Title: Analytical Solution of Non-Linear Partial Differential Equation by using the Extended (G'/G) Expansion Method with Non-Linear Auxiliary Equation

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Declaration

I do hereby declare that the thesis titled "Analytical Solution of Non-Linear Partial Differential Equation by using the Extended (G'/G) Expansion Method with Non-Linear Auxiliary Equation" is submitted to the Department of Mathematics and Natural Sciences of BRAC University in partial fulfilment of the Bachelor of Science in Mathematics. This is my original work and has not been submitted elsewhere for the award of any other degree or any other publication. Every work that has been used as reference for this work has been cited properly.

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Dedication

I would like to dedicate this work to my beloved Parents.

Acknowledgement

First of all, I would like to express all of my reverence and dedication to almighty creator who has made me able to finish this thesis successfully and created the earth in a diverse way by letting us the opportunity to study in different fields.

Now, I will thank my honorable supervisor Dr. Hasibun Naher because she has supported me in a kind and helpful way to complete this thesis step by step. It is not enough to show my all gratitude to her because if she was not beside me then it would be really tough for me to make this thesis paper finished effectively.

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At last, I would like to show my enormous thankfulness to my parents who have supported me in all stages of my life and my family members as well.

ABSTRACT

Among some new methods, these were introduced to find the exact solution of Non-Linear Partial Differential Equations (NLPDEs), (G'/G) expansion method proposed by Mingliang Wang, is straightforward and easy to handle as it gives rich new solutions. On the other hand, Solitons play a dynamic role in the field of engineering applications and, nonlinear science and it delivers more perception into the related nonlinear scientific occurrences by leading to forthcoming scientific research. So, to check the validity and effectiveness of our method we have implemented the extended (G'/G) expansion method to the (2+1) dimensional breaking soliton equation. The outcomes we have found here, are more common, successfully recovered the most of the earlier recognized results which have been established by other sophisticated methods. We have found some new results as well which will lead us to study some new phenomena in future. We have stated the travelling wave solutions here by three types of family. They are the hyperbolic family, the trigonometric family and, the rational family. The results along with the graphical illustration have revealed the high productivity of this algorithm with trustworthiness.

Contents

CHAPTER ONE

INTRODUCTION

1.1 Mathematical Preliminary

A dynamic and significant branch of modern mathematics is organized by the subject of differential equation. From the ancient time of the calculus, the subject has provided the mathematicians a proper neighborhood of countless hypothetical investigation and applied applications. Differential equation first came into the light with the invention of calculus by Newton and Leibnez. Isaac Newton had listed three kinds of differential equations as follows:

$$\frac{dy}{dx} = f(x) \tag{1.1.1}$$

$$\frac{dy}{dx} = f(x, y) \tag{1.1.2}$$

and $x_1 \frac{\partial y}{\partial x_1} + x_2 \frac{\partial y}{\partial x_2} = y$. (1.1.3)

He solved these difficulties (1.1.1, 1.1.2 and 1.1.3) and others connected to these by using infinite series and deliberated about the non-uniqueness of the results.

Jacob Bernoulli recommended an ordinary differential equation, the Bernoulli differential equation in 1695 which was as follows:

$$y' + P(x)y = Q(x)y^n \tag{1.1.4}$$

Differential Equation:

An enormous amount of real world phenomena involve moving quantities like the speed of a missile, the number of bacteria in a medium, the increase of currency, the voltage of an electrical signal, the intensity of an earthquake, the growth rate of population of species and so on. One of the most valuable successes of calculus is its' ability to capture continuous motion mathematically and giving us the opportunity to analyze that motion instantaneously. Differential equations infiltrate the science and let us use it as a tool by which we can try to bring out the laws of motion of nature in an abridged mathematical language. We have heard a lot about differential equation for radioactive decay in nuclear physics[1] but there are also many other numerous differential equations like Newton's law of cooling in thermodynamics[2], the Navier-Stokes equations in general relativity[3], the Blacke-Scloles equation in finance[4], the heat equation in thermodynamics[5], the Cauchy-Riemann equations in complex analysis[6], Schdinger equation in quantum mechanics[7], the wave equation[8], the Lotka-Volterra equation in population dynamics[9], Maxwell's equations in electromagnetism [10], Laplace's equation and Poisson's equation[11], Einstein's field equation in general relativity[12] and so on.

Definition of the differential equation: An equation connecting derivatives or differentials of one or more dependent variable with respect to one or more independent variable is defined as differential equation (DE). For example, an expression as follows -

$$\frac{d}{dx}(uv) = \frac{du}{dx} + \frac{dv}{dx} = 0$$

is the form of differential equation.

Classifications of Differential Equation:

Ordinary Differential Equations (ODEs) and **Partial Differential Equation (PDEs)** are two major classification of Differential equation**.** We will shortly discuss about this two classification shortly in below:

Ordinary Partial Differential Equations (ODE) - A differential equation is said to be an ordinary differential equation (ODE) if it involves ordinary derivatives of one or more dependent variables with respect to only one independent variable. For example, an expression as follows -

$$\frac{dy}{dx}+\frac{dz}{dx}=0$$

is the form of ordinary differential equation where $y=x+\sin x$ and $z=x^2+2x$

Partial Differential Equation (PDE) – A differential equation is said to be a partial differential equation (PDE) if it contains one or more partial derivatives of one or more dependent variable with respect to more than one independent variable. For example, an expression as follows -

$$\frac{dz}{dx}+\frac{dz}{dy}=0$$

is the form of partial differential equation where $z=x^5+\sin y$.

In this paper we will discuss about the field of partial differential equation. So, here we will talk about the classifications of the partial differential equations only. Partial differential equation is classified as **Linear Partial Differential Equations (LPDEs)** and **Non-Linear Partial Differential Equations (NLPDEs).**

Linear Partial Differential Equations (LPDEs) - A partial differential equation will be linear if the power of the dependent variable and each partial derivative contained in the equation is one, and the coefficients of each variable as well as the coefficients of each partial derivative are constants or independent variables. For example, an expression as follows -

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_1(x)y'+a_0(x)y=0$$

is the form of a non-linear partial differential equation.

Non-Linear Partial Differential Equations (NLPDEs) - The equation is said to be non-linear if any of the condition for being linear is not satisfied. For example, an expression as follows-

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_1(x)y'+a_0(x)y+y^2=0$$

is the form of non-linear partial differential equation.

Homogeneous Partial Differential Equations (HPDEs) - If every term of the P.D.E. contains the dependent variable x or one of its derivatives then it is called homogeneous partial differential equation. For example, an expression as follows-

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_1(x)y'+a_0(x)y=0$$

is the form of homogeneous partial differential equation.

Non-homogeneous Partial Differential Equations (IPDEs) - The equation is said to be non-homogeneous if any of the condition for being homogeneous is not satisfied. For example, an expression as follows –

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_1(x)y'+a_0(x)y=b.$$

1. 2 Waves

According to physics, a wave is a fluctuation accompanied by energy transmission. Wave motion actually transfers energy from one place to another by moving particles of the communication medium by amount of little or no related form of transportation. There are two main types of waves and one of these is Mechanical wave and the other one is Electromagnetic wave. Mechanical waves need a medium to continue the molecules travelling where the Electromagnetic wave does not require any medium. Electromagnetic wave is consist of periodic alternations of electrical and magnetic fields that is produced by charged particles.

The modest wave circulation is of the following form:

$$u_{tt} = v^2 u_{xx}$$

where $u(x,t)$ is the amplitude of the wave, and v is the wave speed. This equation can be represented through general d'Alembert' solution and that is-

$$u(x,t) = a(x - vt) + b(x + vt)$$

where a and b are uninformed constraints that denotes the left and right circulating respectively and this two individual waves circulate without altering its uniqueness.

1. 3 Soliton

Many physical occurrences have shown the appearances of solitons and been arisen as the solutions of an extensive session of weakly nonlinear dispersive partial differential equations which describes physical systems. The differences between solitons and solitary waves is indistinct in physical phenomena because soliton like solutions can define solitary waves of nonlinear equations. It describes wave actions in dispersive and dissipative media. A single soliton solution is often referred as a solitary wave, but for appearing more than one solution they are count as soliton. A nonlinear partial differential equation defines a soliton precisely if that shows the following properties:

(i) the solution should establish a wave of stable form;

(ii) the solution is localized, that means the solution either converges to a constant at infinity such as the solitons given by the Sine-Gordon equation or decays exponentially to zero such as the solitons provided by the KdV equation;

(iii) By preserving its own character, the soliton interacts with other solitons.

A solitary wave of basic form is as follows-

$$u(x,t) = f(x - vt).$$

where v is the wave speed of circulation. For $v > 0$, the wave travels in the positive x direction, while the wave travels in the negative x direction for $v < 0$. The solutions of nonlinear equations may be written in the form of $\sec h^2, \sec h, or \arctan\left(e^{\lambda(x-vt)}\right)$ function. Various types of method have been introduced to observe solitons.

1.4 Solitary Waves

The solitary waves was firstly observed by John Scott Russel in 1844. He observed “great wave of translation”, a large bulge of water that was slowly traveling along the Edinburgh-Glasgow canal holding its original shape for a long period. The finding is described here in Scott Russell's (1844) own words:

“I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.”

Russel was inspired by the surprising discovery to conduct physical laboratory experiments so that he can highlight his observance and study these solitary waves.

He empirically derived the relation in the following form:

$$c^2 = g(h+a).$$

here c is the speed of solitary wave, a is the maximum amplitude above the water surface, h is the finite depth and g is the acceleration of the gravity. When solitary wave was observed moving with the group velocity of the wave in the reference frame, the solitary wave was seen circulating without any sequential evolution in shape or size. The cover of the wave includes one universal peak and decays far away from the peak. Solitary waves arise in many circumstances, with the advancement of the surface of water and the intensity of light in optical fibers.

CHAPTER TWO

LITERATURE REVIEW

2.1 Analytical Methods:

The area of differential equation is vast enough to discuss about. Nonlinear occurrences exist in most of the fields of scientific or engineering field like fluid mechanics, optical fibers, chemical kinematics, plasma physics, biology, solid state physics, chemical physics, hydrodynamic, nonlinear optic, chemistry, geochemistry, ocean engineering, and meteorology and so on. In recent years so many effective expansion methods have been proposed to construct, develop and extend exact solutions of nonlinear evolution equation, such as the tanh-function expansion and its various extension[13], the Jacobi elliptic function expansion[14], the F-expansion[15], the sub-ODE method[16], the homogeneous balance method[17], the sine-cosine method[18], the Exp-function method[19,58], inverse scattering method[20], Hirota's bilinear transformation[21], the tanh-coth method [22], extended tanh method[23], the Darboux transformation[24], Backlund transformation[25], Bethe Ansatz method[26], Wrongskian technique[27], truncated Painlevé expansion method[28], symmetry method[29], the generalized Riccati euation method[30,57,59-61], the variational iteration method[31], the direct algebraic method[32], the homotopy perturbation methods[33], the rank analysis method[34], various types of (G'/G) expansion method[35-40,42-56], and others, but we cannot deal with every nonlinear evolution equation with one unified method only. We will discuss about the extended (G'/G) expansion method in this paper for finding out the analytical solution of nonlinear partial differential equation.

2.2 Basic (G'/G) Expansion Methods:

Wang has introduced the (G'/G) Expansion method to solve nonlinear problems and construct traveling wave solutions of different kinds of NLEEs, which is one of the strongest methods. In (G'/G) expansion method a second order linear ordinary differential equation is executed that is

$$G''+\lambda G'+\mu G=0$$

where λ and μ are arbitrary constants.

For showing the effectiveness of the (G'/G) expansion method many researchers have carried out many researches i.e. Zhang has extended the (G'/G) expansion method and named it improved (G'/G) expansion method. The difference between the original and extended (G'/G) expansion method is that,

In original method- $u(\xi)=\sum_{i=0}^{m} a_i (G'/G)^i$, where $a_m \neq 0$, on the other hand

In Zhang's method- $u(\xi)=\sum_{i=-m}^{m} a_i (G'/G)^i$, where $a_m \neq o$ or $a_{-m} \neq o$ but both cannot be zero instantaneously.

The extended (G'/G) expansion method for getting traveling wave solution of the Whithsm-Broer-Kaup-like method and couple Hirota-Satsuma KdV equations are acquaint with Guo and Zhou in the form, $u(\xi)=a_0+\sum_{i=1}^{m}\{a_i(G'/G)^i+b_i(G'/G)^{i-1}\sqrt{\sigma(1+(\frac{1}{\mu})(G'/G)^2)}\}$

Here, the basic (G'/G) expansion method will be described for finding travelling wave solutions of nonlinear evolution equation.

Considering a nonlinear equation of two independent variables x and t, is in the form of-

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0 \qquad (2.2.1)$$

here $u(x,t) = u(\xi)$ is unidentified function, a polynomial function P is in $u = u(x,t)$ and nonlinear terms and the highest order derivatives are related to its several partial derivatives. We will show the basic steps of the (G'/G) expansion method in the following steps.

<u>Step-1:</u> we have joined two independent variables x and t into one variable $\xi = x - vt$ and then we have presume that,

$$u(x,t) = u(\xi),\ \xi = x - vt \qquad (2.2.2)$$

We reduced Eq. (2.2.1) to an ODE with the help of the travelling wave variable (2.2.2) for $u = u(\xi)$

$$P(u, -vu', u', v^2u'', -vu'', u'', \ldots) = 0 \qquad (2.2.3)$$

<u>Step-2-</u> We expressed the ODE (2.2.3) by a polynomial in (G'/G) as follows:

$$u(\xi) = a_m \left(\frac{G'}{G}\right)^m + \ldots \qquad (2.2.4)$$

Where $G = G(\xi)$ satisfies the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0 \qquad (2.2.5)$$

$a_m,...,\lambda$ and μ are constants to be determined later, $a_m \neq 0$, the unwritten part in (2.2.4) is also a polynomial in $\left(\frac{G'}{G}\right)$, but the degree of which is generally equal to or less than $(m-1)$, the positive integer can be observed by considering the homogeneous balance between the highest order derivatives and nonlinear terms performing in ODE (2.2.1).

Step-3- By replacing (2.2.4) into Eq. (2.2.3) and using second order LODE (2.2.5), gathering all terms with the same order of $\left(\frac{G'}{G}\right)$ together, and the left hand side of Eq. (2.2.3) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Calculating each co-efficient of this polynomial to zero, produces a set of algebraic equations for $a_m,...,v,\lambda$ and μ.

Step-4- Supposing that the constraints $a_m,...,v,\lambda$ and μ can be acquired by resolving the algebraic equations in Step 3, meanwhile the general results of the second order LODE (2.2.5) have been well identified for us, then switching $a_m,...,v$ and the general solutions of Eq. (2.2.5) into (2.2.4) we get more travelling solutions of the nonlinear evolution equation (2.2.1)

Chapter Three

Methodology

Extended (G'/G) Expansion Method with Non-Linear Auxiliary Equation:

A NLPDE has been considered as of the following form:

$$P\left(u, u_t, u_x, u_y, u_z, u_{tt}, u_{xt}, u_{xx}, u_{xxx} \ldots\right) = 0, \tag{3.1}$$

here P is a polynomial function of $u(x, y, t)$ whose partial derivative $u = u(x, y, t)$ is an unidentified function of x, y and t that includes the highest number of derivatives and nonlinear terms.

The most significant procedures of the method are as follows:

<u>Step 1</u>- Assume that, ξ is the combination of real variables x, y and t such that

$$u(x, y, t) = u(\xi), \quad \xi = x + y \pm ct \tag{3.2}$$

and c is the speed of the travelling wave.

The travelling wave transformation Eq. (3.2) lets us to transform Eq. (3.1) into an ODE form for $u = u(\xi)$, denoted by:

$$Q(u, u', u'', u''', \ldots) = 0 \tag{3.3}$$

here Q is a polynomial of $u(\xi)$ and the superscripts specify the ordinary derivatives with respect to ξ.

Step 2- Allowing to option, we can integrate Eq. (3.3) for one or more times term by term and that produces constant(s) of integration. We will consider the integral constant possible to zero for easiness.

Step 3- we may accept that, the solution of Eq. (3.3) can be stated in the subsequent form:

$$u(\xi)=\sum_{r=0}^{M}a_r\left[\beta(\xi)\right]^r+\sum_{r=1}^{M}b_r\left[\beta(\xi)\right]^{-r}, \tag{3.4}$$

where $\beta(\xi)=\lfloor d+\lambda(\xi)\rfloor$ and $\lambda(\xi)$ is:

$$\lambda(\xi)=\left(G'(\xi)/G(\xi)\right). \tag{3.5}$$

Here, a_M or b_M may individually be zero, but a_M and b_M cannot be zero at the same time.Here, $a_r\left(r=0,1,2,...,M\right), b_r\left(r=1,2,...,M\right)$ and d are arbitrary constants which will be determined later.

Now, $G=G(\xi)$ satisfies the second order NLODE:

$$AGG''-BGG'-C\left(G'\right)^2-EG^2=0, \tag{3.6}$$

here prime of G signifies the derivative with respect to ξ. A,B,C and E are real parameters.

Step 4- By taking the homogeneous balance between the highest degree nonlinear terms and the highest derivatives acting in Eq. (3.3), we can find the limiting value, M.

Step 5- By switching Eq. (3.4) together with Eq. (3.5) and Eq. (3.6) into Eq. (3.3) and gathering polynomials in $\left(d+\lambda(\xi)\right)^M\left(M=0,1,2,...\right)$ and $\left(d+\lambda(\xi)\right)^{-M}\left(M=1,2,3,...\right)$, we set all coefficients of the resulted polynomials to zero which produces a set of algebraic equations for $a_r\left(r=0,1,2,...,M\right)$, $b_r\left(r=1,2,...,M\right)$, d and c.We can get the values of unknown parameters by resolving the system of algebraic equations.

Step 6- From the general solution of Eq. (3.6), we find the following form,

Family 1- Hyperbolic function:

$$\lambda(\xi)=\left(\frac{G'}{G}\right)=\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}, \tag{3.7}$$

when $B\neq 0, \Psi=A-C$ and $\Omega=B^2+4E(A-C)>0$ and θ_1,θ_2 are arbitrary constants.

Family 2- trigonometric function:

$$\lambda(\xi)=\left(\frac{G'}{G}\right)=\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}, \tag{3.8}$$

when $B\neq 0, \Psi=A-C$ and $\Omega=B^2+4E(A-C)<0$ and θ_1,θ_2 are arbitrary constants.

Family 3- rational form:

$$\lambda(\xi)=\left(\frac{G'}{G}\right)=\frac{B}{2\Psi}+\frac{\theta_2}{\theta_1+\theta_2\xi}, \tag{3.9}$$

when $B\neq 0, \Psi=A-C$ and $\Omega=B^2+4E(A-C)=0$ and θ_1,θ_2 are arbitrary constants.

Family 4- hyperbolic form:

$$\lambda(\xi)=\left(\frac{G'}{G}\right)=\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}, \tag{3.10}$$

when $B=0, \Psi=A-C$ and $\Delta=\Psi E>0$ and θ_1,θ_2 are arbitrary constants.

Family 5- trigonometric form:

$$\lambda(\xi)=\left(\frac{G'}{G}\right)=\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}, \tag{3.11}$$

when $B=0, \Psi=A-C$ and $\Delta=\Psi E<0$ and θ_1,θ_2 are arbitrary constants.

we have got these five types of family for the solution. We will use this families by providing the values of $\left(\frac{G'}{G}\right)$ for finding the result in chapter Four.

Chapter Four

Application of the method

4.1 (2+1) dimensional breaking soliton equations:

Solitons have a vital role in nonlinear science and engineering applications as it provide more insight into the relevant nonlinear science phenomena that leading us to the future scientific features. At present, many mathematicians and physicists have been attracted through breaking soliton equations and to describe the $(2+1)$ dimensional interaction of Riemann wave propagation, these equations have been used with the long-wave propagation. The initial soliton equation was KdV equation which was derived by Kortweg and de Vries to model the evolution of shallow water in 1985. Yet Zabusky and Kruskal had presented the concept of soliton in 1965.

Some of the nonlinear evolution equations are integrable that provide multiple soliton solutions and holds sufficiently large number of conservation laws which plays a vital role in solitary waves theory.

The $(2+1)$ dimensional breaking soliton equation can be written as:

$$u_{xt} - 4u_x u_{xy} - 2u_{xx}u_y + u_{xxxy} = 0 \tag{4.1.1}$$

This equation was used to describe the $(2+1)$ dimensional interaction of the Riemann wave propagation along the $y-$axis with a long wave propagated along the $x-$axis. For $y = x$ and by integrating the resulting Eq. (4.1.1), the equation can be reduced to the KdV equation.

Even, Eq. (4.1.1) is studied using the homogeneous balance principle followed by the Hirota's method.

4.2 Algebraic equations of the (2+1) dimensional breaking soliton equations by the method:

Let us assume that the (2+1)-dimensional Breaking Soliton equations in the form:

$$\begin{aligned} &u_t + \alpha u_{xxy} + 4\alpha u\, v_x + 4\alpha u_x v = 0, \\ &u_y = v_x, \end{aligned} \tag{4.2.1}$$

where $u = u(x, y, t)$, $v = v(x, y, t)$ and α is an arbitrary function.

Eq. (4.2.1) demonstrates the (2+1)-dimensional relations of the Riemann wave circulating along the y-axis with a long wave transmission along the x-axis.

Now, we use the wave transformation Eq. (3.2) into the Eq. (4.2.1), which produces:

$$\begin{aligned} &-cu' + \alpha u''' + 4\alpha u v' + 4\alpha u' v = 0, \\ &u' = v'. \end{aligned} \tag{4.2.2}$$

We can integrate Eq. (4.2.2). After that, setting all the constant of integration to zero and integrating twice with respect to ξ, we can find that:

$$u = v$$

$$-cu + 4\alpha u^2 + \alpha u'' = 0. \tag{4.2.3}$$

Taking the homogeneous stability between the highest order derivative term u'' and the highest non-linear term u^2, we attain that, $M = 2.$

Therefore, the result of Eq. (4.2.3) goes into the following form:

$$u(\xi) = a_0 + a_1\beta(\xi) + a_2\{\beta(\xi)\}^2 + b_1\{\beta(\xi)\}^{-1} + b_2\{\beta(\xi)\}^{-2}, \tag{4.2.4}$$

Where a_0, a_1, a_2, b_1 and b_2 are constants which we will determine later.

We will discuss about the five types of families which is mentioned in methodology section.

Replacing Eq. (4.2.4) together with Eq. (3.5) and (3.6) into Eq. (4.2.3), the left-hand side has transformed into polynomials in $(d+\lambda)^{M}$ $(M=0,1,2,...)$ and $(d+\lambda)^{-M}$ $(M=1,2,3,...)$.

We gather all coefficients of these resulted polynomials to zero, produces a set of algebraic equations for $a_0, a_1, a_2, b_1, b_2, d$ and c are as follows:

$$(d+\frac{G'}{G})^4 : 6\alpha C^2 a_2 - 12\alpha C a_2 A + 6\alpha A^2 a_2 + 4\alpha A^2 a_2^2,$$

$$(d+\frac{G'}{G})^3 : -20\alpha A^2 a_2 d + 2\alpha A^2 a_1 + 10\alpha B a_2 C + 2\alpha C^2 a_1 + 40\alpha C a_2 A d - 4\alpha C a_1 A$$
$$+8\alpha A^2 a_1 a_2 - 20\alpha C^2 a_2 d - 10\alpha B a_2 A,$$

$$(d+\frac{G'}{G})^2 : 8\alpha C a_2 E + 24\alpha C^2 a_2 d^2 - 8\alpha A a_2 E - 48\alpha C a_2 A d^2 + 4\alpha B^2 a_2$$
$$+12\alpha C a_1 A d - 3\alpha B a_1 A - 24\alpha B a_2 C d + 8\alpha A^2 a_0 a_2 + 4\alpha A^2 a_1^2 + 3\alpha B a_1 C$$
$$-cA^2 a_2 + 24\alpha A^2 a_2 d^2 - 6\alpha C^2 a_1 d + 24\alpha B a_2 A d - 6\alpha A^2 a_1 d,$$

$$(d+\frac{G'}{G})^1 : 6\alpha B a_1 A d - 6\alpha B a_1 C d + 6\alpha A^2 d^2 a_1 + 24\alpha C d^3 a_2 A - 12\alpha C d^2 a_1 A + 6\alpha B a_2 E$$
$$-18\alpha B a_2 A d^2 + 2\alpha E a_1 C - 12\alpha C^2 d^3 a_2 - cA^2 a_1 - 12\alpha A^2 d^3 a_2 + 6\alpha C^2 d^2 a_1 + 18\alpha B a_2 C d^2$$
$$+8\alpha A^2 a_0 a_1 + 12\alpha A d a_2 E - 6\alpha B^2 a_2 d + \alpha B^2 a_1 + 8\alpha A^2 a_2 b_1 - 12\alpha C d a_2 E - 2\alpha E a_1 A,$$

$$(d+\frac{G'}{G})^0 : 4\alpha A^2 a_0^2 + 2\alpha C^2 b_2 - cA^2 a_0 + 2\alpha A^2 b_2 + 2\alpha a_2 E^2 + 2\alpha B^2 d^2 a_2 - \alpha B^2 d a_1 - 2\alpha C^2 b_1 d + \alpha C b_1 B$$
$$-4\alpha C b_2 A + 2\alpha C^2 d^4 a_2 - 2\alpha C^2 d^3 a_1 - 2\alpha A^2 b_1 d - \alpha A b_1 B + 2\alpha A^2 d^4 a_2 - 2\alpha A^2 d^3 a_1 + \alpha E a_1 B$$
$$+8\alpha A^2 a_1 b_1 + 8\alpha A^2 a_2 b_2 - 4\alpha C d^4 a_2 A + 4\alpha C d^3 a_1 A + 4\alpha C d^2 a_2 E - 4\alpha A d^2 a_2 E + 2\alpha E a_1 A d - 2\alpha E a_1 C d$$
$$-4\alpha B d^3 a_2 C + 4\alpha B d^3 a_2 A - 3\alpha B d^2 a_1 A + 3\alpha B d^2 a_1 C - 4\alpha B d a_2 E + 4\alpha C b_1 A d,$$

$$(d+\frac{G'}{G})^{-1} : 6\alpha B b_2 C - 12\alpha A^2 b_2 d + 6\alpha A^2 b_1 d^2 + 6\alpha B b_1 A d - 6\alpha B b_2 A + \alpha B^2 b_1 - 12\alpha C^2 b_2 d$$
$$+8\alpha A^2 a_0 b_1 - 12\alpha C b_1 A d^2 + 8\alpha A^2 a_1 b_2 + 24\alpha C b_2 A d - cA^2 b_1 + 6\alpha C^2 b_1 d^2 - 2\alpha A b_1 E - 6\alpha B b_1 C d + 2\alpha C b_1 E,$$

$(d+\frac{G'}{G})^{-2}: 24\alpha A^2 b_2 d^2 + 4\alpha B^2 b_2 + 4\alpha A^2 b_1^2 - 48\alpha C b_2 A d^2 - 6\alpha C d b_1 E - 8\alpha A b_2 E - 3\alpha B^2 b_1 d + 12\alpha C d^3 b_1 A$
$+24\alpha C^2 b_2 d^2 + 8\alpha C b_2 E - 6\alpha C^2 d^3 b_1 - 9\alpha B b_1 A d^2 - cA^2 b_2 + 3\alpha B b_1 E + 8\alpha A^2 a_0 b_2 - 24\alpha B b_2 C d + 6\alpha A d b_1 E$
$+9\alpha B b_1 C d^2 + 24\alpha B b_2 A d - 6\alpha A^2 d^3 b_1$

$(d+\frac{G'}{G})^{-3}: 8\alpha A^2 b_1 b_2 + 20\alpha A d b_2 E + 4\alpha C d^2 b_1 E + 4\alpha B d^3 b_1 A - 4\alpha C d^4 b_1 A + 10\alpha B b_2 E$
$-30\alpha B b_2 A d^2 + 2\alpha b_1 E^2 + 40\alpha C d^3 b_2 A + 2\alpha A^2 d^4 b_1 - 4\alpha B d^3 b_1 C - 20\alpha C^2 d^3 b_2 + 2\alpha B^2 d^2 b_1$
$-4\alpha A d^2 b_1 E - 20\alpha C d b_2 E + 30\alpha B b_2 C d^2 + 2\alpha C^2 d^4 b_1 - 4\alpha B d b_1 E - 10\alpha B^2 b_2 d - 20\alpha A^2 d^3 b_2$

$(d+\frac{G'}{G})^{-4}: -12\alpha B d b_2 E + 6\alpha b_2 E^2 + 6\alpha B^2 d^2 b_2 - 12\alpha B d^3 b_2 C + 12\alpha C d^2 b_2 E + 12\alpha B d^3 b_2 A$
$+6\alpha C^2 d^4 b_2 + 6\alpha A^2 d^4 b_2 - 12\alpha A d^2 b_2 E + 4\alpha A^2 b_2^2 - 12\alpha C d^4 b_2 A$

4.3 Results:

Result 1:

$$a_0 = \frac{-3}{2}\frac{\left(d^2\Psi^2 + Bd\Psi - E\Psi\right)}{A^2},\ a_1 = \frac{3}{2}\frac{\left(2d\Psi^2 + B\Psi\right)}{A^2}, a_2 = \frac{-3}{2}\frac{\Psi^2}{A^2}, b_1 = 0, b_2 = 0,$$
$$c = \frac{\alpha\left(B^2 + 4E\Psi\right)}{A^2}, d = d, \tag{4.3.1}$$

where $\Psi = A - C, A, B, C$,d and E are free constraints.

Result 2:

$$a_0 = \frac{3}{8}\frac{\left(B^2 + 4E\Psi\right)}{A^2},\ a_1 = 0, a_2 = 0, b_1 = 0, b_2 = \frac{-3}{32}\frac{\left(B^2 + 4E\Psi\right)^2}{A^2\Psi^2},$$
$$c = \frac{\alpha\left(B^2 + 4E\Psi\right)}{A^2}, d = \frac{-1}{2}\frac{B}{\Psi}, \tag{4.3.2}$$

where $\Psi = A - C, A, B, C$,d and E are free constraints.

Result 3:

$$a_0 = \frac{-3}{2}\frac{\left(d^2\Psi^2 + Bd\Psi - E\Psi\right)}{A^2},\ a_1 = 0, a_2 = 0, b_1 = \frac{3}{2}\frac{\left(2d^3\Psi^2 + 3d^2B\Psi - 2Ed\Psi + dB^2 - BE\right)}{A^2},$$

$$c = \frac{\alpha\left(B^2 + 4E\Psi\right)}{A^2}, d = d, b_2 = \frac{-3}{2}\frac{\left(d^4\Psi^2 + 2d^3B\Psi - 2Ed^2\Psi + d^2B^2 - 2dBE + E^2\right)}{A^2}, \tag{4.3.3}$$

where $\Psi = A - C, A, B, C$,d and E are free constraints.

Result 4:

$$a_0 = \frac{-1}{4}\frac{\left(B^2 + 4E\Psi\right)}{A^2},\ a_1 = 0, a_2 = \frac{-3}{2}\frac{\Psi^2}{A^2}, b_1 = 0,\ d = \frac{-1}{2}\frac{B}{\Psi},$$
$$c = \frac{-4\alpha\left(B^2 + 4E\Psi\right)}{A^2}, b_2 = \frac{-3}{32}\frac{\left(16E^2\Psi^2 + 8EB^2\Psi + B^4\right)}{A^2\Psi^2}, \tag{4.3.4}$$

where $\Psi = A - C, A, B, C$,d and E are free constraints.

Result 5:

$$a_0 = \frac{3}{4}\frac{\left(B^2 + 4E\Psi\right)}{A^2},\ a_1 = 0, a_2 = \frac{-3}{2}\frac{\Psi^2}{A^2}, b_1 = 0,\ d = \frac{-1}{2}\frac{B}{\Psi},$$
$$c = \frac{4\alpha\left\{B^2 + 4E\Psi\right\}}{A^2}, b_2 = \frac{-3\left\{16E^2\Psi^2 + B^4 + 8E\Psi B^2\right\}}{32A^2\Psi^2}, \tag{4.3.5}$$

where $\Psi = A - C, A, B, C$,d and E are free constraints.

Result 6:-

$$a_0 = -\frac{1}{4}\frac{(B^2 - 2\Psi E + 6\Psi dB + 6\Psi^2 d^2)}{A^2}, a_1 = 0, a_2 = 0, d = d, c = -\alpha\frac{4\Psi E + B^2}{A^2},$$

$$b_1 = \frac{3}{2}\frac{(2\Psi^2 d^3 + 3\Psi d^2 B - 2\Psi dE + B^2 d - BE)}{A^2}, b_2 = -\frac{3}{2}\frac{(\Psi^2 d^4 + 2\Psi B d^3 - 2\Psi dE + E^2 + B^2 d^2 - 2BdE)}{A^2} \tag{4.3.6}$$

where $\Psi = A - C, A, B, C$,d and E are free constraints.

4.4 Solutions of the mentioned equations by applying the method:

Solution 1: Hyperbolic Function solutions:

Substituting Eq. (4.3.1) to Eq. (4.3.6) including Eq. (3.7) into Eq. (4.2.4) and simplifying, we get following travelling wave solutions:

when, $B\neq 0, \Psi = A-C, \Omega = B^2+4E\left(A-C\right)>0, \xi = x+y\pm ct$ and θ_1,θ_2 are arbitrary constants.

$$u1_1(x,y,t) = -\frac{3}{2}\frac{(\Psi dB+\Psi^2d^2-\Psi E)}{A^2}+\frac{3}{2}\frac{(2\Psi^2 d+\Psi B)}{A^2}\left[d+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right]$$

$$-\frac{3}{2}\frac{\Psi^2}{A^2}\left[d+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right]^2-\Psi d\sqrt{\Omega}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}-\frac{B\sqrt{\Omega}}{2}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}$$

$$=\frac{3}{2A^2}\left[\Psi E+\frac{B^2}{4}-\frac{\Omega}{4}\frac{\cosh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\sinh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right]\qquad\left[when\quad \theta_1=0\ \ but\ \ \theta_2\neq 0\right]$$

$$\therefore u1_1(x,y,t)=\frac{3}{2A^2}\left[E\Psi+\frac{1}{4}\left(B^2-\Omega\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)\right]$$

$$v1_1(x,y,t)=u1_1(x,y,t)=\frac{3}{2A^2}\left[E\Psi+\frac{1}{4}\left(B^2-\Omega\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)\right] \qquad (4.4.1)$$

$$u1_2(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3}{32}\frac{\left(B^2+4E\Psi\right)^2}{A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)^{-2}$$

$$\Rightarrow u1_2(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3}{32}\frac{\left(B^2+4E\Psi\right)^2}{A^2\Psi^2}\left(\frac{\sqrt{\Omega}}{2\Psi}\frac{\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)^{-2}\quad\left[when\quad \theta_1=0\ \ but\ \ \theta_2\neq 0\right]$$

$$\therefore u1_2(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}\left(1-\frac{\left(B^2+4E\Psi\right)}{\Omega}\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)$$

$$v1_2(x,y,t)=u1_2(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}\left(1-\frac{1}{4}\frac{\left(B^2+4E\Psi\right)}{\Omega}\tan^2\text{h}\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)\qquad(4.4.2)$$

$$u1_3(x,y,t)=-\frac{3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2}\left(d+\left(\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)\right)^{-1}$$

$$-\frac{3}{2}\frac{\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2}\left(d+\left(\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)\right)^{-2}$$

$$\therefore u1_3(x,y,t)=-\frac{3}{2A^2}\left[\begin{array}{l}\left(d^2\Psi^2+Bd\Psi-E\Psi\right)-\dfrac{2\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{2d\Psi+B+\sqrt{\Omega}\coth\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\\+\dfrac{4\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{\left(2d\Psi+B+\sqrt{\Omega}\coth\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)^2}\end{array}\right]\left[when\quad \theta_1=0\ \ but\ \ \theta_2\neq 0\right]$$

$$v1_3(x,y,t)=u1_3(x,y,t)=-\frac{3}{2A^2}\left[\begin{array}{l}\left(d^2\Psi^2+Bd\Psi-E\Psi\right)-\dfrac{2\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{2d\Psi+B+\sqrt{\Omega}\coth\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\\+\dfrac{4\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{\left(2d\Psi+B+\sqrt{\Omega}\coth\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)^2}\end{array}\right]\qquad(4.4.3)$$

$$u1_4(x,y,t)=-\frac{1}{4}\frac{(B^2+4E\Psi)}{A^2}-\frac{3}{2}\frac{\Psi^2}{A^2}\left(-\frac{1}{2}\frac{B}{\Psi}+\left(\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)\right)^2$$

$$-\frac{3}{32}\frac{(B^4+16E^2\Psi^2+8\Psi B^2E)}{A^2\Psi^2}\left[-\frac{1}{2}\frac{B}{\Psi}+\left(\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)\right]^{-2}$$

$$\therefore u1_4(x,y,t)=-\frac{1}{4}\frac{(B^2+4E\Psi)}{A^2}-\frac{3\Omega}{4A^2}\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)-\frac{3}{8}\frac{(B^2+4E\Psi)^2}{A^2\Omega}\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\qquad[when\quad \theta_1=0\ \ but\ \ \theta_2\neq 0]$$

$$v1_4(x,y,t)=u1_4(x,y,t)=-\frac{1}{4}\frac{(B^2+4E\Psi)}{A^2}-\frac{3\Omega}{4A^2}\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)-\frac{3}{8}\frac{(B^2+4E\Psi)^2}{A^2\Omega}\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\qquad(4.4.4)$$

$$u1_5(x,y,t)=\frac{3}{4}\frac{(4\Psi E+B^2)}{A^2}-\frac{3}{2}\frac{\Psi^2}{A^2}\left(-\frac{1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)^2$$

$$-\frac{3}{32}\frac{(16\Psi^2E^2+8\Psi EB^2+B^4)}{A^2\Psi^2}\left(-\frac{1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u1_5(x,y,t)=\frac{3}{4}\frac{(4\Psi E+B^2)}{A^2}-\frac{3}{8}\frac{\Omega}{A^2}\left(\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)-\frac{3}{8}\frac{(B^2+4E\Psi)^2}{A^2\Omega}\left(\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)\quad[when\quad \theta_1=0\ \ but\ \ \theta_2\neq 0]$$

$$v1_5(x,y,t)=u1_5(x,y,t)=\frac{3}{4}\frac{(4\Psi E+B^2)}{A^2}-\frac{3}{8}\frac{\Omega}{A^2}\left(\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)-\frac{3}{8}\frac{(B^2+4E\Psi)^2}{A^2\Omega}\left(\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)\qquad(4.4.5)$$

$$u1_6(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2 d^2)}{A^2}+\frac{3}{2}\frac{(2\Psi^2 d^3+3\Psi d^2 B-2\Psi dE+B^2 d-BE)}{A^2}\left[d+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right]^{-1}$$

$$-\frac{3}{2}\frac{(\Psi^2 d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{A^2}\left[d+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right]^{-2}$$

$$\therefore u1_6(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2 d^2)}{A^2}+\frac{3}{2}\frac{(2\Psi^2 d^3+3\Psi d^2 B-2\Psi dE+B^2 d-BE)2\Psi}{A^2\left[2d\Psi+B+\sqrt{\Omega}\coth\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right]}$$

$$-\frac{6\Psi^2(\Psi^2 d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{A^2\left[2d\Psi+B+\sqrt{\Omega}\coth\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right]^2}\quad\left[when\quad \theta_1=0\ \ but\ \ \theta_2\neq 0\right]$$

$$v1_6(x,y,t)=u1_6(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2 d^2)}{A^2}+\frac{3}{2}\frac{(2\Psi^2 d^3+3\Psi d^2 B-2\Psi dE+B^2 d-BE)2\Psi}{A^2\left[2d\Psi+B+\sqrt{\Omega}\coth\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right]}$$

$$-\frac{6\Psi^2(\Psi^2 d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{A^2\left[2d\Psi+B+\sqrt{\Omega}\coth\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right]^2}\tag{4.4.6}$$

$$u1_7(x,y,t)=-\frac{3}{2}\frac{(\Psi dB+\Psi^2 d^2-\Psi E)}{A^2}+\frac{3}{2}\frac{(2\Psi^2 d+\Psi B)}{A^2}\left|d+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right|$$

$$-\frac{3}{2}\frac{\Psi^2}{A^2}\left[d+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right]^2$$

$$\therefore u1_7(x,y,t)=\frac{3}{2A^2}\left[E\Psi+\frac{1}{4}\left(B^2-\Omega\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)\right]\qquad\left[when\quad \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$v1_7(x,y,t)=u1_7(x,y,t)=\frac{3}{2A^2}\left[E\Psi+\frac{1}{4}\left(B^2-\Omega\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)\right] \qquad (4.4.7)$$

$$u1_8(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3}{32}\frac{\left(B^2+4E\Psi\right)^2}{A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u1_8(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}\left(1-\frac{1}{4}\frac{\left(B^2+4E\Psi\right)}{\Omega}\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)\left[when\quad \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$v1_8(x,y,t)=u1_8(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}\left(1-\frac{\left(B^2+4E\Psi\right)}{\Omega}\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right) \qquad (4.4.8)$$

$$u1_9(x,y,t)=-\frac{3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2}\left(d+\left(\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)\right)^{-1}$$

$$-\frac{3}{2}\frac{\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2}\left(d+\left(\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)\right)^{-2}$$

$$\therefore u1_9(x,y,t)=-\frac{3}{2A^2}\left[\begin{array}{l}\left(d^2\Psi^2+Bd\Psi-E\Psi\right)-\dfrac{2\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{2d\Psi+B+\sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\\+\dfrac{4\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{\left(2d\Psi+B+\sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)^2}\end{array}\right]$$

$$\left[when\quad \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$vl_9(x,y,t)=ul_9(x,y,t)=-\frac{3}{2A^2}\left[\begin{array}{l}\left(d^2\Psi^2+Bd\Psi-E\Psi\right)-\dfrac{2\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{2d\Psi+B+\sqrt{\Omega}\tanh\left(\dfrac{\sqrt{\Omega}}{2\Psi}\xi\right)}\\+\dfrac{4\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{\left(2d\Psi+B+\sqrt{\Omega}\tanh\left(\dfrac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)^2}\end{array}\right] \tag{4.4.9}$$

$$ul_{10}(x,y,t)=-\frac{1}{4}\frac{(B^2+4E\Psi)}{A^2}-\frac{3}{2}\frac{\Psi^2}{A^2}\left(-\frac{1}{2}\frac{B}{\Psi}+\left(\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)\right)^2$$

$$-\frac{3}{32}\frac{(B^4+16E^2\Psi^2+8\Psi B^2E)}{A^2\Psi^2}\left[-\frac{1}{2}\frac{B}{\Psi}+\left(\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)\right]^{-2}$$

$$\therefore ul_{10}(x,y,t)=-\frac{1}{4}\frac{(B^2+4E\Psi)}{A^2}-\frac{3\Omega}{4A^2}\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)-\frac{3}{8}\frac{(B^2+4E\Psi)^2}{A^2\Omega}\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\left[when\quad \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$vl_{10}(x,y,t)=ul_{10}(x,y,t)=-\frac{1}{4}\frac{(B^2+4E\Psi)}{A^2}-\frac{3\Omega}{4A^2}\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)-\frac{3}{8}\frac{(B^2+4E\Psi)^2}{A^2\Omega}\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right) \tag{4.4.10}$$

$$ul_{11}(x,y,t)=\frac{3}{4}\frac{(4\Psi E+B^2)}{A^2}-\frac{3}{2}\frac{\Psi^2}{A^2}\left(-\frac{1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)^2$$

$$-\frac{3}{32}\frac{(16\Psi^2E^2+8\Psi EB^2+B^4)}{A^2\Psi^2}\left(-\frac{1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right)^{-2}$$

$$\therefore ul_{11}(x,y,t)=\frac{3}{4}\frac{(4\Psi E+B^2)}{A^2}-\frac{3}{8}\frac{\Omega}{A^2}\left(\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)-\frac{3}{8}\frac{(B^2+4E\Psi)^2}{A^2\Omega}\left(\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)\left[when\quad \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$vl_{11}(x,y,t)=ul_{11}(x,y,t)=\frac{3}{4}\frac{(4\Psi E+B^2)}{A^2}-\frac{3}{8}\frac{\Omega}{A^2}\left(\tanh^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right)-\frac{3}{8}\frac{(B^2+4E\Psi)^2}{A^2\Omega}\left(\coth^2\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right) \qquad (4.4.11)$$

$$ul_{12}(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}+\frac{3}{2}\frac{(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)}{A^2}\left[d+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right]^{-1}$$

$$-\frac{3}{2}\frac{(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{A^2}\left[d+\frac{B}{2\Psi}+\frac{\sqrt{\Omega}}{2\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)}\right]^{-2}$$

$$\therefore ul_{12}(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}+\frac{3}{2}\frac{(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)2\Psi}{A^2\left[2d\Psi+B+\sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right]}$$

$$-\frac{3}{2}\frac{(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)4\Psi^2}{A^2\left[2d\Psi+B+\sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right]^2}\left[when\quad \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$vl_{12}(x,y,t)=ul_{12}(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}$$

$$+\frac{3}{2}\frac{(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)2\Psi}{A^2\left[2d\Psi+B+\sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right]}-\frac{3}{2}\frac{(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)4\Psi^2}{A^2\left[2d\Psi+B+\sqrt{\Omega}\tanh\left(\frac{\sqrt{\Omega}}{2\Psi}\xi\right)\right]^2} \qquad (4.4.12)$$

Solution 2: Trigonometric solutions:

Substituting Eq. (4.3.1) to Eq. (4.3.6) including Eq. (3.8) into Eq. (4.2.4), we attain the explicit soliton solutions as follows:

when, $B \neq 0, \Psi = A - C,\ \Omega = B^2 + 4E\left(A - C\right) < 0, \xi = x + y \pm ct$ and θ_1, θ_2 are arbitrary constants.

$$u2_1\left(x,y,t\right) = \frac{-3}{2}\frac{\left(d^2\Psi^2 + Bd\Psi - E\Psi\right)}{A^2} + \frac{3}{2}\frac{\left(2d\Psi^2 + B\Psi\right)}{A^2}\left(d + \frac{B}{2\Psi} + \frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1 \sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right) + \theta_2 \cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1 \cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right) + \theta_2 \sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)$$

$$+\frac{-3}{2}\frac{\Psi^2}{A^2}\left(d + \frac{B}{2\Psi} + \frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1 \sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right) + \theta_2 \cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1 \cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right) + \theta_2 \sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^2$$

$$\Rightarrow u2_1(x,y,t) = \frac{3}{2A^2}\left\lfloor \Psi E + \frac{B^2}{4} + \frac{\Omega}{4}\frac{\cos^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\sin^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right\rfloor \quad \left[if\ \ \theta_1 = 0\ \ but\ \ \ \theta_2 \neq 0\right]$$

$$\therefore u2_1\left(x,y,t\right) = \frac{3}{2A^2}\left\lfloor \Psi E + \frac{1}{4}\left(B^2 + \Omega\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)\right\rfloor$$

$$v2_1\left(x,y,t\right) = u2_1\left(x,y,t\right) = \frac{3}{2A^2}\left\lfloor \Psi E + \frac{1}{4}\left(B^2 + \Omega\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)\right\rfloor \tag{4.4.13}$$

$$u2_2\left(x,y,t\right) = \frac{3}{8}\frac{\left(B^2 + 4E\Psi\right)}{A^2} - \frac{3}{32}\frac{\left(B^2 + 4E\Psi\right)^2}{A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi} + \frac{B}{2\Psi} + \frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1 \sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right) + \theta_2 \cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1 \cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right) + \theta_2 \sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u2_2\left(x,y,t\right) = \frac{3}{8}\frac{\left(B^2 + 4E\Psi\right)}{A^2}\left(1 + \frac{\left(B^2 + 4E\Psi\right)}{\Omega}\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)\left[if\ \ \theta_1 = 0\ \ but\ \ \theta_2 \neq 0\right]$$

$$v2_2(x,y,t)=u2_2(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}\left(1+\frac{\left(B^2+4E\Psi\right)}{\Omega}\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right) \tag{4.4.14}$$

$$u2_3(x,y,t)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-1}$$

$$+\frac{-3}{2}\frac{\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u2_3(x,y,t)=\frac{-3}{2A^2}\left(\begin{array}{l}\left(d^2\Psi^2+Bd\Psi-E\Psi\right)-\dfrac{2\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{2d\Psi+B+\sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\\ +\dfrac{4\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{\left(2d\Psi+B+\sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)^2}\end{array}\right)\quad\left[if\ \ \theta_1=0\ \ but\ \ \theta_2\neq 0\right]$$

$$v2_3(x,y,t)=u2_3(x,y,t)=\frac{-3}{2A^2}\left(\begin{array}{l}\left(d^2\Psi^2+Bd\Psi-E\Psi\right)-\dfrac{2\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{2d\Psi+B+\sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\\ +\dfrac{4\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{\left(2d\Psi+B+\sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)^2}\end{array}\right) \tag{4.4.15}$$

$$u2_4(x,y,t)=\frac{-1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{-3}{2}\frac{\Psi^2}{A^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^2$$

$$+\frac{-3}{32}\frac{\left(16E^2\Psi^2+8EB^2\Psi+B^4\right)}{A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u2_4(x,y,t)=\frac{-1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{3}{8}\frac{\Omega}{A^2}\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\frac{3}{8}\frac{\left(B^2+4E\Psi\right)^2}{A^2\Omega}\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\ \left[if\ \theta_1=0\ but\ \theta_2\neq 0\right]$$

$$v2_4(x,y,t)=u2_4(x,y,t)=\frac{-1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{3}{8}\frac{\Omega}{A^2}\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\frac{3}{8}\frac{\left(B^2+4E\Psi\right)^2}{A^2\Omega}\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right) \qquad (4.4.16)$$

$$u2_5\left(x,y,t\right)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{-3}{2}\frac{\Psi^2}{A^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^2$$

$$+\frac{-3\left\{16E^2\Psi^2+B^4+8E\Psi B^2\right\}}{32A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u2_5\left(x,y,t\right)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{3}{8}\frac{\Omega}{A^2}\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\frac{3\left(B^2+4E\Psi\right)^2}{8A^2\Omega}\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\quad \left[if\ \theta_1=0\ but\ \theta_2\neq 0\right]$$

$$v2_5\left(x,y,t\right)=u2_5\left(x,y,t\right)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{3}{8}\frac{\Omega}{A^2}\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\frac{3\left(B^2+4E\Psi\right)^2}{8A^2\Omega}\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right) \qquad (4.4.17)$$

$$u2_6(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}+\frac{3}{2}\frac{(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-1}$$

$$-\frac{3}{2}\frac{(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u2_6(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}+\frac{3\Psi(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)}{A^2\left(2d\Psi+B+\sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)}$$

$$-\frac{6\Psi^2(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{A^2\left(2d\Psi+B+\sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)^2}\qquad \left[if\ \theta_1=0\ but\ \theta_2\neq 0\right]$$

$$v2_6(x,y,t)=u2_6(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}+\frac{3\Psi(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)}{A^2\left(2d\Psi+B+\sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)}$$

$$-\frac{6\Psi^2(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{A^2\left(2d\Psi+B+\sqrt{-\Omega}\cot\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)^2} \qquad (4.4.18)$$

$$u2_7(x,y,t)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\left(2d\Psi^2+B\Psi\right)}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)$$

$$+\frac{-3}{2}\frac{\Psi^2}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^2$$

$$\therefore u2_7(x,y,t)=\frac{3}{2A^2}\left\lfloor\Psi E+\frac{1}{4}\left(B^2+\Omega\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)\right\rfloor \qquad [if\ \ \theta_1\neq 0\ \ but\ \ \theta_2=0]$$

$$v2_7(x,y,t)=u2_7(x,y,t)=\frac{3}{2A^2}\left\lfloor\Psi E+\frac{1}{4}\left(B^2+\Omega\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)\right\rfloor \qquad (4.4.19)$$

$$u2_8(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3}{32}\frac{\left(B^2+4E\Psi\right)^2}{A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u2_8(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}\left(1+\frac{\left(B^2+4E\Psi\right)}{\Omega}\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right) \qquad [if\ \ \theta_1\neq 0\ \ but\ \ \theta_2=0]$$

$$v2_8(x,y,t)=u2_8(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}\left(1+\frac{\left(B^2+4E\Psi\right)}{\Omega}\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right) \qquad (4.4.20)$$

$$u2_9(x,y,t)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-1}$$

$$+\frac{-3}{2}\frac{\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u2_9(x,y,t)=\frac{-3}{2A^2}\left(\begin{array}{l}\left(d^2\Psi^2+Bd\Psi-E\Psi\right)-\dfrac{2\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{2d\Psi+B-\sqrt{-\Omega}\tan\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\\+\dfrac{4\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2\left(2d\Psi+B-\sqrt{-\Omega}\tan\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)^2}\end{array}\right)\quad\left[if\ \ \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$v2_9(x,y,t)=u2_9(x,y,t)=\frac{-3}{2A^2}\left(\begin{array}{l}\left(d^2\Psi^2+Bd\Psi-E\Psi\right)-\dfrac{2\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{2d\Psi+B-\sqrt{-\Omega}\tan\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\\+\dfrac{4\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{\left(2d\Psi+B-\sqrt{-\Omega}\tan\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)^2}\end{array}\right)\quad(4.4.21)$$

$$u2_{10}(x,y,t)=\frac{-1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{-3}{2}\frac{\Psi^2}{A^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{2}$$

$$+\frac{-3}{32}\frac{\left(16E^2\Psi^2+8EB^2\Psi+B^4\right)}{A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u2_{10}(x,y,t)=\frac{-1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{3}{8}\frac{\Omega}{A^2}\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\frac{3}{8}\frac{\left(B^2+4E\Psi\right)^2}{A^2\Omega}\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\quad\left[if\ \ \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$v2_{10}(x,y,t)=u2_{10}(x,y,t)=\frac{-1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{3}{8}\frac{\Omega}{A^2}\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\frac{3}{8}\frac{\left(B^2+4E\Psi\right)^2}{A^2\Omega}\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\qquad(4.4.22)$$

$$u2_{11}\left(x,y,t\right)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{-3}{2}\frac{\Psi^2}{A^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^2$$

$$+\frac{-3\left\{16E^2\Psi^2+B^4+8E\Psi B^2\right\}}{32A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u2_{11}\left(x,y,t\right)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{3}{8}\frac{\Omega}{A^2}\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\frac{3\left(B^2+4E\Psi\right)^2}{8A^2\Omega}\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\quad\left[if\ \ \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$v2_{11}\left(x,y,t\right)=u2_{11}\left(x,y,t\right)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{3}{8}\frac{\Omega}{A^2}\tan^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\frac{3\left(B^2+4E\Psi\right)^2}{8A^2\Omega}\cot^2\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\qquad(4.4.23)$$

$$u2_{12}(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}+\frac{3}{2}\frac{(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-1}$$

$$-\frac{3}{2}\frac{(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\sqrt{-\Omega}}{2\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u2_{12}(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}+\frac{3\Psi(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)}{A^2\left(2d\Psi+B-\sqrt{-\Omega}\tan\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)}$$

$$-\frac{6\Psi^2(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{A^2\left(2d\Psi+B-\sqrt{-\Omega}\tan\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)^2}\quad\left[if\ \ \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$v2_{12}(x,y,t)=u2_{12}(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}+\frac{3\Psi(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)}{A^2\left(2d\Psi+B-\sqrt{-\Omega}\tan\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)}$$

$$-\frac{6\Psi^2(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{A^2\left(2d\Psi+B-\sqrt{-\Omega}\tan\left(\frac{\sqrt{-\Omega}}{2\Psi}\xi\right)\right)^2} \quad (4.4.24)$$

Solution 3: Rational Form Solutions

Considering Eq. (3.9) composed with Eq. (4.3.1) to Eq. (4.3.6) to the Eq. (4.2.4) the following solutions have been constructed.

when, $B\neq 0,\Psi=A-C,\Omega=B^2+4E\left(A-c\right)=0,\xi=x+y\pm ct$ and θ_1,θ_2 are arbitrary constants.

$$u3_1\left(x,y,t\right)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\left(2d\Psi^2+B\Psi\right)}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\theta_2}{\theta_1+\theta_2\xi}\right)+\frac{-3}{2}\frac{\Psi^2}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\theta_2}{\theta_1+\theta_2\xi}\right)^2$$

$$\therefore u3_1(x,y,t)=-\frac{3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{4}\frac{\left(2d\Psi+B\right)^2\left(\theta_1+\theta_2\xi\right)+\left(2d\Psi+B\right)2\Psi\theta_2}{A^2\left(\theta_1+\theta_2\xi\right)}-\frac{3}{8}\frac{\left(\left(2d\Psi+B\right)\left(\theta_1+\theta_2\xi\right)+2\Psi\theta_2\right)^2}{A^2\left(\theta_1+\theta_2\xi\right)^2}$$

$$v3_1(x,y,t)=u3_1(x,y,t)=-\frac{3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{4}\frac{\left(2d\Psi+B\right)^2\left(\theta_1+\theta_2\xi\right)+\left(2d\Psi+B\right)2\Psi\theta_2}{A^2\left(\theta_1+\theta_2\xi\right)}$$

$$-\frac{3}{8}\frac{\left(\left(2d\Psi+B\right)\left(\theta_1+\theta_2\xi\right)+2\Psi\theta_2\right)^2}{A^2\left(\theta_1+\theta_2\xi\right)^2} \quad (4.4.25)$$

$$u3_2(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{-3}{32}\frac{\left(B^2+4E\Psi\right)^2}{A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\theta_2}{\theta_1+\theta_2\xi}\right)^{-2}$$

$$\therefore u3_2(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}\left(1-\frac{\left(B^2+4E\Psi\right)\left(\theta_1+\theta_2\xi\right)^2}{4\Psi^2\theta_2{}^2}\right)$$

$$v3_2(x,y,t)=u3_2(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}\left(1-\frac{\left(B^2+4E\Psi\right)\left(\theta_1+\theta_2\xi\right)^2}{4\Psi^2\theta_2{}^2}\right) \quad (4.4.26)$$

$$u3_3(x,y,t)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\theta_2}{\theta_1+\theta_2\xi}\right)^{-1}$$
$$+\frac{-3}{2}\frac{\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\theta_2}{\theta_1+\theta_2\xi}\right)^{-2}$$

$$\therefore u3_3(x,y,t)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3\Psi(\theta_1+\theta_2\xi)\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2\left(2d\Psi(\theta_1+\theta_2\xi)+B(\theta_1+\theta_2\xi)+2\Psi\theta_2\right)}$$
$$-\frac{6\Psi^2(\theta_1+\theta_2\xi)^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2\left(2d\Psi(\theta_1+\theta_2\xi)+B(\theta_1+\theta_2\xi)+2\Psi\theta_2\right)^2}$$

$$v3_3(x,y,t)=u3_3(x,y,t)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3\Psi(\theta_1+\theta_2\xi)\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2\left(2d\Psi(\theta_1+\theta_2\xi)+B(\theta_1+\theta_2\xi)+2\Psi\theta_2\right)}$$
$$-\frac{6\Psi^2(\theta_1+\theta_2\xi)^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2\left(2d\Psi(\theta_1+\theta_2\xi)+B(\theta_1+\theta_2\xi)+2\Psi\theta_2\right)^2} \qquad (4.27)$$

$$u3_4(x,y,t)=\frac{-1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{-3}{2}\frac{\Psi^2}{A^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\theta_2}{\theta_1+\theta_2\xi}\right)^2$$
$$+\frac{-3}{32}\frac{\left(16E^2\Psi^2+8EB^2\Psi+B^4\right)}{A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\theta_2}{\theta_1+v_2\xi}\right)^{-2}$$

$$\therefore u3_4(x,y,t)=-\frac{1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3}{2}\frac{\Psi^2\theta_2^{\,2}}{A^2(\theta_1+\theta_2\xi)^2}-\frac{3}{32}\frac{\left(B^2+4E\Psi\right)^2(\theta_1+\theta_2\xi)^2}{A^2\Psi^2\theta_2^{\,2}}$$

$$v3_4(x,y,t)=u3_4(x,y,t)=-\frac{1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3}{2}\frac{\Psi^2\theta_2^{\,2}}{A^2(\theta_1+\theta_2\xi)^2}-\frac{3}{32}\frac{\left(B^2+4E\Psi\right)^2(\theta_1+\theta_2\xi)^2}{A^2\Psi^2\theta_2^{\,2}} \qquad (4.4.28)$$

$$u3_5(x,y,t)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3}{2}\frac{\Psi^2}{A^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\theta_2}{\theta_1+\theta_2\xi}\right)^2-\frac{3\left\{16E^2\Psi^2+B^4+8E\Psi B^2\right\}}{32A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{B}{2\Psi}+\frac{\theta_2}{\theta_1+\theta_2\xi}\right)^{-2}$$

$$\therefore u3_5(x,y,t)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3}{2}\frac{\Psi^2\theta_2{}^2}{A^2\left(\theta_1+\theta_2\xi\right)^2}-\frac{3\left(B^2+4E\Psi\right)^2\left(\theta_1+\theta_2\xi\right)^2}{32A^2\Psi^2\theta_2{}^2}$$

$$v3_5(x,y,t)=u3_5(x,y,t)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3}{2}\frac{\Psi^2\theta_2{}^2}{A^2\left(\theta_1+\theta_2\xi\right)^2}-\frac{3\left(B^2+4E\Psi\right)^2\left(\theta_1+\theta_2\xi\right)^2}{32A^2\Psi^2\theta_2{}^2} \qquad (4.4.29)$$

$$u3_6\left(x,y,t\right)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}+\frac{3}{2}\frac{(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\theta_2}{\theta_1+\theta_2\xi}\right)^{-1}$$
$$-\frac{3}{2}\frac{(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{A^2}\left(d+\frac{B}{2\Psi}+\frac{\theta_2}{\theta_1+\theta_2\xi}\right)^{-2}$$

$$\therefore u3_6(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}+\frac{3\Psi(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)\left(\theta_1+\theta_2\xi\right)}{A^2\left(2d\Psi+B\left(\theta_1+\theta_2\xi\right)+2\theta_2\Psi\right)}$$
$$-\frac{6\Psi^2(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)\left(\theta_1+\theta_2\xi\right)^2}{A^2\left(2d\Psi+B\left(\theta_1+\theta_2\xi\right)+2\theta_2\Psi\right)^2}$$

$$v3_6(x,y,t)=u3_6(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}+\frac{3\Psi(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)\left(\theta_1+\theta_2\xi\right)}{A^2\left(2d\Psi+B\left(\theta_1+\theta_2\xi\right)+2\theta_2\Psi\right)}$$
$$-\frac{6\Psi^2(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)\left(\theta_1+\theta_2\xi\right)^2}{A^2\left(2d\Psi+B\left(v_1+\theta_2\xi\right)+2\theta_2\Psi\right)^2} \qquad (4.4.30)$$

Solution 4: Hyperbolic function

Picking Eqn. (3.10) as a group of Eqn. (4.3.1) to Eqn. (4.3.6) for the Eqn. (4.2.4) the formed travelling wave solutions are as follows:

when, $B=0, \Psi = A-C$, $\Delta = \Psi E > 0, \xi = x+y \pm ct$ and θ_1, θ_2 are arbitrary constants.

$$u4_1(x,y,t) = \frac{-3}{2}\frac{\left(d^2\Psi^2 + Bd\Psi - E\Psi\right)}{A^2} + \frac{3}{2}\frac{\left(2d\Psi^2 + B\Psi\right)}{A^2}\left(d + \frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1 \sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) + \theta_2 \cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1 \cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) + \theta_2 \sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)$$

$$-\frac{3}{2}\frac{\Psi^2}{A^2}\left(d + \frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1 \sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) + \theta_2 \cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1 \cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) + \theta_2 \sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^2$$

$$\therefore u4_1(x,y,t) = \frac{3}{2A^2}\left[E\Psi + B\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) - \Psi^2\coth^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right] \qquad [\text{ if } \theta_1 = 0 \text{ but } \theta_2 \neq 0]$$

$$v4_1(x,y,t) = u4_1(x,y,t) = \frac{3}{2A^2}\left[E\Psi + B\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right) - \Psi^2\coth^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right] \qquad (4.4.31)$$

$$u4_2(x,y,t) \quad \therefore u4_2(x,y,t) = \frac{3}{8A^2}\left(B^2 + 4E\Psi\right)\left[1 - \frac{\left(B^2 + 4E\Psi\right)}{\left(-B + 2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}\right] \quad [\text{ if } \theta_1 = 0 \text{ but } \theta_2 \neq 0]$$

$$v4_2(x,y,t) = u4_2(x,y,t) = \frac{3}{8A^2}\left(B^2 + 4E\Psi\right)\left[1 - \frac{\left(B^2 + 4E\Psi\right)}{\left(-B + 2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}\right] \qquad (4.4.32)$$

$$u4_3(x,y,t)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2}\left(d+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^{-1}$$

$$+\frac{-3}{2}\frac{\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2}\left(d+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u4_3(x,y,t)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2\left(d\Psi+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)}\ [\ if\ \theta_1=0\ but\ \theta_2\neq 0]$$

$$-\frac{3}{2}\frac{\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2\left(d\Psi+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}$$

$$v4_3(x,y,t)=u4_3(x,y,t)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2\left(d\Psi+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)}$$

$$-\frac{3}{2}\frac{\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2\left(d\Psi+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2} \qquad (4.4.33)$$

$$u4_4(x,y,t)=\frac{-1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3}{2}\frac{\Psi^2}{A^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^2$$

$$-\frac{3}{32}\frac{\left(16E^2\Psi^2+8EB^2\Psi+B^4\right)}{A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u4_4(x,y,t)=-\frac{1}{4A^2}\left(B^2+4E\Psi\right)-\frac{3}{8A^2}\left(-B+2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2-\frac{3}{8A^2}\frac{\left(B^2+4E\Psi\right)^2}{\left(-B+2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}\ [\ if\ \theta_1=0\ but\ \theta_2\neq 0]$$

$$v4_4(x,y,t)=u4_4(x,y,t)=-\frac{1}{4A^2}\left(B^2+4E\Psi\right)-\frac{3}{8A^2}\left(-B+2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2-\frac{3}{8A^2}\frac{\left(B^2+4E\Psi\right)^2}{\left(-B+2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2} \quad (4.4.34)$$

$$u4_5(x,y,t)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{-3}{2}\frac{\Psi^2}{A^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^2$$

$$-\frac{3\left\{16E^2\Psi^2+B^4+8E\Psi B^2\right\}}{32A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u4_5(x,y,t)=\frac{3\left(B^2+4E\Psi\right)}{4A^2}-\frac{3\left(-B+2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}{8A^2}-\frac{3\left\{16E^2\Psi^2+B^4+8E\Psi B^2\right\}}{8A^2\left(-B+2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}\ \left[\ if\ \ \theta_1=0\ \ but\ \ \theta_2\neq 0\right]$$

$$v4_5(x,y,t)=u4_5(x,y,t)=\frac{3\left(B^2+4E\Psi\right)}{4A^2}-\frac{3\left(-B+2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}{8A^2}-\frac{3\left\{16E^2\Psi^2+B^4+8E\Psi B^2\right\}}{8A^2\left(-B+2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2} \quad (4.4.35)$$

$$u4_6(x,y,t)==-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}+\frac{3}{2}\frac{(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)}{A^2}\left(d+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^{-1}$$

$$-\frac{3}{2}\frac{(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{A^2}\left(d+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u4_6(x,y,t)=\frac{1}{4}\frac{(-B^2+2\Psi E-6\Psi dB-6\Psi^2 d^2)}{4A^2}+\frac{3\Psi(2\Psi^2 d^3+3\Psi d^2 B-2\Psi dE+B^2 d-BE)}{2A^2\left(d\Psi+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)}$$

$$-\frac{3\Psi^2(\Psi^2 d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{2A^2\left(d\Psi+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}\quad [\ if\ \theta_1=0\ but\ \theta_2\neq 0]$$

$$v4_6(x,y,t)=u4_6(x,y,t)=\frac{1}{4}\frac{(-B^2+2\Psi E-6\Psi dB-6\Psi^2 d^2)}{4A^2}+\frac{3\Psi(2\Psi^2 d^3+3\Psi d^2 B-2\Psi dE+B^2 d-BE)}{2A^2\left(d\Psi+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)}$$

$$-\frac{3\Psi^2(\Psi^2 d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{2A^2\left(d\Psi+\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2} \tag{4.4.36}$$

$$u4_7(x,y,t)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\left(2d\Psi^2+B\Psi\right)}{A^2}\left(d+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)$$

$$-\frac{3}{2}\frac{\Psi^2}{A^2}\left(d+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^2$$

$$\therefore u4_7(x,y,t)=\frac{3}{2A^2}\left[E\Psi+B\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)-\Psi^2\tanh^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right]\ [\ if\ \theta_1\neq 0\ but\ \theta_2=0]$$

$$v4_7(x,y,t)=u4_7(x,y,t)=\frac{3}{2A^2}\left[E\Psi+B\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)-\Psi^2\tanh^2\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right] \tag{4.4.37}$$

$$u4_8(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3}{32}\frac{\left(B^2+4E\Psi\right)^2}{A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u4_8(x,y,t)=\frac{3}{8A^2}\left(B^2+4E\Psi\right)\left[1-\frac{\left(B^2+4E\Psi\right)}{\left(-B+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}\right]\ [\ if\ \theta_1\neq 0\ but\ \theta_2=0]$$

$$v4_8(x,y,t)=u4_8(x,y,t)=\frac{3}{8A^2}\left(B^2+4E\Psi\right)\left[1-\frac{\left(B^2+4E\Psi\right)}{\left(-B+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}\right] \qquad (4.4.38)$$

$$u4_9(x,y,t)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2}\left(d+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^{-1}$$

$$+\frac{-3}{2}\frac{\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2}\left(d+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u4_9(x,y,t)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2\left(d\Psi+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)}$$

$$-\frac{3}{2}\frac{\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2\left(d\Psi+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}\quad\left[\ if\ \ \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$v4_9(x,y,t)=u4_9(x,y,t)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2\left(d\Psi+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)}$$

$$-\frac{3}{2}\frac{\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2\left(d\Psi+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2} \qquad (4.4.39)$$

$$u4_{10}(x,y,t)=\frac{-1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3}{2}\frac{\Psi^2}{A^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^2$$

$$-\frac{3}{32}\frac{\left(16E^2\Psi^2+8EB^2\Psi+B^4\right)}{A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u4_{10}(x,y,t)=-\frac{1}{4A^2}\left(B^2+4E\Psi\right)-\frac{3}{8A^2}\left(-B+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2-\frac{3}{8A^2}\frac{\left(B^2+4E\Psi\right)^2}{\left(-B+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}\ \left[\ if\ \theta_1\neq 0\ but\ \theta_2=0\right]$$

$$v4_{10}(x,y,t)=u4_{10}(x,y,t)=-\frac{1}{4A^2}\left(B^2+4E\Psi\right)-\frac{3}{8A^2}\left(-B+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2-\frac{3}{8A^2}\frac{\left(B^2+4E\Psi\right)^2}{\left(-B+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2} \tag{4.4.40}$$

$$u4_{11}(x,y,t)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{-3}{2}\frac{\Psi^2}{A^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^2$$

$$-\frac{3\left\{16E^2\Psi^2+B^4+8E\Psi B^2\right\}}{32A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u4_{11}(x,y,t)=\frac{3\left(B^2+4E\Psi\right)}{4A^2}-\frac{3\left(-B+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}{8A^2}-\frac{3\left\{16E^2\Psi^2+B^4+8E\Psi B^2\right\}}{8A^2\left(-B+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}\quad\left[\ if\ \theta_1\neq 0\ but\ \theta_2=0\right]$$

$$v4_{11}(x,y,t)=u4_{11}(x,y,t)=\frac{3\left(B^2+4E\Psi\right)}{4A^2}-\frac{3\left(-B+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}{8A^2}-\frac{3\left\{16E^2\Psi^2+B^4+8E\Psi B^2\right\}}{8A^2\left(-B+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2} \tag{4.4.41}$$

$$u4_{12}(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}+\frac{3}{2}\frac{(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)}{A^2}\left(d+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^{-1}$$

$$-\frac{3}{2}\frac{(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{A^2}\left(d+\frac{\sqrt{\Delta}}{\Psi}\frac{\theta_1\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}{\theta_1\cosh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)+\theta_2\sinh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u4_{12}(x,y,t)=\frac{1}{4}\frac{(-B^2+2\Psi E-6\Psi dB-6\Psi^2 d^2)}{4A^2}+\frac{3\Psi(2\Psi^2 d^3+3\Psi d^2 B-2\Psi dE+B^2 d-BE)}{2A^2\left(d\Psi+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)}$$

$$-\frac{3\Psi^2(\Psi^2 d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2 d^2-2BdE)}{2A^2\left(d\Psi+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2}\quad [\,if\;\; \theta_1\neq 0\;\; but\;\; \theta_2=0]$$

$$v4_{12}(x,y,t)=u4_{12}(x,y,t)=\frac{(-B^2+2\Psi E-6\Psi dB-6\Psi^2 d^2)}{4A^2}+\frac{3\Psi(2\Psi^2 d^3+3\Psi d^2 B-2\Psi dE+B^2 d-BE)}{2A^2\left(d\Psi+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)}$$

$$-\frac{3\Psi^2(\Psi^2 d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2 d^2-2BdE)}{2A^2\left(d\Psi+\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\Psi}\xi\right)\right)^2} \qquad (4.4.42)$$

Solution 5: Trigonometric Solutions

Substituting Eq. (4.3.1) to Eq. (4.3.6) into Eq. (4.2.4), along with Eq. (3.11) and simplifying, we get following travelling wave solutions:

when, $B=0,\Psi=A-C,\Delta=\Psi E,\xi=x+y\pm ct$ and θ_1,θ_2 are arbitrary constants.

$$u5_1(x,y,t)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\left(2d\Psi^2+B\Psi\right)}{A^2}\left(d+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)$$

$$-\frac{3}{2}\frac{\Psi^2}{A^2}\left(d+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^2$$

$$\therefore u5_1(x,y,t)=\frac{3}{2A^2}\left(E\Psi+B\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\Delta\cot^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)[\,if\;\;\theta_1=0\;\;but\;\;\theta_2\neq 0]$$

$$v5_1(x,y,t)=u5_1(x,y,t)=\frac{3}{2A^2}\left(E\Psi+B\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\Delta\cot^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right) \tag{4.4.43}$$

$$u5_2\left(x,y,t\right)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{-3}{32}\frac{\left(B^2+4E\Psi\right)^2}{A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u5_2(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}\left[1-\frac{\left(B^2+4E\Psi\right)}{\left(-B+2\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2}\right]\left[if\ \ \theta_1=0\ \ but\ \ \theta_2\neq 0\right]$$

$$v5_2(x,y,t)=u5_2(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}\left[1-\frac{\left(B^2+4E\Psi\right)}{\left(-B+2\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2}\right] \tag{4.4.44}$$

$$u5_3\left(x,y,t\right)=-\frac{3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2}\left(d+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-1}$$

$$-\frac{3}{2}\frac{\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2}\left(d+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u5_3(x,y,t)=\frac{3\left(-d^2\Psi^2-Bd\Psi+E\Psi\right)}{2A^2}+\frac{3\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{2A^2\left[d\Psi+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]}$$

$$-\frac{3\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{2A^2\left[d\Psi+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]^2}\left[if\ \ \theta_1=0\ \ but\ \ \theta_2\neq 0\right]$$

$$v5_3(x,y,t)=u5_3(x,y,t)=\frac{3\left(-d^2\Psi^2-Bd\Psi+E\Psi\right)}{2A^2}+\frac{3\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{2A^2\left[d\Psi+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]}$$

$$-\frac{3\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{2A^2\left[d\Psi+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]^2} \qquad (4.4.45)$$

$$u5_4(x,y,t)=\frac{-1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{-3}{2}\frac{\Psi^2}{A^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^2$$

$$+\frac{-3}{32}\frac{\left(16E^2\Psi^2+8EB^2\Psi+B^4\right)}{A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u5_4(x,y,t)=\frac{-1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3\left(-B+2\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2}{8A^2}-\frac{3\left(B^2+4E\Psi\right)^2}{8A^2\left(-B+2\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2}\ \left[if\ \ \theta_1=0\ \ but\ \ \theta_2\neq 0\right]$$

$$v5_4(x,y,t)=u5_4(x,y,t)=\frac{-1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3\left(-B+2\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2}{8A^2}-\frac{3\left(B^2+4E\Psi\right)^2}{8A^2\left(-B+2\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2} \qquad (4.4.46)$$

$$u5_5(x,y,t)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3}{2}\frac{\Psi^2}{A^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^2$$

$$-\frac{3\left\{16E^2\Psi^2+B^4+8E\Psi B^2\right\}}{32A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u5_5(x,y,t)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3\left[-B+2\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]^2}{8A^2}-\frac{3\left(B^2+4E\Psi\right)^2}{8A^2\left[-B+2\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]^2}\;\left[if\;\;\theta_1=0\;\;but\;\;\theta_2\neq 0\right]$$

$$v5_5(x,y,t)=u5_5(x,y,t)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3\left[-B+2\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]^2}{8A^2}-\frac{3\left(B^2+4E\Psi\right)^2}{8A^2\left[-B+2\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]^2} \qquad (4.4.47)$$

$$u5_6(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}+\frac{3}{2}\frac{(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)}{A^2}\left(d+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-1}$$

$$-\frac{3}{2}\frac{(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{A^2}\left(d+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u5_6(x,y,t)=-\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{4A^2}+\frac{3\Psi(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)}{2A^2\left(d\Psi+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)}$$

$$-\frac{3\Psi^2(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{2A^2\left(d\Psi+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2}\;\left[if\;\;\theta_1=0\;\;but\;\;\theta_2\neq 0\right]$$

$$v5_6(x,y,t)=u5_6(x,y,t)=-\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{4A^2}+\frac{3\Psi(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)}{2A^2\left(d\Psi+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)}$$

$$-\frac{3\Psi^2(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{2A^2\left(d\Psi+\sqrt{-\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2} \qquad (4.4.48)$$

$$u5_7(x,y,t)=\frac{-3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\left(2d\Psi^2+B\Psi\right)}{A^2}\left(d+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)$$

$$-\frac{3}{2}\frac{\Psi^2}{A^2}\left(d+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^2$$

$$\therefore u5_7(x,y,t)=\frac{3}{2A^2}\left(E\Psi-B\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\Delta\tan^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)\left[if\ \ \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$v5_7(x,y,t)=u5_7(x,y,t)=\frac{3}{2A^2}\left(E\Psi-B\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\Delta\tan^2\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right) \tag{4.4.49}$$

$$u5_8(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{-3}{32}\frac{\left(B^2+4E\Psi\right)^2}{A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u5_8(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}\left[1-\frac{\left(B^2+4E\Psi\right)}{\left(-B-2\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2}\right]\left[if\ \ \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$v5_8(x,y,t)=u5_8(x,y,t)=\frac{3}{8}\frac{\left(B^2+4E\Psi\right)}{A^2}\left[1-\frac{\left(B^2+4E\Psi\right)}{\left(-B-2\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2}\right] \tag{4.4.50}$$

$$u5_9(x,y,t)=-\frac{3}{2}\frac{\left(d^2\Psi^2+Bd\Psi-E\Psi\right)}{A^2}+\frac{3}{2}\frac{\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{A^2}\left(d+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-1}$$

$$-\frac{3}{2}\frac{\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{A^2}\left(d+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u5_9(x,y,t)=\frac{3\left(-d^2\Psi^2-Bd\Psi+E\Psi\right)}{2A^2}+\frac{3\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{2A^2\left[d\Psi-\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]}$$

$$-\frac{3\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{2A^2\left[d\Psi-\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]^2}\left[if\ \ \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$v5_9(x,y,t)=u5_9(x,y,t)=\frac{3\left(-d^2\Psi^2-Bd\Psi+E\Psi\right)}{2A^2}+\frac{3\Psi\left(2d^3\Psi^2+3d^2B\Psi-2Ed\Psi+dB^2-BE\right)}{2A^2\left[d\Psi-\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]}$$

$$-\frac{3\Psi^2\left(d^4\Psi^2+2d^3B\Psi-2Ed^2\Psi+d^2B^2-2dBE+E^2\right)}{2A^2\left[d\Psi-\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]^2}\qquad(4.4.51)$$

$$u5_{10}(x,y,t)=\frac{-1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}+\frac{-3}{2}\frac{\Psi^2}{A^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^2$$

$$+\frac{-3}{32}\frac{\left(16E^2\Psi^2+8EB^2\Psi+B^4\right)}{A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u5_{10}(x,y,t)=\frac{-1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3\left(-B-2\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2}{8A^2}-\frac{3\left(B^2+4E\Psi\right)^2}{8A^2\left(-B-2\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2}\left[if\ \ \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$v5_{10}(x,y,t)=u5_{10}(x,y,t)=\frac{-1}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3\left(-B-2\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2}{8A^2}-\frac{3\left(B^2+4E\Psi\right)^2}{8A^2\left(-B-2\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2}\qquad(4.4.52)$$

$$u5_{11}(x,y,t)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3}{2}\frac{\Psi^2}{A^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^2$$

$$-\frac{3\left\{16E^2\Psi^2+B^4+8E\Psi B^2\right\}}{32A^2\Psi^2}\left(\frac{-1}{2}\frac{B}{\Psi}+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u6_{11}(x,y,t)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3\left[-B-2\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]^2}{8A^2}-\frac{3\left(B^2+4E\Psi\right)^2}{8A^2\left[-B-2\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]^2}\left[if\ \ \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$v5_{11}(x,y,t)=u6_{11}(x,y,t)=\frac{3}{4}\frac{\left(B^2+4E\Psi\right)}{A^2}-\frac{3\left[-B-2\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]^2}{8A^2}-\frac{3\left(B^2+4E\Psi\right)^2}{8A^2\left[-B-2\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right]^2} \qquad (4.4.53)$$

$$u5_{12}(x,y,t)=-\frac{1}{4}\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{A^2}+\frac{3}{2}\frac{(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)}{A^2}\left(d+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-1}$$

$$-\frac{3}{2}\frac{(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{A^2}\left(d+\frac{\sqrt{-\Delta}}{\Psi}\frac{-\theta_1\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}{\theta_1\cos\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)+\theta_2\sin\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)}\right)^{-2}$$

$$\therefore u5_{12}(x,y,t)=-\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2d^2)}{4A^2}+\frac{3\Psi(2\Psi^2d^3+3\Psi d^2B-2\Psi dE+B^2d-BE)}{2A^2\left(d\Psi-\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)}$$

$$-\frac{3\Psi^2(\Psi^2d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2d^2-2BdE)}{2A^2\left(d\Psi-\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2}\qquad\left[if\ \ \theta_1\neq 0\ \ but\ \ \theta_2=0\right]$$

$$v5_{12}(x,y,t)=u5_{12}(x,y,t)=-\frac{(B^2-2\Psi E+6\Psi dB+6\Psi^2 d^2)}{4A^2}+\frac{3\Psi(2\Psi^2 d^3+3\Psi d^2 B-2\Psi dE+B^2 d-BE)}{2A^2\left(d\Psi-\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)}$$

$$-\frac{3\Psi^2(\Psi^2 d^4+2\Psi Bd^3-2\Psi dE+E^2+B^2 d^2-2BdE)}{2A^2\left(d\Psi-\sqrt{-\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\Psi}\xi\right)\right)^2} \qquad (4.4.54)$$

These are the solutions we have found by applying this method.

4.5 Numerical Explanation:

We have taken some different values of constraints A, B, C, d, E, θ_1 *and* θ_2 for finding out the values of u and have found the graphical presentation of u in maple. For different types of family we have found different types of graph. Even we have found difference between the two graphs for the same equation when we have changed the values of the arbitrary constants. The graphical figure that have been found in maple for a particular value of the constraints are shown as follows:

Taking the values of A=5.5, B=2.2, C=2.5, d=3.8, E=1.4, $c=\frac{3}{\sqrt{6}}$, y=2.8 and ξ=x+y-ct in Eq. (4.4.3), we get the following graph:

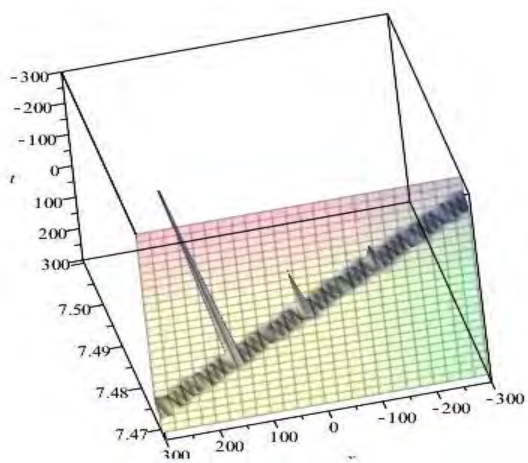

Figure 1: Hyperbolic form of solution of the (2+1) breaking soliton Equation

Then, taking the values of A=5, B=3.2, C=3.5, d=2.8, E=4.4, $c=\frac{4}{\sqrt{6}}$, y=4.2 and ξ=x+y+ct in Eq. (4.4.2), we get the following graph:

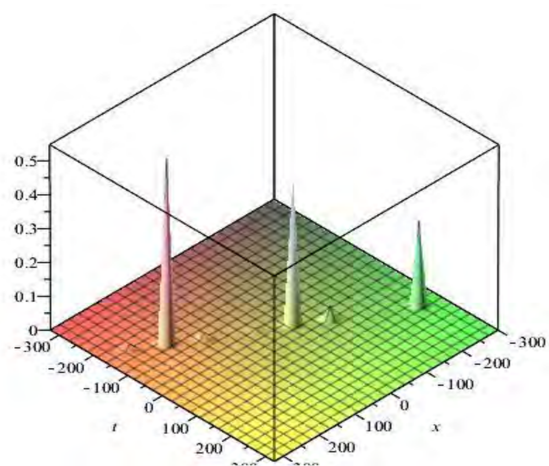

Figure 2: Hyperbolic form of solution of the (2+1) breaking soliton Equation

now, taking the values of A=5.5, B=3.56, C=1.67, d=3.8, E=5.44, $c=\frac{4}{\sqrt{3}}$, y=1.48 and ξ=x+y-ct in Eq. (4.4.3), we get the following graph:

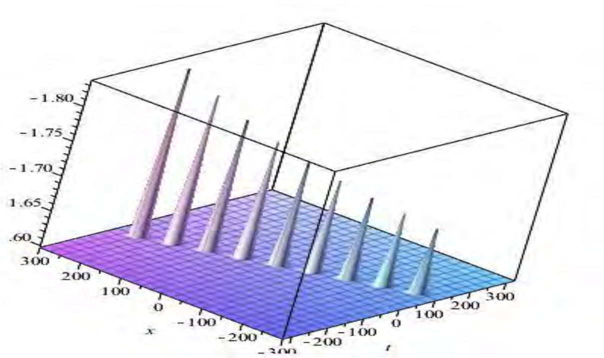

Figure 3: Hyperbolic form of solution of the (2+1) breaking soliton Equation

now, we have taken the values of A=4.33, B=3.63, C=1.75, d=3.8, E=3.53, $c=\frac{4}{\sqrt{8}}$, y=3.13 and ξ=x+y-ct in Eq. (4.4.13), we get the graph:

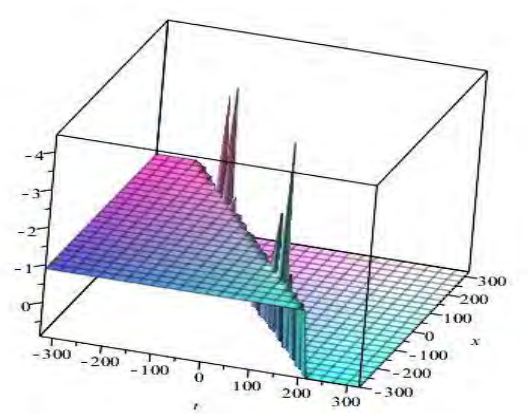

Figure 5: Trigonometric form of solution of the (2+1) breaking soliton Equation

here, taking the values of A=3.47, B=4.68, C=5.88, d=2.96, E=4.65, $c=\frac{8}{\sqrt{7}}$, y=6.35 and ξ=x+y+ct in Eq. (4.4.5), we get the following graph:

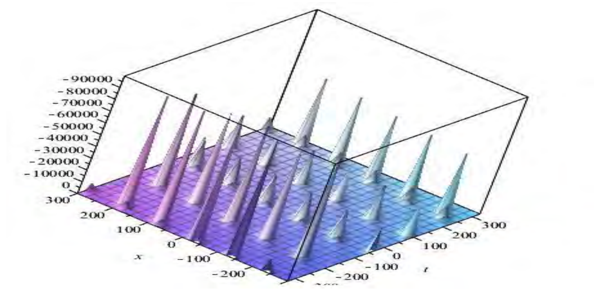

Figure 4: Hyperbolic form of solution of the (2+1) breaking soliton Equation

Then, taking the values of A=6.75, B=4.92, C=3.78, d=1.98, E=3.94, $c=5.44$, y=2.76 and ξ=x+y-ct in Eq. (4.4.14), we get the following graph:

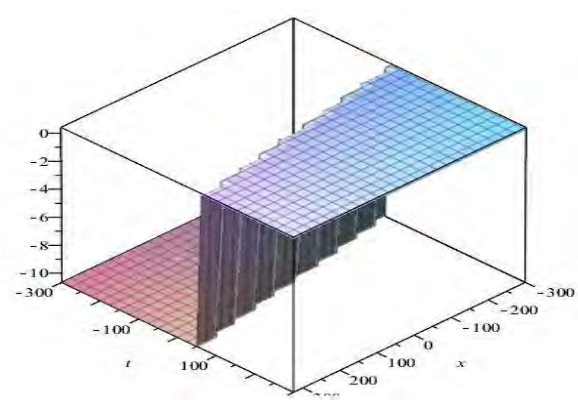

Figure 6: Trigonometric form of solution of the (2+1) breaking soliton Equation

now, taking the values of A=5.56, B=3.46, C=4.98, d=2.69, E=4.65, $c=\frac{4}{\sqrt{3}}$, y=2.43 and ξ=x+y+ct in Eq. (4.4.34), we get the following graph:

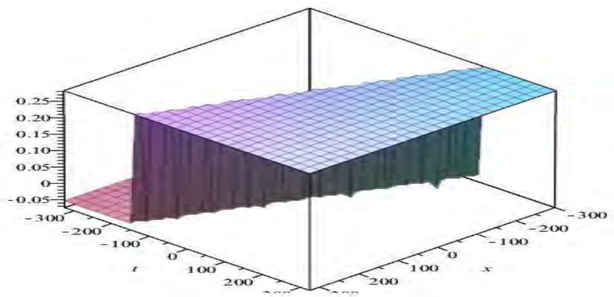

Figure 7: Hyperbolic form of solution of the (2+1) breaking soliton Equation

then, taking the values of A=4.66, B=5.98, C=1.34, d=2.69, E=3.65, $c=\frac{3}{\sqrt{2}}$, y=4.23, $\theta_1=3$, $\theta_2=7$ and ξ=x+y+ct in Eq. (4.4.26),we get the graph:

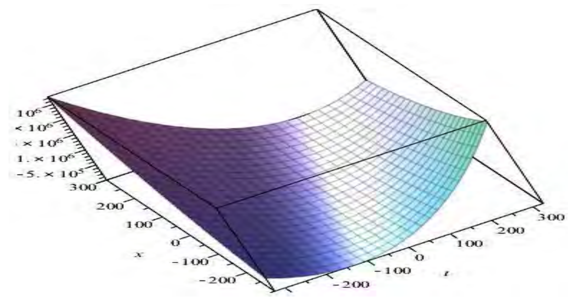

Figure 8: Rational form of solution of the (2+1) breaking soliton Equation

4.6 Comparison

We will compare the extended (G'/G) expansion method that used here to find the exact solutions of (2+1)-dimensional breaking soliton equation with the Bekir and Uygun's [41] acquired exact solutions by basic (G'/G) expansion method.

First, we will discuss about the Bekir and Uygun's found exact solutions of the (2+1)-dimensional breaking soliton equation using basic (G'/G) expansion method.

when $\lambda^2 - 4\mu > 0$,

$$u_1(\xi) = \frac{3}{8}\left(4\mu - \lambda^2\right)\left[\frac{C_1 \sinh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right) + C_2 \cosh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right)}{C_1 \cosh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right) + C_2 \sinh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right)}\right]^2 + \frac{3\lambda^2}{8} - \frac{3\mu}{2}$$

$$v_1(\xi) = \frac{3}{8}\left(4\mu - \lambda^2\right)\left[\frac{C_1 \sinh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right) + C_2 \cosh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right)}{C_1 \cosh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right) + C_2 \sinh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right)}\right]^2 + \frac{3\lambda^2}{8} - \frac{3\mu}{2}, \qquad (4.6.1)$$

where, $\xi = x + y - \alpha\left(\lambda^2 - 4\mu\right)t$.

$$u_2(\xi) = \frac{3}{8}\left(4\mu - \lambda^2\right)\left[\frac{C_1 \sinh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right) + C_2 \cosh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right)}{C_1 \cosh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right) + C_2 \sinh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right)}\right]^2 + \frac{\lambda^2}{8} - \frac{\mu}{2}$$

$$v_2(\xi) = \frac{3}{8}\left(4\mu - \lambda^2\right)\left[\frac{C_1 \sinh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right) + C_2 \cosh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right)}{C_1 \cosh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right) + C_2 \sinh\left(\sqrt{\frac{\lambda^2 - 4\mu}{2}}\,\xi\right)}\right]^2 + \frac{\lambda^2}{8} - \frac{\mu}{2} \qquad (4.6.2)$$

where, $\xi = x + y + \alpha\left(\lambda^2 - 4\mu\right)t$.

When, $\lambda^2 - 4\mu < 0$,

$$u_3(\xi) = \frac{3}{8}\left(\lambda^2 - 4\mu\right)\left[\frac{-C_1 \sin\left(\sqrt{\frac{4\mu - \lambda^2}{2}}\,\xi\right) + C_2 \cos\left(\sqrt{\frac{4\mu - \lambda^2}{2}}\,\xi\right)}{C_1 \cos\left(\sqrt{\frac{4\mu - \lambda^2}{2}}\,\xi\right) + C_2 \sin\left(\sqrt{\frac{4\mu - \lambda^2}{2}}\,\xi\right)}\right]^2 + \frac{3\lambda^2}{8} - \frac{3\mu}{2}$$

$$v_3(\xi)=\frac{3}{8}\left(\lambda^2-4\mu\right)\left[\frac{-C_1\sin\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)+C_2\cos\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)}{C_1\cos\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)+C_2\sin\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)}\right]^2+\frac{3\lambda^2}{8}-\frac{3\mu}{2} \tag{4.6.3}$$

where, $\xi=x+y-\alpha\left(\lambda^2-4\mu\right)t$

$$u_4(\xi)=\frac{3}{8}\left(\lambda^2-4\mu\right)\left[\frac{-C_1\sin\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)+C_2\cos\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)}{C_1\cos\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)+C_2\sin\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)}\right]^2+\frac{\lambda^2}{8}-\frac{\mu}{2}$$

$$v_4(\xi)=\frac{3}{8}\left(\lambda^2-4\mu\right)\left[\frac{-C_1\sin\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)+C_2\cos\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)}{C_1\cos\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)+C_2\sin\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)}\right]^2+\frac{\lambda^2}{8}-\frac{\mu}{2} \tag{4.6.4}$$

where, $\xi=x+y+\alpha\left(\lambda^2-4\mu\right)t$.

When $\lambda^2-4\mu=0$,

$$u_5(\xi)=\frac{3c_2^2}{2\left[c_1+c_2(x+y)\right]^2}+\frac{3\lambda^2}{8}-\frac{3\mu}{2}\ \text{ and, }\ v_5(\xi)=\frac{3c_2^2}{2\left[c_1+c_2(x+y)\right]^2}+\frac{3\lambda^2}{8}-\frac{3\mu}{2}, \tag{4.6.5}$$

$$u_6(\xi)=\frac{3c_2^2}{2\left[c_1+c_2(x+y)\right]^2}+\frac{\lambda^2}{8}-\frac{\mu}{2}\ \text{ and, }\ v_6(\xi)=\frac{3c_2^2}{2\left[c_1+c_2(x+y)\right]^2}+\frac{\lambda^2}{8}-\frac{\mu}{2} \tag{4.6.6}$$

The comparison in table:

Solutions acquired in this process	**Solutions obtained by Bekir and Uygun's**
1.If $A=1,B=-\lambda,C=0,E=-\mu,\Psi=1,C=0,\theta_1=0$ *and* $\theta_2\neq 0$, then we get from equation (4.4.8) $u1_8(\xi)=\frac{3\lambda^2}{8}-\frac{3\mu}{2}+\frac{3}{8}(4\mu-\lambda^2)\coth^2\left(\sqrt{\frac{\lambda^2-4\mu}{2}}\xi\right)$ $v1_8(\xi)=\frac{3\lambda^2}{8}-\frac{3\mu}{2}+\frac{3}{8}(4\mu-\lambda^2)\coth^2\left(\sqrt{\frac{\lambda^2-4\mu}{2}}\xi\right)$ If $A=1,B=-\lambda,C=0,E=-\mu,\Psi=1,C=0,\theta_1\neq 0$ and *and* $\theta_2=0$, then we get from equation (4.4.2) $u1_2(\xi)=\frac{3\lambda^2}{8}-\frac{3\mu}{2}+\frac{3}{8}(4\mu-\lambda^2)\tanh^2\left(\sqrt{\frac{\lambda^2-4\mu}{2}}\xi\right)$ $v1_2(\xi)=\frac{3\lambda^2}{8}-\frac{3\mu}{2}+\frac{3}{8}(4\mu-\lambda^2)\tanh^2\left(\sqrt{\frac{\lambda^2-4\mu}{2}}\xi\right)$	1.If $C_1=0$ *and* $C_2\neq 0$, the solutions of eqn. (4.6.1) become: $u_1(\xi)=\frac{3}{8}(4\mu-\lambda^2)\coth^2\left(\sqrt{\frac{\lambda^2-4\mu}{2}}\xi\right)+\frac{3\lambda^2}{8}-\frac{3\mu}{2}$ $v_1(\xi)=\frac{3}{8}(4\mu-\lambda^2)\coth^2\left(\sqrt{\frac{\lambda^2-4\mu}{2}}\xi\right)+\frac{3\lambda^2}{8}-\frac{3\mu}{2}$ If $C_1\neq 0$ *and* $C_2=0$, the solutions of eqn. (4.6.1) become: $u_1(\xi)=\frac{3}{8}(4\mu-\lambda^2)\tanh^2\left(\sqrt{\frac{\lambda^2-4\mu}{2}}\xi\right)+\frac{3\lambda^2}{8}-\frac{3\mu}{2}$ $v_1(\xi)=\frac{3}{8}(4\mu-\lambda^2)\tanh^2\left(\sqrt{\frac{\lambda^2-4\mu}{2}}\xi\right)+\frac{3\lambda^2}{8}-\frac{3\mu}{2}$
2. If $A=1,B=-\lambda,C=0,E=-\mu,\Psi=1,C=0,\theta_1\neq 0$ and *and* $\theta_2=0$, then we get from equation (4.4.20) $u2_8(\xi)=\frac{3\lambda^2}{8}-\frac{3\mu}{2}+\frac{3}{8}(\lambda^2-4\mu)\cot^2\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)$ $v2_8(\xi)=\frac{3\lambda^2}{8}-\frac{3\mu}{2}+\frac{3}{8}(\lambda^2-4\mu)\cot^2\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)$ If $A=1,B=-\lambda,C=0,E=-\mu,\Psi=1,C=0,\theta_1=0$ and *and* $\theta_2\neq 0$, then we get from equation (4.4.14) $u2_2(\xi)=\frac{3\lambda^2}{8}-\frac{3\mu}{2}+\frac{3}{8}(\lambda^2-4\mu)\tan^2\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)$ $v2_2(\xi)=\frac{3\lambda^2}{8}-\frac{3\mu}{2}+\frac{3}{8}(\lambda^2-4\mu)\tan^2\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)$	2.If $C_1=0$ *and* $C_2\neq 0$, the solutions of eqn.(4.6.3) becomes as follows: $u_3(\xi)=\frac{3}{8}(\lambda^2-4\mu)\cot^2\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)+\frac{3\lambda^2}{8}-\frac{3\mu}{2}$ $v_3(\xi)=\frac{3}{8}(\lambda^2-4\mu)\cot^2\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)+\frac{3\lambda^2}{8}-\frac{3\mu}{2}$ If $C_1\neq 0$ *and* $C_2=0$, the solutions of eqn.(4.6.3) becomes: $u_3(\xi)=\frac{3}{8}(\lambda^2-4\mu)\tan^2\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)+\frac{3\lambda^2}{8}-\frac{3\mu}{2}$ $v_3(\xi)=\frac{3}{8}(\lambda^2-4\mu)\tan^2\left(\sqrt{\frac{4\mu-\lambda^2}{2}}\xi\right)+\frac{3\lambda^2}{8}-\frac{3\mu}{2}$

From the table, it can be said that, from this two method we have got almost the same results for the new extended (G'/G) expansion method. But here in this extended (G'/G) expansion method we have even got more solution than the basic (G'/G) expansion method used by Bekir and Uygun's. So, it can be predicted that, the extended (G'/G) expansion method is better and easier to find more exact solutions of (2+1)-dimensional breaking soliton equation.

4.7 Conclusion

The extended $\left(G'/G\right)$- expansion method provides many new solutions with reliability and simplicity. So, it is more powerful and efficient method to examine the exact solutions of Non-linear Partial Differential Equations. In this work, by using the extended $\left(G'/G\right)$- expansion method, we have found more enriched types of explicit and exact travelling wave solutions of the $\left(2+1\right)$- dimensional breaking soliton systems. The travelling wave solutions have been included through these travelling wave solutions. During the solving procedure we have obtained five types of travelling wave solutions in terms of hyperbolic, trigonometric and rational families to work with and six different types of sets of solution for each of the family. It should be pointed out that some of our solutions are matched with previously published results when parameters are taken in particular values which has authenticated our solutions. The arbitrary functions in obtained solutions indicate that these solutions have rich native configurations. The solutions of the strategic nonlinear evolution equations in this work have many prospective uses in physics and engineering. Finally, the method delivers a strong mathematical instrument to gain more common exact results of an excessive countless nonlinear PDEs in Mathematical physics.

Chapter Five

Future study

The extended $\left(G'/G\right)$ expansion method that is used in this work, is a standard, straight forward and, computerized method which will allow us to solve complex and tedious algebraic calculation in future. Therefore, by choosing the appropriate arbitrary function $\xi\left(x,y,t\right)$, that is included in its solutions, one can study various interesting localized soliton excitations and the wide applications of the solitary theory. Furthermore, this recommended method can be functional to solve various nonlinear PDEs with higher dimensional and higher order nonlinear evolution equation, which often get up in mathematical physics, engineering sciences and many scientific real time application fields. In one words, it will let us to handle more critical NLPDEs for different types of evolution equation in an easy and standard way in future.

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