 1 & : & -0.15 \end{bmatrix} \qquad (13)$$

Next we multiply R3 with (-1) for new R3 of the matrix in (13)

$$\begin{bmatrix} 1 & 1 & 0 & : & 0.70 \\ 0 & 1 & 1 & : & 0.75 \\ 0 & 1 & -1 & : & 0.15 \end{bmatrix} \tag{14}$$

Next we subtract R2 from R3 for new R3 of the matrix in (14):

$$\begin{bmatrix} 1 & 1 & 0 & : & 0.70 \\ 0 & 1 & 1 & : & 0.75 \\ 0 & 0 & -2 & : & -0.60 \end{bmatrix} \tag{15}$$

Next halve R3 and multiply it with $-(1/2)$ for new R3 of the matrix in (15):

$$\begin{bmatrix} 1 & 1 & 0 & : & 0.70 \\ 0 & 1 & 1 & : & 0.75 \\ 0 & 0 & 1 & : & 0.30 \end{bmatrix} \tag{16}$$

We have finally acquired the REF of the original matrix in (11).

Now we reform the pre augmented matrix, that is, acquire the original form of $AX = B$. But the coefficient matrix A has been optimized through Gaussian Elimination to acquire the row echelon form that we have in (16). So we will be using the optimized matrix A in the equation (ii), as shown below:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.70 \\ 0.75 \\ 0.30 \end{bmatrix}$$

Now we use matrix multiplication upon A and:

$$AX = \begin{matrix} a & + & b & & \\ & & b & + & c \\ & & & & c \end{matrix}$$

Finally we reform the individual equations from newly acquired $AX = B$ and as follows:

$$a + b = 0.70 \tag{17}$$

$$b + c = 0.75 \tag{18}$$

$$c = 0.30 \tag{19}$$

After setting up equations as shown above we use the process of Back Substitution to find the values of the individual variables. We start from the bottom and solve our way up as shown below:

From (19) we have:

$$c = 0.30$$

Substitute the value of c in equation (18), we get:

$$b = 0.45$$

Finally we substitute the value of b in (17), we get:

$$a = 0.25$$

The values of the variables that represent an individual grid cell as stated originally can now be used in reference to the **Table 1.** to identify the type of object that is located in the grid cells. Using the data in the table we have:

- Grid cell A contains healthy tissues
- Grid cell B contains bone structure
- Grid cell C contains tumorous tissues

[Solved]

4.2.2 Solution using Gauss-Jordan Elimination (RREF)

This method is an extension of the method shown above with the exception that this method does not require the use of back substitution by adding two extra ERO to the REF form of A achieved in **4.2.1**. That is, we will import (16) of **4.2.1** in this section and apply the extra EROs to achieve the RREF form of A and finally solve for X.

The REF of A from **4.2.1** labeled (16) is given below:

$$\begin{bmatrix} 1 & 1 & 0 & : & 0.70 \\ 0 & 1 & 1 & : & 0.75 \\ 0 & 0 & 1 & : & 0.30 \end{bmatrix} \tag{16}$$

Now we subtract R3 from R2 for new R2 in (16):

$$\begin{bmatrix} 1 & 1 & 0 & : & 0.70 \\ 0 & 1 & 0 & : & 0.45 \\ 0 & 0 & 1 & : & 0.30 \end{bmatrix} \tag{20}$$

Next we subtract R2 from R1 for new R1 in (20):

$$\begin{bmatrix} 1 & 0 & 0 & : & 0.25 \\ 0 & 1 & 0 & : & 0.45 \\ 0 & 0 & 1 & : & 0.30 \end{bmatrix} \tag{21}$$

Hence we have achieved the RREF of A in (21).

We know from the previous example that the form $AX = B$ must be reacquired and after doing so we have the following equations:

$$a = 0.25$$

$$b = 0.45$$

$$c = 0.30$$

And we see that the equations that we found are actually the same solution for the variables that we were supposed to find. The Back Substitution method was not required at all since the matrix was reduced to its RREF using Gauss-Jordan's elimination method.

The values of the variables, which represent an individual grid cell as stated originally can now be used in reference to the **Table 1.** to identify the type of object that is located in the grid cells. Comparing the data with the table we have:

- Grid cell A contains healthy tissues
- Grid cell B contains bone structure
- Grid cell C contains tumorous tissues

[Solved]

4.2.3 Gaussian Elimination (REF) via Python

We can use a code that implements the Gaussian Elimination (REF) and Back Substitution to solve a linear system of equations. Now unlike the procedure we showed in **4.2.1** the code requires the input of the augmented matrix only and the code does all the rest. Below is the code where we will input the augmented matrix formed using the coefficient and constant matrix of Case 1 as shown in (11) of **4.2.1**.

Algorithm. The following code is based on this algorithm given below [8]:

Input. Number of unknowns and equations; augmented matrix $A = [a_{i\,j}]$, where $1 \leq i \leq n$ and $1 \leq j \leq n + 1$.

Output. Solution $x_1, x_2 \ldots x_n$ or message that the linear system has no unique solution.

Step 1 For $i = 1, \ldots, n - 1$ do **Steps 2–4**. (*Elimination process.*)

Step 2 Let p be the smallest integer with $i \leq p \leq n$ and $a_{pi} \neq 0$.
If no integer p can be found
then **Output** ('no unique solution exists');
STOP.

Step 3 If $p \neq i$ then perform $(a_{pi}) \leftrightarrow (a_{ip})$.

Step 4 For $j = i + 1, \ldots, n$ do **Steps 5 - 6**.

Step 5 Set $m_{ji} = a_{ji}/a_{ii}$.

Step 6 Perform $(a_j - m_{ji} * a_i) \rightarrow (a_j)$;

Step 7 If $a_{nn} = 0$ then **Output** ('no unique solution exists');
STOP.

Step 8 Set $x_n = a_{n,n+1}/a_{nn}$. (*Start backward substitution*.)

Step 9 For $i = n - 1, \ldots, 1$ set $x_i = {[a_{i,n+1} - \sum_{j=i+1}^{n} (a_{ij}\, x_i)]}/{a_{ii}}$

Step 10 Output $(x_1, x_2 \ldots x_n)$; (*Procedure completed successfully*.)
STOP.

Input. Coefficient matrix *A* and its corresponding constant matrix *B*

```
'''
Using Gaussian Elimination and Back Substitution to solve a given set of equations:
'''
import numpy as np
import ast

#Elimination Process:

def elim(A):
    m,nc=A.shape            #nc=number of columns,m=number of rows
    n=m
    for i in range(n-1):
        print i
        p=1
        for k in range(i, n):
            print k
            if A[k,i] !=0:
                p=k
                break
```

```
                if p==0:
                    print 'no sol'
                    break
        if p!=i:
            A[[p,i]]=A[[i,p]]
            print 'A\n',A

        for j in range(i+1,n):
            if A[i,i]!=0:
                m1=A[j,i]/A[i,i]
                A[j,:]=A[j,:]-(m1*A[i,:])
                print A
    return A

#Back-Substitution process:

def back_sub(B):
    m,nc=B.shape
    n=m-1
```

```
    if B[n,n]==0:
        print 'no sol'
        return
    x=np.zeros(m)
    x[n]=B[n,n+1]/B[n,n]
    for k in range(m):
        i=m-k-1
        sum=0
        for j in range(i+1,m):
            sum=sum+B[i,j]*x[j]
        x[i]=(B[i,n+1]-sum)/float(B[i,i])
        print "\nx",[i+1],": ",x[i]
    return x
```

```

def matrix_input(n,m):

    s = raw_input("\nEnter space sep. matrix elements: ")
    items  = map(ast.literal_eval, s.split(' '))    # Use literal_eval

    assert(len(items) == n*m)                       # Check to see if it matches
    matrix = np.array(items).reshape((n,m))         # "Convert to numpy array and
                                                    # reshape to the right size"

    return matrix

```

```
if __name__=='__main__':
    n = input('Enter number of rows: ')
    m = input('Enter number of columns: ')
    A=matrix_input(n,m)
    print '\nA: \n',A
    B=elim(A)
    print '\nB: \n',B
    x=back_sub(B)
    print '\nOr\n \nSolution: x[1,2,3,...,n]=',x
```

Code. 1

Output.

```
Enter number of rows: 3
Enter number of columns: 4

Enter space sep. matrix elements: 1 1 0 0.7 1 0 1 0.55 0 1 1 0.75

A:
[[ 1.    1.    0.    0.7 ]
 [ 1.    0.    1.    0.55]
 [ 0.    1.    1.    0.75]]
0
0
[[ 1.    1.    0.    0.7 ]
 [ 0.   -1.    1.   -0.15]
 [ 0.    1.    1.    0.75]]
[[ 1.    1.    0.    0.7 ]
 [ 0.   -1.    1.   -0.15]
 [ 0.    1.    1.    0.75]]
1
1
[[ 1.    1.    0.    0.7 ]
 [ 0.   -1.    1.   -0.15]
 [ 0.    0.    2.    0.6 ]]

B:
[[ 1.    1.    0.    0.7 ]
 [ 0.   -1.    1.   -0.15]
 [ 0.    0.    2.    0.6 ]]

x [3] :  0.3

x [2] :  0.45

x [1] :  0.25

Or

Solution: x[1,2,3,....,n]= [ 0.25  0.45  0.3 ]
```

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.45 \\ 0.30 \end{bmatrix}$$

Now comparing the values with **Table 1.** we have:

- Grid cell A contains healthy tissues
- Grid cell B contains bone structure
- Grid cell C contains tumorous tissues

[Solved]

4.2.4 Using matrix inversion method

This is the same method we discussed and showed in **2.3** which requires a system of the form $AX = B$ to be converted or computed into $X = A^{-1}B$ so we may solve the system and acquire its solutions. Now in the subsequent sections of **2.3**, that is **2.3.1** and **2.3.2** we showed two different methods to find the inverse of A as in this method using only A^{-1}, can we solve a system. But since the procedure in **2.3.2** namely Row Canonical form works the same way as Gauss-Jordan's Elimination method (RREF) with the exception being that the solution column in the augmented matrix is replaced by an Identity matrix and how the process works from there is shown in **2.3.2**. So to solve the case in **4.2** we will only be using the adjoint of a matrix to find the inverse of the matrix to be used by the matrix inversion method.

Using adjoint of a matrix

Let us utilize the matrix form of the case equations from **4.2.1** and so we have the form $AX = B$ where,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad B = \begin{bmatrix} 0.70 \\ 0.55 \\ 0.75 \end{bmatrix}$$

Now we will use the method that utilizes the adjoint of a matrix like we showed in **2.3.1** where we must find the determinant, cofactor and transpose of cofactor of A.

First we calculate the determinant,

Determinant of A = $det(A) = 1(0-1) - 1(1-0) + 0 = -2$

Now we find the cofactor of A in the same way as shown in **2.3.1**,

Cofactor of A= $C_{ij} = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$

Now we compute the transpose of C_{ij},

Adjoint of $A = adj(A) = [C_{ij}]^T = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$

Using the $adj(A)$ and $det(A)$ in (iv) we have,

$$A^{-1} = \frac{1}{|A|} * adj(A) = -\left(\frac{1}{2}\right) * \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 0.50 & 0.50 & -0.50 \\ 0.50 & -0.50 & 0.50 \\ -0.50 & 0.50 & 0.50 \end{bmatrix}$$

Using the acquired A^{-1} in (iii) we can solve for X, that is:

$$X = A^{-1}B = \begin{bmatrix} 0.50 & 0.50 & -0.50 \\ 0.50 & -0.50 & 0.50 \\ -0.50 & 0.50 & 0.50 \end{bmatrix} * \begin{bmatrix} 0.70 \\ 0.55 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.45 \\ 0.30 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\therefore\ a = 0.25\ ;\ b = 0.45\ ;\ c = 0.30$$

Hence this answer matches the last two methods as the problem is under the same case, so comparing the values with **Table 1.** we have:

- Grid cell A contains healthy tissues
- Grid cell B contains bone structure
- Grid cell C contains tumorous tissues

[Solved]

4.2.5 Matrix Inversion method via Python

We can use a code that implements the Matrix Inversion method to solve a linear system of equations. Now unlike the two procedures we showed in **2.3.1** and **2.3.2** where we acquired the inverse of A in two different ways, the code requires just one line to find the inverse of any matrix and so the solution process becomes **less tedious and time consuming**. Hence *line#39* in the code that says: $\boldsymbol{A\_inv = np.linalg.inv(A)}$ does the work for both the adjoint of a matrix and the row canonical form that is otherwise required manually to find the inverse of A.

Below is the code where we will input the coefficient matrix and constant matrix of Case 1 as shown in (11) of **4.2.1.**

Algorithm.

Step 1: Input coefficient matrix A

Step 2: Compute the inverse of A or A^{-1} using the following line:

$$A^{-1} = np.linalg.inv(A)$$

#The code above is specific to Python only, so it will vary on different programs

Step 3: Input constant matrix B

Step 4: Compute the product of the A^{-1} and B:

$$X = A^{-1} . B$$

Step 5: Outcome of the system:

$$X = [x_1, x_2 \dots x_n]$$

STOP

Input. Coefficient matrix A and its corresponding constant matrix B, as prompted by the program.

```
'''
Using Matrix Inversion Method to solve a system of linear equations:
'''
import numpy as np
import ast

def matrix_input(n,m):

    s = raw_input("\nEnter space sep. matrix elements: ")
    print "\n"
    items  = map(ast.literal_eval, s.split(' '))    # Use literal_eval

    assert(len(items) == n*m)                        # Check to see if it matches
    matrix = np.array(items).reshape((n,m))          # 'Convert to numpy array and
                                                     # reshape to the right size'
    #print 'Wrong number of elements have been input'

    return matrix

```

```
def roots(X):
    n,m=X.shape

    x=np.zeros(n)
    #print 'x',x

    for k in range(n):
        i=k
        #print 'i:',k
        x[i]=float(X[i,m-1])
        print "x",[i+1],": ",x[i]
    return x

```

```
if __name__=='__main__':
    n = input('Enter number of rows: ')
    m=n

    A = matrix_input(n,m)
    print 'A: \n',A
    A_inv=np.linalg.inv(A)
    print '\nA_inv: \n',A_inv

    n = input('\nEnter number of rows: ')
    m = input('Enter number of columns: ')
    b = matrix_input(n,m)

    X = np.dot(A_inv,b)     #this represents the form X=A^(-1) . b
    #print 'X: ',X
    x=roots(X)
    print '\nOr\n \nSolution: x[1,2,3,...,n]=',x
```

Code. 2

Output

```
Enter number of rows: 3

Enter space sep. matrix elements: 1 1 0 1 0 1 0 1 1


A:
[[1 1 0]
 [1 0 1]
 [0 1 1]]

A_inv:
[[ 0.5  0.5 -0.5]
 [ 0.5 -0.5  0.5]
 [-0.5  0.5  0.5]]

Enter number of rows: 3
Enter number of columns: 1

Enter space sep. matrix elements: 0.7 0.55 0.75


x [1] :  0.25
x [2] :  0.45
x [3] :  0.3

Or

Solution: x[1,2,3,...,n]= [ 0.25  0.45  0.3 ]
```

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.45 \\ 0.30 \end{bmatrix}$$

Now comparing the values with **Table 1.** we have:

- Grid cell A contains healthy tissues
- Grid cell B contains bone structure
- Grid cell C contains tumorous tissues

[Solved]

4.3 Case 2

(Arbitrary CT report**)**

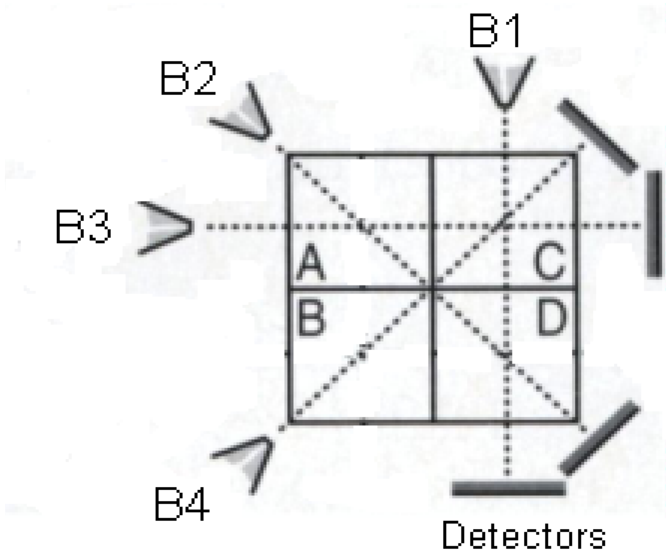


Fig. 7

***The labels for Fig. 7 are the same as Case 1 of 4.2*.**

Suppose the CAT scanner reports that for some patient:

a) B1 is weakened by 0.69 units as it passes through grid cells C and D

b) B2 is weakened by 0.49 units as it passes through grid cells A and D

c) B3 is weakened by 0.55 units as it passes through grid cells A and C

d) B4 is weakened by 0.78 units as it passes through grid cells B and C

Using the above data, a system of linear equations can be obtained as follows:

$$c + d = 0.69$$

$$a + d = 0.49$$

$$a + c = 0.55$$

$$b + c = 0.78$$

Matrix form of the above system is as follows:

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} * \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0.69 \\ 0.49 \\ 0.55 \\ 0.78 \end{bmatrix}$$

Which is in the form $AX = B$, where:

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad X = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \qquad B = \begin{bmatrix} 0.69 \\ 0.49 \\ 0.55 \\ 0.78 \end{bmatrix}$$

Now compared to Case 1, Case 2 has an extra grid cell which allows for more beams to pass through the slice giving more accurate data of the grid contents. Since there is an extra beam for the extra grid the system of linear equations has become larger and so manually solving the system using the Gaussian Elimination (both REF and RREF) or the Matrix Inversion Method is physically, as well as visually tedious. Instead we can simply input the coefficient matrix and its corresponding constant matrix, where necessary in the python codes used in **4.2.3** & **4.2.5** which implements the Gaussian Elimination and Matrix Inversion methods respectively.

4.3.1 Using the python code for Gaussian Elimination

Input. Coefficient matrix A and its corresponding constant matrix B

Code. It is the same as shown in **4.2.3** labeled **Code. 1**

Output

```
Enter number of rows: 4
Enter number of columns: 5

Enter space sep. matrix elements: 0 0 1 1 0.69 1 0 0 1 0.49 1 0 1 0 0.55 0 1 1 0
 0.78

A:
[[ 0.    0.    1.    1.    0.69]
 [ 1.    0.    0.    1.    0.49]
 [ 1.    0.    1.    0.    0.55]
 [ 0.    1.    1.    0.    0.78]]
0
0
1
A
[[ 1.    0.    0.    1.    0.49]
 [ 0.    0.    1.    1.    0.69]
 [ 1.    0.    1.    0.    0.55]
 [ 0.    1.    1.    0.    0.78]]
[[ 1.    0.    0.    1.    0.49]
 [ 0.    0.    1.    1.    0.69]
 [ 1.    0.    1.    0.    0.55]
 [ 0.    1.    1.    0.    0.78]]
[[ 1.    0.    0.    1.    0.49]
 [ 0.    0.    1.    1.    0.69]
 [ 0.    0.    1.   -1.    0.06]
 [ 0.    1.    1.    0.    0.78]]
[[ 1.    0.    0.    1.    0.49]
 [ 0.    0.    1.    1.    0.69]
 [ 0.    0.    1.   -1.    0.06]
 [ 0.    1.    1.    0.    0.78]]
```

```
1
2
3
A
[[ 1.    0.    0.    1.    0.49]
 [ 0.    1.    1.    0.    0.78]
 [ 0.    0.    1.   -1.    0.06]
 [ 0.    0.    1.    1.    0.69]]
[[ 1.    0.    0.    1.    0.49]
 [ 0.    1.    1.    0.    0.78]
 [ 0.    0.    1.   -1.    0.06]
 [ 0.    0.    1.    1.    0.69]]
[[ 1.    0.    0.    1.    0.49]
 [ 0.    1.    1.    0.    0.78]
 [ 0.    0.    1.   -1.    0.06]
 [ 0.    0.    1.    1.    0.69]]
2
2
[[ 1.    0.    0.    1.    0.49]
 [ 0.    1.    1.    0.    0.78]
 [ 0.    0.    1.   -1.    0.06]
 [ 0.    0.    0.    2.    0.63]]

B:
[[ 1.    0.    0.    1.    0.49]
 [ 0.    1.    1.    0.    0.78]
 [ 0.    0.    1.   -1.    0.06]
 [ 0.    0.    0.    2.    0.63]]

x [4] :  0.315

x [3] :  0.375

x [2] :  0.405

x [1] :  0.175

Or

Solution: x[1,2,3,...,n]= [ 0.175  0.405  0.375  0.315]
```

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0.175 \\ 0.405 \\ 0.375 \\ 0.315 \end{bmatrix}$$

Now comparing the values with **Table 1.** we have:

- Grid cell A contains healthy tissues
- Grid cell B contains bone structure
- Grid cell C contains tumorous tissues
- Grid cell D contains tumorous tissues

[Solved]

4.3.2 Using the python code for Matrix Inversion

Input. Coefficient matrix A and its corresponding constant matrix B, as prompted by the program.

Code. It is the same as shown in **4.2.5** labeled **Code. 2**

Output

```
Enter number of rows: 4

Enter space sep. matrix elements: 0 0 1 1 1 0 0 1 1 0 1 0 0 1 1 0


A:
[[0 0 1 1]
 [1 0 0 1]
 [1 0 1 0]
 [0 1 1 0]]

A_inv:
[[-0.5  0.5  0.5  0. ]
 [-0.5  0.5 -0.5  1. ]
 [ 0.5 -0.5  0.5  0. ]
 [ 0.5  0.5 -0.5  0. ]]

Enter number of rows: 4
Enter number of columns: 1

Enter space sep. matrix elements: 0.69 0.49 0.55 0.78


x [1] :  0.175
x [2] :  0.405
x [3] :  0.375
x [4] :  0.315

Or

Solution: x[1,2,3,....,n]= [ 0.175  0.405  0.375  0.315]
```

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0.175 \\ 0.405 \\ 0.375 \\ 0.315 \end{bmatrix}$$

Now comparing the values with **Table 1.** we have:

- Grid cell A contains healthy tissues
- Grid cell B contains bone structure
- Grid cell C contains tumorous tissues
- Grid cell D contains tumorous tissues

[Solved]

Conclusion

In the first chapter, I described the system of linear equations and how solving them gives different types of solutions, of which only the unique solution is preferred in this thesis. Then we talked about how CT works, how the scan takes data which we then use to form a system of linear equations.

In the next chapter we described the methods of solutions and their variations, which will then be used to solve the system of equations acquired using the mechanics shown in the first chapter.

Chapter three describes and explains the mechanics, purposes, etc. that gives a general detailed understanding of X-rays and CT to a non-medical personnel who may not know of these technology beforehand and who is attracted to this thesis mathematically only.

In the last chapter we make use of two arbitrary cases, where a patient is scanned and we have acquired the scan data. Then we combine the knowledge of the first two chapters to acquire a linear system, solve it using one or all the solutions methods and lastly cross reference the acquired solution values with a (collected) table given in the first chapter to identify whether those value represents bones, healthy or tumorous tissues.

In conclusion I was able to show what I wanted to, that is, using a system of linear equations acquired from the CT data to find and identify the objects present in a slice.

References

[1] Gustafsson, K. (2014). *The Role of Linear Algebra in Computed Tomography*.

[2] Lay, D. C., Lay, S. R., & McDonald, J. (2016). *Linear algebra and its applications* (5th ed.)

[3] Jabon, D., Nord, G., Wilson, B. W., Coffman, P., & Nord, J. (1996). Medical Applications of Systems of Linear Equations. *The Mathematics Teacher*, *89*(5), 398–402,408–410.

[4] Kuang, Y., & Kase, E. (2012). *Pre-Calculus For Dummies* (2nd ed.).

[5] Sastry, S. S. (2012-2013). *Introductory Methods of Numerical Analysis* (4th ed.).

[6] http://www.healthofchildren.com/U-Z/X-Rays.html

[7] http://www.healthofchildren.com/C/Computed-Tomography.html

[8] Burden, R. L., & Faires, J. D. (2011). Direct Methods for Solving Linear Systems. In *Numerical Analysis* (pp. 364–365).