



The AdS-CFT correspondence and the black-hole information paradox

Thesis submitted in partial fulfilment of the requirements for the degree of
Bachelor of Science in Physics

by
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Declaration

I, hereby, declare that the thesis titled "**The AdS-CFT correspondence and the black-hole information theory paradox**" is submitted to the Department of Mathematics and Natural Sciences of BRAC University in partial fulfilment of the requirements for the degree of Bachelor of Science in Physics. This is a work of my own and has not been submitted elsewhere for award of any other degree or diploma. Every work that has been used as reference for this thesis has been cited properly.

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ABSTRACT

This thesis discusses the role of AdS/CFT correspondence in high-energy physics. It concentrates on studying the mathematical black holes in various backgrounds, particularly in the anti deSitter(AdS) space. In this thesis, we discuss the emergence of Hawking radiation and the information paradox relating to its entropy. An extensive study of the correlation between the conformal field theory and the anti-deSitter space is given. In addition, we studied the holographic entanglement entropy of a black hole using the theories of quantum field and gauge/gravity duality which was then used to address the existing problem of information theory paradox. Finally, we calculated the entanglement entropy of simple configurations using the reduced matrix formalism.

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Chapter 1

Introduction

In 1687, Sir Issac Newton introduced the concept of absolute time and space and used them to explain the mechanics around us and formulated the law of universal gravitation. According to the Law of Universal Gravitation, every point mass M attracts any other mass m with a gravitational force \mathbf{F} that is proportional to the product of the masses, and inversely proportional to the square of their distances $\mathbf{r} = r\hat{\mathbf{r}}$:

$$\mathbf{F} = -G \frac{Mm}{r^2} \hat{\mathbf{r}}$$

The formulation of classical electrodynamics by Maxwell in 1865 challenged the laws of classical mechanics as given by Sir Issac Newton, precisely the idea of absolute time and space. The concepts of space and time were unified by one of the prominent scientists of the last century, Albert Einstein, who introduced the the idea of *spacetime* as a fundamental notion of his theory of Special Relativity. Einstein proposed in his theory⁶ in 1905. His theory was based on two postulates: (i) the speed of light c is the same in any inertial frame and (ii) the laws of physics are invariant in any inertial frame. The laws of classical mechanics proposed by Newton were then modified by Einstein so that the laws were invariant under Lorentz transformation- as the equations of the classical electrodynamics formulated by Maxwell and also consistent with the postulates of the Special Relativity.

This discrepancies of the theories led Einstein to develop a more general theory of spacetime in 1915⁷. In his theory, Einstein treated gravity no longer as a force, but as a manifestation of the curvature of the spacetime. Spacetime curvature is generated by the presence of matter. Einstein formulated his theory based on two principles: (i) the equivalence principle which states that at every spacetime point in an arbitrary gravitational field, a locally inertial coordinate system can be chosen, such that, within a sufficiently small region of this point, all physical laws take the form of those of special relativity and (ii) the principle of general covariance which states that the equations that express the laws of physics should be generally covariant, i.e. they should preserve their form under a general transformation such as the Lorentz transformation or the Poincare transformation.²¹⁹

General Relativity can be summarized, according to¹⁸³, as follows. Given that the spacetime is a four-dimensional manifold \mathcal{M} endowed with a pseudo-Riemannian metric $g_{\mu\nu}$, the curvature of the spacetime is related to the matter distribution as given by the Einstein equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

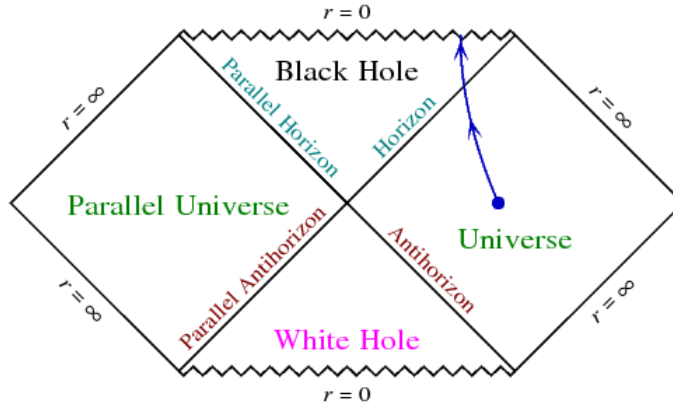


Figure 1.1: Solution of the Einstein's Equation

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci Scalar and the $T_{\mu\nu}$ is the Stress-Energy tensor. The laws of motion formulated by Newton is the slowly moving approximation of the Einstein's equation.

One of the most significant consequences of the general relativity is the existence of black holes, as shown in 1.1 .A black hole is a region of spacetime where the field is so strong that even light cannot escape the horizon, i.e. the boundary of the black hole.¹⁵. Generally, a black hole is formed when the size of the gravitating object of mass M becomes smaller than its gravitational radius $r_s = \frac{2GM}{c^2}$ ⁸, the event-horizon, boundary of black a hole.

Although the discovery of astrophysical black holes gave a solid foundation to the predictions made from Einstein equation yet the theoretical study of black holes and their properties using different mathematical models is currently one of the most interesting topics of interest to scientists. Study of theoretical black holes shed light on Einstein's dream of unifying all the fundamental forces of nature, specially studying black holes using string theory and its correspondence to the quantum field theory one can model particle physics beyond the existent standard model of particle physics.

Another interesting aspects of black hole comes in when we consider black holes as a thermodynamic quantum mechanical objects where entanglement between two quantum states plays a vital role in understanding the undergoing physical phenomena in them. As professor Lenoard Susskind once said,"The phenomenon of entanglement is the essential fact of quantum mechanics, the fact that make it so different from classical physics." In order to understand entanglement in a quantitative sense we calculated entanglement entropy of some ideal system. In chapter 4 where we discussed Hawking radiation and black hole entropy we have seen unlike classical thermodynamical object, black holes entropy is proportional to the area of the black hole rather than the volume. In this thesis, I tried to attempt the following things.

Firstly, I review one of the solutions to Einstein equation, the Schwarzschild solution and then in next couple of chapters I studied about the analytical extension of the solution where I tried to learn about the different algebraic, topological and physical analysis of those solutions. Secondly, after getting a clear picture of black hole horizon from those solution, I focused on studying the horizon in details and I tried to learn about the most celebrated Hawking Radiation and its peculiarity with respect to other thermodynamical objects. In addition, in the forthcoming chapters I studied about AdS spacetime and its mathematical properties and analysis which was a mathematical warm up for the next chapter where I studied conformal field theories and the correlation between conformal field theory and

quantum gravity. Finally, using all these theories in hand, I tried to tackle the problem of information paradox and quantum gravity. In the last chapter I learned about holographic entanglement entropy which can be used to tackle the problem of the information paradox and quantum gravity.

Chapter 2

Black Hole Coordinates

2.1 The Schwarzschild solution

One of the famous analytical solutions to Einstein equation is known as the *Schwarzschild Solution to Einstein Vacuum equation*. This solution is done by idealizing the solution, i.e. suppressing some of the physical possibilities and considering various symmetries. As introduced in the earlier chapter, the famous Einstein equation is written as:

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = \frac{8\pi G}{c^4}T_{ij} \quad (2.1)$$

where R_{ij} is the Ricci Curvature Tensor, R is the Scalar Curvature, g_{ij} is the metric tensor, Λ is the cosmological constant, G is Newton's gravitational constant and c is the speed of light in vacuum, and T_{ij} is the stressenergy tensor.

To solve the given equation we first assume that the cosmological term in the equation 2.1 vanishes, then we use the following equation⁷

$$R_{ij} - \frac{1}{2}Rg_{ij} = \frac{8\pi G}{c^4}T_{ij}$$

where we assume that the solution is static in nature. In addition, to solve the equation we further assume that the solution is of vacuum, otherwise known as the vacuum solution to the Einstein equation, which is given by:

$$R_{ij} - \frac{1}{2}Rg_{ij} = 0$$

where the right hand side of the equation vanishes because the energy-momentum tensor $T_{ij} = 0$. Thus we get the following equation 2.1. Now, to get a solution of the equation we need to assume a distance function that will be used to calculate the remaining terms in the equation, namely the metric g_{ij} , the Ricci curvature tensor, R_{ij} and the Ricci scalar, R . The Ricci curvature tensor and the Ricci scalar is calculated from the symmetric tensor, Riemann curvature tensor.

The distance function is chosen in such a way that the time-like component in it and the distance-like component in it always have the signature $(+, -, -, -)$. Thus the components are chosen to be exponential functions. Moreover, we are assuming the non-vanishing terms in the distance function to be isotropic, homogeneous and time-independent. The distance

function is assumed as⁵:

$$\begin{aligned} ds^2 &= e^{2\lambda} dt^2 - e^{2\mu} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= \eta_{ij}\sigma^i\sigma^j \end{aligned} \quad (2.2)$$

where λ and ν are chosen to be the functions of r and σ^i and σ^j are chosen to the basis of 1-form, which are:

$$\begin{pmatrix} \sigma^t \\ \sigma^r \\ \sigma^\theta \\ \sigma^\phi \end{pmatrix} = \begin{pmatrix} \sigma^0 \\ \sigma^1 \\ \sigma^2 \\ \sigma^3 \end{pmatrix} = \begin{pmatrix} e^\lambda dt \\ e^\mu dr \\ r d\theta \\ r \sin\theta d\phi \end{pmatrix}$$

Now the task is to find the unknown functions in the metric which are μ and λ . In order to do so, we first calculate the torsion free *Levi-Civita* connection which is given by:

$$d\sigma^i = -w_j^i \wedge \sigma^j \quad (2.3)$$

Moreover, using the metric compatibility

$$\begin{aligned} w_r^r &= w_\phi^\phi = 0 \\ w_r^\phi &= -w_\phi^r = 0 \\ w_\phi^r \wedge r d\phi &= 0 \\ dr \wedge d\phi - w_\phi^r \wedge dr &= 0 \end{aligned}$$

we calculate for $i = 0$ or t , which is shown below:

$$\begin{aligned} d\sigma^0 &= -(w_0^0 \wedge \sigma^0 + w_1^0 \wedge \sigma^1 + w_2^0 \wedge \sigma^2 + w_3^0 \wedge \sigma^3) \\ &= -w_r^0 \wedge \sigma^r \\ &= -w_r^0 \wedge e^\mu dr \\ &= \sigma^r \wedge w_r^0 \end{aligned} \quad (2.4)$$

We also calculate $d\sigma^0$ in the following way:

$$\begin{aligned} d\sigma^0 &= d(e^\lambda) \\ &= \lambda' e^\lambda dr \wedge dt \\ &= \sigma^r \wedge (\lambda' e^{-\mu} \sigma^0) \end{aligned} \quad (2.5)$$

And now by comparing equation 2.4 and equation 2.5 we get:

$$w_1^0 = w_r^t = \lambda' e^{-\mu} \wedge \sigma^0 \quad (2.6)$$

Using the properties of the metric used, i.e. η_{ij} , in a similar way we calculate all of the rest *Levi-Civita* connection component for $i = 1, 2, 3 = r, \theta, \phi$ which are listed below¹²

$$\begin{aligned}
 w_1^0 &= \lambda' e^{-\mu} \wedge \sigma^0 = w_0^1 \\
 w_2^0 &= w_3^0 = w_0^2 = w_0^3 = 0 \\
 w_2^1 &= -\frac{1}{r} e^{-\mu} \theta^2 \\
 w_1^2 &= \frac{1}{r} e^{-\mu} \theta^2 \\
 w_3^1 &= -\frac{1}{r} e^{-\mu} \theta^3 \\
 w_1^3 &= \frac{1}{r} e^{-\mu} \theta^3 \\
 w_3^2 &= -\frac{1}{\cot \theta} \theta^3
 \end{aligned} \tag{2.7}$$

Now using the equation 2.7 we have calculated the curvature components, otherwise known as the curvature 2 - form using the following equation:

$$\Omega_j^i = dw_j^i + w_k^i \wedge w_j^k \tag{2.8}$$

Using equation 2.8 and assigning $i = 0$ and $j = 1$ we proceed with our calculations:

$$\begin{aligned}
 \Omega_1^0 &= dw_1^0 + w_0^0 \wedge w_1^0 + w_1^0 \wedge w_1^1 + w_2^0 \wedge w_1^2 + w_3^0 \wedge w_1^3 \\
 &= d(\lambda' e^{-\mu} \wedge \sigma^0) \\
 &= \lambda'' e^{-\mu} \wedge dr \wedge \sigma^0 - \lambda' \mu' e^{-\mu} \wedge dr \wedge \sigma^0 + (\lambda'')^2 e^{-\mu} \wedge dr \wedge \sigma^0 \\
 &= -(\lambda'' - (\lambda')^2 + \lambda' \mu') e^{-2\mu} \wedge \sigma^0 \wedge \sigma^1
 \end{aligned}$$

Similar to this, we calculated the rest of the non-zero curvature components using equation 2.8, which are listed below:

$$\begin{aligned}
 \Omega_2^0 &= -\frac{1}{r} \lambda' e^{-2\mu} \sigma^0 \wedge \sigma^2 \\
 \Omega_3^0 &= -\frac{1}{r} \lambda' e^{-2\mu} \sigma^0 \wedge \sigma^3 \\
 \Omega_2^1 &= \frac{1}{r} \mu' e^{-2\mu} \sigma^1 \wedge \sigma^2 \\
 \Omega_3^1 &= \frac{1}{r} \mu' e^{-2\mu} \sigma^1 \wedge \sigma^3 \\
 \Omega_3^2 &= \frac{1}{r^2} (1 - e^{-2\mu}) \sigma^2 \wedge \sigma^3
 \end{aligned} \tag{2.9}$$

The next part for solving the Einstein equation i.e. to find the unknown functions in the Schwarzschild line element led us to find the Riemann tensor and we can use the relation between the curvature component calculated above and the equation 2.10 below to find them. The equation is given by:

$$\Omega_j^i = \frac{1}{2} R_{mnj}^i \sigma^m \wedge \sigma^n \tag{2.10}$$

All the non-zero Riemann curvature tensor are calculated using the above equation, which are listed below¹²:

$$\begin{aligned}
 R_{011}^0 &= -(\lambda'' - (\lambda')^2 - \lambda'\mu')e^{-2\mu} \\
 R_{022}^0 &= -\frac{1}{r}\lambda'e^{-2\mu} \\
 R_{033}^0 &= -\frac{1}{r}\lambda'e^{-2\mu} \\
 R_{122}^1 &= \frac{1}{r}\mu'e^{-2\mu} \\
 R_{133}^1 &= \frac{1}{r}\mu'e^{-2\mu} \\
 R_{233}^2 &= \frac{1}{r}(1 - e^{-2\mu})
 \end{aligned} \tag{2.11}$$

Then contracting the Riemann curvature tensor and using the approximation that for an isolated system the values of both λ and μ is zero (i.e. as r tends to infinity λ and μ goes to zero). We then calculated the Ricci tensors using $R_{ab} = R_{acb}$, which are listed below:

$$\begin{aligned}
 R_{00} &= e^{2\lambda-2\mu}(\lambda'' + \lambda'\mu' - \frac{2\lambda'}{r} - \lambda'^2) \\
 R_{11} &= \lambda'' + \lambda'^2 - \lambda'\mu' - \frac{2\mu'}{r} \\
 R_{22} &= e^{-2\mu}(1 - e^{2\mu} + r(\lambda' - \mu')) \\
 R_{33} &= \sin^2\theta e^{-2\mu}(1 - e^{2\mu} + r(\lambda' - \mu')) \\
 R_{ij} &= 0 \quad \text{for } i \neq j
 \end{aligned} \tag{2.12}$$

Now equating these equations to zero and using the boundary conditions, locally spherically symmetric and function vanishes at infinity, we get:

$$g_{00} = e^{2\lambda} = \left(1 + \frac{k}{r}\right)$$

Moreover, approximating the value of g_{00} from the Newton's gravitational potential we have:

$$g_{00} \approx (1 + 2\Phi + \Theta(\epsilon^2))$$

Thus we get:

$$\frac{k}{r} = 2\Phi = -\frac{2GM}{r} \tag{2.13}$$

Now substituting the value of equation 2.13 into the equation 2.2 we get the Schwarzschild solution to the Einstein equation, which is:

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{2.14}$$

2.1.1 Properties of the Schwarzschild solution

The coordinates $(x^a) = (x^0, x^1, x^2, x^3) = (t, \vec{r})$ of the metric for a region ($r > 2m$) has t as a time-like and r as a space-like coordinate. It is also seen from equation 2.14 that

the solution if equation is time-symmetric, since it is invariant under the time reflection $t \rightarrow t' = -t$ and time translation invariant, since it is also invariant under the transformation $t \rightarrow t' = t + \text{constant}$. Moreover, using *Birkhoff's Theorem*: "A spherically symmetric vacuum solution in the exterior region is necessarily static" it can be said that the solution is static as well as stationary. Furthermore, taking the limit $r \rightarrow \infty$, we obtain the flat space metric of special relativity in spherical polar coordinate:

$$ds^2 = dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (2.15)$$

Hence it can be further concluded that the Schwarzschild spherically symmetric vacuum solution is asymptotically flat.

2.2 Interpretation of the mass in the solution

From Newtonian theory, a point mass M situated at the origin gives rise to a potential $\phi = -\frac{GM}{r}$, comparing this with the weak-field limit $g_{00} = 1 + \frac{2\phi}{c^2} + O(\frac{v}{c})$ we see that

$$m = \frac{GM}{c^2} \quad (2.16)$$

in non-relativistic units. Therefore, the Schwarzschild solution can also be interpreted as simply the mass of a particle at the origin, but from the solution, it is also seen that that m has a dimension of length⁴. Hence, it is also sometimes referred to as the *geometric mass*.

2.3 Singularities of the Schwarzschild solution

We know that in general a coordinated system associated with a manifold \mathcal{M} covers only a portion of the manifold. In fact, this is also true for the Schwarzschild solution and there exist several singularities of the line element, where the line element degenerates and the metric ceases to be of the rank 4 tensor. The singularities can be of two types:

- Coordinate singularities
- Intrinsic, curvature or real singularities

The coordinate singularities are usually removable by a change in the coordinate system. For example, when $\theta = 0$ in the solution, the Schwarzschild line element becomes degenerate. This degeneracy can be removed by introducing the Cartesian coordinate system, where:

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta \quad (2.17)$$

There is another removable coordinate singularities of the line element at $r = 2M$, which is conventionally also known as the **Schwarzschild Radius**, which can be removed by taking into account the fact the Riemann tensor scalar invariant $R_{abcd}R^{abcd} = 48m^2r^{-6}$ is the same in all coordinate system. The singularity at $r = 0$ is irremovable and therefore it is often termed as the intrinsic or real singularity of the coordinate system. Two other important intrinsic geometric quantities of the horizon are:

- Area of the spatial section

$$A = 4\pi r^2 = 16\pi G_N^2 M^2$$

- Surface Gravity K

$$K = \frac{1}{2}f'(r) = \frac{1}{4G_N M}$$

2.4 Characterizing the coordinate system

The line element of the Schwarzschild solution can be written in the form:

$$g^{00} = \left(1 - \frac{2m}{r}\right)^{-1} \quad g^{11} = -\left(1 - \frac{2m}{r}\right) \quad g^{22} = -\frac{1}{r^2} \quad g^{33} = -\frac{1}{r^2 \sin^2 \theta} \quad (2.18)$$

It follows from this that $x^0 = t$ is time-like and x^1 is space-like as long as $r > 2m$ and both $x^2 = \theta$ and $x^3 = \phi$ are space-like. Again, since there is no cross terms in the metric, we can say that the metric is static and further conclude that t is the invariantly defined world time. By this characterization, it is seen that the manifold \mathcal{M} is divided into two disconnected parts:

- $2m < r < \infty$
- $0 < r < 2m$

Since the line element swaps signs in the region II, i.e t and r reverse their character, hence, t becomes a space- like coordinate and r becomes a time-like coordinate. In addition, we can see that the line element is time reversal invariant, i.e. a transformation of the form $t \rightarrow -t$. Hence it can be said that the Schwarzschild metric does not represent a black that might be formed from a gravitational collapse, it is a mathematical idealization that helps us to study the properties of black holes and its relation to various physical theories.

Chapter 3

Eddington-Finkelstein Coordinate

3.1 Motivation for a new coordinate system

The geodesic equation is given by⁴,

$$\frac{\partial K}{\partial x^a} - \frac{d}{du} \left(\frac{\partial K}{\partial \dot{x}^a} \right) = 0$$

where u is the affine parameter along the geodesic line and $2K = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \text{constant}$. For a radially in-falling particle moving on a time-like geodesic is given by the equations⁴

$$\begin{aligned} \left(1 - \frac{2m}{r}\right) \dot{t} &= k \\ \left(1 - \frac{2m}{r}\right) \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 &= 1 \end{aligned}$$

where the dot represents differentiation with respect to proper time, τ , and k is some constant. Now, if we choose $k = 1$, corresponds to zero initial velocity, we find that

$$\tau - \tau_0 = \frac{2}{3(2m)^{1/2} \left(r_0^{3/2} - r^{3/2} \right)} \quad (3.1)$$

where the particle is at r_0 at proper time τ_0 . No singular behaviour is seen at the Schwarzschild radius and the body falls continuously to $r = 0$ in finite proper time. On contrary, if we now use the Schwarzschild coordinate to describe the motion of the system, i.e use $x^0 = t$, then we get,

$$\begin{aligned} t - t_0 &= - \frac{2}{3(2m)^{1/2}} \left(r^{3/2} - r_0^{3/2} + 6mr^{1/2} - 6mr_0^{1/2} \right) \\ &+ 2m \ln \frac{(r^{1/2} + (2m)^{1/2}) (r_0^{1/2} - (2m)^{1/2})}{(r_0^{1/2} + (2m)^{1/2}) (r^{1/2} - (2m)^{1/2})} \end{aligned}$$

Here we can see that for situations where r_0 and r are much larger than $2m$, equation 2.9 and equation 2.10 are approximately the same. If, however, r is very near to $2m$ then we find:

$$r - 2m = (r_0 - 2m) e^{-(t-t_0)/2m} \quad (3.2)$$

from which it is clear that as $t \rightarrow \infty \Rightarrow r - 2m \rightarrow 0$ so that $r = 2m$ is approached but never passed. From the discussion, it is clear that two reference frames having two different observers see the same situation completely differently. Therefore, it can be concluded that the *Schwarzschild time coordinate* is not appropriate for describing the motion of an radially in-falling particle. In addition, the coordinate system goes bad at $r = 2m$, hence to overcome the ambiguities and to get a well- behaved function to describe the motion we introduce the *Eddington- Finkelstein coordinates*.¹²

3.2 Eddington-Finkelstein solution

As discussed in earlier section, the Schwarzschild time-like coordinate is not suitable for describing an in-falling particle in the black hole because of the difference in observation of the two reference frame. Moreover, there is a coordinate singularity at $r = 2m$ in the Schwarzschild coordinate. Therefore, a transformation of the time-like coordinate is required i.e. $t \rightarrow \bar{t}$, to ensure that the observation, of a radially in-falling particle, made by the observer at infinity and the reference frame of the in-falling particle matches. To begin with, a transformation of the form $t \rightarrow \bar{t} = t + 2m \ln(r - 2m)$ is made so that the ingoing radial null geodesics become straight line. Moreover, using the geodesic equation, we get:

$$t = -(r + 2m \ln|r - 2m| + const.) \quad (3.3)$$

and using the equations given above we get:

$$\bar{t} = -r + const. \quad (3.4)$$

Differentiating \bar{t} we get,

$$d\bar{t} = dt + \frac{2m}{r - 2m} dr \quad (3.5)$$

Now, substituting these we get the Eddington- Finkelstein line element,

$$ds^2 = \left(1 - \frac{2m}{r}\right) d\bar{t}^2 - \frac{4m}{r} d\bar{t}dr - \left(1 + \frac{2m}{r}\right) dr^2 - r^2(d\theta^2 + \sin^2 d\phi^2) \quad (3.6)$$

From the line element, we can see, although, all our purpose of the coordinate transformation is not achieved yet the transformation made our line element continuous in the previously partitioned space; i.e. the transformation equation 2.12 extended the coordinate range to $0 < r < \infty$. Now from this, we see that there is a region, $2m < r < \infty$, which overlaps, and hence they must represent the same solution, if Eddington- Finkelstein line element is used to describe the motion. The solution of this coordinate system is not time- symmetric, as seen in the line element, there is a cross-term. A transformation of $t \rightarrow t^* = t + 2m \ln(r - 2m)$ will help us to get the time- symmetric line element, i.e. the Schwarzschild metric.

3.3 Event horizon: Schwarzschild and Eddington-Finkelstein

The division of the space at $r = 2m$, as seen in the Schwarzschild line element, plays a vital role in studying properties and dynamics of black hole as a whole. At $r = 2m$, only radially

outgoing particles stay where they are, whereas all the rest are dragged inwards. In region II, $r < 2m$, all the particles, even the radially outgoing ones are dragged inwards towards the singularity, $r = 0$, intrinsic singularity. Therefore, it is clear that the surface $r = 2m$ acts as a one-way membrane, i.e. it does not allow any particle or information to pass from the region II to region I ($r > 2m$). This surface is conventionally known as the "event horizon" because this represents the boundary of all events which can be observed, in principle, by an external inertial observer.

The Schwarzschild event horizon is absolute because it seals off all the internal events from the external observer (the internal and the external is referred to as the region divided by $r = 2m$). On the contrary, if we consider the Eddington-Finkelstein event horizon along with a time-like coordinate transformation, $w = t^* - r$, the membrane allows only the past-directed time-like or null curves cross from the outside to the inside.

Chapter 4

The Kruskal Coordinate

4.1 The Kruskal solution

A manifold having a geometry is said to be **maximal** if for every geodesic originating from any arbitrary point of the manifold either can be extended to infinite values along the geodesics in both directions or terminates on any intrinsic or physical singularities⁴. Moreover, if all the originating geodesics from any arbitrary point of the manifold can be extended to both directions, i.e. the geodesic can be extended to infinity and also terminates at the intrinsic singularities then the manifold is said to be **geodesically complete**¹². The Kruskal solution to the Einstein field equation is simply a maximal analytic extension of the Eddington-Finkelstein geometry of a non-rotating black holes. The line element of the Kruskal geometry can be derived by introducing both an advance null coordinate v and a retarded null coordinate w . Thus the Schwarzschild line element in the coordinates (v, w, θ, ϕ) becomes:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dv dw - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (4.1)$$

where r is a function of v and w determined implicit by:

$$\frac{1}{2}(v - w) = r + 2m \ln(r - 2m) \quad (4.2)$$

Moreover, we can see that since $d\Omega^2 = r^2(d\theta^2 + \sin^2 \theta d\phi^2)$ is constant. The 2-space metric is:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dv dw \quad (4.3)$$

Two new parameters x and t are defined such that the above 2-space metric takes the form,

$$ds^2 = \left(1 - \frac{2m}{r}\right)(dt^2 - dx^2) \quad (4.4)$$

where

$$t = \frac{1}{2}(v + w) \quad x = \frac{1}{2}(v - w) \quad (4.5)$$

The 2- space metric has $\theta = \text{const.}$ and $\phi = \text{const.}$ and hence it suffice the necessary conditions to be a conformally flat space. The most general coordinate transformation which leaves the 2-space in such conformally flat double null coordinates is:

$$v \rightarrow v' = v'(v) \quad w \rightarrow w' = w'(w) \quad (4.6)$$

where v' and w' are arbitrary, which leads to:

$$ds^2 = \left(1 - \frac{2m}{r}\right) \frac{dv}{dv'} \frac{dw}{dw'} dv' dw' \quad (4.7)$$

Introducing

$$t' = \frac{1}{2}(v' + w') \quad x' = \frac{1}{2}(v' - w') \quad (4.8)$$

we can write the 2-space metric in the most general form

$$ds^2 = F^2(t', x')(dt'^2 - dx'^2) \quad (4.9)$$

A particular choice of v' and w' will then determine the precise form of the line element. For the Kruskal line element the functions are chosen to be:

$$\begin{aligned} v' &= e^{\frac{v}{4m}} \\ w' &= -e^{-\frac{w}{4m}} \end{aligned} \quad (4.10)$$

The radial coordinate r is to be considered a function of t' and x' determined implicitly by the equation:

$$t'^2 - x'^2 = -(r - 2m)e^{\frac{r}{2m}} \quad (4.11)$$

and the F that was introduced as a general function in the line element earlier takes the following form:

$$F^2 = \frac{16m}{r} e^{\frac{-r}{2m}} \quad (4.12)$$

Using all these transformations, the **Kruskal-line element** takes the form:

$$ds^2 = \left(\frac{16m^2}{r}\right) \exp\left(\frac{-r}{2m}\right) dt'^2 - \left(\frac{16m^2}{r}\right) \exp\left(\frac{-r}{2m}\right) dx'^2 - d\Omega^2 \quad (4.13)$$

4.2 Features of the Kruskal solution

The incoming and outgoing radial null geodesics are straight lines in Kruskal coordinates which can be derived from the line element equation considering $ds = 0$. This tells us that the line cone in Kruskal coordinate system will look the same as in Minkowski space time coordinate. Moreover, a signal originating at the event horizon ($r = 2m$) would remain in the horizon at all times. The equation $t'^2 - x'^2 = -(r - 2m)e^{\frac{r}{2m}}$, it is also seen that the space-time is bounded by two hyperbolas representing the intrinsic singularity at $r = 0$. These two hyperbolas are known as the *past singularity* and the *future singularity*. The asymptotes of the hyperbolas represent the event horizons corresponding to $r = 2m$. These asymptotes divide the space-time region into four regions, these are labelled as I, II, I' and II' in the figure. The regions I and II correspond to the Eddington- Finkelstein solution with region I corresponding to the Schwarzschild solutions for $r > 2m$.

4.3 Compactification of the Kruskal solution

In the previous section, only a two dimensional solution of the Kruskal is drawn. In fact, the Kruskal solution is time symmetric with respect to t' . At $t' = 0$, the Kruskal manifold

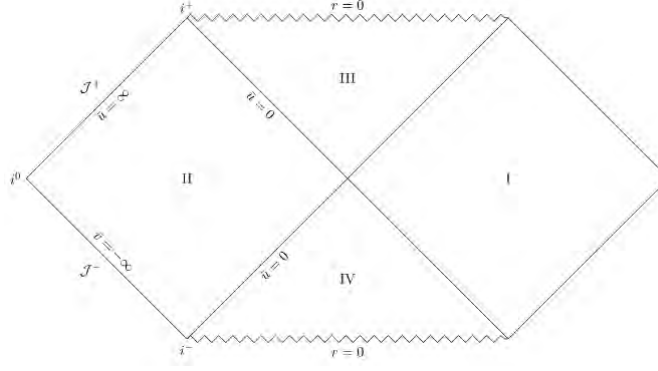


Figure 4.1: Penrose Diagram of the Kruskal solution

can be thought of as being formed by two distinct but asymptotically flat Schwarzschild manifolds joined at $r = 2m$. At $r = 0$ these two distinct universes are connected and is thought hypothetically to be connected via an **Einstein-Rosen bridge**². As t' increases the the two flat universes get completely separated each containing a singularity at $r = 0$. Although, the Kruskal solution is very informative near the horizon and the solution reveals a lot of information about the black hole, yet as $x' \rightarrow \infty$ very less is known and the Kruskal solution does not seem to reveal a lot of information.

A compactification of the Kruskal solution known as the Conformal compactification⁴ of the Kruskal can be obtained by defining new advanced and retarded null coordinates in terms of the previously defined null coordinates v' and w' . The new coordinates are defined as follows:

$$\begin{aligned} v'' &= \tan^{-1} \left(\frac{v'}{(2m)^{1/2}} \right) \\ w'' &= \tan^{-1} \left(\frac{w'}{(2m)^{1/2}} \right) \end{aligned} \quad (4.14)$$

for the coordinate range

$$\begin{aligned} -\frac{1}{2}\pi &< v'' < \frac{1}{2}\pi \\ -\frac{1}{2}\pi &< w'' < \frac{1}{2}\pi \\ -\pi &< v'' + w'' < \pi \end{aligned} \quad (4.15)$$

These transformation of the Kruskal Solution is drawn in figure 4.1 and this diagram is known as the **Penrose Diagram of the Kruskal Solution** which is the conformally compactified space time diagram of the Kruskal solution. Similar to the space time diagram of the previous Kruskal solution, the regions I and II are there in the Penrose diagram. In addition, the region I' and II' are redefined to be III and IV. The regions I and II represent the geometry of the a real black hole and the regions III and IV represents a different kind of hypothetical or mathematical hole known as the white hole. At $r = 2m$, an outward radial null geodesic ends up at \mathcal{S}^+ but an inward radial null geodesic ends up at the future singularity. Also, any point lying inside $r = 2m$, both the outward and inward radial null geodesics end up to the future singularity.

Chapter 5

Hawking Radiation

5.1 Introduction

The event horizon or the Schwarzschild radius sets a limit of the radius of the black hole. As seen in the earlier chapters that nothing can get out of a black hole or radius of the Schwarzschild black hole. The spacetime is wrapped in such a way that even light rays cannot make a way out of it once it is inside the horizon of the black hole, seen on the Penrose diagram of the Kruskal solution to the Einstein's equation. This holds true only when we are considering classical physical phenomena around the event horizon. However, this does not hold true when we incorporate the quantum characteristics, for instance particle popping out of a vacuum, of the black hole². This is depicted in the figure 5.1. Black holes radiate as black bodies in thermodynamics, each with a temperature characteristic of the specific black hole. The temperature of the radiation, known as the Hawking temperature T_H of the black holes can be estimated purely by using dimensional analysis. Considering the gravitational field around an object of mass M , the mass of the black hole, and the Newtonian universal gravitational constant G being proportional to the gravitational field we assume the constant GM to be proportional to it. In natural units (only $c = \hbar = 1$), the combination of GM is a length and hence it is an inverse mass. Moreover, temperature has the dimension of energy of a mass with $c = 1$. Hence, it can be said that:

$$T_H \approx \frac{1}{GM} \approx \frac{\hbar c^3}{GM} \quad (5.1)$$

5.2 Consequence of the Hawking radiation

Although the dimensional analysis was trivial yet it has unprecedented consequences. It suggests that the black hole radiates energy, as M goes down, T_H goes up and thus black hole radiates even faster. The radiative mass accelerates. According to the second law of thermodynamics, $dE = TdS$ where E and is related to the mass of the black hole by:

$$\frac{dS}{dE} = \frac{1}{T} \approx GM \quad (5.2)$$

Since the radius of the black hole is R_s and $R \approx GM$ thus the surface area of the black, $A \approx R_s^2$ we can say:

$$S \approx \frac{R_s^2}{G} \approx \frac{A}{l_p^2} \quad (5.3)$$

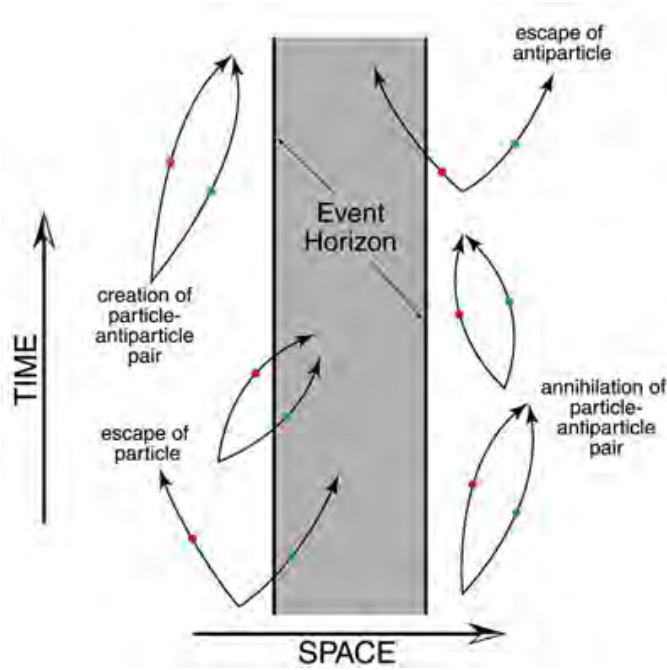


Figure 5.1: Schematic Diagram of the Hawking Radiation

where l_p is the Planck length. This concludes that black hole has an entropy proportional to its surface area rather than its volume which is the general case for any other thermodynamical black body.

5.3 Vacuum fluctuations near a black hole

Using the Schwarzschild metric on Einstein equations, at the horizon $R_s = 2GM$, we see the coefficients of the dt^2 and the dr^2 interchanges sign, indicating the interchanging of time and the spatial coordinates, thus the interchange of energy and the momentum. If we assume that a pair of electron- positron pops out near the horizon and during the short time of their existence one of the particles falls through the horizon, at which point its energy becomes a momentum component and the other particle is liberated from the constrain of energy conservation and Heisenberg's principle and can exist forever. In Kruskal diagram, at $R_s = 0$ the particle which crossed the horizon reaches singularity and the other one escapes toward \mathcal{I}^+ . For the conservation of energy-momentum, the black hole would lose a bit of energy and with recoil mass M much greater than the typical energy of the escaping particle, these effects will be negligible. These fluctuations occur around the horizon and the black hole keeps on radiating¹⁵. Incidentally, this leads to the "Black hole information paradox". What happens to the information contained in the material that fell into and became a part of the black hole. The material end up becoming thermal radiation, which according to standard considerations, does not contain information at all. But the law of quantum mechanics does not permit a pure state to be transformed to a thermal state by any unitary operator. Thus there appears to be a basic contradiction with quantum mechanics and statistical mechanics, hence is the black hole information paradox.

5.4 Determining the Hawking temperature: mathematical approach

Let $|F\rangle$ and $|I\rangle$ be the final state and the initial quantum state of a system respectively. Heisenberg's formulation of a quantum state after a time T is governed by the evolution operator e^{-iHT} with H being the Hamiltonian of the system. The probability amplitude for the transformation is given by the partition function:

$$Z = \langle F | e^{-iHT} | I \rangle \quad (5.4)$$

According to thermodynamics, the relative probability of a state $|n\rangle$ of energy E_n occurring is given by $e^{-\beta E_n}$ where $\beta \equiv \frac{1}{\text{'Temperature'}}$. The partition function of a quantum mechanical system with the Hamiltonian is then defined as:

$$\begin{aligned} Z &= \sum_n \langle n | e^{-\beta H} | n \rangle \\ &= \sum_n e^{-\beta E_n} \\ &= \text{Tr}(e^{\beta H}) \end{aligned} \quad (5.5)$$

where $e^{-\beta H}$ is regarded as a matrix.

Let us now consider an electromagnetic field governed by the action:

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \right)$$

propagating in the Schwarzschild spacetime described by

$$ds^2 = - \left(1 - \frac{R_s}{r} \right) dt^2 + \left(1 - \frac{R_s}{r} \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

Near the horizon,

$$ds^2 \approx \frac{r - R_s}{R_s} dt^2 + \frac{R_s}{r - R_s} dr^2 + r^2 \sin^2 \theta d\theta d\phi^2$$

when $r \rightarrow \rho$ where $\rho^2 \equiv 4R_s(r - R_s)$

$$\begin{aligned} \rho d\rho &= 2R_s dr \\ \Rightarrow \rho^2 d\rho^2 &= 4R_s^2 dr^2 \\ \Rightarrow (r - R_s) d\rho^2 &= R_s dr^2 \end{aligned} \quad (5.6)$$

then,

$$\begin{aligned} ds^2 &= -\frac{\rho^2}{4R_s^2} dt^2 + d\rho^2 + r^2 \sin^2 \theta d\theta d\phi^2 \\ ds^2 &= \frac{\rho^2}{4R_s} dt_E^2 + d\rho^2 + r^2 \sin^2 \theta d\theta d\phi^2 \end{aligned}$$

where $t \equiv -it_E$ which is imaginary and cyclic. Again if $t_E = 2R_s\psi$ then

$$ds^2 \approx d\rho^2 + \rho^2 d\psi^2 + r^2 \sin^2 \theta d\theta d\phi^2$$

The first two terms in the equation given above describes a plane with polar radius ρ and the polar angle ψ apart from the already existing solid angle in the distance function. The $(3 + 1)$ dimensional spacetime has been analytically continued in to 4-dimensional Euclidean space consisting of a plane, at every point of which is attached a sphere of radius R_s . Moreover, ψ is an angular variable and it seen that $t_E = 2R_s\psi$ has a period of $2R_s(2\pi = 4\pi R_s)$. Using the concept that the inverse of the recurrence period β^{11} , temperature can be determined and is equal to

$$T_H = \frac{1}{4\pi R_s} = \frac{1}{8\pi GM} = \frac{\hbar c^3}{8\pi GM} \quad (5.7)$$

which is the Hawking temperature.

5.5 Black hole thermodynamics

Black holes are interesting thermodynamical objects. In the past, it was thought that black holes violated the second law of thermodynamics but the theoretical understanding of Hawking radiation made us think otherwise. Hawking radiation is the emission of a black body radiation which can be thought of as emission of a black body. One of the possible explanations for the existence of this radiation is the Unruh effect¹¹, which tells us that an observer who moves with constant acceleration observes black body radiation coming from a vacuum while a stationary observer sees nothing. The Unruh effect gives rise to a black body spectrum of the form:

$$n(E) = \frac{1}{\exp\left[\frac{E}{T_U}\right] - 1}$$

where E is the energy of the emitted radiation and T_U is given by:

$$T_U = \frac{a}{2\pi}$$

is the *Unruh temperature* and a is the 'proper' acceleration. This acceleration is provided the surface gravity in the vicinity of the black hole horizon. The surface gravity is given as in⁴ by:

$$\kappa = \frac{1}{4m}$$

where m is the mass of the black hole. Therefore the temperature takes the following form:

$$T_H = \frac{1}{8\pi m}$$

which is the same as the expression derived for the Hawking temperature earlier in the chapter. Now using the first law of thermodynamics we can write the black hole entropy as:

$$dE = dm = T_H dS_{BH}$$

where dS_{BH} is the black hole entropy or the Bekenstein-Hawking entropy. Now we calculate $dS = 8\pi m dm$ and then we can get the expression for entropy as:

$$S = 4\pi m^2 = \frac{1}{4} \frac{k_B c^3}{\hbar G} A \sim \frac{A}{4} \quad (5.8)$$

and again we get the same result for the entropy as in our earlier calculations.

There are several problems with this theoretical notion of black hole entropy. Two of the most interesting them are:

- Where does the entropy of black hole comes from?
- Why is the entropy of a black hole dependent on the area rather than the volume?

To give answer to these and several other questions, black holes have been interpreted in different models and in different geometric spaces; one of them being the black hole entropy as the multiplicity of horizon gravitational states or with string theory. Having that in mind, we try to study in the upcoming chapters about one of the most promising candidates known as the AdS/CFT correspondence²¹ and about the entanglement in quantum field theory.

Chapter 6

DeSitter and Anti-deSitter Spacetime

6.1 Introduction

A d -dimensional sphere S^d of radius L is defined as the set of all point $(X^1, X^2, \dots, X^{d+1})$ in a $(d + 1)$ dimensional Euclidean space E^{d+1} , a space with the distance function defined as:

$$ds^2 = (dX^1)^2 + (dX^2)^2 + \dots + (dX^{d+1})^2 = L^2$$

Similarly, a d -dimensional de Sitter spacetime dS^d with length scale L is the set of all points (X^0, X^1, \dots, X^d) in a $(d + 1)$ dimensional Minkowskian space $M^{d,1}$, a spacetime with the distance function defined as:

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + \dots + (dX^d)^2 = L^2$$

Only by renaming X^{d+1} as X^0 and turning it into a timelike coordinate we have designed a Minkowskian version of the sphere which lives in the Minkowskian spacetime and this spacetime is known as the de Sitter spacetime²⁰. The difference in the sign of the timelike and the spatial coordinate is very crucial and it forms a $(d - 1)$ dimensional sphere S^{d-1} defined by:

$$(dX^1)^2 + \dots + (dX^d)^2 = L^2 + (X^0)^2$$

The time coordinate X^0 goes from $-\infty$ to $+\infty$, the radius $\sqrt{L^2 + (X^0)^2}$ of S^{d-1} starts at infinity and contracts to a minimum value of L and then again expands to infinity as shown in figure 6.1. The d -dimensional anti de Sitter spacetime AdS^d is analogously defined as the set of all points (X^0, X^1, \dots, X^d) in a $(d + 1)$ dimensional Minkowskian type spacetime $M^{(d-1),2}$, a spacetime with two timelike coordinates, having the distance function as:

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + \dots + (dX^{d-1} - (dX^d)^2)^2 = L^2$$

satisfying the following equation:

$$-(X^0)^2 + (X^1)^2 + \dots + (X^d) - (X^d)^2 = -L^2$$

which can also be written using the summation notation as in equation 6.1.

$$(X^0)^2 - \sum_{i=1}^{d-1} (X_i)^2 + (X^d)^2 = L^2 \tag{6.1}$$

In the figure 6.2 we can see the differences between a dS^d spacetime and the AdS^d spacetime.

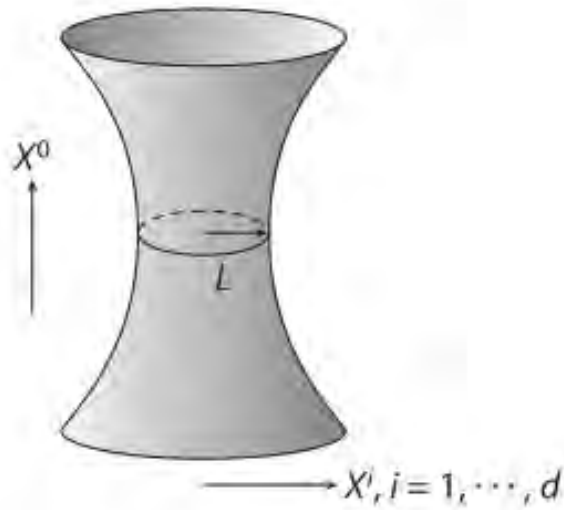


Figure 6.1: The d-dimensional de Sitter Spacetime²⁰

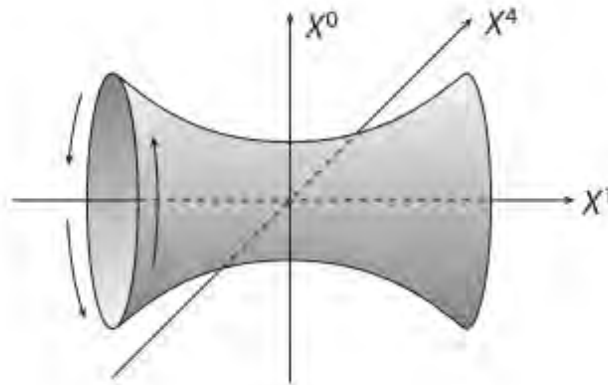


Figure 6.2: The d-dimensional anti de Sitter Spacetime

6.2 Maximal symmetry of the deSitter and anti-deSitter space

6.2.1 Isometries of the spacetime

An isometry between two groups is defined to be a bijective map between two metric spaces that preserves the distance function²⁰. The isometry group of a sphere S^d is $\mathbb{SO}(d+1)$, the rotation group of the embedding space E^{d+1} which has the Killing generators $-X^N \frac{\partial}{\partial X^M}$ where $M, N = 1, 2, \dots, X^{d+1}$.

6.2.2 Coset manifold

If we assume a Lie group \mathfrak{G} and a subgroup \mathfrak{H} of \mathfrak{G} and the group elements g_1 and g_2 which belong to the group \mathfrak{G} . Then, we can consider g_1 and g_2 to be equivalent if there exists an element h of \mathfrak{H} such that the following relation is maintained⁹.

$$g_1 = g_2 h$$

This relation allows us to define equivalence classes. Hence, we can define a space or manifold by associating each equivalence class with a point in the space. The resulting manifold is known as the *coset manifold* and is expressed by $\mathfrak{G}/\mathfrak{H}$. The sphere S^d can therefore be considered as the coset manifold: $\mathbb{SO}(d+1)/\mathbb{SO}(d)$ where the quotient group $\mathbb{SO}(d)$ is the subgroup of the parent group $\mathbb{SO}(d+1)$ which leaves a point on the sphere S^d invariant. The isometry group of the deSitter spacetime dS^d is $\mathbb{SO}(d+1)$ and the Lorentz group of the embedding space $M^{d,1}$. Thus the Killing generators fall into two sets, d-dimensional rotations and boosts, they are:

$$X^M \frac{\partial}{\partial X^N} - X^N \frac{\partial}{\partial X^M} \qquad X^M \frac{\partial}{\partial X^0} - X^0 \frac{\partial}{\partial X^M}$$

for all $M, N = 1, 2, \dots, d$.

Similar to the sphere S^d , deSitter spacetime is also a coset manifold: $dS^d = \mathbb{SO}(d, 1)/\mathbb{SO}(d-1, z)$. The group $\mathbb{SO}(d, 1)$ rotates the point on dS^d around but keeps the distance function invariant. Hence it can be concluded that just like the sphere, deSitter spacetime is also maximally symmetric. Comparing between the distance function, as in equation 6.2 of the deSitter spacetime with the anti deSitter spacetime:

$$\begin{aligned} (X^0)^2 - \sum_{i=1}^{d-1} (X_i)^2 + (X^d)^2 &= L^2 \quad \text{for the } AdS^d \text{ and} \\ - (X^0)^2 + \sum_{i=1}^{d-1} (X_i)^2 + (X^d)^2 &= L^2 \quad \text{for the } dS^d \end{aligned} \tag{6.2}$$

and going through the same argument, as for the deSitter spacetime and the sphere, we can say that the AdS is also maximally symmetric. However, there is a difference between the symmetry groups of the deSitter and the anti deSitter spacetimes. The dS spacetime has the isometry group $\mathbb{SO}(d, 1)$ but the isometry group of the AdS spacetime is $\mathbb{SO}(d-1, 2)$ because there exist two timelike coordinates in the AdS spacetime. Consequently, the AdS^d

is the coset manifold of $\mathbb{SO}(d-1, 2)/\mathbb{SO}(d-2, 2)$. For instance for a particular value of d , say $d = 6$, AdS would be the coset manifold of:

$$AdS^5 = \frac{\mathbb{SO}(4, 2)}{\mathbb{SO}(4, 1)}$$

6.3 Reimann curvature tensor for dS^d and the AdS^d

A maximally symmetric space has $\frac{1}{2}D(D+1)$ constrain on the Reimann curvature tensor¹⁸. This $\frac{1}{2}D(D+1)$ constrain is enough to uniquely determine a Reimann curvature tensor for any d -dimensional space. In addition, for such space the Reimann curvature tensor

$$R_{\mu\nu\lambda\sigma} = K(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda})$$

where K is some constant that depends on the choice of the dimensions of the d -dimensional space. If the deSitter coordinates is chosen to have the dimension of length and $g_{\mu\nu}$ is normalized to be dimensionales, then by dimensional analysis, the Riemann curvature must be:

$$R_{\mu\nu\lambda\sigma} = \frac{1}{L^2} (g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda})$$

Now, if we coordinatized deSitter spacetime with $W = X^d$ ad use X^μ with $\mu = 0, 1, \dots, d-1$ as the coordinates then we have $W^2 = L^2 - X.X$ then

$$\begin{aligned} (WdW)^2 &= (-X.dX)^2 \\ \Rightarrow dW^2 &= \frac{(X.dX)^2}{W^2} \\ \Rightarrow dW^2 &= \frac{(X.dX)^2}{L^2 - X.X} \end{aligned} \tag{6.3}$$

Therefore, we can write the distance function as:

$$\begin{aligned} ds^2 &= \eta_{\mu\nu}dX^\mu dX^\nu + dW^2 \\ &= \eta_{\mu\nu}dX^\mu dX^\nu + \frac{(X.dX)^2}{L^2 - X.X} \\ &= \left(\eta_{\mu\nu} - \frac{\eta_{\mu\lambda}\eta_{\nu\rho}X^\lambda X^\rho}{X.X - L^2} \right) dX^\mu dX^\nu \end{aligned} \tag{6.4}$$

Now when $X \rightarrow 0$ in the equation 6.4, the metric becomes locally flat at $X^\mu = 0$. Now if we expand the metric at that point using the Taylor expansion we get:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{L^2} \eta_{\mu\lambda}\eta_{\nu\rho} X^\lambda X^\rho$$

By defining a new tensor as follows:

$$B_{\mu\nu,\lambda\rho} \equiv \frac{1}{L^2} (\eta_{\mu\lambda}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\lambda})$$

And using the fact that for a locally flat tensor the Riemann tensor is:

$$\begin{aligned}
 R_{\tau\rho\mu\nu} &= B_{\tau\nu,\mu\rho} - B_{\rho\nu,\mu\tau} - B_{\tau\mu,\nu\rho} - B_{\rho\mu,\nu\tau} \\
 &= B_{\tau\nu,\mu\rho} - B_{\rho\nu,\mu\tau} - B_{\tau\mu,\nu\rho} + B_{\rho\mu,\tau\nu} \\
 &= (B_{\tau\nu,\mu\rho} + B_{\rho\mu,\tau\nu}) - (B_{\rho\nu,\mu\tau} + B_{\tau\mu,\nu\rho}) \\
 &= \frac{1}{L^2} (\eta_{\tau\mu}\eta_{\rho\nu} - \eta_{\tau\nu}\eta_{\rho\mu})
 \end{aligned} \tag{6.5}$$

Similar to the dS^d spacetime we can construct the Riemann tensor for the AdS^d spacetime. The dS and the AdS differs only by a sign, therefore to treat dS and AdS together we can introduce a constant $\sigma = \pm 1$ and hence we can unify both the metric and then the metric can be written simply as:

$$\eta_{\mu\nu}X^\mu X^\nu + \sigma(X^D)^2 = \sigma L^2$$

with $\mu, \nu = 0, 1, \dots, d-1$ and $\sigma = +1$ for the dS and $\sigma = -1$ for the AdS . For constructing the Riemann curvature tensor again, $W^2 = L^2 - \sigma X.X$, this time along with the defines constant $\sigma = \pm 1$. Similarly,

$$\begin{aligned}
 WdW &= -\sigma X.dX \\
 \Rightarrow dW^2 &= \frac{(X.dX)^2}{L^2 - \sigma X.X}
 \end{aligned} \tag{6.6}$$

Thus the distance function becomes:

$$\begin{aligned}
 ds^2 &= \eta_{\mu\nu}dX^\mu dX^\nu + \sigma dW^2 \\
 &= \left(\eta_{\mu\nu} - \frac{\eta_{\mu\lambda}\eta_{\nu\rho}X^\lambda X^\rho}{X.X - \sigma L^2} \right) dX^\mu dX^\nu
 \end{aligned} \tag{6.7}$$

The Riemann tensor for the AdS space thus becomes:

$$R_{\mu\nu\lambda\sigma} = -\frac{1}{L^2} (g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda})$$

Since the general metric can be approximated as:

$$g_{\mu\nu} \simeq \eta_{\mu\nu} + \frac{\sigma}{L^2} \eta_{\mu\lambda}\eta_{\nu\rho}X^\lambda X^\rho$$

the metric is locally flat as $X \rightarrow 0$ at X^μ then the Riemann tensor becomes:

$$R_{\tau\rho\mu\nu} = \frac{\sigma}{L^2} (\eta_{\tau\mu}\eta_{\rho\nu} - \eta_{\tau\nu}\eta_{\rho\mu})$$

which is similar to the previously derived dS Riemann curvature tensor and only differs by the sign of the σ , which was previously defined to be $\sigma = +1$ for the dS space and $\sigma = -1$ for the AdS space. Now, for solving the Einstein equation we can use the contraction law for tensors and get the *Ricci tensors* required for both the dS space and the AdS space. Furthermore, it is also possible for us to contract the Ricci tensor further to get the *Ricci scalar*.

6.4 Schwarzschild space from the AdS space

The anti deSitter metric can be mathematically manipulated²⁰ in order to derive the Schwarzschild metric from it. The Minkowskian spacetime $M^{d,1}$ of the special relativity has one timelike coordinate. In contrast, the locally flat AdS^d spacetime can be thought of as the $M^{d-1,2}$ Minkowskian spacetime with two timelike coordinates. The timelike coordinates can be mathematically manipulated using a number of ways, for instance the *Wick rotation*. In order to get the Schwarzschild metric hidden in AdS metric, we need to work with a particular value of d in the AdS^d metric. Let us consider the value of d to be 3. Now, using the equation:

$$(X^0)^2 - \sum_{i=1}^{d-1} (X^i)^2 + (X^d)^2 = L^2$$

and choosing $X^0 = T$, $X^1 = T$, $X^2 = W$ and $X^3 = Y$ the metric for AdS^3 takes the form:

$$(T^2 + W^2) - (X^2 + Y^2) = L^2$$

Since we have now two timelike Minkowskian metric, the signature of the metric is $\eta = (-1, +1, +1, -1)$ of the embedding space $M^{2,2}$ and the distance function becomes:

$$ds^2 = -(dT^2 + dW^2) + (dX^2 + dY^2)$$

By using the replacement of $(T, W) \rightarrow (R, t)$ where (R, t) are the polar coordinates:

$$\begin{aligned} T &= R \cos t \\ W &= R \sin t \end{aligned} \tag{6.8}$$

and also by replacing (X, Y) by the polar coordinates (r, θ) , i.e.

$$\begin{aligned} X &= r \cos \theta \\ Y &= r \sin \theta \end{aligned} \tag{6.9}$$

then, the distance function takes the form:

$$ds^2 = -(dR^2 + R^2 dt^2) + (dr^2 + r^2 d\theta^2)$$

Now if we consider $L = 1$, we can see that the apparent temporal coordinate R is not independent of the spatial coordinate since there is a constrain, to be specific $R^2 - r^2 = 1$. Also, we have $RdR = r dr$ and hence:

$$\begin{aligned} dR^2 - dr^2 &= \left(\frac{r^2}{R^2} - 1 \right) dr^2 \\ &= -\frac{1}{1+r^2} dr^2 \end{aligned}$$

Substituting all these transformation into the distance function gives us:

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\theta^2$$

Thus we end up with only one timelike coordinate. Hence we can say that AdS^3 is more generalized spacetime metric that can be used to figure out the already established metric as in the above given equation. It can be further concluded that AdS^d can be used to obtain any metric which are spherically symmetric and the distance function can be written as:

$$ds^2 = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\Omega_{d-2}^2$$

This form of the the metric is similar to that of the Schwarzschild metric:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{d-2}^2$$

For the particular case of $d = 3$ we have seen that $f(r) = 1 + r^2$. One of the most important features of this metric is that it does not have a coordinate singularity at the horizon since $f(r) = 1 + r^2$, therefore it is always positive and hence it does not even change the signature of the metric and also does not become zero to give us an undefined distance function.

6.5 Conformal coordinates for AdS spacetime

Setting the value of $d = 3$ we get the metric for AdS^3 where $\eta = (-1, +1+1, -1)$ and defining $X^0 = T, X^1 = X, X^2 = W$ and $X^3 = Y$ we get the distance function to be:

$$ds^2 = -(dT^2 + dW^2) + (dX^2 + dY^2)$$

In addition, setting the value of $L = 1$ in the AdS metric and doing the following coordinate transformation:

$$\begin{aligned} X &= r \cos \theta & T &= R \cos \theta \\ Y &= r \sin \theta & W &= R \sin t \end{aligned} \tag{6.10}$$

we get:

$$\begin{aligned} ds^2 &= -(dR^2 + R^2 dt^2) + (dr^2 + r^2 d\theta^2) \\ ds^2 &= -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\theta^2 \end{aligned}$$

where we hvae taken the constrain $R^2 - r^2$ into account. Now if we do another transformation of coordinates by setting:

$$\begin{aligned} r &= \tan \psi \\ r^2 &= \tan^2 \psi \\ (dr)^2 &= (\sec^2 \psi d\psi)^2 \end{aligned}$$

Then

$$\begin{aligned} ds^2 &= -(1 + \tan^2 \psi)dt^2 + \frac{\sec^2 \psi d\psi^2}{1 + \tan^2 \psi} + \tan^2 \psi d\theta^2 \\ &= -\frac{1}{\cos^2 \psi} dt^2 + \frac{1}{\cos^2 \psi} d\psi^2 + \frac{\sin^2 \psi}{\cos^2 \psi} d\Omega_{d-2}^2 \\ &= \frac{1}{\cos^2 \psi} (-dt^2 + d\psi^2 + \sin^2 \psi d\Omega_{d-2}^2) \end{aligned}$$

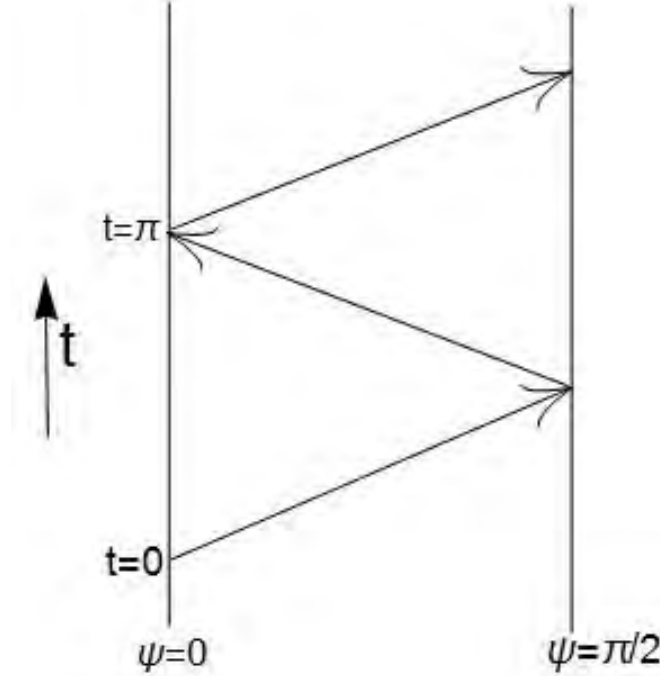


Figure 6.3: Figure Showing how the Conformal AdS is Bounded in the Spatial Direction

Or more compactly it can be written as:

$$ds^2 = \frac{1}{\cos^2\psi} (-dt^2 + d\Omega_{d-1}^2) \quad (6.11)$$

In the equation 6.11, the timelike coordinate is unbounded, it has the values from $-\infty$ to $+\infty$, this is shown in the figure 6.3, the time strip extends upto ∞ whereas the the spatial coordinates are bounded between $\psi = 0$ to $\psi = \pi/2$. Therefore, this transformation can be regarded as the *conformal compactification*²⁰ of metric in the spatial direction with a compactification factor of $\frac{1}{\cos^2\psi}$. The spatial coordinates of the AdS metric is bounded by S^{d-2} which is similar to the Euclidean space E^{d-2} with the spatial infinity identified at a single point. If we take the time coordinate into consideration, we go to the Minkowskian space, $M^{d-2,1}$ from E^{d-2} . Hence we can say that by this conformal compactification of the AdS spacetime becomes bounded by the Minkowskian space $M^{d-2,1}$, Thus it can be further implied that we might be living on the boundary of a $(4 + 1)$ dimensional AdS^5 spacetime where the usual $M^{3,1}$ spacetime is embedded. The conformal group for $M^{3,1}$ is $\mathbb{SO}(4, 2)$ and the isometry group for AdS^5 is also $\mathbb{SO}(4, 2)$, therefore, it can also be concluded that the conformal group is the manifestation of the isometry group on the boundary of the spacetime.

6.6 Poincaré coordinate for anti deSitter spacetime

Using the equation $(T^2 - X^2) + (W^2 - Y^2) = 1$ and relabelling them as:

$$\begin{aligned} T &= \frac{t}{w} \\ X &= \frac{x}{w} \\ Y &= \frac{1}{2} \left(\frac{x^2 - t^2}{w} + w - \frac{1}{w} \right) \\ Y &= \frac{1}{2w} (x^2 - t^2 + w^2 - 1) \\ W &= \frac{1}{2} \left(\frac{x^2 - t^2}{w} + w + \frac{1}{w} \right) \\ W &= \frac{1}{2w} (x^2 - t^2 + w^2 + 1) \end{aligned}$$

Then we get:

$$\begin{aligned} T^2 - X^2 &= \left(\frac{t}{w} \right)^2 - \left(\frac{x}{w} \right)^2 \\ &= \frac{t^2 - x^2}{w^2} \end{aligned}$$

and then we get:

$$W^2 - Y^2 = 1 + \frac{x^2 - t^2}{w^2}$$

Now differentiating we get:

$$\begin{aligned} dT &= \frac{dt}{w} \quad \Rightarrow (dT)^2 = \frac{dt^2}{w^2} \\ dX &= \frac{dx}{w} \quad \Rightarrow (dX)^2 = \frac{dx^2}{w^2} \end{aligned}$$

Differentiating Y parameter of the metric we get:

$$\begin{aligned} Y &= \frac{1}{2w} (x^2 - t^2 + w^2 - 1) \\ dY &= \frac{1}{w} (x^2 - t^2 + w^2 - 1) dw \\ dY^2 &= \frac{1}{w^2} (x^2 - t^2 + w^2 - 1)^2 dw^2 \end{aligned}$$

Similarly differentiating W and differentiating Y parameter of the metric we get:

$$\begin{aligned} W &= \frac{1}{2w} (x^2 - t^2 + w^2 + 1) \\ dW &= \frac{1}{w} (x^2 - t^2 + w^2 + 1) dw \\ dW^2 &= \frac{1}{w^2} (x^2 - t^2 + w^2 + 1)^2 dw^2 \end{aligned}$$

Substituting all the equations into the metric $ds^2 = -dT^2 + dX^2 - dW^2 + dY^2$ we get:

$$\begin{aligned} ds^2 &= -\frac{dt^2}{w^2} + \frac{dx^2}{w^2} + \frac{1}{w^2} \left[(x^2 - t^2 + w^2 + 1)^2 - (x^2 - t^2 - w^2 - 1)^2 \right] dw^2 \\ &= \frac{1}{w^2} (-dt^2 + dx^2 + dw^2) \end{aligned} \quad (6.12)$$

The equation 6.12 given above is the Minkowskian version of the Poincaré half plane. Now, defining the lightcone coordinates of the metric 6.12 as:

$$\begin{aligned} W^+ &\equiv W + Y = \frac{1}{w}(x^2 - t^2) + w \\ W^- &\equiv W - Y = \frac{1}{w} \end{aligned}$$

The equation $(T^2 - X^2) + (W^2 - Y^2) = 1$ can be written as $T^2 - X^2 + W^+W^- = 1$. Generalizing this concept further for $d = 4$, i.e. AdS^4 and writing:

$$\begin{aligned} T &= \frac{t}{w} \\ X &= \frac{x}{w} \\ Y &= \frac{y}{w} \end{aligned}$$

and then lightcone coordinates becomes:

$$\begin{aligned} W^+ &\equiv W + Z = \frac{1}{w}(x^2 + y^2 - t^2) + w \\ W^- &\equiv W - z = \frac{1}{w} \end{aligned}$$

Similarly for $d = 5$, i.e. for AdS^5 the metric would be:

$$ds^2 = \frac{1}{w^2} (-dt^2 + dx^2 + dy^2 + dz^2 + dw^2)$$

Thus we can see that a slice of 5-dimensional spacetime at some specific value of w , for instance w_0 with the metric given above is the 4-dimensional Minkowskian spacetime where:

$$ds^2 = \frac{1}{w_0^2} (-dt^2 + dx^2 + dy^2 + dz^2)$$

Since we know a metric is invariant under the scaling and Lorentz transformations, thus the factor of $\frac{1}{w_0^2}$ will keep it the same.

6.7 Motion of photons and matter in Poincaré coordinates

Since the metric is conformally equivalent to the Minkowskian metric $d\tilde{s}^2 = (-dt^2 + dx^2 + dy^2 + dz^2)$, so the path followed by a light particle is determined by $ds = 0 = d\tilde{s}$. Now, for instance, if we consider a light beam sent by an observer located at

$w = w_0$ toward the boundary $w = 0$. The time taken for the particle to come back would be $t_{return} = 2w_0$, if by some mechanism we made it reflect back to the origin. However, if a massive particle is projected towards the boundary in $(t - w)$ plane, using definition of proper time:

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -1$$

we get:

$$\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dw}{d\tau}\right)^2 = w^2$$

Moreover, the isometry under $t \rightarrow t^+$ we get:

$$\frac{d}{d\tau} \left(w^{-2} \frac{dt}{d\tau} \right) = 0 \quad (6.13)$$

and

$$\frac{dt}{d\tau} = \frac{b^2}{b}$$

Thus we get:

$$\left(\frac{dw}{dt}\right)^2 + \frac{b^2}{w^2} = 1$$

Hence we get the potential in the plane $V(w) = +\frac{b^2}{w^2}$ and we see that the massive particle cannot reach the boundary but turns back at $w_{return} = b$ with b determined by its initial position and the speed.

6.8 Stereographic projection for AdS spacetime

Anti deSitter spacetime can be stereographically projected by mapping, for example the set of *AdS* coordinates (X^0, X^1, X^3, X^4) can be projected into (X^0, X^1, X^3) by the following transformation:

$$X^M = \frac{1}{1 - \frac{x^2}{4L^2}} \delta^M_\mu x^\mu \quad (6.14)$$

where $M = 0, 1, 2, 3$ and

$$X^4 = L \left(\frac{1 + \frac{x^2}{4L^2}}{1 - \frac{x^2}{4L^2}} \right)$$

where $x^2 \equiv -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$. Now by using equation 6.14 we see that:

$$\begin{aligned} X^0 &= \frac{1}{1 - \frac{x^2}{4L^2}} (\delta_0^0 x^0) &\Rightarrow \frac{x^0}{1 - \frac{x^2}{4L^2}} \\ X^1 &= \frac{1}{1 - \frac{x^2}{4L^2}} (\delta_1^1 x^1) &\Rightarrow \frac{x^1}{1 - \frac{x^2}{4L^2}} \\ X^2 &= \frac{1}{1 - \frac{x^2}{4L^2}} (\delta_2^2 x^2) &\Rightarrow \frac{x^2}{1 - \frac{x^2}{4L^2}} \\ X^3 &= \frac{1}{1 - \frac{x^2}{4L^2}} (\delta_3^3 x^3) &\Rightarrow \frac{x^3}{1 - \frac{x^2}{4L^2}} \end{aligned}$$

Thus if

$$x^2 = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$$

then the metric becomes:

$$ds^2 = \left(\frac{1}{1 - \frac{x^2}{4L^2}} \right)^2 \eta_{\mu\nu} dx^\mu dx^\nu$$

Again we can see that the stereographic projection is similar to the Minkowskian metric with a conformal factor. Again, it can be concluded that anti deSitter space is conformally a flat space.

6.9 Isomorphism between AdS^3 and $\mathbb{SL}(2, \mathfrak{R})$

A general 2×2 matrix with all real entries can be written as:

$$U = \begin{pmatrix} T + X & Y + W \\ Y - W & T - X \end{pmatrix}$$

If U is constrained to have a determinant $\det U = +1$ which implies $T^2 - X^2 - Y^2 + W^2 = 1$. Under multiplication, the set of all matrices with real entries and unit determinant clearly generates a group known as $\mathbb{SL}(2, \mathfrak{R})$. Therefore, it can be said that AdS^3 is isomorphic to the universal cover of the $\mathbb{SL}(2, \mathfrak{R})$: there is a 1-to-1 correspondence between the points of AdS^3 to that of the elements of $\mathbb{SL}(2, \mathfrak{R})$. Moreover, if V and Z are the group elements of $\mathbb{SL}(2, \mathfrak{R})$ we can define $U' \equiv VUZ$ which should also have $\det = +1$ and thus an element of $\mathbb{SL}(2, \mathfrak{R})$ which corresponds to another point on AdS^3 . To be precise, the isometry group of AdS^3 is $\mathbb{SL}(2, \mathfrak{R}) \times \mathbb{SL}(2, \mathfrak{R})$. Thus we know that the isometry group of AdS^3 is $\mathbb{SO}(2, 2)$. It can be concluded then $\mathbb{SL}(2, \mathfrak{R}) \times \mathbb{SL}(2, \mathfrak{R})$ must be isomorphic to $\mathbb{SO}(2, 2)$, i.e.

$$\mathbb{SO}(2, 2) = \mathbb{SL}(2, \mathfrak{R}) \times \mathbb{SL}(2, \mathfrak{R})$$

Chapter 7

Conformal Field and Correlation Function

7.1 Introduction

In theoretical physics, when two different concepts are related to each other at the fundamental level, they are often described as dual to each other. One of such dualities is the gauge/gravity duality, otherwise known as the AdS/CFT correspondence, which is a comparatively new type of duality realized by Maldacena in 1997. This correspondence claims that quantum field theory on flat spacetime is correlated to string theory. Furthermore, the correspondence is also a cognizance of the holographic principle. The principle states that in a gravitational theory the number of degrees of freedom in a given volume V scales as the surface area δV of that volume. To elaborate¹⁴, "in the context of semi-classical considerations for quantum gravity, the holographic principle asserts that the information stored in a volume of the dimension V_{d+1} is encoded in its boundary area A_d measured in in units of the Planck area l_p^d ." The theory of quantum gravity involved in the AdS/CFT correspondence is defined on a manifold of the form $AdS \times \chi$, where AdS is the anti-deSitter spacetime and the χ is the compact space or the boundary. The quantum field theory is assumed to be defined on the conformal boundary of the compact space of the AdS spacetime. Hence, in order to study such correspondence, the AdS/CFT conjecture, we need to start by studying the symmetries in a field theory and their associated transformation laws and algebras that arise because of the continuous symmetries in the field theory.

7.2 Role of symmetries in field theories

Symmetries play an important role in mathematics and physics, it helps us to identify the underlying physical meaning of a theory. Symmetries can be classified in a number of ways, but for our purpose we will be taking about two fundamental classification of symmetries, the local and global symmetries of a field and the continuous and discrete symmetries. In this chapter, firstly, we will be taking about the symmetries of quantum field theories which will eventually lead us to conformal symmetry and supersymmetry. In particular, we will discuss the tensor and spinor representation of the Lorentz algebra. Moreover, we will be talking about the massless and massive states within the Poincare algebra. Both these symmetries are related to the study of the spacetime of special relativity. The Lorentz

symmetry and the Poincare symmetry which lead us to conformal symmetry is useful because these symmetries constrain the *correlation function* of the Conformal Field theory which gives us the fundamental picture of the conjecture, AdS-CFT correspondence or the gauge/gravity duality. The fundamental idea behind supersymmetry is to add spinorial charges to the Poincare algebra, extension of the Poincare algebra, which satisfies the anti-commutation relation of the Poincare algebra.

7.3 Lorentz group and its algebra

A Lorentz transformation is of the form $x^\mu \rightarrow x'^\mu = \Lambda(\omega)^\mu_\nu x^\nu$ that leaves a spacetime coordinate and the line element of the spacetime invariant. For an infinitesimal transformation we can expand $\Lambda(\omega)$ as:

$$\Lambda(\omega)^\mu_\nu = \delta^\mu_\nu + \eta^{\mu\nu} \omega_{\rho\nu}$$

where $\omega_{\rho\nu}$ is antisymmetric under the exchange of two indices ρ and ν because the transformation needs to satisfy the relation, $\Lambda\eta\Lambda^T = \eta$. A finite transformation can be constructed by exponentiation the infinitesimal form and by introducing the generators of the Lorentz group $J_{\mu\nu}$, which are $d \times d$ matrices such that:

$$\Lambda(\omega)^\mu_\nu = \delta^\mu_\nu + \frac{i}{2} \omega^{\rho\sigma} (J_{\rho\sigma})^\mu_\nu$$

The components of $J_{\rho\sigma}$ are specified by:

$$(J_{\rho\sigma})^\mu_\nu = i (\eta_{\rho\nu} \delta^\mu_\sigma - \eta_{\sigma\nu} \delta^\mu_\rho)$$

and must satisfy the commutation relation of the Lie algebra $\sim \mathfrak{so}(d-1, 1)$ which is:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i (\eta_{\mu\rho} J_{\nu\sigma} + \eta_{\nu\sigma} J_{\mu\rho} - \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\nu\rho}) \quad (7.1)$$

The generators J_{kl} with $k, l = 1, \dots, d-1$ corresponds to rotations and J_{0k} corresponds to the boosts of the spacetime coordinates. The generators of the rotation may be Hermitian but the generators of the boosts must be anti-Hermitian, i.e.

$$(J_{kl})^\dagger = J_{kl} \quad (J_{0k})^\dagger = -J_{0k}$$

A finite transformation under Lorentz symmetry can be written in the form:

$$\Lambda(\omega) = \exp\left(\frac{i}{2} \omega_{\mu\nu} J^{\mu\nu}\right) \quad (7.2)$$

In addition, a field $\phi(x)$ transforms under a infinitesimal Lorentz transformation as:

$$\delta\phi^a = \frac{i}{2} \omega_{\mu\nu} (\mathcal{J}^{\mu\nu})^a_b \phi^b \quad (7.3)$$

where \mathcal{J} satisfies the Lorentz algebra 7.1 and a finite transformation of the field follows

$$\phi'(x) = D(\Lambda(\omega))^a_b \phi^b(\Lambda^{-1}x) \quad (7.4)$$

with

$$D(\Lambda(\omega)) = \exp\left(\frac{i}{2} \omega_{\mu\nu} J^{\mu\nu}\right)$$

The $\mathcal{J}^{\mu\nu}$ satisfy the commutation relations 7.1 and the matrices $\mathcal{J}^{\mu\nu}$ form the representation of the Lorentz algebra. A representation can either be classified as reducible representation or an irreducible representation. The irreducible representation of a group is of our primary interests since they correspond to the elementary fields.

7.3.1 Tensor representation

One of the important finite dimensional irreducible representation of the Lorentz algebra is $\sim \times(d-1, 1)$. This representation can be used to describe objects like scalars, vectors or even a n -dimensional objects, which are otherwise known as the *tensors*. The scalar representation is also known as the trivial or singlet and the vector space associated with the singlet or scalars is known as the one dimensional vector space or field, ϕ . The generators of this representation are defined to be $\mathcal{J}_1^{\rho\sigma} = 0$ which satisfies the algebra of the group 7.1. The second type of object with a single index is known as the vector representation of the Lorentz algebra which has a dimension of d . The field associated with the vectors ϕ has d components, hence represented as ϕ^d where $d = 0, \dots, d-1$. The generators of the algebra are the $d \times d$ matrices $\mathcal{J}_d^{\rho\sigma}$ which is given by:

$$(\mathcal{J}_d^{\rho\sigma})_\nu^\mu = i(\delta_\nu^\rho \eta^{\mu\sigma} - \delta_\nu^\sigma \eta^{\mu\rho})$$

Similarly, for the representation of a field, $\phi_{\mu_1, \dots, \mu_n}$ we need to consider a n tensor product of their individual vector representations. The representation of the resulting object are in general reducible since they can be break down into a symmetric part and an anti-symmetric part. Using the language of tensor product, \otimes we can denote the decomposition of a tensor using vectors, for instance the rank two tensor product representation $d \otimes d$ can be broken down into a direct sum of a symmetric rank two tensor and an anti-symmetric rank two tensor as follows:

$$dd = (d \otimes_S d) \oplus (d \otimes_A d)$$

where S and A represent symmetric and anti-symmetric parts respectively. The dimension of the symmetric part is given by $\frac{1}{2}d(d+1)$ and the dimension of the anti-symmetric part is given by $\frac{1}{2}d(d-1)$. In general, it is not always true that either $d \otimes_S d$ or $d \otimes_A d$ is irreducible. The representation can be reduced further by contracting the indices using an invariant tensor like the metric tensor of the Minkowskian space. Particularly for the group $\mathbb{SO}(1, n)$, the metric $\eta_{\mu\nu}$ or its inverse $\eta^{\mu\nu}$ are the invariant tensors that can be used to get the irreducible representation of the group. The symmetric rank two tensor can be decomposed into a traceless symmetric rank two tensor, denoted by S and its trace part as:¹⁴

$$d \otimes_S d = 1 \oplus S$$

Similarly we can decompose a rank two antisymmetric tensor into its self-dual and its anti-self dual, as explained in¹⁴ in four spacetime dimensions as:

$$4 \otimes_A 4 = 3^+ \oplus 3^-$$

7.3.2 Spinor representation

The other group of irreducible representation of the Lorentz group can be constructed using the *Clifford algebra* which are known as the spinor representation of the Lorentz group. The Clifford algebra⁹ is given by:

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu \equiv \gamma_\mu, \gamma_\nu = -2\eta_{\mu\nu} \mathbb{1} \quad (7.5)$$

Mathematically speaking, spinors are representations of the *spin group* which is the double cover of the Lorentz group which implies that spinors are projective representation of

the Lorentz group¹⁴. The γ_μ are known as the Dirac matrices. The anti-commutation relation in equation 7.5 can be used and the following values of the matrices can be written down as:

$$(\gamma_0)^2 = \mathbb{1} \quad (\gamma_k)^2 = -\mathbb{1}$$

where $k = 1, \dots, d-1$. Thus it is seen that the eigenvalues of the γ_0 matrices are ± 1 while γ_k has the eigenvalues $\pm i$. Hence we can say that γ_0 are Hermitian and γ_k are anti-Hermitian. The Dirac matrices can be used to construct the representation of the Lorentz algebra:

$$\mathcal{J}^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \quad (7.6)$$

where $\gamma^\mu = \eta^{\mu\nu}\gamma_\nu$ and are known as the *Dirac spinor representation*. This Dirac spinor representation, in case of odd d gives us two inequivalent complex irreducible representations of the Clifford algebra and in case of even d gives us one complex irreducible representation of the Clifford algebra. For both of the cases, the irreducible complex representation is of the complex dimension $2^{\lfloor d/2 \rfloor}$. Up to a similarity transformation:

$$\mathcal{B}\gamma_\mu\mathcal{B}^{-1} = (\gamma_\mu)^\dagger \quad (7.7)$$

where $\mathcal{B} = \gamma_0$, the other Dirac matrices γ_μ forms a unique irreducible representation of the Clifford algebra. In order to relate $-\gamma_\mu^T$ to γ_μ we can introduce a matrix \mathcal{C} , known as the charge conjugation and the following relation can be used to relate them as:

$$\mathcal{C}\gamma_\mu\mathcal{C}^{-1} = -\gamma_\mu^T$$

Furthermore, using these similarity transformations it is possible to define projection conditions on spinors. The two most commonly known projection conditions are:

- Weyl Spinors
- Majorana spinors

A Dirac spinor Ψ can be projected on a complex two component left and right handed Weyl spinors, Ψ_L and Ψ_R defined by:

$$\Psi_L = \begin{pmatrix} \Psi_L \\ 0 \end{pmatrix} = \mathcal{P}_+\Psi \quad \Psi_R = \begin{pmatrix} 0 \\ \Psi_R \end{pmatrix} = \mathcal{P}_-\Psi$$

where \mathcal{P}_\pm is given by:

$$\mathcal{P}_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_5)$$

Similarly, we can derive the following relationship:

$$\mathcal{B}\mathcal{C}\mathcal{J}^{\mu\nu}(\mathcal{B}\mathcal{C}^{-1}) = -(\mathcal{J}^{\mu\nu})^*$$

The complex conjugated Dirac spinor Ψ^* transforms in the same way as $\mathcal{B}\mathcal{C}\Psi$ under Lorentz transformation and imposing the reality condition $\Psi^* = \mathcal{B}\mathcal{C}\Psi$. This two dimensional representation is defined as the *Majorana spinor*

7.4 Poincaré group and its algebra

Poincaré algebra can be constructed by an extension to the Lorentz algebra, i.e. along with the generator of Lorentz algebra, $J^{\mu\nu}$ we have another generator of infinitesimal translations, P_μ for the Poincaré algebra. The generators need to satisfy the commutation relations as in equation 7.8:

$$[J_{\mu\nu}, P_\rho] = i(\eta_{\mu\nu}P_\rho - \eta_{\nu\rho}P_\mu), \quad [P_\mu, P_\nu] = 0 \quad (7.8)$$

The generators P_ρ transforms as a vector under Lorentz transformations and the momenta commutes. The extension of the Lorentz group, the Poincaré group- a non-compact group is a semi-direct product of translations and Lorentz transformations.

The Poincaré group is not compact because boosts and translations are non-compact transformations. A non-compact group does not have a finite-dimensional representation. Therefore, the representation have to be parametrized by a continuous parameters. This labelling is done in Poincaré algebra by the momentum, p^μ . In four spacetime dimensions, the different infinite-dimensional representations of the Poincaré algebra corresponds to *massive* and *massless* particle states. In case of massive particles, we will always be able to find a reference frame in which the momentum four vector takes the form:

$$p^\mu = (m, 0, 0, 0)$$

Then we can define something called the little group which leaves the momentum vector, p^μ invariant, in particular $\text{SO}(3)$ for this. In the case of massless particles, it is not possible to boost to a reference frame where all the spatial components is zero. However, we can boost to a frame where the momentum four vector takes the form:

$$p^\mu = (E, 0, 0, E)$$

If we generalize the argument to $d \neq 4$ spacetime dimensions, then for a massive particle we can boost to the rest-frame and hence the little group is $\text{SO}(d-1)$ while for massless particles the little group turns out to be $\text{SO}(d-2)$ instead of $\text{SO}(d-1)$.

7.5 Ward identities

In *quantum field theory* the presence of symmetries lead to the relations between the *correlation function*. These relations are known as the Ward identities. A generating functional $Z[J]$ under the change of variables $\phi(x) \rightarrow \tilde{\phi}(x) = \phi(x) + \delta\phi(x)$ remains invariant, i.e. $\mathcal{D}\tilde{\phi} = \mathcal{D}\phi$ and thus we obtain:

$$0 = \delta Z[J] = i \int \mathcal{D}\phi e^{i(S + \int d^d x J(x)\phi(x))} \int d^d x \left(\frac{\delta S}{\delta\phi(x)} + J(x) \right) \delta\phi(x) \quad (7.9)$$

Taking functional derivatives with respect to $J(x_i)$ and subsequently setting J to zero, we obtain the **Schwinger- Dyson** equation:

$$0 = i \left\langle \frac{\delta S}{\delta\phi(x)} \phi(x_1) \dots \phi(x_n) \right\rangle + \sum_{j=1}^n \langle \phi(x_1) \dots \phi(x_{j-1}) \delta(x - x_j) \phi(x_{j+1}) \dots \phi(x_n) \rangle \quad (7.10)$$

If we now apply the Schwinger-Dyson equations to continuous symmetry transformations $\phi(x) \rightarrow \phi(x) + \delta\phi(x)$, the variation of the Lagrangian gives us:

$$\begin{aligned}\delta L &= \frac{\partial L}{\partial\phi(x)}\delta\phi(x) + \frac{\partial L}{\partial(\partial_\mu\phi(x))}\partial_\mu\delta\phi(x) \\ &= \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\phi(x))}\delta\phi(x)\right) + \frac{\delta S}{\delta\phi(x)}\delta\phi(x)\end{aligned}$$

If $\delta\phi$ corresponds to the symmetry that leaves the Lagrangian invariant, then the Noether's current is given by:

$$\partial_\mu\mathcal{J}^\mu(x) = \frac{\delta S}{\delta\phi(x)}\delta\phi(x)$$

Now using the equation 7.10, we obtain the Ward identities of the form:

$$\partial_\mu\langle\mathcal{J}^\mu(x)(x_1)\dots\phi(x_n)\rangle - i\sum_{j=1}^n\langle\phi(x_1)\dots\delta(x_j)\delta(x-x_j)\dots\phi(x_n)\rangle = 0 \quad (7.11)$$

If we know that $\delta\phi$ does not involve time derivatives and if we know that the Noether's charge, $Q = \int d^{d-1}x \vec{x} \mathcal{J}$ then we can represent $\delta\phi$ using the following commutation relation:

$$[\hat{Q}, \hat{\phi}(x)] = i\delta\hat{\phi}(x)$$

In general if we now consider a quantum field $\phi(x)$ which is not necessarily a scalar field, then the change of the field $\phi(x)$ is given by:

$$\delta\phi(x) = \tilde{\phi}(x) - \phi(x) = e^{i\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}\phi(\Lambda^{-1}x) - \phi(x) \quad (7.12)$$

Hence the corresponding infinitesimal transformation at $x = 0$ is

$$\delta\phi(0) = \frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\phi(0)$$

Because the field that we are considering is a quantum field we do the following change

$$J^{\mu\nu} \rightarrow \hat{J}^{\mu\nu}$$

and then the infinitesimal change $\delta\phi(x)$ takes the following form:

$$\delta\phi(0) = -\frac{i}{2}\omega_{\mu\nu}[\hat{J}^{\mu\nu}, \phi(0)]$$

Hence we can conclude that

$$[\hat{J}^{\mu\nu}, \phi(0)] = -\mathcal{J}^{\mu\nu}\phi(0)$$

The generator $J^{\mu\nu}$ acts on the spacetime coordinates and the corresponding operator \mathcal{J} acts on the Hilbert space of the quantum field. Similarly, for translations the infinitesimal transformation $\phi(x)$ we get the commutation relation

$$[\hat{P}_\mu, \phi(x)] = -i\partial_\mu\phi(x)$$

and similarly we define $\mathcal{P}_\mu = -i\partial_\mu$ to be acting on the Hilbert space. Now, using the $\delta\phi$ on the Ward identity 7.11, we see that the n-point correlation functions depends on the differences $(x_i - x_j)^2$. To be precise, the one-point function has to be a constant and the two-point function takes the form

$$\langle\phi(x_1)\phi(x_2)\rangle = f((x_1 - x_2)^2) \quad (7.13)$$

where for any function f the equation 7.13 is satisfied.

7.6 Conformal group and its algebra

A conformal transformation is one in which there exists an angle preserving transformation. In Minkowskian space, a conformal transformation is one in which the *causality* is preserved, i.e. the spacelike components are mapped into another spacelike components, the timelike components are mapped into another timelike components and the lightlike seperated points will remain lightlike seperated points in the space. A conformal transformation leave the metric $g_{[\mu\nu]}$ invariant up to an arbitrary positive spacetime dependent scale factor, i.e. conformal transformation are those transformation for which the following relationship is satisfied:

$$g_{\mu\nu} \mapsto \Omega(x)^{-2} g_{\mu\nu}(x) \equiv e^{2\sigma(x)} g_{\mu\nu}(x)$$

and hence the distance function transforms as $ds'^2 = e^{2\sigma(x)} ds^2$. Although the distance function is changed but the transformation leaves the angles invariant locally and preserve the causal structure of the spacetime. For a flat spacetime metric where $g_{\mu\nu} = \eta_{\mu\nu}$ an infinitesimal conformal transformation has to satisfy:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = 2\sigma(x) \eta_{\mu\nu}$$

where $\Omega(x) = 1 - \sigma(x) + \mathcal{O}(\sigma)^2$ and now if we contract both sides of the equation given above using the invariant tensor we get:

$$\partial \cdot \epsilon = \sigma(x) \cdot d$$

in d-dimension, therefore the infinitesimal transformation is conformal only if it satisfies

$$(\eta_{\mu\nu} \partial_\rho \partial^\rho + (d-2) \partial_\mu \partial_\nu) \partial \cdot \epsilon = 0 \quad (7.14)$$

Equation 7.14 simplifies for $d = 2$, hence we divide conformal transformation into two categories, for $d > 2$ and for $d = 2$.

7.6.1 Conformal transformation for $d > 2$

Equation 7.14, when solved for values of $d > 2$ gives us $\epsilon(x)$ (upto second order terms)

$$\epsilon^\mu(x) = a^\mu + \omega^\mu_\nu x^\nu + \lambda x^\mu + b^\mu x^\mu_\mu - 2(b_\mu x^\mu) x^\mu \quad (7.15)$$

The parameters $a^\mu, \omega^\mu_\nu, \lambda$ and b^μ have finite number of components, therefore, the conformal algebra associated with the symmetry group is finite dimensional. The geometric interpretation of the transformation of the conforaml group is listed below:

Name	$\epsilon^\mu(x)$	$\sigma(x)$	Operator
Translation	a^μ	0	P_μ
Lorentz transformation	$\omega^\mu_\nu, \omega_{\mu\nu} = -\omega_{\nu\mu}$	0	$J_{\mu\nu}$
Dilatation	λx^μ	λ	D
Special conformal transformation	$b^\mu x^\mu x_\mu - 2(b_\mu x^\mu) x^\mu$	$-2(b^\mu_\mu)$	K_μ

The conformal algebra consisting of all these generators is given by the commutation relations of the Poincaré algebra as well as the following commutation relation:

$$\begin{aligned} [J_{\mu\nu}, K_\rho] &= i(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu) & [D, P_\mu] &= iP_\mu \\ [D, K_\mu] &= iK_\mu & [D, J_{\mu\nu}] &= 0 \\ [K_\mu, K_\rho] &= 0 & [K_\mu, P_\nu] &= -2i(\eta_{\mu\nu} D - J_{\mu\nu}) \end{aligned} \quad (7.16)$$

7.6.2 Conformal transformation for $d = 2$

For the case where $d = 2$, the condition in equation 7.14 gives the form of conformal transformation as:

$$\partial_0 \epsilon_1 = -\partial_1 \epsilon_0 \quad \partial_0 \epsilon_0 = \partial_1 \epsilon_1 \quad (7.17)$$

Now using complex coordinates $z = x^0 + iX^1$ and $\bar{z} = x^0 - iX^1$ we can write ϵ as a function of z as $\epsilon = \epsilon^0 + i\epsilon^1$. Now if we expand $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ as:

$$\begin{aligned} \epsilon(z) &= - \sum_{n \in \mathbb{Z}} \epsilon_n z^{n+1} \\ \bar{\epsilon}(\bar{z}) &= - \sum_{n \in \mathbb{Z}} \bar{\epsilon}_n \bar{z}^{n+1} \end{aligned}$$

Hence we can say that the infinitesimal transformation given by $z \mapsto z' = z + \epsilon(z)$ and $\bar{z} \mapsto \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z})$ is conformal and the generators are given by:

$$l_n = -z^{n+1} \partial_z \quad \bar{l}_n = -\bar{z}^{n+1} \partial_{\bar{z}}$$

The counterpart of these commutators, takes the following form in Hilbert spaces:

$$[L_n, L_m] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \quad (7.18)$$

The commutation relation in equation 7.18 is known as the *Virasoro* algebra, where c is the central charge.

7.7 Correlation function from conformal transformation

The path integral formalism of quantum mechanics states that we need to sum up all possible paths of propagation. In particular, the path integral sums over all the possible paths which start at some position q_i at a time t_i and end at a position q_f at time t_f . The extension of this idea is seen in quantum field theory. In quantum field theory this procedure gets translated to summing over all field configurations ϕ in the configuration space. Mathematically,

$$\mathcal{D}\phi \propto \prod_{t_i \leq t \leq t_f} \prod_{\vec{x} \rightarrow \mathbb{R}^{d-1}} d\phi(t, \vec{x})$$

The transition from initial state $|\phi_i, t_i\rangle$ to final state $|\phi_f, t_f\rangle$ where $\hat{\phi}(t_i, \vec{x})|\phi_i, t_i\rangle = \phi_i(\vec{x})|\phi_i, t_i\rangle$ and $\hat{\phi}(t_f, \vec{x})|\phi_f, t_f\rangle = \phi_f(\vec{x})|\phi_f, t_f\rangle$ is then given by:

$$\langle \phi_f, t_f | \phi_i, t_i \rangle = N \int \mathcal{D}\phi \exp \left[i \int_{t_i}^{t_f} dt \int_{\mathbb{R}^{d-1}} d^{d-1} \vec{x} \mathcal{L}_{free}(\phi, \partial\phi) \right]$$

where N is a normalization factor. In strict mathematical sense, this integral might not exist hence a common trick to improve the convergence of the path integral is used. The mass m^2 in the Lagrangian is replaced by $m^2 - i\epsilon$ and then at the end of the calculation, ϵ is taken to

be zero, i.e. we take the limit $t_i \mapsto -\infty$ and $t_f \mapsto +\infty$ and we consider $\phi_i(\vec{x}) = \phi_f(\vec{x}) = 0$ which is called the vacuum transition amplitude and is written as:

$$\langle 0|0\rangle = N \int \mathcal{D}\phi \exp \left[i \int d^d x \mathcal{L}_{free}(\phi, \partial\phi) \right]$$

where N is chosen such that $\langle 0|0\rangle = 1$. Similar to this we are particularly interested in a correlation function¹⁴ of the form:

$$\langle 0|\mathcal{T}\hat{\phi}(x_1)\hat{\phi}(x_2)\dots\hat{\phi}(x_n)|0\rangle \equiv \langle \hat{\phi}(x_1)\hat{\phi}(x_2)\dots\hat{\phi}(x_n)\rangle \equiv G^{(n)}(x_1, \dots, x_n) \quad (7.19)$$

where \mathcal{T} denotes the time ordering prescription which states that a product of operators $\hat{\phi}(x_1)\hat{\phi}(x_2)\dots\hat{\phi}(x_n)$ to the right of \mathcal{T} has to be ordered such that fields at later times stand to the left of those at earlier times¹⁴. In particular, for two operators $\hat{\phi}(x)\hat{\phi}(y)$ the time ordering is given by:

$$\mathcal{T}\hat{\phi}(x)\hat{\phi}(y) \equiv \Theta(x^0 - y^0)\hat{\phi}(x)\hat{\phi}(y) + \Theta(y^0 - x^0)\hat{\phi}(y)\hat{\phi}(x)$$

where Θ is the step function. The conformal symmetries of a field as introduced in the previous section imposes restrictions on the correlation functions of the form as introduced in equation 7.19. Particularly it determines the form of the two and three point correlation which applies to both forms of conformal symmetries, i.e. when $d > 2$ and $d = 2$. The invariance under dilatation, two-point function of two scalar conformal primary operators ϕ_1 and ϕ_2 with scaling dimensions Δ_1 and Δ_2 transforms as:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{\phi_1\phi_2}}{(x_1 - x_2)^{\Delta_1 + \Delta_2}}$$

The denominator of the equation given above can be written as $((x_1 - x_2)^2)^{(\Delta_1 + \Delta_2)/2}$ then the equation takes the form:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_{\phi_1\phi_2}}{((x_1 - x_2)^2)^{(\Delta_1 + \Delta_2)/2}}$$

We can constrain the correlation function by applying an inversion which states that the two point function is zero unless both fields have the same scaling dimensions Δ . In addition, since the function is real and symmetric under the exchange of the ϕ 's we can diagonalize the constant C in the space of scalar primary operators \mathcal{O} such that C is only non-zero for conjugated operators \mathcal{O} and $\bar{\mathcal{O}}$ and then by setting $C = 1$ we get the the correlation two point function for a scalar conformal primary operator \mathcal{O} of scaling dimension Δ :

$$\langle \mathcal{O}(x_1)\hat{\mathcal{O}}(x_2)\rangle = \frac{1}{(x_1 - x_2)^{2\Delta}}$$

Similarly, we can get the *three-point correlation function* for the scalar conformal primary operators \mathcal{O}_i , ($i = 1, 2, 3$) with scaling dimension Δ_i which is:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}}{(x_1 - x_2)^{\Delta_1 + \Delta_2 + \Delta_3}(x_2 - x_3)^{-\Delta_1 + \Delta_2 + \Delta_3}(x_1 - x_3)^{\Delta_1 - \Delta_2 + \Delta_3}} \quad (7.20)$$

with $C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$ determined by the field content. A general conformal primary operator transforms as:

$$\mathcal{O}^i(x) \mapsto \mathcal{O}^{x'} = \Omega(x)^\Delta D(\mathcal{R}(x))_j^i \mathcal{O}^j(x)$$

where $\Omega(x)$ is the scale factor, Δ is the conformal dimensions and $D(\mathcal{R}(x))$ is the local Lorentz transformation. Now we can construct a conformally covariant expression for the two point function of the conformal primary operators. In irreducible representations of the conformal primary operator takes the following form:

$$\langle \mathcal{O}^i(x) \bar{\mathcal{O}}_j(y) \rangle = \frac{C_{\mathcal{O}}}{(x-y)^{2\Delta}} D(I(x-y))^i_j \quad (7.21)$$

where $C_{\mathcal{O}}$ is an overall constant scale factor which can be modified by redefining the set.

7.8 Correlation function using gravity

Let us assume a scalar operator \mathcal{O} having conformal dimension Δ on the field theory side which is dual to a scalar field ϕ on the $d+1$ dimensional gravity side. In Euclidean signature, the action of the gravity side $S[\phi]$ is given by:

$$S[\phi] = \frac{C}{2} \int dz d^d x \sqrt{g} (g^{mn} \partial_m \phi \partial_n \phi + m^2 + \phi^2) \quad (7.22)$$

in which the mass of the scalar is such that $m^2 L^2 = \Delta(\Delta - d)$ and we consider the Euclidean AdS metric in $d+1$ dimension which is given by

$$ds^2 = \frac{L^2}{z^2} (dz^2 + \delta_{\mu\nu} dx^\mu dx^\nu) \quad (7.23)$$

The equation of motion can be deduced by taking the extremum of the action in equation 7.22 which is:

$$(\square_g - m^2)\phi = 0, \quad \square_g \phi = \frac{1}{\sqrt{g}} \partial_m (\sqrt{g} g^{mn} \partial_n \phi)$$

The solution ϕ satisfies the equation of motion, therefore, we can say that the action is just a boundary term which can be written as:

$$S[\phi] = -\frac{C}{2} \int d^d x \sqrt{g} g^{zz} \phi(z, x) \partial_z \phi(z, x)|_z = \epsilon \quad (7.24)$$

Assuming that the interior is regular it can be ensured that the integrand above vanishes for $z \mapsto \infty$. When $z = 0$, the expression in the integrand $\sqrt{g} g^{zz} = (L/z)^{d-1}$ is divergent and hence we are required to regularize the action, $S[\phi]$. This can be done by excluding the region $0 < x < \epsilon$ and imposing all the boundary conditions at $z = \epsilon$. Since we are restricting z to $z \geq \epsilon$, therefore the isometries of the AdS spacetime cannot be used in order to find the solution of the ϕ . However, we can do a Fourier transformation along the boundary while keeping the radial direction z in the configuration space. The Fourier transformation is given by:

$$\phi(z, x) = \int \frac{d^d p}{(2\pi)^d} e^{ipx} \phi(z, p)$$

where p is the momentum along the direction of the field and obeys the relation $p \cdot x = \delta_{\mu\nu} p^\mu x^\nu$. The function $\phi(z, p)$ satisfies:

$$z^2 \partial_z^2 \phi_p(z) - (d-1)z \partial_z \phi_p(z) - (m^2 L^2 + p^2 z^2) \phi_p(z) = 0$$

with $p^2 = \delta_{\mu\nu} p^\mu p^\nu$ where $|p| = \sqrt{p^2}$. Considering the necessary boundary conditions, as in¹⁴, the normalized solution for ϕ is given as:

$$\phi(z, p) = \frac{z^{d/2} K_\nu(z|p|)}{\epsilon^{d/2} K_\nu(\epsilon|p|)} \phi_{(0)}(p) \epsilon^{d-\Delta} \quad (7.25)$$

where $K_\nu(z) \sim z^{-\nu}$ and $\phi_{(0)}$ is the zeroth component of the Fourier transformation. Then we determine the on-shell action by inserting the equation given above in equation 7.24 and then we get:

$$S[\phi] = -\frac{CL^{d-1}}{2\epsilon^{d-1}} \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} (2\pi)^d \delta^d(p+q) \phi(z, p) \partial_z(z, q) \Big|_{z=\epsilon} \quad (7.26)$$

We can see that $\phi(z, p)$ can be expressed in terms of the zeroth component and hence it can be said that the action depends only on this. In addition, by using the AdS/CFT conjecture that classical supergravity action is the generating functional for a connected Green's function of composite gauge invariant operators¹⁴. By introducing all composite operators \mathcal{O}_i on the field theory side the corresponding sources $\phi_{(0)}^t$, we can write down the correlation function from the generating functional $W[\phi_{(0)}^t]$ by taking the derivatives with respect to the sources $\phi_{(0)}^t$, we get:

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \dots \mathcal{O}_n(x_n) \rangle_{CFT,c} = -\frac{\delta^n W}{\delta \phi_{(0)}^1(x_1) \delta \phi_{(0)}^2(x_2) \dots \delta \phi_{(0)}^n(x_n)} \Big|_{\phi_{(0)}^t=0} \quad (7.27)$$

Now by using the equation given above and equation 7.26 we can write down the two-point correlation functions for the dual CFT operators as:

$$\begin{aligned} \langle \mathcal{O}(p) \mathcal{O}(q) \rangle_\epsilon &= (2\pi)^{2d} \frac{\delta^2 S[\phi_0]}{\delta \phi_0(-p) \delta \phi_0(-q)} \\ &= -\frac{(2\pi)^d \delta^d(p+q) CL^{d-1}}{\epsilon^{2\Delta-d-1}} \frac{d}{dz} \ln(z^{d/2} K_\nu(z|p|)) \Big|_{z=\epsilon} \\ &= -\frac{(2\pi)^d \delta^d(p+q) CL^{d-1}}{\epsilon^{2\Delta-d}} \left(\frac{d}{2} + \frac{\epsilon|p| K'_\nu(\epsilon|p|)}{K_\nu(\epsilon|p|)} \right) \end{aligned} \quad (7.28)$$

In this equation, when we take the limit $\epsilon \mapsto 0$, we obtain the two point function. Then using the *Bessel expansion* as in¹⁴, we realize that the conformal dimension associated with the CFT operator \mathcal{O} has the conformal dimension of $\Delta = \nu + \frac{d}{2}$. Additionally, using the result of the expansion of the Bessel modes¹⁴, we obtain:

$$\begin{aligned} \langle \mathcal{O}(p) \mathcal{O}(q) \rangle_\epsilon &= (2\pi)^d \delta^d(p+q) CL^{d-1} \left(\frac{\beta_0 + \beta_1 \epsilon^2 |p|^2 + \dots + \beta_\nu (\epsilon|p|)^{2(\nu-1)}}{\epsilon^{2\Delta-d}} \right. \\ &\quad \left. - \frac{2\nu b_o}{a_o} |p|^{2\nu} \ln(\epsilon|p|) (1 + \mathcal{O}(\epsilon^2)) \right) \end{aligned}$$

where β_i are ratios of the a_k and b_k which are the coefficient of the Bessel expansion and thus is a function of ν . Now, when we take the limit $\epsilon \mapsto 0$, only the term involving the logarithm of the momentum remains. Thus, we obtain the non-local result for the correlator

$$\langle \mathcal{O}(p) \mathcal{O}(q) \rangle = -(2\pi)^d \delta^d(p+q) CL^{d-1} \frac{(-1)^{\nu+1}}{2^{2(\nu-1)} \Gamma(\nu)^2} |p|^{2\nu} \ln(\epsilon|p|) \quad (7.29)$$

Finally by transforming the non-local contribution proportional to $|p|^{2\nu} \ln |p|$ back to the position space we the result that is independent of the ϵ which is

$$\mathcal{O}(x)\dagger\rangle = CL^{d-1} \frac{\Gamma(\Delta)}{\Gamma(\Delta - d/2)} \frac{2\Delta - d}{\pi^{d/2} |x - y|^{2\Delta}} \quad (7.30)$$

The equation that was derived in the last section, equation 7.21 agrees with the equation derived above 7.30. Hence, we can find the correlation between the gravity side of a theory to its corresponding field theory by calculating the the values of C in the equation 7.30. One such example of such AdS/CFT correspondence if the $N = 4$ Super Yang-Mills theory.

Chapter 8

Entanglement Entropy and Holography

8.1 Introduction

Entanglement entropy is a measure of how the quantum information of a system is encoded in a quantum state. For a holographic system this means that the entanglement entropy is stored in the geometric features of the bulk geometry. Although entanglement entropy can be defined in quantum field theory and can be used to get an insight about the nature of the renormalization group. Mathematically, entanglement entropy is the *von Neumann entropy* of the reduced density matrix, i.e.

$$S_A \equiv -\text{tr } \rho_A \log \rho_A$$

where the reduced density matrix of a subsystem is defined as:

$$\rho_A = \text{tr}_B \rho$$

where A and B is used to denote a *bipartite system* with Hilbert space equal to the direct product of the two such that:

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

and the state (pure or mixed) of the full system is given by ρ . The system of two states is said to be maximally entangled if the reduced density matrix ρ is proportional to the identity matrix, i.e. the resulting density matrix is a diagonal matrix.

The idea of holography has played a major role in the recent developments of String theory. Since we know that the entropy of a black hole is not proportional to its volume but its area (Hawking Radiation) of the event horizon given by:

$$S_{BH} = \frac{\text{Area}(\gamma)}{4G_N} \tag{8.1}$$

where γ denotes the area of the horizon, G_N denotes the Newton constant and S_{BH} is known as the *Bekenstein-Hawking Entropy*. In addition, we know that the AdS/CFT correspondence claims that quantum gravity on $(d + 2)$ dimensional anti-deSitter spacetime is equivalent to a certain conformal field theory in $d + 1$ dimensional CFT. Hence we can say that holography is manifestly realized in AdS/CFT correspondence¹³. However, most of the recent work in

the correspondence is done, i.e. the theories are formulated using specific operator and for this reason it was unable to answer which region of the AdS is responsible to particular information in the dual CFT. This problem can only be solved if we formulate holography in terms of a universal observable and for this and this is one of the prime reason for studying the *entanglement entropy*¹⁷. The entanglement entropy in quantum field theories or many body quantum system is a non-local quantity as opposed to correlation functions.

8.1.1 Properties of entanglement entropy

1. For two system ρ and σ we can define relative entropy as $S(\rho||\sigma) \equiv \text{tr } \rho \log \rho - \text{tr } \rho \log \sigma$ which obey the inequality $S(\rho||\sigma) \geq 0$.
2. Entanglement entropy of two system obey the general triangular inequality law, $|S_A - S_B| \leq S_{AB}$.
3. If we define something called *mutual information* as $I(A, B) \equiv S_A + S_B - S_{AB}$ then it obeys the relationship $I(A, B) = S(\rho_{AB}||\rho_A \otimes \rho_B) \geq 0$.
4. In a pure state of A and B , the correlation of A and B comes from entanglement of the system and in a mixed state the classical contributions of information from the system also plays a role.
5. For a system with three or more subsystem, the strong subadditivity inequality is maintained which states that for a tripartite system, $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ the entropy follows $S_{ABC} + S_B \leq S_{AB} + S_{BC}$

8.1.2 Schmidt decomposition

Suppose $|\psi\rangle$ is a pure state of a composite system, AB . Then there exist orthonormal states $|i_A\rangle$ for system A and orthonormal states $|i_B\rangle$ of system B such that:

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle \quad (8.2)$$

where λ_i re non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as the Schmidt coefficients and this decomposition of states are known as the Schmidt decomposition¹.

8.2 Calculation of entanglement entropy

As discussed earlier, we say that two quantum mechanically described are in an entangled states if the description of one of the state is incomplete without the description of the other one. *Bell- spin state*, is one of many such examples which is given by:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|\downarrow_A \otimes \downarrow_B\rangle + |\uparrow_A \otimes \uparrow_B\rangle) \quad (8.3)$$

Although the system described in 8.3 are independent of each other yet a measurement on the system A will allow us to predict the state of system B without any calculations on B . Hence a measurement performed, collapsing the wave function of A , will also in turns collapse the wave function of B . Thus, we can say that the two systems are entangled to each other. This collapse of the wave function of system B after measuring system A happens instantaneously which was referred as *spooky action at a distance* by famous physicist Albert Einstein.

For a more rigorous description of entanglement, we choose to write the equation 8.3 as follows:

$$|\Phi^+\rangle = |\phi\rangle_A \otimes |\phi\rangle_B \quad (8.4)$$

where $|\phi\rangle_A$ and $|\phi\rangle_B$ are arbitrary states:

$$\begin{aligned} |\phi\rangle_A &= a|\uparrow\rangle + b|\downarrow\rangle \\ |\phi\rangle_B &= c|\uparrow\rangle + d|\downarrow\rangle \end{aligned}$$

where a, b, c and d are some arbitrary constants. However, it turns out that there exist no such values of these coefficients which satisfy the Bell state equation in 8.3. Hence we can say that neither the description of system A nor the description of system B will be complete without their counterparts which implies that the systems are entangled to each other.

It is often useful to define a mathematical quantity if we want to measure something and in this case, it is the *entanglement entropy* that helps us to deduce to what extent two states are entangled.

In general, an entangled state cannot be described by a single state, rather the system is described using a density matrix, usually denoted by ρ ¹¹. The density matrix gives us a probabilistic description of the entangled system which can be represented as:

$$\rho = \begin{pmatrix} \rho_1 & 0 & \cdots & 0 \\ 0 & \rho_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho_N \end{pmatrix}$$

where ρ_i 's are the probabilities of the system being in the state $|i\rangle$ or the eigenvalues of the diagonal density matrix.

The density matrix can be used to express the system in terms of a single state vector and the density matrix is calculated as follows:

$$\rho = |\psi\rangle\langle\psi|$$

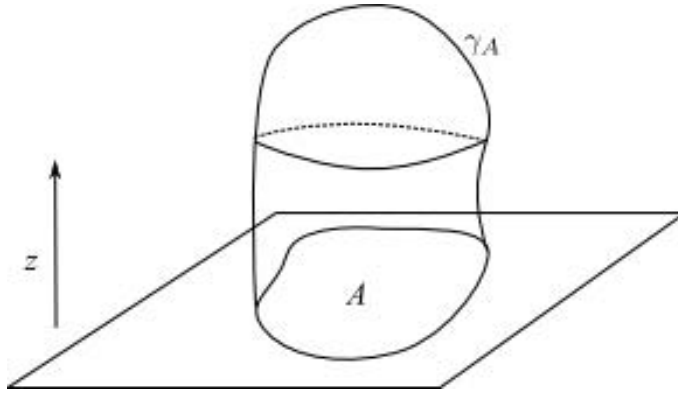
and then it can be generalized to write the generalized description of a mixed state as:

$$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$$

where p_n are the probabilities for the states ψ_n . The density matrix can be used to calculate the expectation value $\langle O \rangle$ of any operator O using $\langle O \rangle = \text{Tr}(\rho O)$.

A density can be simplified further in order to describe parts of the composite or mixed system which is done using the reduced density matrix formalism. Let us define a system consisting of two subsystems and define the reduced density matrix as:

$$\begin{aligned} \rho_1 &\equiv \text{Tr}_2(\rho) \\ \rho_2 &\equiv \text{Tr}_1(\rho) \end{aligned}$$


 Figure 8.1: Boundary of an Anti-deSitter Spacetime¹¹

The reduced density matrix contains all the information about only one subsystem which can be shown by calculating the expectation value of an operator acting on the composite state. For example, if operator O works on system 1 alone we get,

$$\begin{aligned}\langle O \rangle_\rho &= \text{Tr}[\rho(O \otimes I)] \\ &= \langle O \rangle_{\rho_1} = \text{Tr}(O\rho_1)\end{aligned}$$

As we are using tensor product it implies that the operator O is acting on the first subsystem while the identity operator I is acting on the second thus keeping it invariant. Reduced density matrices give us an insight about the correlation between the variables of the subsystem 1 and 2 used in the example. If we define $\rho_{12} \equiv \rho_1 \otimes \rho_2$ and see that $\rho = \rho_{12}$ then we can conclude that the systems are not correlated. Otherwise, the systems are entangled and ρ contains certain amount of information of the correlation between the two system, thus the systems are entangled.

8.3 Holographic entanglement entropy

For a system, if we assume that the CFT has a large number of degrees of freedom and is in a state with a geometric dual of itself then the entanglement entropy is given by the holographic entanglement entropy which is:

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}$$

which is the same as the equation 8.1. The area is a spacelike extremal surface in the dual geometry attached to the AdS boundary, as shown in figure 8.1. The boundary must satisfy two conditions¹¹. Firstly, the extremal surfaces γ_A must be homologous, i.e. these surfaces must be continuously deformable to region A, as in figure 8.1. Secondly, if there are multiple extremal surfaces satisfying the first condition then we need to look for the one that has the minimal area.

8.4 Calculating the entanglement entropy of 1d-coupled harmonic oscillators

As in our calculation of entanglement entropy for two harmonic oscillators, the Hamiltonian of the system is given as:

$$H = \frac{1}{2}[p_1^2 + p_2^2 + k_0(x_1^2 + x_2^2) + k_1(x_1 - x_2)^2] \quad (8.5)$$

As we want to diagonalize the system, we need to introduce a variable substitution. Let us assume the following:

$$\begin{aligned} x_+ &= \frac{x_1 + x_2}{\sqrt{2}} \\ x_- &= \frac{x_1 - x_2}{\sqrt{2}} \\ \omega_+ &= \sqrt{k_0} \\ \omega_- &= \sqrt{k_0 + 2k_1} \end{aligned} \quad (8.6)$$

Using the substitution 8.6 we can introduce the following momenta operators:

$$\begin{aligned} p_+ &= i \frac{d}{dx_+} = \frac{dx_1}{dx_+} p_1 + \frac{dx_2}{dx_+} p_2 \\ p_- &= i \frac{d}{dx_-} = \frac{dx_1}{dx_-} p_1 + \frac{dx_2}{dx_-} p_2 \end{aligned} \quad (8.7)$$

and then we can write the usual momenta operators as:

$$\begin{aligned} p_1 &= i \frac{d}{dx_1} = \frac{dx_+}{dx_1} p_+ + \frac{dx_-}{dx_1} p_- \\ &= \frac{1}{\sqrt{2}}(p_+ + p_-) \end{aligned} \quad (8.8)$$

$$\begin{aligned} p_2 &= i \frac{d}{dx_2} = \frac{dx_+}{dx_2} p_+ + \frac{dx_-}{dx_2} p_- \\ &= \frac{1}{\sqrt{2}}(p_+ - p_-) \end{aligned} \quad (8.9)$$

Finally we get:

$$p_1^2 + p_2^2 = p_+^2 + p_-^2$$

Moreover, using 8.6 and 8.7 we can write down the Hamiltonian as:

$$H = \frac{1}{2}(p_+^2 + p_-^2 + \omega_+^2 x_+^2 + \omega_-^2 x_-^2) \quad (8.10)$$

Now using the derivation from¹⁰ we can write down the associated ground state wave functions of the two independent wave functions as:

$$\begin{aligned} \psi_0^+ &= \left(\frac{\omega_+}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\omega_+ x_+^2} \\ \psi_0^- &= \left(\frac{\omega_-}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\omega_- x_-^2} \end{aligned} \quad (8.11)$$

$$\psi_0 = \pi^{-1/2}(\omega_+\omega_-)^{(1/4)} e^{-\frac{1}{2}(\omega_+x_+^2 + \omega_-x_-^2)} \quad (8.12)$$

writing ψ_0 as a state vector in the position basis, we now have:

$$|\psi_0\rangle = \int_{-\infty}^{+\infty} dx_1 dx_2 \psi_0(x_1 x_2) |x_1\rangle |x_2\rangle \quad (8.13)$$

The density formalism is used now and the density matrix is written as:

$$\begin{aligned} \rho &= |\psi_0\rangle \langle \psi_0| \\ &= \int_{-\infty}^{+\infty} dx_1 dx_2 dx'_1 dx'_2 \psi_0(x_1 x_2) \psi_0^*(x'_1 x'_2) |x_1\rangle |x_2\rangle \langle x'_1| \langle x'_2| \end{aligned} \quad (8.14)$$

Next, we calculate the reduced density matrix for one oscillator, $\rho_{red} = \text{Tr}(\rho)$ where:

$$\begin{aligned} \rho_{red} &= \int_{-\infty}^{+\infty} dx \langle x | \rho | x \rangle \\ &= \int_{-\infty}^{+\infty} dx dx_1 dx_2 dx'_1 dx'_2 \psi_0(x_1, x_2) \psi_0^*(x'_1, x'_2) \langle x | x_1 \rangle |x_2\rangle \langle x'_1 | \langle x'_2 | x \rangle \\ &= \int_{-\infty}^{+\infty} dx dx_1 dx_2 dx'_1 dx'_2 \psi_0(x_1, x_2) \psi_0^*(x'_1, x'_2) \langle x | x_1 \rangle |x_2\rangle \langle x'_2 | \langle x'_1 | x \rangle \\ &= \int_{-\infty}^{+\infty} dx dx_1 dx_2 dx'_1 dx'_2 \psi_0(x_1, x_2) \psi_0^*(x'_1, x'_2) \delta(x - x_1) \delta(x - x'_1) |x_2\rangle \langle x'_2| \\ &= \int_{-\infty}^{+\infty} dx_2 dx'_2 \psi_0(x_1, x_2) \psi_0^*(x'_2, x'_2) |x_2\rangle \langle x'_2| \end{aligned} \quad (8.15)$$

Now we can calculate the diagonal matrix element of the reduced density matrix as:

$$\rho_{red}(x_2, x'_2) = \langle x_2 | \rho_{red} | x'_2 \rangle = \int_{-\infty}^{+\infty} dx_1 \psi_0(x_1, x_2) \psi_0^*(x_1, x'_2) \quad (8.16)$$

Now, we solve the following equation:

$$\begin{aligned} \frac{\sqrt{\omega_+\omega_-}}{\pi} \int_{-\infty}^{+\infty} dx_1 \exp \left[-\frac{1}{2}\omega_+x_1^2 - \frac{1}{4}\omega_+(x_2^2 + x_2'^2)^2 - \frac{1}{2}\omega_+x_1(x_2 + x_2') \right. \\ \left. - \frac{1}{2}\omega_-x_1^2 - \frac{1}{4}\omega_-(x_2^2 + x_2'^2)^2 + \frac{1}{2}\omega_-x_1(x_2 + x_2') \right] \end{aligned} \quad (8.17)$$

to get this:

$$\rho_{red}(x_2, x'_2) = \pi^{-1/2}(\gamma - \beta) e^{-\frac{\gamma}{2}(x_2^2 + x_2'^2) + \beta x_2 x_2'} \quad (8.18)$$

where the symbols β and γ are used to denote the following functions of the variables of the coupled oscillator system, $\beta = \frac{\frac{1}{4}(\omega_+ - \omega_-)^2}{\omega_+ + \omega_-}$ and $\gamma - \beta = \frac{2\omega_+\omega_-}{\omega_+ + \omega_-}$. The eigenvalues of this density matrix are the solutions to:

$$\int_{-\infty}^{+\infty} dx' \rho_{red}(x, x') f_n(x') = p_n f_n(x) \quad (8.19)$$

where p_n and f_n are guessed as in¹⁶ as:

$$\begin{aligned} p_n &= (1 + \xi)\xi^n \\ f_n(x) &= H_n(\alpha^{1/2}x) e^{-\frac{\alpha x^2}{2}} \end{aligned} \quad (8.20)$$

Lastly we compute the entropy of the subsystem using:

$$\begin{aligned} S &= - \sum_{n=1}^{\infty} \rho_n \log \rho_n \\ &= - \sum_{n=1}^{\infty} (1 - \xi)\xi^n \log[(1 - \xi)\xi^n] \\ &= -(1 - \xi) \sum_{n=1}^{\infty} \xi^n [\log(1 - \xi) + n \log \xi] \\ &= -(1 - \xi) \left[\log(1 - \xi) \sum_{n=1}^{\infty} \xi^n + \log \xi \sum_{n=1}^{\infty} n \xi^n \right] \\ &= -(1 - \xi) \left[\log(1 - \xi) \frac{1}{1 - \xi} + \log \xi \frac{\xi}{(1 - \xi)^2} \right] \\ &= -\log(1 - \xi) - \frac{\xi}{1 - \xi} \log \xi \end{aligned} \quad (8.21)$$

8.5 Calculating entanglement entropy of N-coupled harmonic oscillators

The Hamiltonian for N coupled harmonic oscillator is given by:

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 + \frac{1}{2} \sum_{i,j=1}^N x_i K_{ij} x_j$$

where K is a real symmetric matrix, representing a eigenvector, with positive eigenvalues representing all spring constants that are included in the system. The matrix K can be written in the form $K = U^T K_D U$, hence we can say that K is diagonalizable matrix when U is an orthogonal matrix. Therefore, we can say that the Hamiltonian of the system can be diagonalized using the following transformations:

$$\begin{aligned} x &\mapsto \tilde{x} = Ux \\ x^T &\mapsto \tilde{x}^T = U^T x^T \end{aligned} \quad (8.22)$$

Using equation 8.22 the Hamiltonian of the system takes the form:

$$\begin{aligned} H &= \frac{1}{2} \sum_{i=1}^N p_i^2 + \frac{1}{2} \sum_{i,j=1}^N x_i K_{ij} x_j \\ &= \frac{1}{2} \sum_{i=1}^N p_i^2 + \frac{1}{2} \sum_{i,j=1}^N U^T x_i K_{ij} U x_j \\ &= \frac{1}{2} \sum_{i=1}^N p_i^2 + \frac{1}{2} \sum_{i,j=1}^N x_i K_D^{ij} x_j \end{aligned}$$

Again using the idea from¹⁰ and the diagonalized Hamiltonian, the full ground state wavefunction can be written as:

$$\begin{aligned}
 \psi_0 &= \prod_i \psi_o^i \\
 &= \prod_i \left(\frac{\sqrt{K_D^{ii}}}{\pi} \right)^{1/4} \exp \left[-\frac{1}{2} x_i \sqrt{K_D^{ii}} x_i \right] \\
 &= \prod_i \left(\frac{\Omega_D^{ii}}{\pi} \right)^{1/4} \exp \left[-\frac{1}{2} x_i \Omega_D^{ii} \right] \\
 &= \pi^{-\frac{N}{4}} (\det \Omega)^{1/4} \exp \left[-\frac{1}{2} \vec{x} \dot{\Omega}_D \dot{x} \right]
 \end{aligned} \tag{8.23}$$

where $\Omega = U^T \sqrt{K_D} U$ and \vec{x} is an N -vector. For the purpose of calculating the reduced density matrix we trace over the system consisting of $n \leq N$ oscillators, which gives us:

$$\begin{aligned}
 \text{red} &= \int_{-\infty}^{+\infty} \prod_{j=1}^n d\vec{x}_j \langle x_j | \rho | x_j \rangle \\
 &= \int_{-\infty}^{+\infty} \prod_{j=1}^n dx_i \psi_0(x_1, \dots, x_n, x_{n+1}, \dots, x_N) \psi_0^*(x_1, \dots, x_n, x'_{n+1}, \dots, x'_N)
 \end{aligned} \tag{8.24}$$

To compute equation 8.24 we need to introduce a series of vectors as below. At first we construct the vector:

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} \vec{y} \\ \vec{z} \end{pmatrix}$$

then we construct the vector:

$$\vec{x}' = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x'_{n+1} \\ \vdots \\ x'_N \end{pmatrix} = \begin{pmatrix} \vec{y}' \\ \vec{z}' \end{pmatrix}$$

and finally we construct a vector:

$$\Omega = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

where A is a $n \times n$, C is a $N \times N$ and both B and B^T are $(N-n) \times (N-n)$ matrices. Then, the reduced density matrix can then be written using these matrices as:

$$\rho_{\text{red}}(\vec{z}, \vec{z}') = \int_{-\infty}^{+\infty} d\vec{y} \exp \left[-\frac{1}{2} (\vec{x} \cdot \Omega \cdot \vec{x} + \vec{x}' \cdot \Omega \cdot \vec{x}') \right]$$

where the pre-factors have been ignored and then we calculated the following:

$$\begin{aligned}\vec{x}.\Omega.\vec{x} &= \vec{y}.A.\vec{y} + \vec{y}.B.\vec{z} + \vec{z}.B^T.\vec{y} + \vec{z}.C.\vec{z} \\ \vec{x}'.\Omega.\vec{x}' &= \vec{y}.A.\vec{y} + \vec{y}.B.\vec{z}' + \vec{z}'.B^T.\vec{y} + \vec{z}'.C.\vec{z}'\end{aligned}\tag{8.25}$$

and then we can write reduced density matrix as:

$$\begin{aligned}\rho &\sim \int_{-\infty}^{+\infty} d\vec{y} \exp \left[-\vec{y}.A.\vec{y} - \vec{y}.B.\vec{z} - \vec{y}.B.\vec{z}' - \frac{1}{2}(\vec{z}.C.\vec{z} + \vec{z}'.C.\vec{z}') \right] \\ &= \exp \left[\frac{1}{4} \left(z.B^T A^{-1} B.z + z'.B^T A^{-1} B.z' + z.B^T A^{-1} B.z' + z'.B^T A^{-1} B.z - 2z.C.z - 2z'.C.z' \right) \right] \\ &= \exp \left[-\frac{1}{2} z.\gamma.z - \frac{1}{2} z'.\gamma.z' + z.\beta.z' \right] \\ &= \exp \left[-\frac{1}{2} (x.\gamma.x + x'.\gamma.x') \right] \\ &\sim \rho_{\text{red}}(x, x')\end{aligned}\tag{8.26}$$

where in the last we exchanged the dummy variable z and renamed it x and also we have omitted the arrow over the vectors. In addition, we introduced (similar to coupled oscillator) the coefficients $\gamma = C - \beta$ and $\beta = \frac{1}{2}B^T A^{-1}B$. Finally we assumed, like we did for the coupled oscillator, the eigenvector equation as:

$$\int_{-\infty}^{+\infty} d\vec{x} \rho_{\text{red}}(\vec{x}, \vec{x}') f_n(\vec{x}') = p_n f_n(\vec{x})$$

Now using the transformations below where γ_D is an orthogonal matrix

$$\begin{aligned}\gamma &= V^T \gamma_D V \\ x &= V^T \gamma_D^{-1/2} y \\ \beta' &= \gamma_D^{-1/2} V \beta V^T \gamma_D^{-1/2}\end{aligned}$$

we can rewrite the reduced density matrix as:

$$\begin{aligned}\rho_{\text{red}} &\sim \exp \left[-\frac{1}{2} (x.\gamma.x + x'.\gamma.x') + x.\beta.x' \right] \\ &= \exp \left[-\frac{1}{2} \left(V \gamma_D^{-1/2} y V^T \gamma_D V V^T \gamma_D^{-1/2} y + V \gamma_D^{-1/2} y' V^T \gamma_D V V^T \gamma_D^{-1/2} y' \right) + V \gamma_D^{-1/2} y \beta V^T \gamma_D^{-1/2} y' \right] \\ &= \exp \left[-\frac{1}{2} (y.y + y'.y') + y. \left(\gamma_D^{-1/2} V \beta V^T \gamma_D^{-1/2} \right) .y' \right] \\ &= \exp \left[-\frac{1}{2} (y.y + y'.y') + y.\beta'.y' \right]\end{aligned}\tag{8.27}$$

Now by setting $y = Wz$, where W is orthogonal and $W^t\beta W$ is diagonal we get:

$$\begin{aligned}
 \rho &\sim \exp \left[-\frac{1}{2} (W^T z . W z + W^T z' . W z') + W^T z . \beta' . W z' \right] \\
 &= \exp \left[-\frac{1}{2} (z . z + z' . z') + z . W^T \beta' W . z' \right] \\
 &= \prod_{i=1}^N \exp \left[-\frac{1}{2} (z_i^2 + z_i'^2) + \beta'_i z_i z_i' \right]
 \end{aligned} \tag{8.28}$$

Now comparing equation 8.28 with equation 8.17 we can see that for every value of i when $\beta = \beta'$ we get the entropy for a coupled oscillator. Hence we can conclude that if we now the index i from 1 to N and then if we sum over all the entropy, using the equation $S = \sum_i S(\xi_i)$, we will get the entropy for N - coupled oscillator as:

$$\xi_i = \frac{\beta'_i}{1 + \sqrt{1 - \beta_i'^2}}$$

For instance, for $i = 3$ we will have:

$$S = \xi_1 + \xi_2 + \xi_3 = \frac{\beta'_1}{1 + \sqrt{1 - \beta_1'^2}} + \frac{\beta'_2}{1 + \sqrt{1 - \beta_2'^2}} + \frac{\beta'_3}{1 + \sqrt{1 - \beta_3'^2}} \tag{8.29}$$

Chapter 9

Conclusion

The information loss problem as predicted by Hawking is based on the fact that the black hole taken into account was formed from a collapse which eventually evaporates. However, in anti deSitter space this is only true for small black holes which are not in thermal equilibrium and is therefore very difficult to address properly using the gauge/gravity duality. The conformal field theory is unitary, hence if we are to construct a duality, assuming there is a correlation, then the evaporation process should also be unitary. Unitarity should be preserved and locality or some other tenet of effective field theory should be violated. This suggests that that local effective field theory is not quite right in non-perturbative quantum gravity and this problem is not really understood and we are yet to figure out the exact method of characterizing the breakdown. This characterization can be understood to a greater extent if further research is being carried out.

In chapter 8, we talked about Entanglement entropy and its application in Holographic description of AdS space. We introduced the concept of density matrix formalism of the quantum theory and used this formalism to calculate entropy of 'toy system' such as the coupled harmonic oscillator. The purpose of this exercise was to show the scope of the idea of entanglement entropy and its possible extent. If nurtured properly, this can be used to describe the physical phenomena inside a black hole such as the black hole entropy and might someday us to solve the mystery of the information with holographic entanglement entropy. Further research will help us to predict the functions (as we have done in the chapter) more precisely and will bring new insight in mathematical physics.

In the concluding part of the thesis, we talked about the entanglement entropy which is a measure of how quantum information is spatially organized in quantum state. For quantum field theory, in general, it is extremely complicated to calculate the entanglement entropy. However, in holographic entanglement entropy becomes a topological problem and can easily be coupled with CFTs. This implies, for a strong coupling, the organization of quantum information approaches a simplified and universal form via the emergent geometry. Although much of the information about how the coupling is related to the emergent geometry is still unknown. Hence this opens door to several other theoretical researches that could be undertaken to unravel the mysterious relations

Appendix A

Symbolic calculation using *Mathematica*: Schwarzschild metric satisfies Einstein equation

The following page is a notebook file created using *Mathematica 11*. For the purpose of symbolic calculation, *xAct* and *xCoba* packages has been used. The notebook file is given to shown how the calculations has been performed. At the end of the calculation, we see that the Einstein tensor vanishes, thus ensuring the fact that the *Schwarzschild metric* is a solution to the Einstein's field equation.

<< xAct`xTensor` (*Adding the package*)

Package xAct`xPerm` version 1.2.3, {2015, 8, 23}

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Connecting to external MinGW executable...

Connection established.

Package xAct`xTensor` version 1.1.2, {2015, 8, 23}

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These packages come with ABSOLUTELY NO WARRANTY; for details type

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it under certain conditions. See the General Public License for details.

DefManifold[M4, 4, { α , β , γ , μ , ν , λ , σ , η }] (*Defining the Four Dimension Manifold*)

** DefManifold: Defining manifold M4.

** DefVBundle: Defining vbundle TangentM4.

DefMetric[-1, metric[- α , - β], CD, {";", "\n"}, PrintAs \rightarrow "g"]

**(*Defining the metric with the abstract properties of the metric,
individual component is not yet assigned*)**


```

** DefTensor: Defining symmetric metric tensor metric[- $\alpha$ , - $\beta$ ].
** DefTensor: Defining antisymmetric tensor epsilonmetric[- $\alpha$ , - $\beta$ , - $\gamma$ , - $\eta$ ].
** DefTensor: Defining tetrametric Tetrametric[- $\alpha$ , - $\beta$ , - $\gamma$ , - $\eta$ ].
** DefTensor: Defining tetrametric Tetrametric†[- $\alpha$ , - $\beta$ , - $\gamma$ , - $\eta$ ].
** DefCovD: Defining covariant derivative CD[- $\alpha$ ].
** DefTensor: Defining vanishing torsion tensor TorsionCD[ $\alpha$ , - $\beta$ , - $\gamma$ ].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelCD[ $\alpha$ , - $\beta$ , - $\gamma$ ].
** DefTensor: Defining Riemann tensor RiemannCD[- $\alpha$ , - $\beta$ , - $\gamma$ , - $\eta$ ].
** DefTensor: Defining symmetric Ricci tensor RicciCD[- $\alpha$ , - $\beta$ ].
** DefCovD: Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarCD[].
** DefCovD: Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor EinsteinCD[- $\alpha$ , - $\beta$ ].
** DefTensor: Defining Weyl tensor WeyLCD[- $\alpha$ , - $\beta$ , - $\gamma$ , - $\eta$ ].
** DefTensor: Defining symmetric TFRicci tensor TFRicciCD[- $\alpha$ , - $\beta$ ].
** DefTensor: Defining Kretschmann scalar KretschmannCD[].
** DefCovD: Computing RiemannToWeylRules for dim 4
** DefCovD: Computing RicciToTFRicci for dim 4
** DefCovD: Computing RicciToEinsteinRules for dim 4
** DefTensor: Defining weight +2 density Detmetric[]. Determinant.

```

```
<< xAct`xCoba` (*Including the package xCoba*)
```

```
-----
Package xAct`xCoba` version 0.8.3, {2015, 8, 23}
```

```
Copyright (C) 2005–2014, David Yllanes
and Jose M. Martin-Garcia, under the General Public License.
```

```
-----
These packages come with ABSOLUTELY NO WARRANTY; for details type
Disclaimer[]. This is free software, and you are welcome to redistribute
it under certain conditions. See the General Public License for details.
-----
```

```
$DefInfoQ = False;
```

```
$PrePrint = ScreenDollarIndices;
```

```
$CVSimplify = Simplify;
```

```
DefChart[cb, M4, {0, 1, 2, 3}, {t[], r[],  $\theta$ [],  $\phi$ []}]
(*Assigning the symbols of the coordinate system*)
```

```
cb /: CIndexForm[0, cb] := "t";
```

```
cb /: CIndexForm[1, cb] := "r";
```

```
cb /: CIndexForm[2, cb] := "θ";
```

```
cb /: CIndexForm[3, cb] := "φ";
```

```
DefConstantSymbol[G] (*Defining the Constant in the Schwarzschild Metric*)
```

```
DefConstantSymbol[M]
```

```
MatrixForm[met = DiagonalMatrix[{{1 -  $\frac{2M}{r[]}$ ,  $\left(1 - \frac{2M}{r[]}\right)^{-1}$ , r[]2, r[]2 Sin[θ[]]2}}]]
```

```
(*Assigning the components of the metric using the symbols in the defined chart*)
```

$$\begin{pmatrix} 1 - \frac{2M}{r} & 0 & 0 & 0 \\ 0 & \frac{1}{1 - \frac{2M}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2[\theta] \end{pmatrix}$$

```
MetricInBasis[metric, -cb, met] // TableForm
```

```
(*Using the diagonal matrix defined above defining the metric in a specific way*)
```

Added independent rule $g_{tt} \rightarrow 1 - \frac{2M}{r}$ for tensor metric

Added independent rule $g_{tr} \rightarrow 0$ for tensor metric

Added independent rule $g_{t\theta} \rightarrow 0$ for tensor metric

Added independent rule $g_{t\phi} \rightarrow 0$ for tensor metric

Added dependent rule $g_{rt} \rightarrow g_{tr}$ for tensor metric

Added independent rule $g_{rr} \rightarrow \frac{1}{1 - \frac{2M}{r}}$ for tensor metric

Added independent rule $g_{r\theta} \rightarrow 0$ for tensor metric

Added independent rule $g_{r\phi} \rightarrow 0$ for tensor metric

Added dependent rule $g_{\theta t} \rightarrow g_{t\theta}$ for tensor metric

Added dependent rule $g_{\theta r} \rightarrow g_{r\theta}$ for tensor metric

Added independent rule $g_{\theta\theta} \rightarrow r^2$ for tensor metric

Added independent rule $g_{\theta\phi} \rightarrow 0$ for tensor metric

Added dependent rule $g_{\phi t} \rightarrow g_{t\phi}$ for tensor metric

Added dependent rule $g_{\phi r} \rightarrow g_{r\phi}$ for tensor metric

Added dependent rule $g_{\phi\theta} \rightarrow g_{\theta\phi}$ for tensor metric

Added independent rule $g_{\phi\phi} \rightarrow r^2 \text{Sin}[\theta]^2$ for tensor metric

$$\begin{array}{llll}
 g_{tt} \rightarrow 1 - \frac{2M}{r} & g_{tr} \rightarrow 0 & g_{t\theta} \rightarrow 0 & g_{t\phi} \rightarrow 0 \\
 g_{rt} \rightarrow 0 & g_{rr} \rightarrow \frac{1}{1 - \frac{2M}{r}} & g_{r\theta} \rightarrow 0 & g_{r\phi} \rightarrow 0 \\
 g_{\theta t} \rightarrow 0 & g_{\theta r} \rightarrow 0 & g_{\theta\theta} \rightarrow r^2 & g_{\theta\phi} \rightarrow 0 \\
 g_{\phi t} \rightarrow 0 & g_{\phi r} \rightarrow 0 & g_{\phi\theta} \rightarrow 0 & g_{\phi\phi} \rightarrow r^2 \text{Sin}[\theta]^2
 \end{array}$$

TensorValues@metric

FoldedRule[$\{ \{ g_{rt} \rightarrow g_{tr}, g_{\theta t} \rightarrow g_{t\theta}, g_{\theta r} \rightarrow g_{r\theta}, g_{\phi t} \rightarrow g_{t\phi}, g_{\phi r} \rightarrow g_{r\phi}, g_{\phi\theta} \rightarrow g_{\theta\phi} \},$
 $\{ g_{tt} \rightarrow 1 - \frac{2M}{r}, g_{tr} \rightarrow 0, g_{t\theta} \rightarrow 0, g_{t\phi} \rightarrow 0, g_{rr} \rightarrow \frac{1}{1 - \frac{2M}{r}},$
 $g_{r\theta} \rightarrow 0, g_{r\phi} \rightarrow 0, g_{\theta\theta} \rightarrow r^2, g_{\theta\phi} \rightarrow 0, g_{\phi\phi} \rightarrow r^2 \text{Sin}[\theta]^2 \}$]

MetricCompute[metric, cb, "Weyl"[-1, -1, -1, -1]]

g = CTensor[met, {-cb, -cb}];

(*Defining a tensor that will be used later to compute the covariant derivatie*)

SetCMetric[g, -cb];

MetricCompute[metric, cb, "Weyl"[-1, -1, -1, -1]]

cd = CovDOfMetric[g] (*Defining the Covariant Derivative*)

```
CCovD[PDcb, CTensor[{{{{0, - $\frac{M}{2Mr-r^2}$ , 0, 0}, {- $\frac{M}{2Mr-r^2}$ , 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
  {{ $\frac{M(2M-r)}{r^3}$ , 0, 0, 0}, {0,  $\frac{M}{2Mr-r^2}$ , 0, 0}, {0, 0, 2M-r, 0}, {0, 0, 0, (2M-r) Sin[ $\theta$ ]^2}},
  {{0, 0, 0, 0}, {0, 0,  $\frac{1}{r}$ , 0}, {0,  $\frac{1}{r}$ , 0, 0}, {0, 0, 0, -Cos[ $\theta$ ] Sin[ $\theta$ ]}}},
  {{0, 0, 0, 0}, {0, 0, 0,  $\frac{1}{r}$ }, {0, 0, 0, Cot[ $\theta$ ]}, {0,  $\frac{1}{r}$ , Cot[ $\theta$ ], 0}}}, {cb, -cb, -cb}, 0],
CTensor[{{{1 -  $\frac{2M}{r}$ , 0, 0, 0}, {0,  $\frac{1}{1 - \frac{2M}{r}}$ , 0, 0}, {0, 0, r^2, 0}, {0, 0, 0, r^2 Sin[ $\theta$ ]^2}},
  {-cb, -cb}, 0]]
```

Christoffel[CD, PDcb][α , - β , - γ]

(*Defining the Christoffel Symbols using the Covariant Derivative*)

$\Gamma[\nabla, \mathcal{D}]^{\alpha}_{\beta\gamma}$

Part[TensorValues@ChristoffelCDPDcb, 2] // TableForm

$$\begin{aligned}
 \Gamma[\nabla, \mathcal{D}]^{t_{tt}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{t_{tr}} &\rightarrow -\frac{M}{2M r - r^2} \\
 \Gamma[\nabla, \mathcal{D}]^{t_{t\theta}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{t_{t\phi}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{t_{rr}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{t_{r\theta}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{t_{r\phi}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{t_{\theta\theta}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{t_{\theta\phi}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{t_{\phi\phi}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{r_{tt}} &\rightarrow \frac{M(2M-r)}{r^3} \\
 \Gamma[\nabla, \mathcal{D}]^{r_{tr}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{r_{t\theta}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{r_{t\phi}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{r_{rr}} &\rightarrow \frac{M}{2M r - r^2} \\
 \Gamma[\nabla, \mathcal{D}]^{r_{r\theta}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{r_{r\phi}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{r_{\theta\theta}} &\rightarrow 2M - r \\
 \Gamma[\nabla, \mathcal{D}]^{r_{\theta\phi}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{r_{\phi\phi}} &\rightarrow (2M - r) \sin[\theta]^2 \\
 \Gamma[\nabla, \mathcal{D}]^{\theta_{tt}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{\theta_{tr}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{\theta_{t\theta}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{\theta_{t\phi}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{\theta_{rr}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{\theta_{r\theta}} &\rightarrow \frac{1}{r} \\
 \Gamma[\nabla, \mathcal{D}]^{\theta_{r\phi}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{\theta_{\theta\theta}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{\theta_{\theta\phi}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{\theta_{\phi\phi}} &\rightarrow -\cos[\theta] \sin[\theta] \\
 \Gamma[\nabla, \mathcal{D}]^{\phi_{tt}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{\phi_{tr}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{\phi_{t\theta}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{\phi_{t\phi}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{\phi_{rr}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{\phi_{r\theta}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{\phi_{r\phi}} &\rightarrow \frac{1}{r} \\
 \Gamma[\nabla, \mathcal{D}]^{\phi_{\theta\theta}} &\rightarrow 0 \\
 \Gamma[\nabla, \mathcal{D}]^{\phi_{\theta\phi}} &\rightarrow \cot[\theta] \\
 \Gamma[\nabla, \mathcal{D}]^{\phi_{\phi\phi}} &\rightarrow 0
 \end{aligned}$$

riemann = Riemann[cd] (*Defining the Riemann Curvature Tensor*)

```
CTensor[{{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
  {{{0,  $\frac{2M(-2M+r)}{r^4}$ , 0, 0}, { $\frac{2M}{(2M-r)r^2}$ , 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
  {{{0, 0,  $\frac{M(2M-r)}{r^4}$ , 0}, {0, 0, 0, 0}, { $\frac{M}{r}$ , 0, 0, 0}, {0, 0, 0, 0}},
  {{{0, 0, 0,  $\frac{M(2M-r)}{r^4}$ }, {0, 0, 0, 0}, {0, 0, 0, 0}, { $\frac{M\text{Sin}[\theta]^2}{r}$ , 0, 0, 0}}},
  {{{0,  $\frac{2M(2M-r)}{r^4}$ , 0, 0}, { $-\frac{2M}{(2M-r)r^2}$ , 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
  {{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
  {{{0, 0, 0, 0}, {0, 0,  $\frac{M}{(2M-r)r^2}$ , 0}, {0,  $\frac{M}{r}$ , 0, 0}, {0, 0, 0, 0}},
  {{{0, 0, 0, 0}, {0, 0, 0,  $\frac{M}{(2M-r)r^2}$ }, {0, 0, 0, 0}, {0,  $\frac{M\text{Sin}[\theta]^2}{r}$ , 0, 0}}},
  {{{0, 0,  $\frac{M(-2M+r)}{r^4}$ , 0}, {0, 0, 0, 0}, { $-\frac{M}{r}$ , 0, 0, 0}, {0, 0, 0, 0}},
  {{{0, 0, 0, 0}, {0, 0,  $-\frac{M}{(2M-r)r^2}$ , 0}, {0,  $-\frac{M}{r}$ , 0, 0}, {0, 0, 0, 0}},
  {{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}},
  {{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0,  $\frac{2M}{r}$ }, {0, 0,  $-\frac{2M\text{Sin}[\theta]^2}{r}$ , 0}}},
  {{{0, 0, 0,  $\frac{M(-2M+r)}{r^4}$ }, {0, 0, 0, 0}, {0, 0, 0, 0}, { $-\frac{M\text{Sin}[\theta]^2}{r}$ , 0, 0, 0}},
  {{{0, 0, 0, 0}, {0, 0, 0,  $-\frac{M}{(2M-r)r^2}$ }, {0, 0, 0, 0}, {0,  $-\frac{M\text{Sin}[\theta]^2}{r}$ , 0, 0}},
  {{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0,  $-\frac{2M}{r}$ }, {0, 0,  $\frac{2M\text{Sin}[\theta]^2}{r}$ , 0}},
  {{{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}}, {-cb, -cb, -cb, cb}, 0]
```

Ricci [cd] [-α, -β] (*Defining the Ricci Tensor which is zero*)

0

rs = RicciScalar[cd][[1]] (*Defining the Ricci Scalar*)

Part: Part specification Zero[[1]] is longer than depth of object.

Part: Part specification Zero[[1]] is longer than depth of object.

Zero[[1]]

Einstein[cd] [-α, -β] (*Defining the Einstein Tensor*)

0

```
eineq[a_, b_] := Einstein[cd][{a, -cb}, {b, -cb}] // FullSimplify
(*Defining a general function to figure out the different component of the Einstein*)
eineq[0, 0] (*Specific values to show that it vanished*)
0
eineq[1, 1]
0
```

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