



Review of Einstein Field Equation Solution

“Rotational and non-rotational black holes”

This thesis is submitted as the requirement for completing the degree of Bachelor of Science in
Physics (Department of Mathematics and Natural Sciences)

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Abstract

The origin of gravitational theory of relativity includes the discussion of Sir. Isaac Newton and Einstein's theory of relativity. A brief derivation is introduced about the spherically symmetric solution of Einstein's field equation, the Schwarzschild solution, and the axisymmetric solution, the Kerr solution. We have also presented the generalizations of both of the solutions followed by inter-related different types of coordinate systems like: Eddington-Finkelstein coordinate system and Kruskal coordinate system. The notion of black hole formation, rotational and non-rotational black hole structure is also shown with mathematical calculations and few graphical figures computation.

Keywords: Relativity, Gravity, Spherical symmetry solution, Axisymmetry solution, Singularity, Black holes, Accretion disk

Acknowledgement

I would like to thank my supervisor and Professor Dr. Amin Hasan Kazi and Faculty Mohammad Hassan Murad for their patience and dedication during the course of this thesis, and for not hesitating to meet me several times a week during periods when I was progressing quickly. Their directions along with the courses formed my knowledge and skills to accomplish this thesis. I highly recommend them to anyone who is planning to do their thesis in the upcoming years ahead.

With the deepest gratitude and honour, I would like to mention Professor A. A. Ziauddin Ahmad and late Professor Mofiz Uddin Ahmed as they are the people who planted the seed of physics in my heart and I took the step of continuing double major in Physics fearlessly.

An acknowledgement is also due to my faculties Muhammad Lutfor Rahman , Mahabobe Shobahani Sir for continuously motivating me and the organization I work for, named *It's Humanity Foundation* for providing complete support during my academic period.

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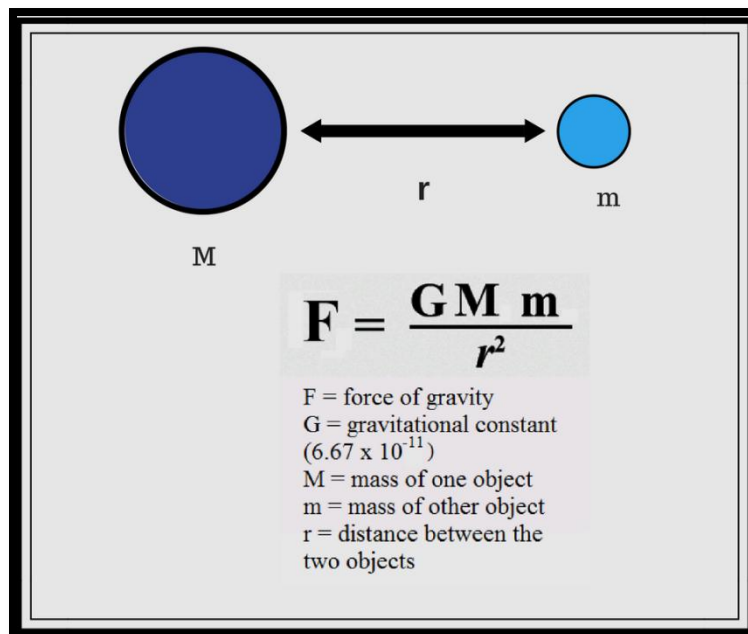
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1. Introduction to theory of gravitation & relativity

The scientific view of the physical universe was initiated with the formulation of the theory of Gravity by Sir Isaac Newton in 1687. This universal law of gravitation gave remarkable support to the perception of underlying simplicity and unity in nature by defining the gravitational force and explaining both falling bodies and astronomical motions.

1.1 Newton's universal law of gravitation

Newton's universal law of gravitation states that every particle in the universe attracts every other particle with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.



Picture: Newton's Universal law of Gravitation

In Newtonian mechanics the universe was considered as an infinite Euclidean space of three dimensions without any spatial boundary. It means any incident in the universe could be illustrated by three spatial co-ordinates, generally written as (x, y, z) based on the assumptions that there existed an absolute time, t which was homogeneous for every observer. Moreover, according to Newton, there was an absolute frame of reference and all motion took place with respect to this space. For him, gravity worked instantaneously. But, even after all these laws he was not able to give any reasons behind the mechanism of gravity as he formulated the idea based solely on observations. Newton said "I do not frame hypothesis on the nature of gravity. I have no idea why it happens; all I can do is mathematically describe how it happens." Even in a

letter to Bentley (a great friend and admirer of Newton) he wrote “You sometimes speak of gravity as an essential and inherent to Matter. Pray do not ascribe that notion to me; for the cause of gravity is what I do not pretend to know and therefore would make more time to consider it.” In another letter Newton wrote “It is inconceivable, that inanimate brute matter, should, without that mediation of something else, which is not material, operate upon and affect other matter without mutual contact. That gravity should be innate, inherent and essential to matter, so that one body may act upon another at distance through a vacuum, without the mediation of anything else. Gravity must be caused by an agent acting constantly according to certain laws; but whether this agent is material or immaterial, I have left to the consideration of my readers.”

This dilemma regarding the mechanism behind gravity came to an end with the advent of Einstein’s general theory of relativity. In this theory he gave a geometric interpretation to explain gravity as the manifestation of the curvature of space and time.

1.2 Einstein’s special theory of relativity

In 1905 Albert Einstein introduced the special theory of relativity in his paper “On the Electrodynamics of Moving Bodies”. He showed that space and time are very well connected with each other. According to this theory any event can be described by four numbers. Three numbers describe the position of the event in miles/latitude/longitude/distance from the center of the galaxy etc. and the fourth number is the incident time. In this way space and time can be considered together, as a four-dimensional entity (x, y, z, t) called space-time. But as in this theory there was absence of gravitational field.

The special theory of relativity was based on two main postulates:

1. The Principle of Relativity:
The physical laws of nature are the same in every inertial frame of reference.

Here the frame of reference means a frame used for any observational perspective which consists of coordinate system, observer and clock to measure time and distance. If two observers O_1 and O_2 are in frame of references of constant speed v . The observer at rest in O_1 will see the objects at rest in O_2 are moving with respect to O_1 . Then again the observer at rest in O_2 will see the objects at rest in O_1 are moving with respect to O_2 . But the law of inertia and other physical laws are equally applicable in all kinds of reference frames.

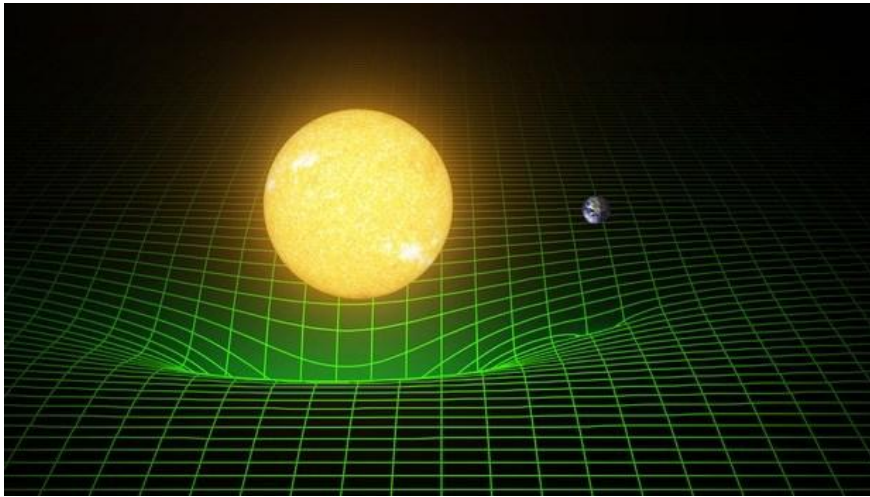
2. Constancy of Speed of Light:

The speed of light is the same in every inertial frame of reference.

Speed of light, $c = 3.00 \times 10^8 \text{ m/s}$ is independent of the relative motion of the source.

1.3 Einstein's general theory of relativity

In 1915, after having research of another 10 years Einstein successfully demonstrated that the effects of gravity could be explained by considering that space-time as warped or distorted due to the matter and energy in it. In his theory, he determined that massive objects cause a distortion in space-time, which is felt as gravity. Einstein named this finding of the gravity as general theory of relativity to distinguish from the earlier one which he called special theory of relativity.



Picture: In Einstein's view of the world, gravity is the curvature of space-time caused by massive objects. Image source: T.Pyle/Caltech/MIT/LIGO Lab.

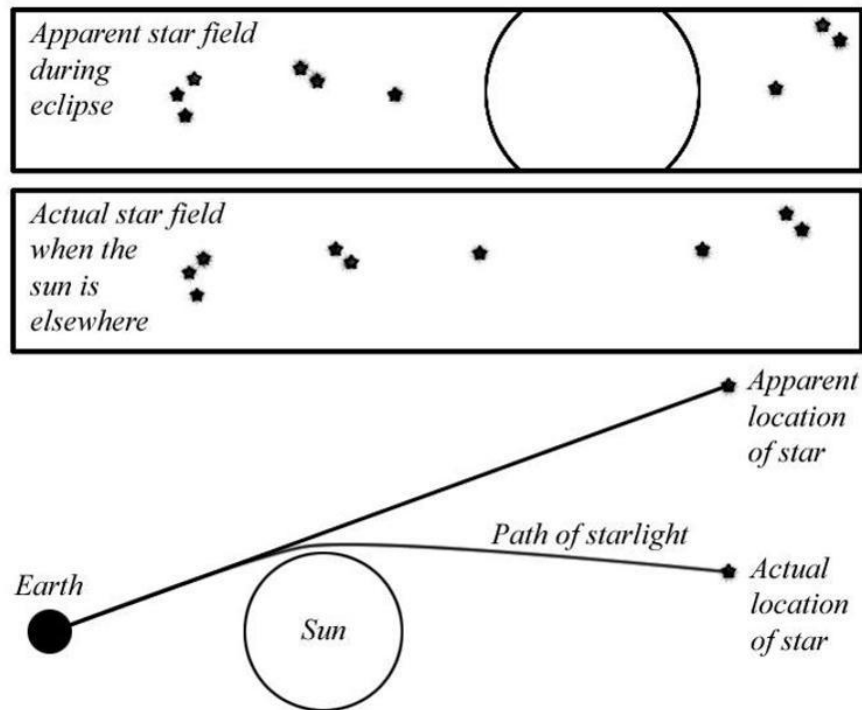
According to the theory of general relativity, matter causes space to curve and it is accepted that gravitation is not a force, as understood by Newtonian physics, but a curved *field* which is basically an area of space under the influence of a force. It is also shown by Einstein that all the celestial objects are still moving along a straight line, but due to a distortion in space-time, the straight line now acting like a spherical path. The shortest line segment between two points in a curved space is called a **geodesic**. So two objects that were moving along a flat plane are now moving along a spherical plane. And two straight paths along that sphere end in a single point. Einstein contended that this theory could be tested by measuring the deflection of starlight traveling near the sun; he correctly asserted that light deflection would be twice that expected by Newton's laws.

Einstein's formation of relativity is somewhat more involved, even if his starting point was in many respects the same. His ground-breaking theories take into account the speed of light, the structure of space-time and the equivalence of acceleration and gravity. They have led to some remarkable consequences, including the dilation of time, the contraction of length, mass-energy equivalence and the bending of light, as well as the prediction of the existence of black holes, wormholes and the "birth" of the universe in a Big Bang.

1.4 Gravitational light deflection

If Einstein's space-time curvature is valid the light will travel in a curved path rather than a straight line. The bending will be in large angle if the force of gravity is very strong. Since earlier only the stars were considered as the sources of strong gravitational field and due to the extremely bright light of sun it was very difficult to observe the bending of light coming from distant stars.

In 1919, finding out a way of the solution Eddington run an experiment during a total eclipse, when for few minutes the distant stars closure to sun are visible. And it was clearly seen that due to the Sun's gravity these stars' original positions were shifted slightly rather than other times we saw them in the sky.. The closer the star appears to the Sun during totality, the bigger the shift would be. It's shown in the picture below:



Picture: Light deflection experiment during the total eclipse (Collected)

2. Einstein's Field Equations

Einstein formulated gravity as geometry of space-time. This space time describes or indicates on which way to move and the mass matter on mass influences the curvature of space-time.

If a mass M is located at the origin of a 3-dimensional system (x, y, z) with a position vector \mathbf{X} , then the force \mathbf{F} on a particle of mass m located at \mathbf{x} is,

$$\mathbf{F} = \frac{Mm}{r^3} \mathbf{r}$$

$$= -\frac{Mm}{r^2} \mathbf{r}$$

We know from Newton's second law,

$$\mathbf{F} = m \frac{d^2 \mathbf{x}}{dt^2}$$

$$\frac{d^2 \mathbf{x}}{dt^2} = -\frac{M}{r^2} \mathbf{r}$$

Now, the gravitational potential energy is equal to the work done in space which is defined by ϕ_r

$$\phi_r = \frac{w}{m}$$

$$w = \int_{\infty}^n \mathbf{F} \cdot d\mathbf{r}$$

$$= -\frac{GmM}{r}$$

$$\phi_r = \frac{-\frac{GmM}{r}}{m}$$

$$= -\frac{GM}{r}$$

Since, $G = 1$,

$$\phi_r = -\frac{M}{r}$$

Gravitational potential in empty space is zero defined by Laplace's equation,

$$\Delta f = \frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} + \frac{\delta^2 f}{\delta z^2} = 0$$

The general form of the metric tensor is denoted by g_{uv} which is referred to as the metric tensor for Minkowski space.

The space-time which was formulated by Minkowski is called Minkowski space. Considering two events and the space-time would be,

$$s^2 = G - (x_a^0 - x_b^0)^2 + (x_a^1 - x_b^1)^2 + (x_a^2 - x_b^2)^2 + (x_a^3 - x_b^3)^2$$

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

This interval is invariant in other coordinate system $(x^{10}, x^{11}, x^{12}, x^{13})$

$$ds^2 > \text{spacelike interval}$$

$$ds^2 < 0 = \text{timelike interval}$$

$$ds^2 = \text{lightlike interval}$$

We can also write

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

For μ and ν values of $\{0, 1, 2, 3\}$

Implementing Einstein's summation notation

$$ds^2 = \sum_{\mu=0}^3 \sum_{\nu=0}^3 \eta_{\mu\nu} dx^\mu dx^\nu$$

As $\Gamma_{\mu\nu}$ can be shown as $G_{\mu\nu}$ from this and we can write $G_{\mu\nu}$ as

$$= \begin{bmatrix} g_0 & g_1 & g_2 & g_3 \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the metric coefficients $G_{\mu\nu}$ we find the Reimann curvature tensor.

Einstein's field equation can be written as:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$$

Here $R_{\mu\nu}$ = Ricci tensor, R = Scalar curvature, $T_{\mu\nu}$ = Stress-energy tensor

So basically the Schwarzschild solution is a solution to the Einstein's vacuum field equation

$$G_{\mu\nu} = R_{\mu\nu} = 0$$

which is valid to the absence of matter in a non-gravitational field ($T_{\mu\nu}=0$)

Starting from the general expression for the line element

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu$$

The only rational variant of space-like coordinates x^i and their differentials are,

$$\vec{x} \cdot \vec{x} = r^2, d\vec{x} \cdot d\vec{x}, \vec{x} \cdot d\vec{x}$$

$$\vec{x} = (x^1, x^2, x^3), x^0 \rightarrow t$$

3. Spherical Symmetry Solution

Before spherical symmetrical solution given by Karl Schwarzschild, Einstein assumed that field equations are not solvable. Spherical symmetrical solution brought an exact solution of Einstein's field Equation. This review will focus on some details of the solution by Schwarzschild.

3.1 Schwarzschild Solution

In this solution gravitational field is considered as static. Considering the line element of Schwarzschild we get,

$$ds^2 = A(t, r)dt^2 - B(t, r)dt \cdot \vec{x} \cdot d\vec{x} - c(t, r)(\vec{x} \cdot d\vec{x})^2 - D(t, r)d\vec{x}^2$$

Where A, B, C, D are arbitrary functions of the coordinates t and r

Let us now transform to the spherical polar coordinates (t, r, θ, φ)

$$x^1 = r \sin \theta \cos \varphi$$

$$x^2 = r \sin \theta \sin \varphi$$

$$x^3 = r \cos \theta$$

We have,

$$x \cdot x = r^2$$

$$d\vec{x} \cdot d\vec{x} = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$

$$dx^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$

Absorbing factors of r into our functions,

$$ds^2 = A(t, r)dt^2 - B(t, r)dt dr - c(t, r)dr^2 - D(t, r)(r^2 d\theta^2 + \sin^2 \theta d\varphi^2)$$

$$r^2 = D(t, r)$$

$$ds^2 = A(t, r)dt^2 - B(t, r)dt dr - c(t, r)d\vec{r} - (t, \vec{r})dr^2 - \vec{r}^2(d\theta^2 + \sin^2 \theta d\varphi^2)$$

$$dt^2 = \varphi(t, \vec{r}) \left[A(t, \vec{r})dt - \frac{1}{2} B(t, \vec{r})d\vec{r} \right]$$

$$dt^2 = \varphi^2 \left(A^2 dt^2 - AB dt d\vec{r} + \frac{1}{4} B^2 d\vec{r}^2 \right)$$

$$A dt^2 - B dt dr = \frac{1}{A\varphi^2} dt^2 - \frac{B}{4A} d\vec{r}^2$$

$$A = \frac{1}{A\varphi^2}, B = c + \frac{B}{4A}$$

$$ds^2 = A(t, r)dt^2 - B(r)dr^2 - (r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2)$$

Now In,

$$g_{\mu\nu} = g_{00} = A(r)$$

$$g_{11} = -B(r)$$

$$g_{22} = -r^2$$

$$g_{33} = -r^2 \sin^2 \theta$$

And

$$g^{\mu\nu}, g^{00} = \frac{1}{A(r)}$$

$$g^{11} = -\frac{1}{B(r)}$$

$$g^{22} = -\frac{1}{r^2}$$

$$g^{33} = -\frac{1}{r^2} \sin^2 \theta$$

The empty field equation,

$$R_{\mu\nu} = 0$$

$$R_{\mu\nu} = \partial_\nu \Gamma_{\mu\sigma}^\sigma - \partial_\nu \Gamma_{\mu\vartheta}^\sigma + \Gamma_{\mu\sigma}^\sigma \Gamma_{\nu\rho}^\sigma - \Gamma_{\mu\nu}^\rho \Gamma_{\rho\sigma}^\sigma$$

$$\Gamma_{\mu\nu}^\sigma = \frac{1}{2} g^{\sigma\rho} (\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu})$$

So,

$$R_{00} = -\frac{A''}{2B} + \frac{A'}{4B} \left(\frac{A'}{A} + \frac{B'}{B} \right)$$

$$R_{11} = \frac{A''}{2A} + \frac{A'}{4A} \left(\frac{A'}{A} + \frac{B'}{B} \right) - \frac{B'}{rB}$$

$$R_{22} = \frac{1}{B} - 1 + \frac{r}{2B} \left(\frac{A'}{A} - \frac{B'}{B} \right)$$

$$R_{33} = R_{22} \sin^2 \theta$$

Adding $\frac{B}{A}$ times R_{00} and R_{11} we get,

$$-\frac{A''B}{2AB} + \frac{A''B}{2A^2} + \frac{A'B}{4BA} \left(\frac{A'}{A} + \frac{B'}{B} \right) - \frac{A'B}{4A^2} \left(\frac{A'}{A} + \frac{B'}{B} \right) - \frac{A'B}{rBA} - \frac{B'B}{rBA} = 0$$

Where,

$$\frac{A'}{A} + \frac{B'}{B} = 0$$

$$\frac{A'B + AB'}{4AB} = 0$$

Hence,

$$A'B + AB' = 0$$

which implies that, $AB = \text{constant}$

$$AB = \alpha \text{ [Since, constant} = \alpha \text{]}$$

$$B = \frac{\alpha}{A}$$

Now, putting this in R_{22}

$$\begin{aligned} R_{22} &= \frac{1}{\frac{\alpha}{A}} - 1 + \frac{r}{2\frac{\alpha}{A}} \left(\frac{A'}{A} - \frac{B'}{\frac{\alpha}{A}} \right) \\ &= \frac{A}{\alpha} - 1 + \frac{A'r}{2\alpha} - \frac{A^2 B' r}{2\alpha^2} \end{aligned}$$

$$= \frac{2A\alpha - 2\alpha^2 + A' r \alpha - A^2 B' r}{2\alpha^2}$$

Now,

$$A + rA' = \alpha$$

$$\frac{d(rA)}{dr} = \alpha$$

$$\int \frac{d(rA)}{dr} = \int \alpha$$

$$rA = \alpha(r + k)$$

$$A(r) = \alpha \left(1 + \frac{k}{r}\right)$$

$$B(r) = \frac{\alpha}{A} = \left(1 + \frac{k}{r}\right)^{-1}$$

Now considering the weak field,

$$\frac{A(r)}{c^2} \rightarrow 1 + \frac{2\varphi}{c^2}$$

Hence,

$$A(r) \rightarrow c^2 \left(1 + \frac{2\varphi}{c^2}\right)$$

We found earlier,

$$\varphi = -\frac{GM}{r}$$

$$k = -\frac{2GM}{c^2}$$

$$\alpha = c^2$$

then, Schwarzschild metric become,

$$ds^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2$$

Now,

If we consider the classical Newtonian physics, then kinetic energy = $\frac{1}{2} mv^2$

Gravitation potential energy =

$$-\frac{GMm}{r}$$

While, $v = c$

$$\frac{1}{2} mc^2 = \frac{GMm}{r}$$

$$rc^2 = 2GM$$

$$r = \frac{2GM}{c^2}$$

Now, we can say that the metric is singular at $r = 0$ and at $r = \frac{2GM}{c^2}$

So, here appear two types of singularities,

- 1) At $r = 0$, from g_{00} which is known as essential singularity.
- 2) At $r_s = \frac{2GM}{c^2}$ from g_{11} which is known as coordinate singularity. Here, $r_s = \frac{2GM}{c^2}$, the radius is called **Schwarzschild radius**.

Coordinate singularity means a location where a chosen coordinate cannot describe the geometry properly.

As, earlier we have seen Schwarzschild radius,

$$r_s = \frac{2GM}{c^2}$$

Now, r_s for a proton is,

$$r_s = \frac{2GM_p}{c^2} = 10^{-52} M$$

Here, M_p = Mass of proton

This can be described effectively by world-line. **World-line** means the set of all past and future events of a single object.

Considering an emitter at fixed spatial coordinates $(r_E, \theta_E, \varphi_E)$ which emits a photon that travel radial to the distant observer. The equation for space-time interval (ds^2)

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 d\Omega^2$$

where, $\mu = \frac{GM}{c^2}$ and $d\Omega$ is an element of solid angle.

$$d\Omega = (d^2\theta + \sin^2\theta d\varphi^2)$$

$$ds^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

As the space-time interval (ds^2) of events on a photon's world-line is zero, $ds^2 = 0$
Hence we can write the equation as below,

$$0 = c^2 \left(1 - \frac{2\mu}{r}\right) dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

$$c^2 \left(1 - \frac{2\mu}{r}\right) dt^2 = \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

For events along the path of such a photon, $d\theta = d\varphi = 0$

Now,

$$c^2 \left(1 - \frac{2\mu}{r}\right) dt^2 = \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2$$

$$\left(1 - \frac{2\mu}{r}\right) dt^2 = \frac{1}{c^2} \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2$$

$$\frac{dt^2}{dr^2} = \frac{1}{c^2} \frac{(1 - \frac{2\mu}{r})^{-1}}{(1 - \frac{2\mu}{r})}$$

$$\frac{dt^2}{dr^2} = \frac{1}{c^2} \frac{1}{(1 - \frac{2\mu}{r})^2}$$

$$\frac{dt}{dr} = \pm \frac{1}{c} (1 - \frac{2\mu}{r})$$

Here the (+) sign resembles the outgoing proton and the (-) sign resembles the incoming proton. Now integrating the equation,

$$\frac{dt}{dr} = \pm \frac{1}{c} (1 - \frac{2\mu}{r})^{-1}$$

$$cdt = 1 - \frac{2\mu}{r} dr$$

$$\int c dt = \int (1 - \frac{2\mu}{r})^{-1} dr$$

$$\int c dt = \int (1 - \frac{2\mu}{r})^{-1} dr$$

$$c \int dt = \int \frac{1}{(1 - \frac{2\mu}{r})} dr$$

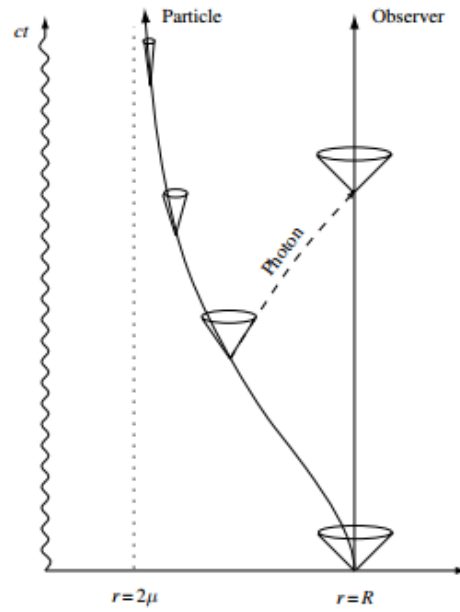
$$ct = r + 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| + \text{constant [outgoing proton]}$$

$$ct = -r - 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| + \text{constant [incoming proton]}$$

It measures the slope of the light cones on a space-time diagram of the (ct, r) plane.

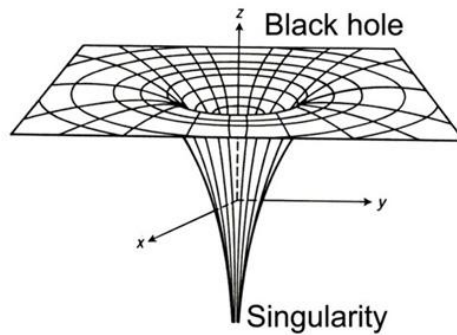
For large r the slope is ± 1 ,

The problem with this current coordinate is that $t \rightarrow \infty$, which approach $r = 2GM$, as a result in r direction the progress is slower and it will take an infinite time for an incoming signal to reach the Schwarzschild radius.



Here, in the above shown picture (collected) when $r = 2\mu$ the photon worldline is vertical, that's why the observer will receive the photon at $t \rightarrow \infty$, and in this way the particle appears infinite time to reach the horizon.

This solution in the normal spherical coordinates has a singularity.



In the given picture(collected) it can be seen that Singularity is a point which contains a huge mass in an infinitely small space. In this point density, gravity infinite and space-time curves infinitely, physics laws don't operate. It is unclear what happens for $r < r_s$, but without singularity photons will be pulled to $r = 0$ and which is the phenomenon of the black hole.

Combined energy for the radial particle worldlines-in r -coordinate,

$$\dot{r}^2 + \frac{h^2}{r^2} \left(1 - \frac{2GM}{c^2 r}\right) - \frac{2GM}{r} = c^2 (k^2 - 1)$$

This energy equation is used to discuss radial free fall and the stability of orbits. While radial motion, $\Phi = \text{constant}$, $h = 0$

Then,

$$\dot{r}^2 = c^2 (k^2 - 1) + \frac{2GM}{r}$$

$$\frac{d\dot{r}^2}{d\tau} = \frac{d}{d\tau} \{c^2 (k^2 - 1)\} + \frac{2GM}{r} \frac{d}{d\tau}$$

$$\dot{r} = 0 + 2 Gmr^{-1} \frac{d}{d\tau}$$

$$\ddot{r} = - GM r^{-2}$$

$$\ddot{r} = - \frac{GM}{r^2}$$

$$k^2 = 1 - \frac{2GM}{c^2 r}$$

Putting, $GM = \mu$ and differentiating,

$$\left(1 - \frac{2\mu}{r}\right) \dot{t} = k$$

Putting $k = 1$,

$$\left(1 - \frac{2\mu}{r}\right) \frac{dt}{d\tau} = 1$$

$$\frac{dt}{d\tau} = \frac{1}{\left(1 - \frac{2\mu}{r}\right)}$$

$$\frac{dt}{d\tau} = \left(1 - \frac{2\mu}{r}\right)^{-1}$$

$$\frac{dt}{d\tau} = \left(\frac{2\mu c^2}{r}\right)^{\frac{1}{2}}$$

Now integrating,

$$\int \frac{dt}{d\tau} = \int - \left(\frac{2\mu c^2}{r}\right)^{\frac{1}{2}}$$

$$\tau = \frac{2}{3} \sqrt{\frac{r_o^3}{2\mu c^2}} - \frac{2}{3} - \sqrt{\frac{r^3}{2\mu c^2}}$$

Mapping out the trajectory of the particle in the (t, r) coordinate plane:

$$\begin{aligned} \frac{dr}{dt} &= \frac{dr}{d\tau} \frac{d\tau}{dt} \\ &= -\left(\frac{2\mu c^2}{r}\right)^{\frac{1}{2}} \left(1 - \frac{2\mu}{r}\right) \end{aligned}$$

After integrating we find,

$$t = \frac{2}{3} \left(\sqrt{\frac{r_o^3}{2\mu c^2}} - \sqrt{\frac{r^3}{2\mu c^2}} \right) + \frac{4\mu}{c} \left(\sqrt{\frac{r_o}{2\mu}} - \sqrt{\frac{r}{2\mu}} \right) + \frac{2\mu}{c} \ln \left| \left(\frac{\sqrt{\frac{r}{2\mu}} + 1}{\sqrt{\frac{r}{2\mu}} - 1} \right) \left(\frac{\sqrt{\frac{r}{2\mu}} - 1}{\sqrt{\frac{r}{2\mu}} + 1} \right) \right|$$

So, it is impossible to have a stationary observer at $r \leq 2\mu$

3.2 Eddington- Finkelstein Coordinate System

Eddington–Finkelstein coordinates use the same position coordinates as Schwarzschild coordinates, only the time coordinate is transformed, so first consider how to define Schwarzschild coordinates in a physical way. This coordinate system will help to remove the coordinate singularity at $r = 2\mu$. Earlier we have seen that the world line of a radially ingoing photon is given by:

$$ct = -r - 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| + \text{constant}$$

Considering this constant as another coordinate:

$$ct + r + 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| = + \text{constant}$$

Now letting the *constant*= p ,

$$p = ct + r + 2\mu \ln \left| \frac{r}{2\mu} - 1 \right|$$

Here p is a null coordinate.

Now, we get by differentiating,

$$\begin{aligned} dp &= cdt + \frac{1}{\left(\frac{r}{2\mu} - 1 \right)} dr \\ &= cdt + \frac{1}{\left(\frac{r - 2\mu}{r} \right)} dr \\ &= cdt + \frac{r}{r - 2\mu} dr \end{aligned}$$

Substituting dt in the Schwarzschild line element, we find in terms of p the line element:

$$ds^2 = \left(1 - \frac{2\mu}{r} \right) dp^2 - 2 dp dr - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

In radial null geodesics:

$$ds = d\theta = d\phi = 0$$

$$\left(1 - \frac{2\mu}{r} \right) \left(\frac{dp}{dr} \right)^2 - 2 \frac{dp}{dr} = 0$$

Through which we get two solutions:

- 1) Incoming radial null geodesic:

Here, $p = \text{constant}$.

$$\frac{dp}{dr} = 0$$

- 2) Outgoing radial null geodesic:

$$\frac{dp}{dr} = 2 \left(1 - \frac{2\mu}{r}\right)^{-1}$$

$$p = 2r + 4\mu \ln \left| \frac{r}{2\mu} - 1 \right| + \text{constant}$$

Now working with a related time-like coordinate t' ,

$$\begin{aligned} ct' &\equiv p - r \\ &= \\ ct + 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| \end{aligned}$$

Then the line elements take the form:

$$ds^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt'^2 - \frac{4\mu c}{r} dt' dr - \left(1 + \frac{2\mu}{r}\right) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

Here the line element isn't invariant with respect to coordinates t' , $-t'$

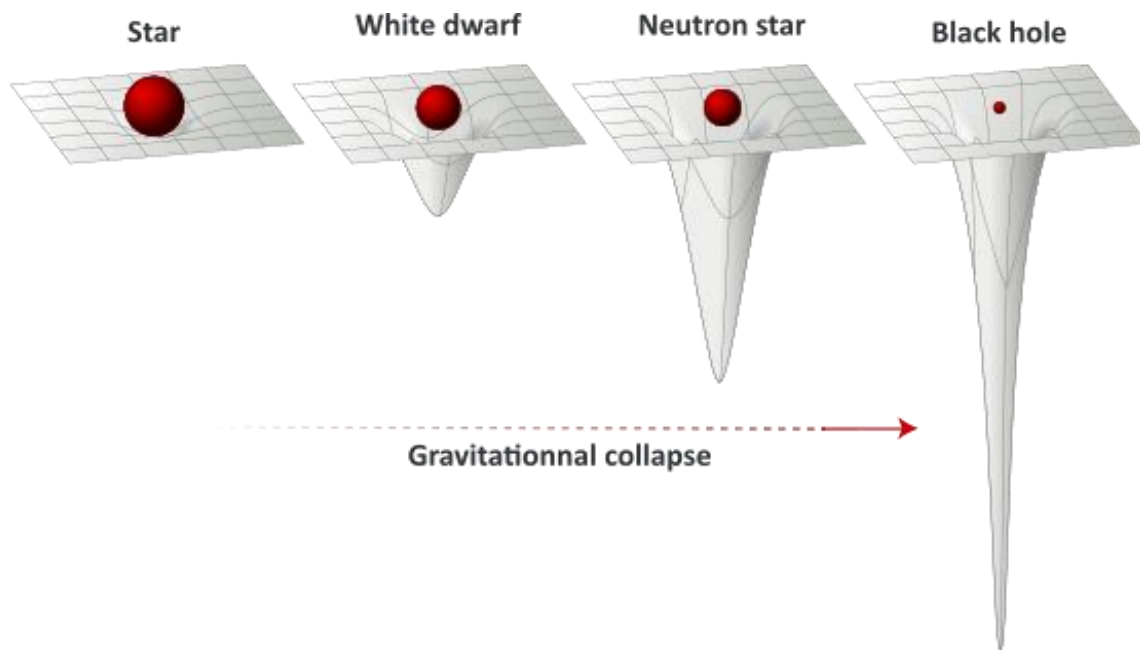
$$ct' = -r + \text{constant}$$

$$ct' = r + 4\mu \ln \left| \frac{r}{2\mu} - 1 \right| + \text{constant}$$

So, ingoing photons are continuous straight lines across $r = 2\mu$ with an angle 45° . As a result, the light-cone structure changes at the Schwarzschild radius $r = 2\mu$, which is also known as the boundary of no return or event horizon. Any particle must fall into singularity $r=0$, if crosses the event horizon. Otherwise, for outgoing null geodesics any photon emitted at $r < 2\mu$ never gets to cross the event horizon and will never be visible to observer. And an object with Schwarzschild radius is called a black hole.

3.3 Basic concept of black-hole formation

The gravitational collapse of a very massive star is considered to be the main reason behind the existence of black-holes. Since a star is formed with the mixture of gas and radiation pressure, this energy is produced by the fusion of light nuclei into heavier ones. But after a certain time all this nuclear fuel will be all used up then the star will start cooling and collapse under its own gravity. This is known as white-dwarf. According to Chandrasekhar realized that the more massive a white- dwarf, the denser it must be and the gravitational field will be that much strong. For white dwarfs it is believed that if the maximum mass exceeds $3M_{\odot}$ which is known as the Oppenheimer- Volkoff limit, no stable neutron-star configuration is possible. As a result, stars with more mass than this limit collapse to form black holes. According to Penrose's "singularity theorem" in the event horizon a singularity will appear and there the curvature will diverge. And Schwarzschild geometry could prove this.



Picture: Singularity Point of different gravitational field's sources (collected)

3.4 Spherical symmetric collapse

The spherically symmetric collapse of a massive star to form a Schwarzschild black hole and the viewer will be a stationary observer at large radius. The particles on the surface will follow the radial geodesics and the energies as observed by the emitter at A and by the receiver at B are respectively given below-

$$E(A) = p(A) \cdot \mu_E(A) = p_\mu(A) \mu^\mu_E(A)$$

$$E(B) = p(B) \cdot \mu_R(B) = p_\mu(B) \mu^\mu_R(B)$$

As, $E = h\nu$

The ratio of the frequencies of a photon at emission and reception is,

$$\frac{V_R}{V_E} = \frac{\mu^\mu_R p_\mu(R)}{\mu^\mu_E p_\mu(E)}$$

Here, the photon 4-momentum p at any point is tangent to the geodesic and parallel-transported along the path.

$p_\mu(R)$ = reception

$p_\mu(E)$ = emission

Since, p is null

$$g^{\mu\nu} p_\mu p_\nu = 0$$

$$\frac{1}{c^2} \left(1 - \frac{2\mu}{r}\right)^{-1} (p_0)^2 - \left(1 - \frac{2\mu}{r}\right) (p_1)^2 = 0$$

Now, $r \rightarrow 2\mu, V_R \rightarrow 0$,

$$\frac{V_R}{V_E} \approx \frac{r - 2\mu}{4\mu}$$

Near the event horizon the time of reception,

$$\frac{V_R}{V_E} \approx \exp\left(-\frac{ct}{4\mu}\right)$$

On the event horizon, except the radial outwards photons all the rest are dragged inwards.

But on the event horizon the force works strongly is called the tidal force. Tidal force means the strength of the gravitational field between two points. Naturally it was thought that anyone

falling into a stationary black hole would be crushed to death in its central singularity. But here the concept of tidal effect was ignored which is the main reason of this vulnerability.

Spaghettification or this tidal force is an incident which occurs only in black holes due to the enormous gravitational force near the event horizon.

This tidal force is usually measured by tetrad frame which forms locally inertial frame at every point and it makes easier to reflect important physical aspects of the space-time. A tetrad is usually specified by its coefficients e^μ with respect to a coordinate basis,

$$(\hat{e}_0)^\mu = \frac{1}{c} \mu^\mu = \frac{1}{c} \left(1 - \frac{2\mu}{r} \right)^{-\frac{1}{2}} \delta_0^\mu$$

$$(\hat{e}_1)^\mu = \left(1 - \frac{2\mu}{r} \right)^{\frac{1}{2}} \delta_1^\mu$$

$$(\hat{e}_2)^\mu = \frac{1}{2} \delta_2^\mu$$

$$(\hat{e}_3)^\mu = \frac{1}{r \sin \theta} \delta_3^\mu$$

In Newtonian gravity, gravity has produced a tidal force in the sphere of particles that results an elongation in the transverse directions.

In free fall considering two non interacting particles moving along the time-like geodesics $x^\mu \tau$ and $\bar{x}^\mu \tau$ respectively, where τ is the proper time experienced by the first particle. Defining a small separation vector between the two particle worldliness by tetrad frame and considering the equation of geodesic deviation,

$$\xi^\mu(\tau) = \bar{x}^\mu(\tau) - x^\mu(\tau)$$

$$\frac{D^2 \xi^\mu}{D\tau^2} = \delta_v^\mu \xi^v$$

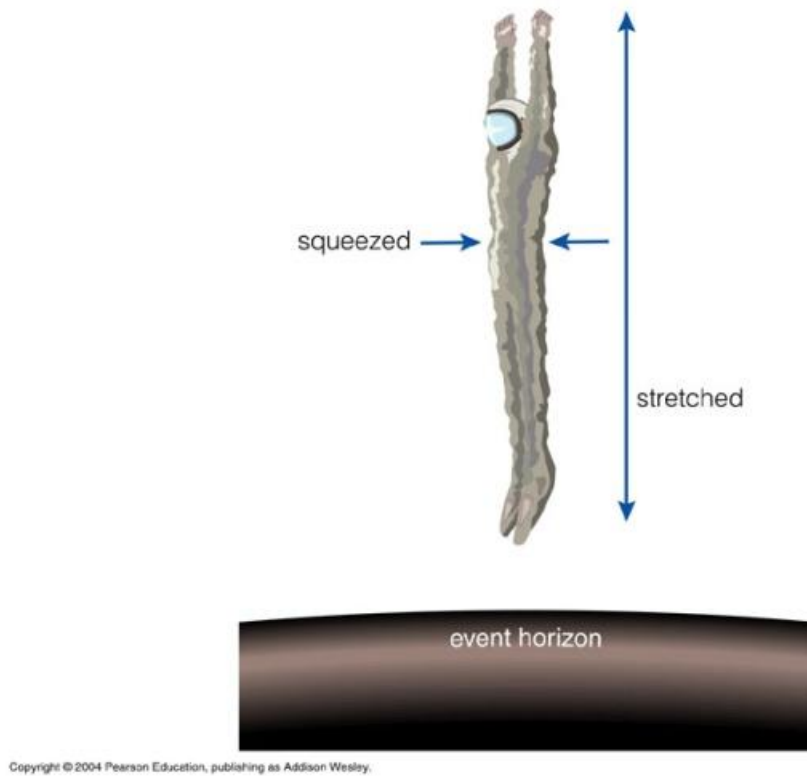
$$\delta_v^\mu = R^\mu \sigma h \rho v u^\rho u^v$$

In which,

$u^\sigma = \frac{Dx^\sigma}{D\tau}$ is the 4-velocity of the first particle.

This equation of the geodesic deviation represents the appropriate acceleration between two points which are distant by a small distance $\delta \xi_m$ and here ξ_m is the locally inertial coordinates of the tetrad frame [A frame of one time like and three space like vectors]. Between the two points when gravity gets stronger unfortunately tidal force becomes stronger as well. As a result, due to this force any object/body bear this will be pulled out into long thin strips like piece of spaghetti - hence the term spaghettification.

Going closure to the gravitational source causes tidal forces, as when a body gets closure to the source of the gravitational field then different parts of the body are close to the force in a different manner. Such in case of human legs go close to the singularity first than the head.



Picture: Body stretching due to tidal force (collected)

And the body will be stretched like the given picture above and later body parts will be torn apart which will cause the ultimate death of any living object at least for human.

3.5 Kruskal Coordinate System

Martin Kruskal formulated a co-ordinate system where both incoming and outgoing radial photon geodesics are continuous straight-lines. It also deduces the removal of the co-ordinate singularity at the Schwarzschild radius. Taking both advanced null coordinates p and the retarded null coordinate q . In the coordinates (p, q, θ, Φ) the line element:

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) dpdq - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

r defined implicitly in terms of \vec{u} and \vec{v} by,

$$\frac{1}{2}(\vec{u} - \vec{v}) = r + 2\mu \ln \left| \frac{r}{2\mu} - 1 \right|$$

$$-u = -\infty, v = -\infty$$

$$u' = \left(\frac{r}{2\mu} - 1\right)^{\frac{1}{2}} e^{\frac{r+t}{4\mu}}$$

$$v' = \left(\frac{r}{2\mu} - 1\right)^{\frac{1}{2}} e^{\frac{r-t}{4\mu}}$$

In this (u', v', θ, Φ) system the Schwarzschild metric,

$$ds^2 = -\frac{16 G^3 M^3}{r} e^{-\frac{r}{2\mu}} (du' dv' + dv' du') + r^2 d\Omega^2$$

Working in a system where one co-ordinate is time-like and the rest are like space-like,

$$\begin{aligned} u &= \frac{1}{2} (u' - v') \\ &= \left(\frac{r}{2\mu} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4\mu}} \cosh \frac{t}{4\mu} \end{aligned}$$

And,

$$v = \frac{1}{2} (u' + v')$$

$$= \left(\frac{r}{2\mu} - 1 \right)^{\frac{1}{2}} e^{\frac{r}{4\mu}} \sinh \frac{t}{4\mu}$$

Now the metric:

$$ds^2 = \frac{32 \mu^3}{r} e^{-\frac{r}{2\mu}} (-dv^2 + du^2) + r^2 d\Omega^2$$

where r is defined implicitly,

$$(u^2 - v^2) = \left(\frac{r}{2\mu} - 1 \right) e^{\frac{r}{2\mu}}$$

Where (u, v, θ, φ) are known as Kruskal Co-ordinate System.

The radial null curves look like:

$$v = \pm u + \text{constant}$$

The event horizon, $r = 2\mu$ is not infinitely far away, which is defined by,

$$v = \pm u$$

Considering the surfaces, $r = \text{constant}$ which is a 2-sphere

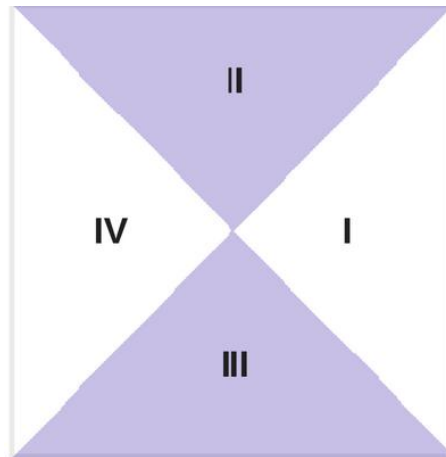
$$(u^2 - v^2) = \text{constant}$$

In this way it appears as hyperbolae in the u - v plane. And the constant is denoted as t . at 45°

$$\frac{u}{v} = \tan \frac{t}{4\mu} [r < 2\mu]$$

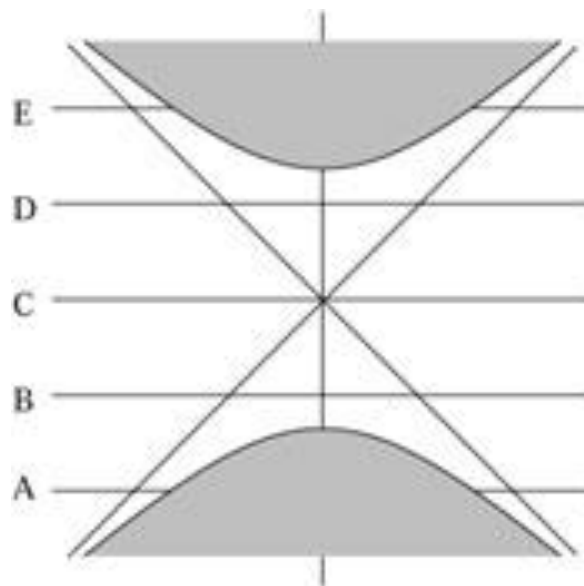
$$\frac{v}{u} = \tanh \frac{t}{4\mu} [r > 2\mu]$$

Now, the extended Schwarzschild space-time is given below by dividing this in 4 different regions where straight lines correspond to the constant time t . However, at the two 45° diagonal lines represent a limiting case where a time-like line overlaps with a space-like line.

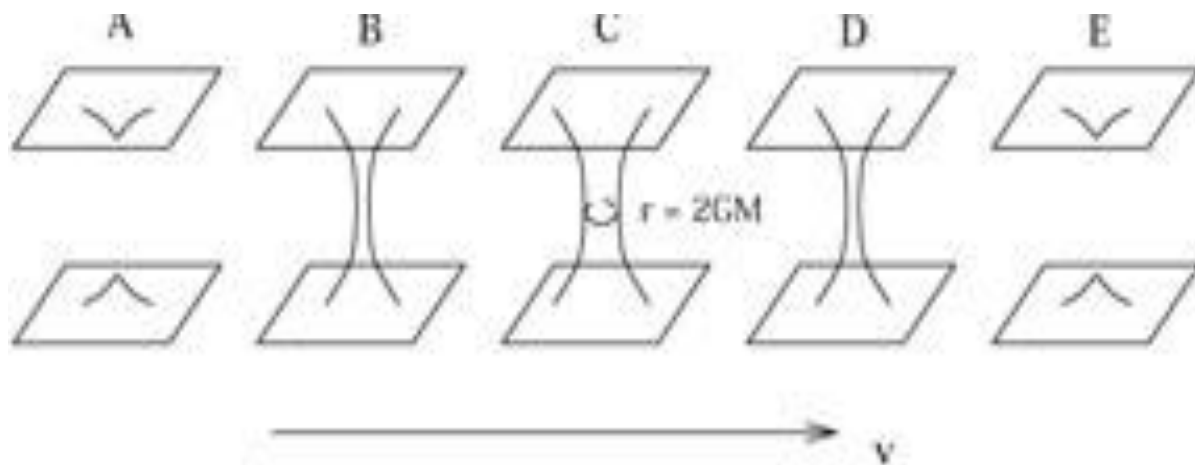


Here region II is the black hole and once anything travels from this region I into II, it can never return. Even every path in region II ends up at the singularity $r = 0$. As approaching to the singularity the tidal force will be infinite. Region III is simply the time reverse of region II. The boundary of region III is sometimes called the past event horizon. Meanwhile, region IV is the mirror image of ours. It can be thought of being connected to region I by a “wormhole”.

Slicing up the Kruskal diagram into space-like surfaces of constant v ,



(Collected)



Here the two flat regions which reach toward each other, are joined together via a wormhole for a while and then disconnect. This wormhole closes up too quickly for any time-like observer to cross it from one region into the next.

4. Axisymmetric Solution

It is believed that practically Schwarzschild black-holes are highly unlikely to be found in the universe, these types of static (non-rotating) black holes can only exist theoretically. Because every celestial object in the universe is rotating and there is angular momentum with this rotation. Even in case of tiny rotation the angular momentum is preserved which makes the smaller objects rotation faster around the core body and due to the amiable rotational ratio of the larger objects it might seem the object as non-rotating. This occurrence is known as ***tidal locking***.

4.1 Introduction to Kerr Metric

In 1963, Roy Kerr published a paper in which he presented a metric which can describe about a rotational object with angular momentum per mass, a . The metric introduced by Roy Kerr is known as Kerr metric which is also applicable in outside of rotating axisymmetric body. The line-element of Kerr geometry is formed in terms of Boyer-Lindquist coordinates (t, r, θ, Φ) .

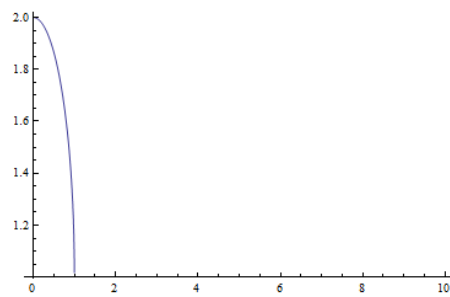
$$ds^2 = -\left(\frac{1-2Mr}{\rho^2}\right) dt^2 + \frac{4Mar \sin^2\theta}{\rho^2} dt d\varphi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2 + \frac{2Mr a^2 \sin^2\theta}{\rho^2}) \sin^2\theta d\varphi^2$$

$$\text{with, } \rho^2 = r^2 + a^2 + a^2 \cos^2\theta$$

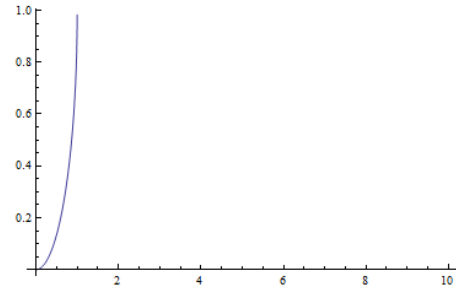
$$\text{and } \Delta = r^2 - 2Mr + a^2$$

And when $\rho = 0$, $\Delta = 0$, $r = 0$, $\theta = \pi/2$ it signifies to the coordinate singularity. Then, we get the solution

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 \cos^2\theta}$$



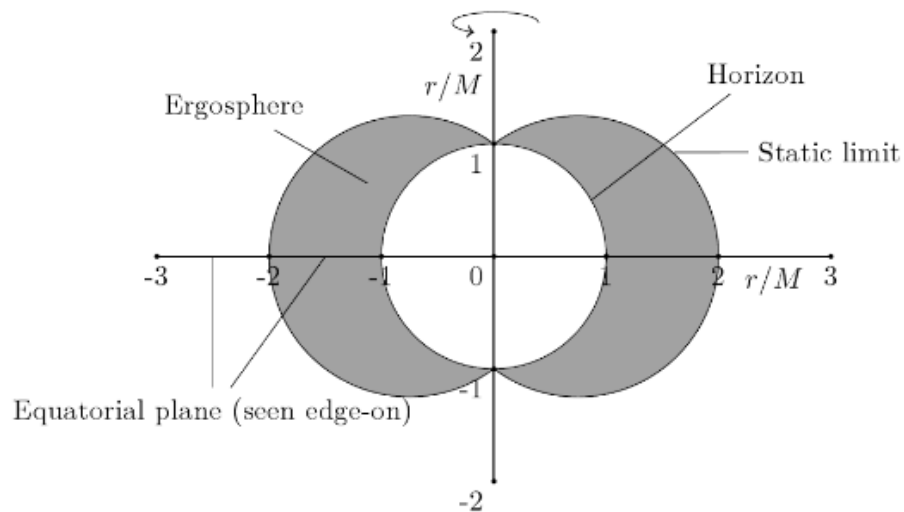
Picture: r_+ = the outer horizon



Picture: r_- = the inner horizon

Here, r_+ = the outer horizon and r_- = the inner horizon of the Kerr metric. So, unlike the non-rotating black holes the Kerr black holes have two event horizons.

4.2 Structure of Kerr Parameter



Picture: Structure of Kerr-rotational black hole

Here the angular momentum per mass of black holes is inversely proportional to the radius of the event horizon. These horizons are only valid if,

$$a < M$$

But whenever,

$$a > M$$

It defines that the event horizons of the black hole will be vanished and the ring singularity will be visible to the observers. That's why this ring singularity is also called "naked singularity". But this type of unreal singularity is disagreed by the principle of cosmic censorship which was derived by Roger Penrose (1965): "All physically reasonable space-times are globally hyperbolic, i.e. no singularity is ever visible to any observer".

In case of extreme rotational black hole $a = M$ or when $a \approx 0.998 M$, Kerr metric becomes:

$$ds^2 = -\left(\frac{1-2Mr}{\rho^2}\right) dt^2 + \frac{4M^2 r \sin^2 \theta}{\rho^2} dt d\varphi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \\ + \left(r^2 + M^2 + \frac{2rM^3 \sin^2 \theta}{\rho^2}\right) \sin^2 \theta d\varphi^2$$

As matters fall into the rotating black hole, it appears as the accretion disc and from the disc spirals inward with increasing angular momentum and falls into the black hole. That means, all observers within r_s must orbit the black hole in the same direction in which it rotates. This critical surface is called the static limit

Moreover,

In Kerr metric when, $g_{tt} = \left(\frac{1-2Mr}{\rho^2}\right) = 0$, it means

the surface of the axisymmetric is of the radius 2μ at equator which fully encircle the outer horizon r_+ and within the inner horizon the surface also overlap with the ring singularity. Due to this between the two horizons the surface is called **ergoregion**. [Shown in the figure-6, page-42]

$$r_{s\pm} = M \pm \sqrt{M^2 - a^2 \cos^2 \theta}$$

As a result, the horizons drop inside the interval $[r_{s+}, r_{s-}]$

$$r_{s-} < r_- < r_+ < r_{s+} \\ \text{Or,} \\ r_{s-} < r < r_{s+}$$

Now, the region $r = r_{s+}$ is known as the ergosphere with respect to an observer at infinite. At this region the killing vectors for the observer will be $(1,0,0,\Omega)$

Angular velocity of the observer becomes,

$$\Omega_H \equiv \omega(r_+, \theta) = \frac{a}{2Mr_+}$$

In the ring singularity for any observer a new surface rises which is independent of time, t with closed time like curves. The line element becomes:

$$ds^2 = -(r^2 + a^2 + \frac{2Ma^2}{r})d\varphi^2$$

And energy per unit mass is also the preserved extent,

Energy,

$$e = -g_{\mu\nu}\xi^\mu u^\nu = 1$$

So, the energy equation becomes:

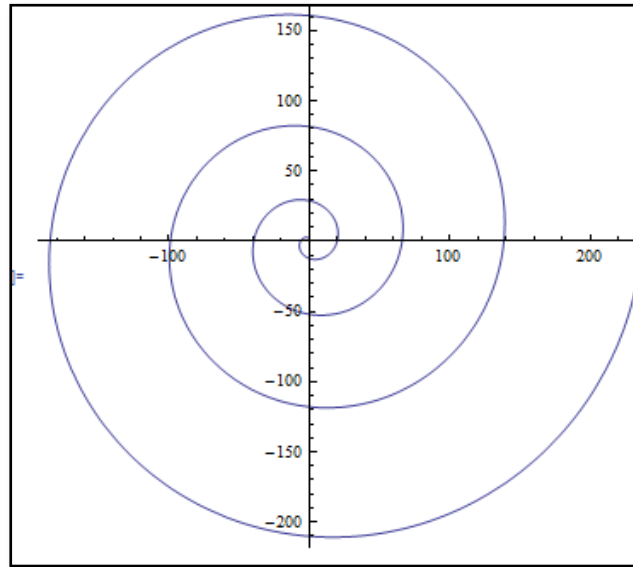
$$\frac{1}{2} (e^2 - 1) = \dot{r}^2 + V_{eff}$$

Where, V_{eff} is called effective potential energy per unit mass:

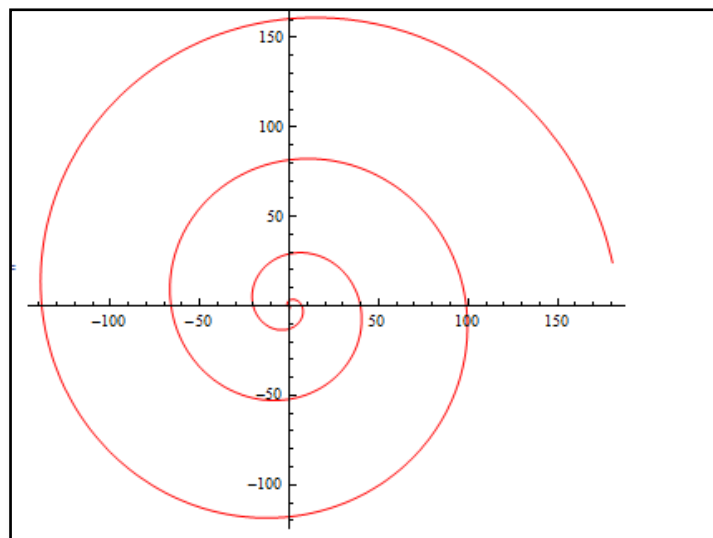
$$V_{eff} = -\frac{M}{r} + \frac{h^2 - a^2(e^2 - 1)}{2r^2} - \frac{M(h - ae^2)}{r^3}$$

4.3 Visualization of Kerr Parameter

Kerr parameter's orbit that barely have similarity between the co-rotation and counter-rotation. the co-rotating particle can maintain a stable orbit while the counter-rotating one falls into the horizon



Picture: Co-rotational View of the Kerr orbit



Picture: Co-rotational View of the Kerr orbit

And in all Kerr space-time close to the static limit the co-rotational and counter-rotational orbits coalesce. (Shown in the figure 3, page-41)

5. Black hole accretion disc

A falling object from infinity is always caught by the space time curvature of the black hole and as per their rotational path we can say if they are co-rotating" or "counter-rotating". Due to strong angular momentum of black hole objects moving in counter-rotation are easy to be caught.

Celestial objects like, gas, dust and other stellar remains that has come close to a black hole forms a flattened band of rotating objects around the event horizon called the accretion disc. So this disc is shaped because of the angular momentum of the incoming gas. Accretion discs are visible since rotating objects release huge heat including strong x-rays and gamma rays due to the extreme gravity of the black hole. This disc is the brightest and gigantic and when no objects are available to be caught this disc which is also known as quasar dim down and become invisible to the observers. The internal edge of accretion disk around a black hole is set by Inner "innermost stable circular orbit (ISCO)". Objects from the accretion disc falls id spirally when ISCO gets unstable.

There are two types of discs:

1. Low temperature disc which is called thick disc
2. High temperature disc which is called thin disc

The Accretion rate = \dot{M} and viscosity parameter = α , radius= r , $t_{r\phi}$ = viscous stress tensor, v =coefficient of the viscosity.

Mass Conservation equation is: $\dot{M} = 4\pi r h \rho v$

The radial momentum equation is:

$$v \frac{dv}{dr} = -\frac{1}{\rho} \frac{dp}{dr} + (\Omega^2 - \Omega_k^2)r$$

The equation for Angular momentum:

$$\frac{\dot{M}}{4\pi} \frac{dl}{dr} + \frac{d}{dr} (r^2 h t_{r\phi}) = 0$$

The equation of the state of accreting object is:

$$P = P_{gas} + P_{rad}$$

$$P_{gas} = \rho RT$$

$$P_{rad} = \frac{\alpha T^4}{3} \left(1 + \frac{4}{3\tau_0}\right) \left(1 + \frac{4}{3\tau_o} + \frac{2}{3\tau_1^2}\right)^{-1}$$

The internal energy density becomes:

$$\rho E = \frac{3}{2} P_{gas} + 3P_{rad}$$

This equation of state is combined of both the gas pressure and radiation pressure.

The apparent productivity of black hole accretion is radiation, which conveys energy and momentum to the surrounding interstellar by photo ionization ,electron scattering, and inclusion by dust crumb.

Gravitational binding energy per unit time is,

$$\dot{E}_g = -\left(\frac{GMm}{2r}\right)$$

The total energy per time in a ring is

$$\dot{E}_v = -\frac{GMm}{r^2} dr$$

Now we can get total released luminosity in the ring by adding \dot{E}_g and \dot{E}_v

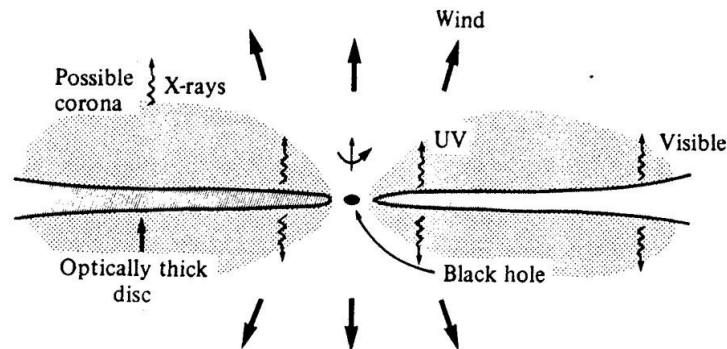
$$dL = \dot{E}_g + \dot{E}_v$$

Advective accretion disks around non-rotating and rotating black hole are a continuous transition between the effectively optically thick outer and optically thin inner disk region.

Accretion rate at the Eddington limiting luminosity:

$$L_{Edd} = 1.3 \times 10^{38} M_{sun} \frac{\text{erg}}{\text{sec}}$$

The first physical model of a disc was presented by Shakura and Sunyaev in 1973. According to this model the disc is optically thick and also let the maximum amount of heat to radiate away from the surface of the disc as the viscosity of the disc turns the gravitational potential energy into heat to radiate.



Picture: Optically thick disc (collected)

6. Concluding Observation

In this thesis we have presented a review of complete derivations of the solutions of Einstein's field equation which include Schwarzschild's spherical symmetry solution and axisymmetry solution of Kerr.

In the first chapter, we discussed the history behind the concept of gravity by introducing Newton's universal law of gravity, Einstein's special and general theory of relativity, gravitational light deflection method to support the evidence of Einstein's space-time curvature.

In the second chapter, Einstein's field equation is introduced with brief derivation and in the third chapter complete derivations are shown for the understanding of the Schwarzschild's static black-hole and its proper structure. They are extended to include different coordinate systems of Eddington-Finkelstein and Kruskal. Through which the concept of singularity point of black holes was discussed. This also directed to the understanding of the gravitational collapse, the reason of black-hole formation.

Another solution is given in chapter four on axisymmetric black-hole by deriving the Kerr metric. This is also known as the rotational black-hole. There we found the evidence of practical existence of rotational black-holes. The structure of the Kerr black hole is also discussed with mathematical visualization of Kerr metric by using mathematica tool.

In the last chapter, we also discussed accretion disc near black hole to make the understanding of black-hole's feature more successfully.

We hope the discussions over here will be helpful for an understanding of the solutions of the field equation more effectively with virtualization.

7. Symbolic calculation of Kerr metric using Mathematica

```

e[r_]:= (1-2/(3 r))^0.5
l[r_]:= (0.75 e[r]+(r/(3^0.5)))
Plot [l[r],{r,0,50}]

PolarPlot [ ((l[r]^2)-0.75^2 (e[r]^2-1)/2 (r^2))-(1 (l[r]-0.75
e[r]))^2/r^3)-(1/r),{r,1,25}]

Plot[ (l[r]^2/2 (r^2))-(1/r)-((l[r]^2)/(r^3)),{r,0,50}]
c1=PolarPlot[{1/r^2+(3 l[r]^2)/r^4-l[r]^2/r^3},{r,-3,-25}]

```

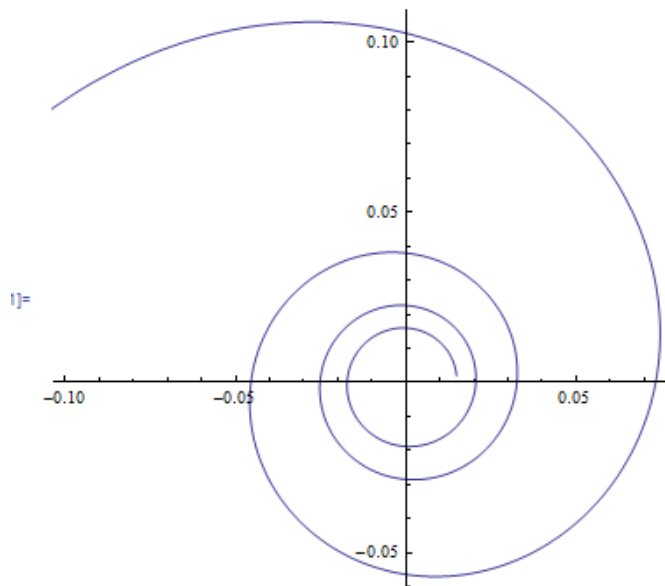


Figure:1

```

c2=PolarPlot[{1/r^2+(3 l[r]^2)/r^4-l[r]^2/r^3},{r,3,25},PlotStyle->
Green]

```

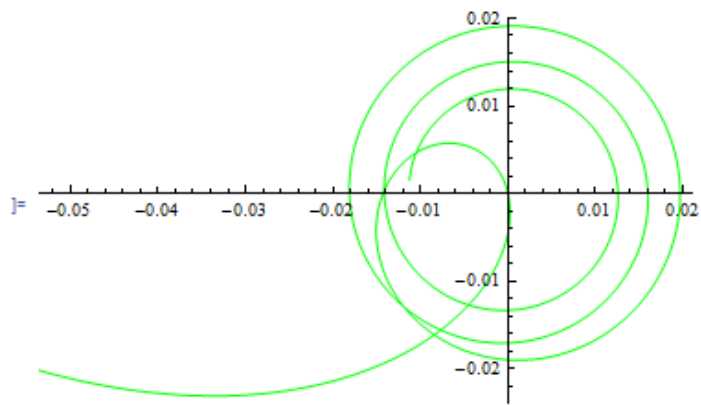


Figure:2

Show[c1,c2]

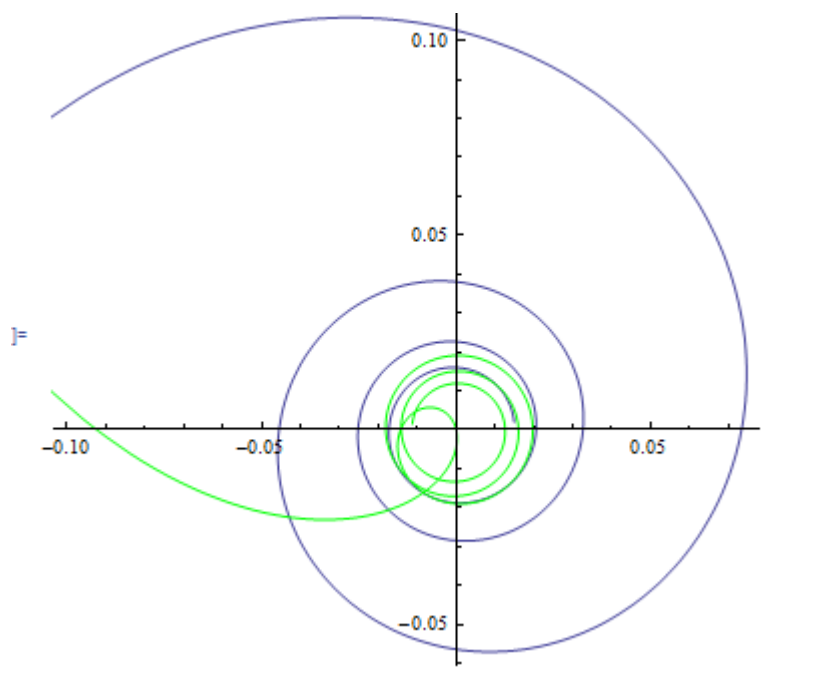


Figure:3

```
s1=Plot[1+Sqrt[1-r^2],{r,0,1},PlotStyle→ Red]
```

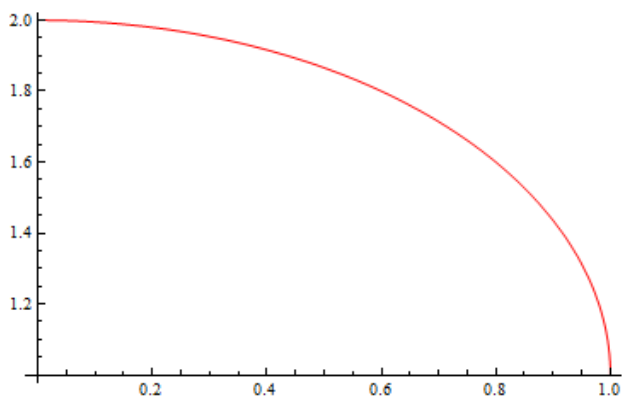


Figure:4

```
s2=Plot[1-Sqrt[1-r^2],{r,0,1},PlotStyle→ Green]
```

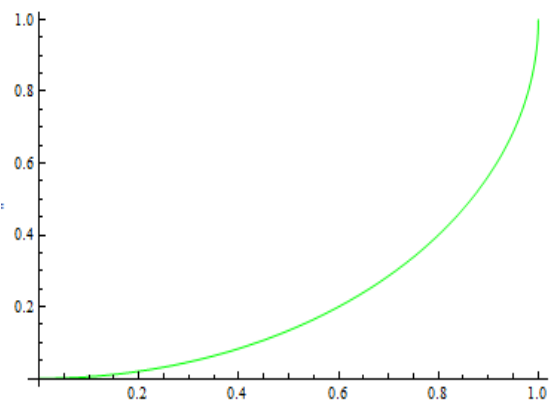


Figure:5

```
Show[s1,s2]
```

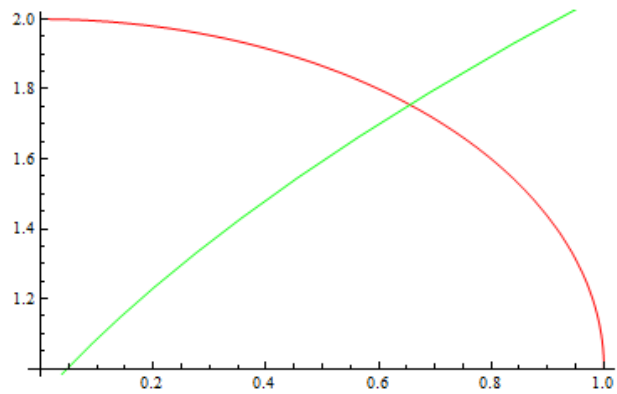


Figure: 6

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