

**Calculating the pion contribution to nucleon self-energy
using chiral effective thermal field theory**

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for the degree of Bachelor of Science in Physics



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Declaration

I hereby declare that the thesis titled "Calculating the pion contribution to nucleon self-energy using chiral effective thermal field theory" is submitted to the Department of Mathematics and Natural Sciences of BRAC University in partial fulfilment of the requirements for the degree of Bachelors of Science in Physics. This work is a product of my own research and has not been submitted elsewhere for any other degree or diploma. Every work that has been used as reference for this work has been cited properly.

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Lastly, I dedicate this to my beloved parents who, although wished their son would pursue a different academic path, have supported me in this intellectual pursuit through all these years. Father and mother, I hope I can make you proud one day!

Abstract

Effective field theories have proven to be efficient tools for studying physics at length scales longer than that which is typical. Chiral Perturbation Theory is one of these EFTs which physicists rely on heavily to study low energy QCD processes. The primary objective of this thesis is to calculate the nucleon self-energy which arises from the one-pion exchange. Chiral symmetries and spontaneous symmetry breaking schemes are discussed along with the emergence of pions as pseudo-Nambu-Goldstone bosons. Using the nonlinear σ -model, a chiral effective pion-nucleon Lagrangian is derived, which contains the necessary information to produce the nucleon self-energy diagram to the lowest order. In this work, the nucleon self-energy is evaluated using a somewhat simplistic numerical implementation and compared with the result from another source. It is seen that $\Sigma(p)$ increases with momentum p , however, with large deviations from what should be expected due to the rudimentary computational methods used.

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Chapter 1

Introduction

1.1 The need for a quantum field theory in nuclear physics

Nowadays, an increasing amount of attention is being given to quantum field theory (QFT) by nuclear physicists. Nuclear physics aims at developing a consistent understanding of the properties of nuclei that can be extended to the domain of varying densities, temperatures and other such properties. It used to be a field of physics where non-relativistic quantum mechanics and certain many body approximations were used along with static potentials. However, this picture was not without its problems. As Yukawa predicted the theory of nuclear interaction via meson exchange in the 30's, it became apparent that at some larger length scales the previous approach would prove to be inadequate. Here, rather than working with the static potentials, the meson fields become the relevant degrees of freedom [17]. Moreover, most of the contemporary experiments, even the medium energy ones, occur at the GeV energy scales which require us to include the effects of special relativity. In order to avoid problems regarding causality, the adoption of a field theoretic approach becomes paramount. Therefore, the introduction of QFT becomes inevitable when we want to combine two of the greatest discoveries of physics- quantum mechanics and special relativity.

Moreover, in this particular work, the methods of thermal field theory (TFT) are applied in conjunction with chiral perturbation theory (ChPT). While conventional quantum field theory is formulated at zero temperature and has proved

to be sufficient for calculations involving cross-sections and scattering amplitudes in particle accelerators, there are phenomena which require the inclusion of thermal backgrounds- examples of which include the study of quark-gluon plasma, astrophysical objects with extremely high densities like neutron stars or even the beginning of the universe[18][12]. At high energies, the strong coupling constant α_s is small and QCD is perturbative, leading to asymptotic freedom. ChPT becomes an indispensable tool once we start to study the low-energy dynamics of QCD (QCD becomes non-perturbative at low energy). This fact is understood from the strong coupling constant

$$\alpha_s \approx \frac{1}{\beta_0 \ln(k^2/\Lambda^2)} \quad (1.1)$$

which depends of the momentum transfer, becoming very large at low energies. In such low-energy regimes, the hadrons and mesons, rather than the quarks, become the relevant degrees of freedom because the quarks and gluons are confined together forming bound states [1]. The non-perturbative calculations at low energies are either done using lattice QCD or ChPT. Lattice QCD is basically carried out using Monte Carlo simulations. ChPT, on the other hand, is essentially an effective field theory which is constructed out Lagrangians incorporating all the necessary symmetries and degrees of freedom necessary for such a study of non-perturbative QCD.

In this thesis, the nucleon self-energy arising from one-pion exchange is calculated using TFT and ChPT for isospin-symmetric nuclear matter. Chapters 1 and 2 serve as introductions to relevant topics in QFT and TFT necessary for this work. Chapter 3 contains the detailed calculations of the self-energy. Here, the natural units are used where $c = \hbar = k_B = 1$ and the metric signature is $\text{diag}(+, -, -, -)$. When the Euclidean metric is used in thermal calculations, $\text{diag}(+, +, +, +)$. A 4-vector is, for example, $a^\mu = (a^0, \mathbf{a})$ and wherever the Feynman slash notation is used, it implies $\not{a} = \gamma_\mu a^\mu$.

Also, the Dirac representation for the gamma matrices is used.

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}$$

$\mathbb{1}_2$ are the 2×2 identity matrix and σ^k are the Pauli matrices.

1.2 Chiral Symmetry

Symmetries are of vast importance in physics- not only because they make problems simpler to analyse, they also lead to conserved quantities called *currents* (enshrined in the famous Noether's theorem). A symmetry transformation also leaves the Lagrangian invariant ($\delta\mathcal{L} = 0$) which amounts to stating that the equations of motion remain the same. Symmetries are also fascinating because they can be broken (explicitly or spontaneously) leading to interesting phenomena. Some symmetries are only approximate to certain theories. Since ChPT is being used, chiral symmetry naturally enters into the fold. A transformation (or group of transformations) under which the left and right handed components of the Dirac field transform independently is called a chiral symmetry of the Lagrangian. Explicit calculations show that chiral symmetry is only an approximate symmetry for QCD since the finite masses of the quarks tend to break the symmetry explicitly. Nevertheless, it can be used to a very good degree of accuracy since the up and down quark masses are small compared to the typical hadronic mass scales [13].

1.2.1 Massless fermion fields

The full Lagrangian for QCD is the following

$$\mathcal{L}_{QCD} = \bar{\psi}i\mathcal{D}\psi - \bar{\psi}\mathcal{M}\psi - \frac{1}{4}\mathcal{G}_{\mu\nu}^a\mathcal{G}^{a\mu\nu} \quad (1.2)$$

where \mathcal{D} is the gauge covariant derivative, \mathcal{M} is the mass matrix and $\mathcal{G}_{\mu\nu}^a$ is the gluon field strength tensor. If we consider a quark field of only two flavours, $\psi = (u, d)^T$, the mass matrix takes the form

$$\mathcal{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \quad (1.3)$$

Chiral symmetry in the limit of vanishing quark masses can be studied, without loss of generality, once we can root out some of the terms that do not contribute to the chiral transformations, for example- terms like $\mathcal{G}_{\mu\nu}^a$. To this end, the gauge covariant derivative can also be replaced by the simple partial derivative. Then, a much simpler QCD Lagrangian can be written (for $\mathcal{M}=0$)

$$\mathcal{L}_{QCD} = i\bar{\psi}^u\partial\psi^u + i\bar{\psi}^d\partial\psi^d \quad (1.4)$$

Now, to include chiral symmetry in QCD, consider the following transformations

$$\psi_L \longrightarrow \psi'_L = e^{-i\frac{\tau}{2}\cdot\Theta_L}\psi_L \quad (1.5)$$

$$\psi_R \longrightarrow \psi'_R = e^{-i\frac{\tau}{2}\cdot\Theta_R}\psi_R \quad (1.6)$$

where $\psi_{L/R}$ are left and right-handed components of the quark field and τ are the Pauli isospin matrices. These two transformations leave \mathcal{L} invariant, forming the symmetry group $SU(2)_L \times SU(2)_R$, in the limit of vanishing quark masses and are chiral symmetries of QCD. The quark field under consideration is a doublet and each of the u and d components have their own right and left-handed components. Hence,

$$\begin{aligned} \bar{\psi}^i \not{\partial} \psi^i &= \bar{\psi}_R^i \not{\partial} \psi_R^i + \bar{\psi}_L^i \not{\partial} \psi_L^i \\ &= (\bar{u}_R \not{\partial} u_R + \bar{u}_L \not{\partial} u_L) + (\bar{d}_R \not{\partial} d_R + \bar{d}_L \not{\partial} d_L) \end{aligned}$$

It can be easily shown from calculations that \mathcal{L} remains invariant upto the linear orders of $\Theta_{R/L}$. Therefore, it is expected that a conserved current exists for the $SU(2)_R \times SU(2)_L$ symmetry group.

$$J_{R/L}^{i\mu} = \bar{\psi}_{R/L} \gamma^\mu \frac{\tau_i}{2} \psi_{R/L} \quad (1.7)$$

with the corresponding charges

$$Q_{R/L}^i = \int d^3x J_{R/L}^{i0} \quad (1.8)$$

However, a more interesting approach to study chiral symmetry is to consider linear combinations of the conserved currents. There are two possible (real) linear combinations of $J_{R/L}^{i\mu}$ and they are defined as follows-

$$V_i^\mu = J_R^{i\mu} + J_L^{i\mu} \quad (1.9)$$

$$A_i^\mu = J_R^{i\mu} - J_L^{i\mu} \quad (1.10)$$

V_i^μ and A_i^μ are called vector and axial currents respectively because of how each of them transform under parity transformations. The charges (Q_i) can also be similarly combined that can produce its own Lie algebra of $SU(2)_R \times SU(2)_L$ called chiral algebra. Computation of these linear combinations reveal-

$$V_i^\mu = \bar{\psi} \gamma^\mu \frac{\tau_i}{2} \psi \quad (1.11)$$

$$A_i^\mu = \bar{\psi} \gamma^\mu \gamma_5 \frac{\tau_i}{2} \psi \quad (1.12)$$

Moreover, these conserved currents reveal the nature of the vector and axial transformations which bring about these currents in the first place. The vector transformation is described in the following way

$$\Lambda_V : e^{-\frac{i}{2} \boldsymbol{\tau} \cdot \boldsymbol{\Theta}} \approx 1 - \frac{i}{2} \boldsymbol{\tau} \cdot \boldsymbol{\Theta} \quad (1.13)$$

In Eq. 1.13, $\boldsymbol{\tau}$ are the isospin Pauli matrices and, hence, we work in the isospin space. The fermion field can be considered to be $\psi = (u, d)^T$. To check whether the Dirac Lagrangian remains invariant under Λ_V , the field and its conjugate are acted upon by the transformation and entered into the Lagrangian. Under the action of Λ_V , \mathcal{L} becomes

$$i\bar{\psi} \not{\partial} \psi \longrightarrow i\bar{\psi} \not{\partial} \psi - i\boldsymbol{\Theta} \cdot \left(i\bar{\psi} \not{\partial} \frac{\boldsymbol{\tau}}{2} \psi - i\bar{\psi} \not{\partial} \frac{\boldsymbol{\tau}}{2} \psi \right) = i\bar{\psi} \not{\partial} \psi \quad (1.14)$$

Λ_V is indeed a symmetry of the Lagrangian, leading to the conserved vector current

$$V_\mu^a = \bar{\psi} \gamma_\mu \frac{\tau^a}{2} \psi \quad (1.15)$$

One can arrive at the result by using the expression for the conserved current in Noether's theorem. The axial transformation is constructed in a similar manner

$$\Lambda_A : \exp\left(-\frac{i}{2} \gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\Theta}\right) \approx 1 - \frac{i}{2} \gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\Theta} \quad (1.16)$$

Showing that Λ_A is a symmetry for massless fermions is done in a similar way with the additional knowledge of the anti-commutation relations of some of the gamma matrices. It is already known that $\{\gamma_0, \gamma_5\} = \{\gamma_\mu, \gamma_5\} = 0$. Then, under the action of Λ_A

$$i\bar{\psi} \not{\partial} \psi \longrightarrow i\bar{\psi} \not{\partial} \psi - i\boldsymbol{\Theta} \cdot \left(\bar{\psi} i \partial_\mu \gamma^\mu \gamma_5 \frac{\boldsymbol{\tau}}{2} \psi + \bar{\psi} \gamma_5 \frac{\boldsymbol{\tau}}{2} i \partial_\mu \gamma^\mu \psi \right) \quad (1.17)$$

By virtue of the fact that $\gamma_\mu \gamma_5 = -\gamma_5 \gamma_\mu$, \mathcal{L} remains invariant under the action of Λ_A , leading to a conserved axial current: $A_\mu^a = \bar{\psi} \gamma_\mu \gamma_5 \frac{\tau^a}{2} \psi$ [9]. Hence, chiral symmetry is an exact symmetry of QCD with vanishing quark masses. Indeed,

chiral symmetry is used to a very good approximation in low energy QCD. The typical hadronic mass scale is close to 1 GeV while the masses of the up and down quarks are 2.5 ± 0.8 MeV and 5 ± 0.9 MeV respectively [13]. Therefore, it is not a big leap to carry out calculations treating the fields massless. In the next section, however, it will be shown that the quark masses explicitly breaks this symmetry.

1.2.2 Massive fermion fields

In the case of massive fermion fields, we include an additional term to Eq. 1.4, $\delta\mathcal{L} = -m\bar{\psi}\psi$ which is referred to as the mass term. For a massive fermion field, we then have the following Lagrangian

$$\mathcal{L} = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi \quad (1.18)$$

Comparing this with Eq. 1.14 and 1.17, it is immediately evident that the first part of \mathcal{L} remains invariant under the actions of both Λ_V and Λ_A [9]. It is the mass term that breaks the symmetry explicitly. A symmetry is explicitly broken when there is an additional term in the Lagrangian that removes the invariance (as opposed to spontaneous symmetry breaking for which no such additional terms are required). While the vector transformation is a symmetry, the axial transformation fails to remain so. Thus, it is also called axial-symmetry breaking. To check this

$$m\bar{\psi}\psi \longrightarrow m\bar{\psi}\psi - 2i\Theta \cdot \left(\bar{\psi} \frac{\boldsymbol{\tau}}{2} \gamma_5 \psi \right)$$

Although axial symmetry is explicitly broken, for all intents and purposes, Λ_A is treated as an approximate symmetry leading to a partially conserved axial current (PCAC) as long as quark masses are small. Moreover, as long as the symmetry breaking is small, the theory can be described perturbatively which is the basis for ChPT.

1.3 Spontaneous Symmetry Breaking

As mentioned in the previous chapter, symmetries can be broken explicitly or spontaneously. An example of explicit symmetry breaking can be explained by the Hydrogen atom. In the absence of any external fields, the system possesses

rotational invariance leading to degenerate states and the conservation of angular momentum. But, if the atom is placed in an external magnetic field the system loses its rotational invariance and picks up a preferred axis (in the direction of the \mathbf{B} field). The action of the external \mathbf{B} field also lifts the degeneracy of the energy levels. This is the famous Zeeman effect. An analogous process occurs in the presence of an external electric field which is called the Stark effect. In a more mathematical language, in explicit breaking, $[Q^i, \mathcal{L}_1] \neq 0$ where the Q^i 's are the generators of certain transformations and \mathcal{L}_1 is the explicit breaking term.

Spontaneous symmetry breaking, on the otherhand, requires no such external fields. A very good illustration is the ferromagnetic transition. Ferromagnets possess permanent magnetism due to the alignment of magnetic dipoles in different magnetic domains. As the temperature reaches a certain critical temperature T_C , called the Curie temperature, a continuous phase transition occurs at which point the material loses its magnetism and all the magnetic dipoles become randomly oriented and the net dipole moment is effectively zero. T_C is around 1000 K for iron and above this temperature, the system is rotationally invariant. Spontaneous symmetry breaking is also of utmost importance in particle physics in the theory of weak interaction (Glashow-Salam-Weinberg model) and explains the Higgs mechanism which is the result of the spontaneous breaking of the $SU(2) \times U(1)$ gauge symmetry [7]. Simply put, if the Lagrangian is invariant under a transformation which the ground state of the system fails to observe, the symmetry is spontaneously broken.

1.3.1 Spontaneously broken discrete symmetry

Spontaneous symmetry breaking can be very easily explained by considering a simple scalar field with a quartic interaction with a Lagrangian of the form

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \mu^2 \phi^2 - \frac{1}{4} \lambda \phi^4 \quad (1.19)$$

This, of course, is a classical system. Since the Lagrangian is independent of any odd powers of ϕ , it is invariant under $\phi \rightarrow -\phi$ (parity invariant). Particularly, this symmetry is encoded by the Z_2 group and $Z^{-1} \phi(x) Z = -\phi(x)$. The ϕ^4 is a self-interaction term in the Lagrangian with λ as a coupling. While $\lambda > 0$ for all intents and purposes, there are two possible scenarios involving μ .

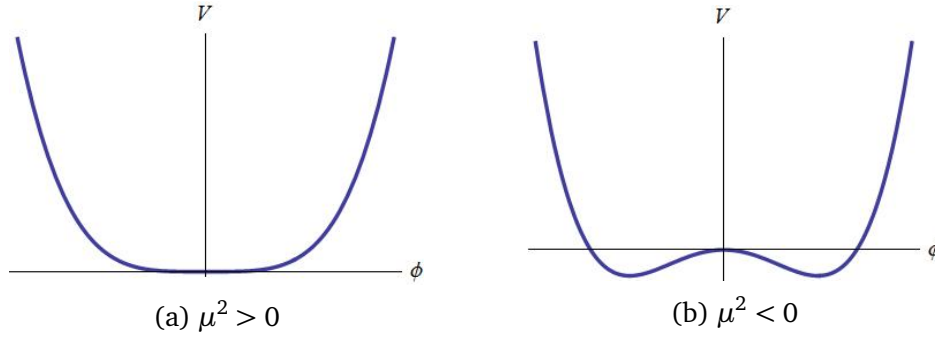


Figure 1.1: The potential $V(\phi)$ for the two possible ranges of values of μ^2

- Case 1: $\mu^2 > 0$
- Case 2: $\mu^2 < 0$

Case 1, represented by Fig. 1.1 (a), is not very interesting and of very little importance. It simply describes a self-interacting scalar field of mass μ . The vacuum state occurs at $\phi = 0$ and obeys the reflection symmetry of the Lagrangian [7]. When this theory is quantized, a unique ground state $|0\rangle$ is associated with the minimum.

Case 2, however, is very interesting. There are now two minima as shown in Fig. 1.1 (b). With $V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4$, extremization of V for $\mu^2 < 0$ gives the values of ϕ at which these minima occur.

$$\phi = \pm v$$

where,

$$v = \sqrt{-\frac{\mu^2}{\lambda}} \quad (1.20)$$

This is problematic because, where as the vacuum expectation value (VEV) for case 1 is zero, it is not true for case 2

$$\langle 0 | \phi(x) | 0 \rangle = v \neq 0 \quad (1.21)$$

This is indeed surprising since, from QFT, the VEV of all operators are supposed to be zero because of the Fourier decomposition of operators into creation and annihilation operators (this is because $\hat{a}_p |\phi\rangle = 0$) and also due to normal

ordering [7][10]. $\phi(x)$ simply cannot be treated as a quantum field for the case of $\mu^2 < 0$. The problem can be avoided by considering a quantum field $\eta(x)$ which can be expressed as a combination of creation and annihilation operators (thus a VEV equal to zero) and adding it to v which is to be treated as a quantum fluctuation. What we end up with is a newly defined quantum field

$$\phi(x) = v + \eta(x) \quad (1.22)$$

Plugging this into Eq. 1.19,

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \partial^\mu \eta \partial_\mu \eta - \frac{1}{2} (\mu^2 + 3\lambda v^2) \eta^2 - \lambda v \eta^3 - \frac{\lambda}{4} \eta^4 \\ &= \frac{1}{2} \partial^\mu \eta \partial_\mu \eta - \lambda v^2 \eta^2 - \lambda v \eta^3 - \frac{\lambda}{4} \eta^4 \end{aligned} \quad (1.23)$$

This new Lagrangian represents a field with $\mu^2 = -\lambda v^2$ and hence a mass of $\sqrt{2\lambda v^2}$. Of course, eqs. 1.19 and 1.23 represent the same system but we see that the choice of the ground state $\phi = v$ breaks the parity invariance that the system originally had (because of the presence of the η^3 term). The same is also true for the other ground state at $\phi = -v$. Therefore, the symmetry has been broken spontaneously as evident because of the lack of any other external agents.

1.3.2 Spontaneously broken $U(1)$ symmetry: The Goldstone bosons

In the previous section, the spontaneous breaking of a discrete global symmetry was discussed. What is more interesting is when a continuous global symmetry is spontaneously broken. A theory has global $U(1)$ symmetry when the transformation $\phi(x) \rightarrow e^{i\theta} \phi(x)$ doesn't change the Lagrangian. Consider the complex scalar field theory-

$$\mathcal{L} = (\partial^\mu \phi^\dagger)(\partial_\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 \quad (1.24)$$

The ϕ in eq. 1.24 is a two-component scalar field usually of the form $\phi_1 + i\phi_2$. It is clear that eq. 1.24 is invariant under the global $U(1)$ symmetry. Now, like before, we consider the case where $\mu^2 < 0$ and $V(\phi) = \mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2$ is the potential term in the Lagrangian but, unlike eq. 1.19, there is an infinite

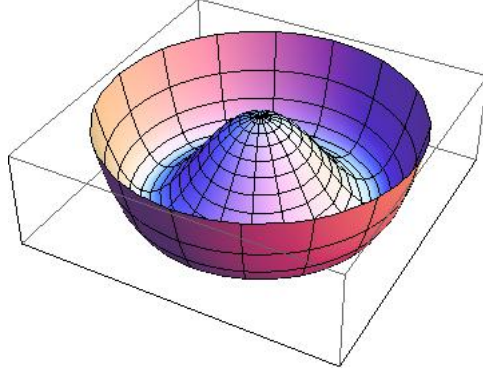


Figure 1.2: The Mexican hat potential

number of minima all lying on the circle $\phi_1^2 + \phi_2^2 = v^2$ such that

$$v = \pm \sqrt{-\frac{\mu^2}{\lambda}} \quad (1.25)$$

The same conundrum arises yet again, i.e. the VEV of the ground state is non zero and $\phi(x)$ cannot yet be considered as a quantum field. So, we add to ϕ two other fields $\eta(x)$ and $\zeta(x)$ such that

$$\langle 0 | \eta(x) | 0 \rangle = 0 \quad (1.26)$$

and,

$$\langle 0 | \zeta(x) | 0 \rangle = 0 \quad (1.27)$$

Hence,

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x) + i\zeta(x)) \quad (1.28)$$

Once the original Lagrangian is rewritten in terms of the new fields. we obtain

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \zeta)(\partial_\mu \zeta) + \frac{1}{2}(\partial^\mu \eta)(\partial_\mu \eta) + \mu^2 \eta^2 + \text{const.} + \mathcal{O}(\eta^3, \zeta^3, \eta^4, \zeta^4) \quad (1.29)$$

Eq. 1.29 reveals something extremely intriguing and yet beautiful. Other than the kinetic terms of both η and ζ fields, the ζ field appears to have no mass term. However, on examination of the potential it is easy to grasp the fact that, while excitations along the radial direction cost energy, excitations along the azimuthal one require no expenditure of energy giving rise to these

massless particles known as Nambu-Goldstone bosons (or simply Goldstone bosons) [10]. Any time a continuous global symmetry is spontaneously broken, the result manifests itself in the production of massless Goldstone bosons. This is the statement of the Goldstone theorem. To this, if we add a small explicit symmetry breaking term, the Goldstone bosons that would have emerged as massless excitations now acquire a small mass. This is exactly what happens to the chiral symmetry in ChPT: it is explicitly broken since the quarks have a finite mass and also broken spontaneously. The bosons that emerge and then called the pseudo-Goldstone bosons [13][1], which are actually the pions in this case that contribute to the nucleon self-energy. And more importantly, the pions have a small mass of their own.

1.3.3 Spontaneously broken chiral symmetry

The fact that chiral symmetry is spontaneously broken is obtained experimentally. As already seen in the previous chapters, the vector and axial currents have conserved charges associated with them which are Q_i^V and Q_i^A . Evidence that chiral symmetry is spontaneously broken comes from the hadron spectrum.

The behaviour of the vector and axial charges under parity transformation is as follows

$$Q_i^V \longrightarrow Q_i^V \quad Q_i^A \longrightarrow -Q_i^A \quad (1.30)$$

Also, for a particular eigenstate $|\psi\rangle$, $Q_i^V |\psi\rangle$ and $Q_i^A |\psi\rangle$ have the same energy but opposite parities. Hence, one could expect that for each positive parity state there exists a negative parity state of equal mass and vice versa. However, these parity doublets are not observed in the hadron spectrum. An example of this is the ρ -meson, which is a vector meson of negative parity ($J^P = 1^-$). The mass of the ρ -meson is 776 MeV. There is a vector meson, the a_1 , having $J^P = 1^+$ but having a mass of 1230 MeV and hence cannot be thought of being degenerate with the ρ -meson. There are also these light pseudoscalar ($J^P = 0^-$) mesons, (π, K, η) which are considerably lighter than the scalar ($J^P = 0^+$) counterparts [15].

The solution to this problem is provided by the Nambu-Goldstone realization of chiral symmetry. The hadron spectrum manifests isospin invariance. On the

other hand, axial symmetry is broken. As a consequence, the QCD vacuum is invariant under vector transformations, i.e., $Q_i^V |0\rangle = 0$ while $Q_i^A |0\rangle \neq 0$. The chiral $SU(2)_R \times SU(2)_L$ symmetry of QCD is spontaneously broken down to $SU(2)_V$. Section 1.3.2 mentions that a spontaneously broken global symmetry results in the formation of massless Nambu-Goldstone bosons. But since the finite quark masses already breaks chiral symmetry explicitly, we end up with the pseudo-Nambu-Goldstone bosons which have a small mass, which are the pions.

1.4 Chiral effective Lagrangians

The study of low energy QCD using EFT consists of creating the most general Lagrangian that incorporates the broken symmetries discussed in the previous chapter. Callan, Coleman, Wess and Zumino developed a theory (CCWZ formalism) of non-linear realization of these symmetries with the feature that whenever functions of the Goldstone bosons appear in the Lagrangian, they are always accompanied with atleast one space-time derivative.

In this theory the relevant degrees of freedom are the pions (pseudo-Goldstone bosons) and the nucleons. Also at zero momentum transfer and in the chiral limit ($m_\pi \rightarrow 0$), the interactions of the pions must vanish. The low energy expansion of the Lagrangian is arranged in powers of derivatives and pion masses. With the scale of chiral symmetry breaking being $\Lambda \approx 1$ GeV, the expansion is done in terms of powers of Q/Λ where Q is the momentum transfer. This is what chiral perturbation theory is all about.

The most general effective Lagrangian is written as

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\pi\pi} + \mathcal{L}_{\pi N} + \dots \quad (1.31)$$

where $\mathcal{L}_{\pi\pi}$ deals with the dynamics among the pions and $\mathcal{L}_{\pi N}$ deals with the interaction between pions and nucleons. After the first two, there are terms which involve more pions and nucleons which are unimportant. The individual components of the effective Lagrangian themselves can be further broken down depending on the number of derivatives they contain. These will be addressed in the next subsection where the concept of non-linear sigma model is discussed.

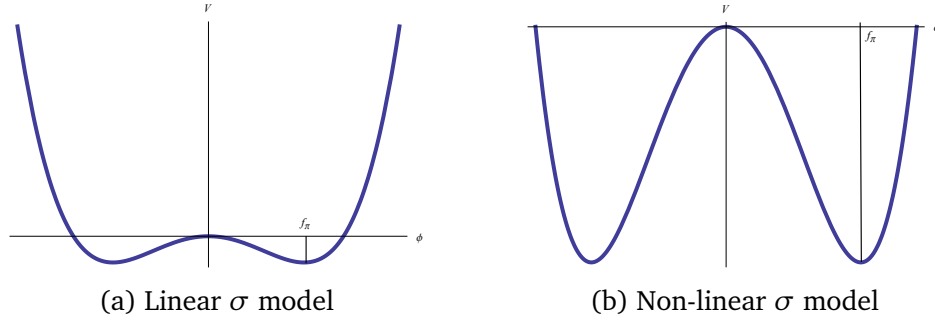


Figure 1.3: Transition from linear to non-linear

1.4.1 The non-linear sigma model

The sigma models (both the linear and non-linear) are physical models for examining spontaneous symmetry breaking which use chirally invariant Lagrangians [9]. The models were first introduced by Gell-Mann and Lévy in 1960. In the linear sigma model, a potential of the form $V(\sigma^2 + \pi^2)$ is used and a problem arises due to the presence of the additional σ field, the excitation of which is very small in the domain of low energy theories. In the non-linear realization ¹ of the theory, the σ field is integrated out- which is done by taking the mass of the field to be infinity and, thus, removing it as a dynamical variable. This process can be achieved by considering a very large coupling for the field σ [9][4]. The profile of the potential from the linear to the non-linear realization is depicted in Fig. 1.3. As a result of this, the so called "Mexican hat" potential becomes infinitely steep in the σ direction, putting a restriction on the dynamics and confining it within the chiral circle describe by the equation

$$\sigma^2 + \pi^2 = f_\pi^2 \quad (1.32)$$

This condition removes one degree of freedom and we are left with only the pionic excitations. It also allows us to express the fields in the following fashion:

$$\sigma(x) = f_\pi \cos\left(\frac{\Phi(x)}{f_\pi}\right) = f_\pi + \mathcal{O}(\Phi^2)$$

¹First published in 1960 by Murray Gell-Mann and Maurice Lévy in their paper "The Axial Vector Current in Beta Decay" [6]

$$\pi(x) = f_\pi \hat{\Phi} \sin\left(\frac{\Phi(x)}{f_\pi}\right) = \Phi(x) + \mathcal{O}(\Phi^3) \quad (1.33)$$

Now, the field $\Phi(x)$ can be identified with the pion field to the first order. This ansatz does not violate the constraint condition in eq. 1.32. There exists, however, a much more convenient and compact notation that incorporates the fields. Constructing a unitary operator $U(x) = \exp\left(i\frac{\tau \cdot \Phi}{f_\pi}\right)$. Upon expansion in terms of sines and cosines, we arrive at-

$$U(x) = \cos\left(\frac{\Phi(x)}{f_\pi}\right) + i\tau \cdot \hat{\Phi}(x) \sin\left(\frac{\Phi(x)}{f_\pi}\right) = \frac{1}{f_\pi}(\sigma + i\tau \cdot \pi) \quad (1.34)$$

The equation of constraint can be immediately recovered by computing the trace of the product of U and its Hermitian conjugate. It is very easy to show that

$$\frac{1}{2} \text{Tr}(U^\dagger U) = \frac{1}{f_\pi^2}(\sigma^2 + \pi^2) = 1 \quad (1.35)$$

For a chirally invariant \mathcal{L} , U requires a particular transformation rule, i.e. U must transform as a bidoublet such that $U \rightarrow g_L U g_R^\dagger$ where g_L and g_R are elements of $SU(2)_L$ and $SU(2)_R$ respectively [13]. Because chiral symmetry involves rotations around the chiral circle, structures of the form

$$\text{Tr}(U^\dagger U), \text{Tr}(\partial_\mu U^\dagger \partial^\mu U), \dots \quad (1.36)$$

remain invariant. The Lagrangian for the non-linear sigma model is really a modified rendition of the linear sigma Lagrangian. The modification becomes necessary since we have redefined the pion field as Φ and removed σ as a dynamical variable and parameterizing the unitary matrix U in terms of the newly defined field. Then, it becomes a matter of expressing the linear sigma model Lagrangian in terms of Φ and U . In the linear model, \mathcal{L} has five terms-nucleon, π and σ kinetic terms, a π - σ nucleon interaction term and the π - σ potential.

$$\begin{aligned} \mathcal{L}_{L.S} = & i\bar{\psi}_N \not{\partial} \psi_N + \frac{1}{2} \partial_\mu \pi \cdot \partial^\mu \pi + \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma \\ & - g_\pi (i\bar{\psi}_N \gamma_5 \boldsymbol{\tau} \psi_N \cdot \boldsymbol{\pi} + \bar{\psi}_N \psi_N \sigma) - \frac{\lambda}{4} V(\sigma^2 + \pi^2) \end{aligned} \quad (1.37)$$

1.4.2 Writing an effective Lagrangian

It is a matter of relatively simple calculations to show that the meson kinetic terms in the Lagrangian hold the following form:

$$\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi} = \frac{f_\pi^2}{4}\text{Tr}(\partial_\mu U^\dagger\partial^\mu U) \quad (1.38)$$

Turning our attention to the pion-nucleon coupling term [16] and using the fact that $\sigma = \cos\left(\frac{\Phi}{f_\pi}\right)$ and $\boldsymbol{\pi} = \hat{\boldsymbol{\Phi}}\sin\left(\frac{\Phi}{f_\pi}\right)$

$$\begin{aligned} -g_\pi(i\bar{\psi}_N\boldsymbol{\gamma}_5\boldsymbol{\tau}\psi_N\cdot\boldsymbol{\pi} + \bar{\psi}_N\psi_N\sigma) &= -g_\pi\bar{\psi}_N f_\pi \left[\cos\left(\frac{\Phi}{f_\pi}\right) + i\boldsymbol{\gamma}_5\boldsymbol{\tau}\cdot\hat{\boldsymbol{\Phi}}\sin\left(\frac{\Phi}{f_\pi}\right) \right] \psi_N \\ &= -g_\pi\bar{\psi}_N \left(f_\pi e^{i\boldsymbol{\gamma}_5\frac{\boldsymbol{\tau}\cdot\hat{\boldsymbol{\Phi}}}{f_\pi}} \right) \psi_N \\ &= -g_\pi f_\pi \bar{\psi}_N \Lambda \Lambda \psi_N \end{aligned} \quad (1.39)$$

and $\Lambda = \exp\left(i\boldsymbol{\gamma}_5\frac{\boldsymbol{\tau}\cdot\hat{\boldsymbol{\Phi}}}{f_\pi}\right)$ Now, redefining the fields in the following way:

$$\begin{aligned} \Psi &= \Lambda\psi_N \\ \bar{\Psi} &= \psi_N^\dagger\Lambda^\dagger\gamma^0 = \psi_N^\dagger\gamma^0\Lambda = \bar{\psi}_N\Lambda \end{aligned} \quad (1.40)$$

This allows us to rewrite the meson-nucleon interaction terms as

$$-g_\pi f_\pi \bar{\psi}_N \Lambda \Lambda \psi_N = -g_\pi f_\pi \bar{\Psi} \Psi = -M_N \bar{\Psi} \Psi \quad (1.41)$$

In terms of the newly defined field Ψ , this interaction terms has been reduced to the nucleon mass term. Similarly, the nucleon kinetic term can be redefined. Noting the fact that Λ is unitary-

$$\begin{aligned} \bar{\psi}_N i\not{\partial}\psi_N &= \bar{\psi}_N \Lambda \Lambda^\dagger i\not{\partial}\Lambda^\dagger \Lambda \psi_N \\ &= \bar{\Psi} \Lambda^\dagger i\not{\partial}\Lambda^\dagger \Psi \end{aligned} \quad (1.42)$$

Also, since $\{\boldsymbol{\gamma}_\mu, \boldsymbol{\gamma}_5\} = 0$,

$$\Lambda^\dagger i\not{\partial}\Lambda^\dagger = i\boldsymbol{\gamma}_\mu\Lambda\partial^\mu\Lambda^\dagger \quad (1.43)$$

Now, defining a new auxiliary field

$$\xi = e^{i\frac{\boldsymbol{\tau}\cdot\hat{\boldsymbol{\Phi}}}{2f_\pi}} \text{ such that } U = \xi\xi \quad (1.44)$$

Λ can now be expressed in terms of the new auxiliary field.

$$\begin{aligned}\Lambda &= \frac{1}{2}(\xi + \xi^\dagger) + \frac{1}{2}\gamma_5(\xi - \xi^\dagger) \\ \Lambda^\dagger &= \frac{1}{2}(\xi + \xi^\dagger) - \frac{1}{2}\gamma_5(\xi - \xi^\dagger)\end{aligned}\quad (1.45)$$

The above modifications will allow us to express eq. 1.42 in a more convenient way, using combinations of ξ to define the vector and axial currents V_μ and A_μ .

$$\bar{\Psi}\Lambda^\dagger i\not{\partial}\Lambda^\dagger\Psi = \bar{\Psi}(i\not{\partial} + \gamma^\mu V_\mu + \gamma^\mu\gamma^5 A_\mu)\Psi \quad (1.46)$$

V_μ and A_μ have been defined in the following way:

$$V_\mu = \frac{i}{2}(\xi^\dagger\partial_\mu\xi + \xi\partial_\mu\xi^\dagger) \quad \text{and} \quad A_\mu = \frac{i}{2}(\xi^\dagger\partial_\mu\xi - \xi\partial_\mu\xi^\dagger) \quad (1.47)$$

The $V(\sigma^2 + \pi^2)$ potential does not require a transformation because it vanishes on the chiral circle due to the constraint imposed. All the simplifications can be put together to form the non-linear sigma model Lagrangian often termed as the Weinberg Lagrangian. It reads-

$$\mathcal{L}_W = \bar{\Psi}(i\not{\partial} + \gamma^\mu V_\mu + \gamma^\mu\gamma^5 A_\mu - M_N)\Psi + \frac{f_\pi}{4}\text{Tr}(\partial_\mu U^\dagger\partial^\mu U) \quad (1.48)$$

This Lagrangian is non-linear on Φ . \mathcal{L}_W can be expanded for small fluctuations around the ground state.

$$\begin{aligned}\xi &\simeq \left(1 + i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\Phi}}{2f_\pi} + \frac{\boldsymbol{\Phi}^2}{8f_\pi^2}\right) \\ \xi^\dagger &\simeq \left(1 - i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\Phi}}{2f_\pi} + \frac{\boldsymbol{\Phi}^2}{8f_\pi^2}\right)\end{aligned}\quad (1.49)$$

Then, the currents become

$$\begin{aligned}V_\mu &\simeq -\frac{\boldsymbol{\tau}\cdot(\boldsymbol{\Phi}\times\partial_\mu\boldsymbol{\Phi})}{4f_\pi^2} \\ A_\mu &\simeq -\frac{\boldsymbol{\tau}\cdot\partial_\mu\boldsymbol{\Phi}}{2f_\pi}\end{aligned}\quad (1.50)$$

To the leading order, the Weinberg Lagrangian (which is also $\mathcal{L}_{\pi N}^{(1)}$) now reads

$$\begin{aligned} \mathcal{L}_{\pi N}^{(1)} \simeq & \bar{\Psi} (i\partial - M_N) \Psi + \frac{1}{2} (\partial_\mu \Phi)^2 + \frac{1}{2f_\pi} (\bar{\Psi} \gamma_\mu \gamma_5 \tau \Psi) \cdot \partial^\mu \Phi \\ & - \frac{1}{4f_\pi^2} (\bar{\Psi} \gamma_\mu \tau \Psi) \cdot (\Phi \times \partial^\mu \Phi) \end{aligned} \quad (1.51)$$

In the above Lagrangian, Φ can be identified with the pion field. Unlike the linear sigma model Lagrangian, σ field has vanished and the coupling between nucleons and pions has been changed into a pseudovector one containing the derivative of the pion field. The term proportional to $1/2f_\pi$ is the axial-vector coupling of one pion to the nucleon while the non-linear one proportional to $1/4f_\pi^2$ is known as the Weinberg-Tomozawa coupling. For the purposes of our work, the Weinberg-Tomozawa coupling term will be ignored to give the final leading order term

$$\mathcal{L}_{\pi N}^{(1)} = \bar{\Psi} (i\partial - M_N) \Psi + \frac{1}{2} (\partial_\mu \Phi)^2 + \frac{1}{2f_\pi} (\bar{\Psi} \gamma_\mu \gamma_5 \tau \Psi) \cdot \partial^\mu \Phi \quad (1.52)$$

The Lagrangian of the system can be stated as $\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{NN}$. The last term in the expression refers to the four-fermion interaction. The Lagrangian will give rise to two distinct contributions that will be discussed in a later chapter.

Chapter 2

Thermal Field Theory

Conventional quantum field theory allows us to compute amplitudes and cross sections for systems at absolute zero. In real experiments however, one has to contend with finite temperatures and, thus, a modification of quantum theories was in order. Thermal field theory has found its place in statistical physics, the study of condensed matter and even in high energy physics- in the study of systems where the assumption of zero temperature fails to hold. It has proved a remarkable tool in computing thermal expectation values and is an indispensable formalism in the study of phase transition, the physics of the early universe, degenerate astrophysical objects and heavy-ion collisions. But perhaps the most simple example is the study of thermal systems in equilibrium [18]. This chapter is aimed at reviewing some of the important tools of thermal field theory that will be used in the evaluation of self energy.

2.1 Imaginary time formalism

A statistical ensemble in equilibrium at a finite temperature of $\frac{1}{\beta}$ has a partition function of the form

$$Z = \text{Tr} \rho = \text{Tr} e^{-\beta H} \quad (2.1)$$

where ρ is the density operator and H is the Hamiltonian. In the canonical ensemble, H remains as it is but in the grand canonical case, where particle exchange is allowed, $H \rightarrow H - \mu N$. Much like quantum mechanics, expectation values are calculated in statistical mechanics, with the difference being expectation values are now ensemble averages. For any observable \mathcal{O} , the ensemble

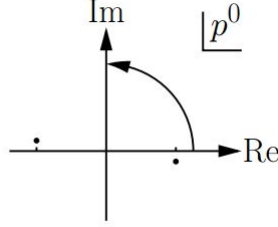


Figure 2.1: Wick rotation involves a rotation of $-\frac{\pi}{2}$ in the complex plane

average is defined as

$$\langle \mathcal{O} \rangle_\beta = \frac{1}{Z} \text{Tr} \rho \mathcal{O} \quad (2.2)$$

The cyclic permutation property of the trace creates some interesting properties when a two-point function comes into consideration. Take for example $\langle \mathcal{O}_1(t) \mathcal{O}_2(0) \rangle_\beta$. According to the cyclic property of traces,

$$\text{Tr} ABC = \text{Tr} CAB = \text{Tr} BCA$$

Then,

$$\begin{aligned} \langle \mathcal{O}_1(t) \mathcal{O}_2(0) \rangle_\beta &= \frac{1}{Z} \text{Tr} e^{-\beta H} \mathcal{O}_1(t) \mathcal{O}_2(0) \\ &= \frac{1}{Z} \text{Tr} \mathcal{O}(t) e^{-\beta H} e^{\beta H} \mathcal{O}(0) e^{-\beta H} \\ &= \frac{1}{Z} \text{Tr} \mathcal{O}(t) e^{-\beta H} e^{i(-i\beta H)} \mathcal{O}(0) e^{-i(-i\beta H)} \\ &= \langle \mathcal{O}(-i\beta) \mathcal{O}(t) \rangle_\beta \end{aligned} \quad (2.3)$$

From eq. 2.3, it is evident that the imaginary temperature is in an equal footing with time in finite temperature field theory. Now we carry out a procedure called *Wick rotation* (Fig. 2.1) through the following-

$$\tau = it \quad t = -i\tau \quad (2.4)$$

Hence, the two-point correlation function can be expressed more succinctly as

$$\langle \mathcal{O}(\tau)(0) \rangle_\beta = \langle \mathcal{O}(\beta) \mathcal{O}(\tau) \rangle_\beta \quad (2.5)$$

Eq. 2.5 is called the Kubo-Martin-Schwinger relation (or the KMS relation) which generalizes to all statistical averages and plays a crucial role in the study

of finite temperature field theory. It follows from the KMS relation that for a field ϕ

$$\phi(\mathbf{x}, 0) = \pm \phi(\mathbf{x}, \beta) \quad (2.6)$$

The \pm signifies whether the fields commute or anti-commute with each other, i.e. whether the fields are bosonic or fermionic in nature. Moreover, the KMS relation gives an indication of whether the fields are periodic or anti-periodic with β . The time dimension in thermal field theory is no longer continuous and, therefore, the fields are often represented using a Fourier expansion, summing over what are known as Matsubara frequencies [18][12].

$$\phi(\mathbf{x}, \tau) = \sum_n \phi(\mathbf{x}, \omega_n) e^{i\omega_n \tau} \quad (2.7)$$

Due to the commuting/anti-commuting constraints over the interval $[0, \beta]$

$$\begin{aligned} \omega_n &= \frac{2\pi n}{\beta} && \text{bosonic fields} \\ \omega_n &= \frac{2\pi(n+1)}{\beta} && \text{fermionic fields} \end{aligned}$$

2.2 Scalar propagators

In quantum mechanics and quantum field theory, when dealing with transitions from one state to another, we are really talking about probability amplitudes. Of utmost importance in quantum field theory, also in the path integral approach in quantum mechanics, is the propagator. The propagator gives the probability amplitude for a particle to travel from one point in spacetime to another and are different for particles of different spin. There exists a more subtle interpretation to what a propagator is in quantum field theory. There is an interacting ground state $|\Omega\rangle$ and a particle is created at the spacetime point x^μ . After interacting with the system and subsequently being annihilated at y^μ , does the system remain in the same ground state $|\Omega\rangle$? If so, the amplitude

$$D(y, x) = \langle \Omega | (\text{Particle annihilated at } y^\mu) (\text{Particle created at } x^\mu) | \Omega \rangle$$

$D(y, x)$ gives the probability amplitude that the system will remain in its ground state after the creation, interaction and subsequent annihilation of the particle

[11]. Propagator is really a fancy term for Green functions which are in fact the two-point functions mentioned in the previous chapter. We define the following propagators

$$D^+(x, y) = \langle \phi(x)\phi(y) \rangle_\beta \quad (2.8)$$

$$D^-(x, y) = \langle \phi(y)\phi(x) \rangle_\beta = D^+(x, y) \quad (2.9)$$

In their definition, the \pm simply refer to their time-retarded ($y^0 < x^0$) and time-advanced ($x^0 < y^0$) natures. Through some straight forward calculations, it can be shown that

$$D^+(x, y) = \frac{1}{Z} \sum_{n,m} e^{-\beta E_n} e^{i(p_n - p_m)(x-y)} |\langle n | \phi(0) | m \rangle|^2 \quad (2.10)$$

which reveals that $D^+(x, y)$ is a function of $(x - y)$. A close examination will prove that, considering $x^i = y^i = 0$, $D^+(x, y)$ is defined for the interval $\beta \leq \text{Im}(x^0 - y^0) \leq 0$. Moreover, its Fourier transformation turns out to be

$$D^+(k) = \int d^4x e^{ikx} D^+(x) = \frac{1}{Z} \sum_{n,m} e^{-\beta E_n} |\langle n | \phi(0) | m \rangle|^2 (2\pi)^4 \delta^{(4)}(k - p_m - p_n) \quad (2.11)$$

Since the propagators are two-point functions by nature, we can invoke the KMS relation and find that

$$D^+(t - i\beta) = D^-(t) \quad (2.12)$$

This can be verified by using merely the x^0 component of D^+ and D^- . Now, in k -space

$$D^+(k^0) = \int dt e^{ik^0 t} D^+(t) \quad (2.13)$$

$$D^-(k^0) = \int dt e^{ik^0 t} D^-(t) = \int dt e^{ik^0 t} D^+(t - i\beta) \quad (2.14)$$

Using the substitution $t \rightarrow t - i\beta$ in eq. 2.14, we can arrive at

$$D^-(k^0) = e^{-\beta k^0} D^+(k^0) \quad (2.15)$$

The two-point function may appear in various forms: imaginary time, real time, advanced or retarded. However, all of these versions have a common entity called the spectral density, $\rho(k^0)$. We define the spectral density as

$$\rho(k^0) = D^+(k^0) - D^-(k^0) = (e^{\beta k^0} - 1) D^+(k^0) \quad (2.16)$$

which leads to the following redefinitions

$$D^+(k^0) = [1 + n_B(k^0)]\rho(k^0) \quad (2.17)$$

$$D^-(k^0) = n_B(k^0)\rho(k^0) \quad (2.18)$$

where $n_B(k^0)$ is the Bose-Einstein distribution.

$$n_B(k^0) = \frac{1}{e^{\beta k^0} - 1} \quad (2.19)$$

Now, referring back to eqs. 2.8 and 2.9, where D^\pm were defined, it can be concluded that the spectral density in the time domain is merely the thermal expectation of the commutator of fields.

$$D^+(t) - D^-(t) = \langle [\phi(t), \phi(0)] \rangle_\beta \quad (2.20)$$

We can avail the use of the Fourier space propagators to take the time derivative of the eq. 2.20. Thus

$$\begin{aligned} \frac{d}{dt} (D^+(t) - D^-(t)) &= \frac{d}{dt} \int \frac{k^0}{2\pi} e^{-ik^0 t} (D^+(k^0) - D^-(k^0)) \\ &= -i \int \frac{dk^0}{2\pi} k^0 e^{-ik^0 t} \rho(k^0) \end{aligned} \quad (2.21)$$

Since the term $D^+ - D^-$ was found to be simply the thermal expectation of the field commutator, eq. 2.21 can be further simplified by noting that $\frac{d}{dt} \langle [\phi(t), \phi(0)] \rangle_\beta = -\langle [\phi(0), \pi(t)] \rangle_\beta$, making use of the fact that conjugate momentum $\pi(t)$ of the field $\phi(t)$ is simply its time derivative. The, with the knowledge that the equal time commutator of a field and its conjugate momentum is $[\phi(t), \pi(t)] = i$, in the limit of $t \rightarrow 0$, eq. 2.21 reduces to-

$$\int \frac{dk^0}{2\pi} k^0 \rho(k^0) = 1 \quad (2.22)$$

The fact that we are dealing with free scalar fields, it is possible to define a spectral density for such fields. We, again, make use of eq. 2.20 in its definition.

$$\begin{aligned} \rho(k^0) &= \frac{1}{Z} \int dt e^{ik^0 t} \sum_n e^{\beta E_n} \langle n | \phi(t)\phi(0) - \phi(0)\phi(t) | n \rangle \\ &= \frac{\pi}{E_{\mathbf{k}}} [\delta(k^0 - E_{\mathbf{k}}) - \delta(k^0 + E_{\mathbf{k}})] \\ &= 2\pi \epsilon(k^0) \delta((k^0)^2 - E_{\mathbf{k}}^2) \end{aligned} \quad (2.23)$$

Referring back to section 2.1, the imaginary time formalism was introduced in which all the calculations of thermal field theory are carried out. Hence, it is rather obvious that these propagators should behave nicely in such a framework. Indeed, the propagators used in thermal field theory employ the imaginary time formalism via the Wick rotation. It is relatively straightforward to make such a transition: one simply requires the transformation $t \rightarrow i\tau$. Everything else falls into place naturally. Then, the Matsubara propagator and its Fourier transform are defined thusly-

$$\Delta(\tau) = \frac{1}{Z} \sum_n \langle n | e^{\beta H} \phi(\tau) \phi(0) | n \rangle \quad (2.24)$$

and,

$$\Delta(i\omega_n) = \int_0^\beta dt e^{i\omega_n t} \Delta(\tau) \quad (2.25)$$

A relationship exists between the imaginary and real time propagators. This can be noted by some simple manipulations-

$$\begin{aligned} \Delta(\tau) &= D^+(t = -i\tau) \\ &= \int \frac{dk^0}{2\pi} e^{-k^0 \tau} [1 + n_B(k^0)] \rho(k^0) \end{aligned} \quad (2.26)$$

from which we can find,

$$\begin{aligned} \Delta(i\omega_n) &= \int_0^\beta d\tau e^{i\omega_n \tau} \int \frac{dk^0}{2\pi} e^{-ik^0 \tau} [1 + n_B(k^0)] \rho(k^0) \\ &= \int \frac{dk^0}{2\pi} \int_0^\beta d\tau e^{(i\omega_n - k^0)\tau} [1 + n_B(k^0)] \rho(k^0) \\ &= - \int \frac{dk^0}{2\pi} \frac{\rho(k^0)}{i\omega_n - k^0} \end{aligned} \quad (2.27)$$

We can find the form of the Matsubara propagator by substituting eq. 2.23 into the preceding equation. This gives us

$$\Delta(i\omega_n) = \frac{1}{\omega_n^2 + E_{\mathbf{k}}^2} \quad (2.28)$$

Eq. 2.28 represents the scalar propagator and, as the name suggests, governs mechanics of scalar (spin 0) particles like the pions that appear in the spontaneous chiral symmetry breaking and the non-linear sigma model. The scalar propagator in thermal field theory can be derived in a much simpler way just by working with the scalar propagator in the real time formalism. However, this chapter was aimed at a much more formal exposition. As such, there are propagators for other particles with different spins but, for the purposes of this thesis, only one other propagator will be required- that of the fermions- which will be introduced in the following segment.

2.3 Electron propagator

The steps taken to derive the electron propagator are, not surprisingly, similar to those involved in the derivation of the scalar propagator. However, there are some obvious differences: the Fermi-Dirac distribution will make an appearance, instead of the Bose-Einstein, since electrons are fermions. Moreover, the anticommuting nature of the fermionic fields will place a slight modification on the definition of the propagator.

The two-point functions for electrons are defined as follows [18][12]:

$$S_{ab}^+(x, y) = \langle \psi_a(x) \bar{\psi}_b(y) \rangle_\beta \quad (2.29)$$

$$S_{ab}^-(x, y) = -\langle \bar{\psi}_b(y) \psi_a(x) \rangle_\beta = -S_{ab}^-(y, x) \quad (2.30)$$

As for bosons, the KMS relation yields a similar relation for electrons-

$$S^+(t - i\beta) = -S^-(t) \quad (2.31)$$

Similar to eq. 2.15,

$$S_{ab}^-(p^0) = -e^{-\beta p^0} S_{ab}^+(p^0) \quad (2.32)$$

The spectral density is defined in the same way, like most of the other components in the derivation

$$\rho_{ab}(p^0) = S_{ab}^+(p^0) - S_{ab}^-(p^0) \quad (2.33)$$

With the spectral density, it is possible to express the two correlators in terms of the Fermi-Dirac distribution.

$$S_{ab}^+(p^0) = [1 - n_F(p^0)]\rho_{ab}(p^0) \quad (2.34)$$

$$S_{ab}^-(p^0) = -n_F(p^0)\rho_{ab}(p^0) \quad (2.35)$$

Also, there exists a similar expression for the free spectral density for electrons, albeit some differences. The free spectral density provides the means to express the electron propagator in terms of $\Delta(i\omega_n)$. For electrons-

$$S_{ab}(p^0) = 2\pi\epsilon(p^0)\delta\left((p^0)^2 - E_{\mathbf{p}}^2\right)(\not{p} + m)_{ab} \quad (2.36)$$

Not unlike eq. 2.27, the electron propagator until now is exactly similar to the scalar one.

$$S(\omega_n, \mathbf{p}) = - \int \frac{dp^0}{2\pi} \frac{\rho(p^0)}{i\omega_n - p^0} \quad (2.37)$$

Inserting eq. 2.36 in eq. 2.37,

$$S(i\omega_n, \mathbf{p}) = - \frac{\not{p} - m}{\omega_n^2 + E_{\mathbf{p}}^2} \quad (2.38)$$

This is the electron propagator and, apart from the numerator, it is the scalar propagator in its entirety. This propagator will serve well as the nucleon propagator, with modifications, that will finally appear in the self energy calculations.

Chapters 1 and 2 were aimed at exploring the more crucial concepts that would finally lead to the main objective of this thesis- calculating the pion contribution to the nucleon self energy. Although the subject in itself is vast, I feel the topics discussed is sufficient to serve as an introduction, making way for the calculations that are performed in the subsequent chapter.

Chapter 3

Pion contribution to the self energy

In the end section of the first chapter, the lowest order term in the chiral Lagrangian, $\mathcal{L}_{\pi N}^{(1)}$, was derived (eq. 1.51). What was not discussed, however, were the two distinct leading order contributions arising from the chiral Lagrangian (ignoring the Weinberg-Tomozawa coupling) [9][13]. The first of the terms correspond to the non-interacting system while the second one describes the four-fermion interaction [2].

Fig. 3.1 encodes all the information of the $\mathcal{L}_{\pi N}^{(1)}$ required to evaluate the one-pion contribution to the nucleon self-energy. The total Lagrangian consists of a nucleon-nucleon term, giving rise to the four-fermion interaction, which will not be considered. In evaluating the self-energy, the imaginary time formalism will be used with the propagators that were developed earlier.

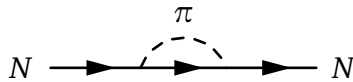


Figure 3.1: One-pion contribution to the nucleon self-energy

3.1 Self-energy as corrections to the propagator

Perturbative analysis of QFT allows us to write down the propagator as an infinite series of one-loop corrections. Such a method is analogous to the techniques used in the Born approximation used in the study of quantum mechanical scattering where an incident particle goes through an increasing number of scattering centres to give a better approximation of the underlying physical process (the Dyson equation). The electron self-energy best serves as an illustration since this remains the most prevalent example in the discussion.

As a rule, perturbation theory allows the following expansion of a generic two-point function in momentum space $G(p) = \frac{i}{p^2 - m^2}$ [5].

$$G = G_0 + G_0 \Sigma G_0 + G_0 \Sigma G_0 \Sigma G_0 + \dots \quad (3.1)$$

This is a geometric series, and a closed expression for the sum exists

$$G(p) = \frac{i}{p^2 - m^2 - \Sigma(p)} \quad (3.2)$$

The quantity Σ in the expansion is the self-energy. We see that there is a shift in the mass term in the propagator- $m^2 \rightarrow m^2 + \Sigma$ which is a quantity measured experimentally (renormalized mass).

An electron propagator $\langle \Omega | T \psi(x) \bar{\psi}(y) | \Omega \rangle$ is, diagrammatically and upto the leading order, a sum of the terms:

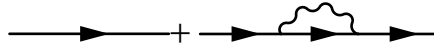


Figure 3.2: One-loop correction to electron propagator

The figure on the right represents the electron self-energy diagram that arises due to the one-loop correction of the electron propagator. Using the Feynman rules for QED, the electron self-energy can be easily calculated by

taking note of the vertices and the propagators involved. One finds that

$$-i\Sigma(p) = (-ie)^2 \int \frac{d^4}{(2\pi)^4} \gamma^\mu \frac{i(\not{k} + m)}{k^2 - m^2} \frac{i}{(p-k)^2 - \mu^2} \quad (3.3)$$

where a small photon mass μ has been added to avoid divergences. The same technique applies in the calculation of the pion contribution to nucleon self-energy, however, with changes to the vertex contributions and the types of propagators used.

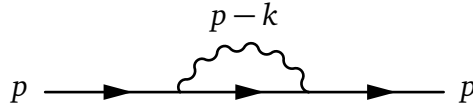


Figure 3.3: Electron self-energy

3.2 Leading order contribution

In section 2.2, the scalar propagator was derived to be

$$\Delta(i\omega_n) = \frac{1}{\omega_n^2 + E_q^2} \quad (3.4)$$

The propagator can be decomposed using the method of partial fractions into the following:

$$\Delta(i\omega_n) = \frac{1}{2E_q} \left(\frac{1}{i\omega_n - E_q} - \frac{1}{i\omega_n + E_q} \right)$$

In the imaginary time formalism, $q^\mu = (q^0, \mathbf{q}) = (i\omega_n, \mathbf{q})$. For notational convenience, $E_q^2 \rightarrow \omega_q^2 = \mathbf{q}^2 + m_\pi^2$. In this notation, the scalar propagator is

$$\Delta(q) = \frac{1}{2\omega_q} \left(\frac{1}{i\omega_n - \omega_q} - \frac{1}{i\omega_n + \omega_q} \right) \quad (3.5)$$

The nucleon propagator can be decomposed in a similar manner, albeit some modifications. With a non-zero chemical potential (μ), the Hamiltonian of the system changes from \hat{H} to $\hat{H} - \mu\hat{Q}$ [8], with \hat{Q} being a conserved charge. Also, the zeroth component of the energy-momentum four vector must be adjusted. In the presence of a non-zero chemical potential:

$$ik_n \rightarrow ik_n + \mu = \frac{(2n+1)\pi}{\beta}i + \mu \quad (3.6)$$

Thus, the nucleon propagator takes the following form:

$$S(k) = \frac{\Lambda_k^+ \gamma_0}{ik_n + \mu - E_k} + \frac{\Lambda_k^- \gamma_0}{ik_n + \mu + E_k} \quad (3.7)$$

Here $k^\mu = (ik_n, \mathbf{k})$ and Λ_k^\pm is a projection operator that projects the state into positive and negative energy ones.

$$\Lambda_k^\pm = \frac{1}{2E_k} = [E_k \pm (\boldsymbol{\alpha} \cdot \mathbf{k} + m\gamma_0)] \quad (3.8)$$

with the projection operators following the completeness relation

$$\Lambda_k^+ + \Lambda_k^- = \mathbb{1} \quad (3.9)$$

where $E_k = \mathbf{k}^2 + m^2$ and $\boldsymbol{\alpha} = \gamma_0 \boldsymbol{\gamma}$. Now, with all the available information, we can begin to write down the amplitude that is used to calculate the self-energy. From quantum field theory, it is known that amplitudes are integrated over all the four dimensions. Due to the fact that we are working with the imaginary time formalism

$$\int \frac{d^4k}{(2\pi)^4} \rightarrow \sum_{ik_n} \int \frac{d^3\mathbf{k}}{(2\pi)^3}$$

Hence, the complete expression for the nucleon self-energy reads:

$$\Sigma(p) = -\frac{3g_A^2}{4f_\pi^2\beta} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_{ik_n} \Delta(p-k) \gamma_5 (\not{p} - \not{k}) S(k) \gamma_5 (\not{p} - \not{k}) \quad (3.10)$$

The summation is carried out over the Matsubara frequencies, $ik_n = \frac{2(n+1)\pi}{\beta}$ where $n \in \mathbb{Z}$. This means that there is an infinite number of poles along the imaginary axis and the methods of calculus of residues need to be invoked to perform said summation.

3.2.1 An aside: summing over Matsubara frequencies

The purpose is to evaluate summations of the form

$$S = \frac{1}{\beta} \sum_{i\omega_n} f(i\omega_n) \quad (3.11)$$

If f has no poles located on the imaginary axis, a suitable weight function is required which has poles at the desired locations on the imaginary axis, i.e. at $i\omega_n$. Let the function be $u_\eta(z)$. The weight functions are different for bosons and fermions, owing to the different nature of their Matsubara frequencies. This allows us to express the infinite sum as a contour integral

$$S = \frac{1}{\beta} \sum_{i\omega_n} f(i\omega_n) = \frac{1}{2\pi i \beta} \oint f(z)u_\eta(z)dz \quad (3.12)$$

As shown in fig. 3.2a, the weight function generates the poles indicated by the red crosses. The contour can be further deformed as shown in fig. 3.2b to enclose only the poles of the function $f(z)$. The summation is calculated by

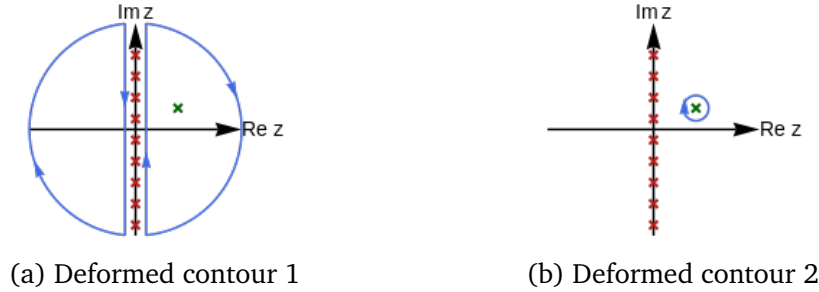


Figure 3.4: The process of deforming the contour

summing the residues of the function $f(z)u_\eta(z)$ over the poles of $f(z)$ [14]. Then,

$$S = \frac{1}{\beta} \sum_{i\omega_n} f(i\omega_n) = -\frac{1}{\beta} \sum_{z_0 \in f(z) \text{ poles}} \text{Res } f(z_0)u_\eta(z_0) \quad (3.13)$$

The choices of the weight function with which the summation can be calculated are relatively simple ones involving the n_B and n_F distributions. For bosons, such a function is

$$u_B(z) = \beta n_B(z)$$

and for fermions

$$u_F(z) = -\beta n_F(z)$$

Using the methods developed, the summation in eq. 3.10 can be carried out. After carrying out the process, it will be seen that $\Sigma(p)$ breaks up into two components- a vacuum part independent of temperature and a genuinely thermal part (dependent on β). The vacuum part will contain divergences that are usual in quantum field theory, which are dealt with using the process of renormalization. The temperature dependent part, though, will be completely free from ultraviolet divergences. Finite temperature field theory does not introduce additional ultraviolet divergences [3]. Nevertheless, for all intents and purposes, only the finite temperature part of $\Sigma(p)$ will be considered in all the calculations.

Now, the actual sequence of calculations involved in the evaluation of the summation is extremely lengthy. Hence, a small portion of the calculation will be explicitly worked out with the tacit understanding that the rest of them are carried out in a similar fashion. It can be understood that once the term $\Delta(p-k)\gamma_5(\not{p}-\not{k})S_0(k)\gamma_5(\not{p}-\not{k})$ is broken down, four other terms appear- of which all of them need to be summed over the Matsubara frequencies. Then,

$$\Delta(p-k)\gamma_5(\not{p}-\not{k}) = \frac{1}{2\omega_{pk}} \left(\frac{\gamma_5(\not{p}-\not{k})}{ip_n - ik_n - \omega_{pk}} - \frac{\gamma_5(\not{p}-\not{k})}{ip_n - ik_n + \omega_{pk}} \right)$$

$$S_0(k)\gamma_5(\not{p}-\not{k}) = \frac{\Lambda_k^+ \gamma_0 \gamma_5(\not{p}-\not{k})}{ik_n + \mu - E_k} + \frac{\Lambda_k^- \gamma_0 \gamma_5(\not{p}-\not{k})}{ik_n + \mu + E_k}$$

From the two expressions above, it can be concluded that the first term under consideration is

$$f_1(z) = \frac{\gamma_5(\not{p}-\not{k})\Lambda_k^+ \gamma_0 \gamma_5(\not{p}-\not{k})}{(ip_n - z - \omega_{pk})(z + \mu - E_k)} \quad (3.14)$$

$f_1(z)$ fails to be analytic at $z = ip_n - \omega_{pk}$ and at $E_k - \mu$. These two points are simple poles for the function $f_1(z)$. There will be two residues associated with these two simple poles, the sum of which will produce the first sum, S_1 . Using the formula derived in eq. 3.13, the first term can be summed over the ip_n 's and, since the Matsubara frequencies ip_n are fermionic, the appropriate weight function considered is $u_F(z) = -\beta n_F(z)$.

$$\text{Res}(z_0 = ip_n - \omega_{pk}) = \frac{\gamma_5(\not{p} - \not{k})\Lambda_k^+\gamma_0\gamma_5(\not{p} - \not{k})}{ip_n - \omega_{pk} + \mu - E_k} \Big|_{k^0=ip_n-\omega_{pk}} \beta(1 + n_B(\omega_{pk})) \quad (3.15)$$

and,

$$\text{Res}(z_0 = E_k - \mu) = - \frac{\gamma_5(\not{p} - \not{k})\Lambda_k^+\gamma_0\gamma_5(\not{p} - \not{k})}{ip_n - \omega_{pk} + \mu - E_k} \Big|_{k^0=E_k-\mu} \beta n_F(E_k - \mu) \quad (3.16)$$

Eq. 3.12 composed entirely of the temperature dependent part while the other contains a vacuum contribution in addition. Ultimately, summing these two residues will produce the desired result for S_1 . However, the complete sum is

$$S = \sum_{i=1}^4 S_i$$

and the other 'partial' sums are carried out in the same manner. Once all the residues have been calculated, the individual terms containing the β dependence are added to produce the following expression for the self-energy:

$$\begin{aligned} \Sigma_T(p) = & \frac{3g_A^2}{4f_\pi^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{pk}} \\ & \left[\frac{1}{ip_n + \mu - E_k - \omega_{pk}} - \frac{1}{ip_n + \mu - E_k + \omega_{pk}} \right] \gamma_5(\not{p} - \not{k})\Lambda_k^+\gamma_0\gamma_5(\not{p} - \not{k}) \Big|_{k^0=E_k-\mu} n_F(E_k - \mu) \\ & - \left[\frac{1}{ip_n + \mu + E_k - \omega_{pk}} - \frac{1}{ip_n + \mu + E_k + \omega_{pk}} \right] \gamma_5(\not{p} - \not{k})\Lambda_k^-\gamma_0\gamma_5(\not{p} - \not{k}) \Big|_{k^0=-E_k-\mu} n_F(\tilde{E}_k) \\ & - \gamma_5(\not{p} - \not{k}) \left[\frac{\Lambda_k^+\gamma_0}{ip_n + \mu - E_k - \omega_{pk}} + \frac{\Lambda_k^-\gamma_0}{p_n + \mu + E_k - \omega_{pk}} \right] \gamma_5(\not{p} - \not{k}) \Big|_{k^0=ip_n-\omega_{pk}} n_B(\omega_{pk}) \\ & - \gamma_5(\not{p} - \not{k}) \left[\frac{\Lambda_k^+\gamma_0}{ip_n + \mu - E_k + \omega_{pk}} + \frac{\Lambda_k^-\gamma_0}{ip_n + \mu + E_k + \omega_{pk}} \right] \gamma_5(\not{p} - \not{k}) \Big|_{k^0=ip_n+\omega_{pk}} n_B(\omega_{pk}) \end{aligned} \quad (3.17)$$

where $\tilde{E}_k = E_k + \mu$ and $\omega_{pk} = (\mathbf{p} - \mathbf{k})^2 + m_\pi^2$. Analytic continuation is used to obtain the retarded self-energy, i.e. via $ip_n \rightarrow p_0 + i0^+$. At sufficiently low temperatures, the occupation number of antiparticles is so low that their contributions can be neglected. Moreover, we assume that at the temperature

and density of interest, there is no macroscopic occupation of pionic modes in nuclear matter [2][8]. Then, only the first term in eq. 3.17 remains

$$\Sigma_T(P) = \frac{3g_A^2}{4f_\pi^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{pk}} \left[\frac{1}{p_0 - k_0 - \omega_{pk} + i0^+} - \frac{1}{p_0 - k_0 + \omega_{pk} + i0^+} \right] \gamma_5(\not{p} - \not{k})\Lambda_k^+\gamma_0\gamma_5(\not{p} - \not{k}) \Big|_{k_0=E_k-\mu} n_F(k_0) \quad (3.18)$$

with $k_0 = E_k - \mu$. Eq. 3.18 can be further simplified by introducing a new set of parameters- s and q^μ . One of our goals is to express $\Sigma_T(p)$ in terms of the Lorentz components:

$$\Sigma(p) = \Sigma_s(p) + \gamma_5 \Sigma_{ps}(p) + \gamma^\mu \Sigma_\mu(p) + \gamma_5 \gamma^\mu \Sigma_\mu^A(p) + \sigma^{\mu\nu} \Sigma_{\mu\nu}(p) \quad (3.19)$$

the nature of each of the components being-

- $\Sigma_s(p)$: scalar
- $\gamma_5 \Sigma(p)$: pseudo-scalar
- $\gamma^\mu \Sigma_\mu(p)$: vector
- $\gamma_5 \gamma^\mu \Sigma_\mu^A(p)$: pseudo-vector
- $\sigma^{\mu\nu} \Sigma_{\mu\nu}(p)$: tensor

In reality, $\{ \mathbb{1}, \gamma^5, \gamma^\mu, \gamma^\mu \gamma^5, \sigma^{\mu\nu} \}$ forms a basis set for any 4×4 matrix. Additionally, it is required of the self-energy to be invariant under parity, translation, rotation and time-reversal transformations. Once these symmetries are taken into account, only the scalar and vector components of the self-energy remain. Evidently, eq. 3.18 must take the following form:

$$\Sigma_T(p) = \Sigma_s(p) + \gamma^\mu \Sigma_\mu(p) \quad (3.20)$$

In order to transform eq. 3.18 into its desired form in terms of the Lorentz components, we can recognize that $\gamma_5(\not{p} - \not{k})\Lambda_k^+\gamma_0\gamma_5(\not{p} - \not{k})$ can be decomposed into a scalar and a vector term.

$$\gamma_5(\not{p} - \not{k})\Lambda_k^+\gamma_0\gamma_5(\not{p} - \not{k}) = \gamma_5(\not{p} - \not{k}) \frac{1}{2E_k} (E_k + \boldsymbol{\alpha} \cdot \mathbf{k} + m\gamma_0) \gamma_0 \gamma_5(\not{p} - \not{k})$$

Going term by term:

1.

$$\begin{aligned}\gamma_5(\not{p} - \not{k}) \frac{m\gamma_0}{2E_k} \gamma_0 \gamma_5(\not{p} - \not{k}) &= -\frac{m}{2E_k} (\not{p} - \not{k})^2 \\ &= -\frac{m}{2E_k} (p - k)^2 = s\end{aligned}$$

Here we use the fact that $(\gamma^0)^2 = (\gamma^5)^2 = \mathbb{1}$ and $\{\gamma^5, \gamma^\mu\} = 0$.

2.

$$\begin{aligned}\frac{1}{2} \gamma_5(\not{p} - \not{k}) \gamma_0 \gamma_5(\not{p} - \not{k}) &= \frac{1}{2} (p - k) \gamma_0 (\not{p} - \not{k}) \\ &= \gamma_0 \left(\frac{1}{2} (p_0 - k_0)^2 + \frac{1}{2} (\mathbf{p} - \mathbf{k})^2 \right) - \boldsymbol{\gamma} \cdot (\mathbf{p} - \mathbf{k}) (p_0 - k_0)\end{aligned}$$

3.

$$\begin{aligned}\frac{1}{2E_k} \gamma_5(\not{p} - \not{k}) \gamma_0 \boldsymbol{\gamma} \cdot \mathbf{k} \gamma_0 \gamma_5(\not{p} - \not{k}) &= -\frac{1}{2E_k} (\gamma_0 (p_0 - k_0) - \boldsymbol{\gamma} \cdot (\mathbf{p} - \mathbf{k})) \boldsymbol{\gamma} \cdot \mathbf{k} \\ &\quad \times (\gamma_0 (p_0 - k_0) - \boldsymbol{\gamma} \cdot (\mathbf{p} - \mathbf{k})) \\ &= \frac{1}{2E_k} (p - k)^2 \boldsymbol{\gamma} \cdot \mathbf{k} - \frac{1}{E_k} \gamma_0 (p_0 - k_0) \mathbf{k} \cdot (\mathbf{p} - \mathbf{k}) \\ &\quad - \frac{1}{E_k} (\mathbf{p} - \mathbf{k}) \cdot \mathbf{k} \boldsymbol{\gamma} \cdot (\mathbf{p} - \mathbf{k})\end{aligned}$$

Introducing a new variable $q^\mu = (q_0, \mathbf{q})$, where

$$q_0 = \frac{1}{2} (p_0 - k_0)^2 + \frac{1}{2} (\mathbf{p} - \mathbf{k})^2 - \frac{1}{E_k} (p_0 - k_0) \mathbf{k} \cdot (\mathbf{p} - \mathbf{k}) \quad (3.21)$$

$$\mathbf{q} = -\frac{1}{2E_k} \left[(p - k)^2 \mathbf{k} + 2(\mathbf{p} - \mathbf{k}) \cdot \mathbf{k} (\mathbf{p} - \mathbf{k}) \right] + (p_0 - k_0) (\mathbf{p} - \mathbf{k}) \quad (3.22)$$

As expected, the expression for self-energy has been decomposed into a scalar and vector components so that the following can be written

$$\Sigma_T(p) = \frac{3g_A^2}{4f_\pi^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{pk}} \left[\frac{1}{p_0 - k_0 - \omega_{pk} + i0^+} - \frac{1}{p_0 - k_0 + \omega_{pk} + i0^+} \right] (s + \boldsymbol{q}) n_F(k_0) \quad (3.23)$$

Because of the introduction of the new parameters, $\Sigma_T(p)$ can be projected onto the required bases.

$$\begin{aligned}\Sigma_s(p) &= -\frac{3g_A^2}{4f_\pi^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{pk}} \left[\frac{1}{p_0 - k_0 - \omega_{pk} + i0^+} - \frac{1}{p_0 - k_0 + \omega_{pk} + i0^+} \right] \\ &\quad \times n_F(k_0) \frac{m}{2E_k} (p - k)^2 \\ \Sigma_0(p) &= \frac{3g_A^2}{4f_\pi^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{pk}} \left[\frac{1}{p_0 - k_0 - \omega_{pk} + i0^+} - \frac{1}{p_0 - k_0 + \omega_{pk} + i0^+} \right] n_F(k_0) \\ &\quad \times \left[\frac{1}{2}(p_0 - k_0)^2 + \frac{1}{2}(\mathbf{p} - \mathbf{k})^2 - \frac{1}{E_k}(p_0 - k_0)\mathbf{k} \cdot (\mathbf{p} - \mathbf{k}) \right] \quad (3.24)\end{aligned}$$

$$\begin{aligned}|\mathbf{p}|\Sigma_v(p) &= \frac{3g_A^2}{4f_\pi^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{pk}} \left[\frac{1}{p_0 - k_0 - \omega_{pk} + i0^+} - \frac{1}{p_0 - k_0 + \omega_{pk} + i0^+} \right] n_F(k_0) \\ &\quad \times \left\{ \frac{1}{2E_k} \left[(p - k)^2 \mathbf{k} + 2(\mathbf{p} - \mathbf{k}) \cdot \mathbf{k}(\mathbf{p} - \mathbf{k}) \right] + (p_0 - k_0)(\mathbf{p} - \mathbf{k}) \right\} \cdot \hat{\mathbf{p}} \quad (3.25)\end{aligned}$$

Eq. 3.23 - 3.25 give the final expressions for the one-pion exchange contribution to the nucleon self-energy. As mentioned earlier, certain symmetry requirements allowed for a reduction of the number of terms involved in the calculation. However, for other processes or conditions the excluded terms can be computed in a similar way if they play contributing roles.

3.3 Computational simplifications

The simplifications done here would be the most logical step if one tries to evaluate numerically the one-pion contribution to the nucleon self-energy. Nevertheless, it is rather interesting to investigate the processes involved without delving into the number crunching. Although the introduction of the term $s + \not{q}$ in the self-energy went a long way to simplify the earlier, more tedious expression, the factor can be subjected to further simplifications and approximations. We start with the following

$$\begin{aligned}2E_k \not{q} &= 2E_k (q_0 \gamma_0 - \mathbf{q} \cdot \boldsymbol{\gamma}) = \mu \left[(p_0 - k_0)^2 + (\mathbf{p} - \mathbf{k})^2 \right] \gamma_0 \\ &\quad + 2(p - k) \cdot k \not{p} - (p^2 - k^2) \not{k} - 2\mu(p_0 - k_0)(\mathbf{p} - \mathbf{k}) \cdot \boldsymbol{\gamma} \quad (3.26)\end{aligned}$$

With the added constraint of low temperature and the particular density of interest, the dynamics of the nucleons will be contained within the Fermi surface. The nucleons will possess a momentum equal to the Fermi momentum p_F . The total momentum vector, however, will contain a residual term

$$\mathbf{p} = p_F \hat{\mathbf{n}} + \delta \mathbf{p} \quad (3.27)$$

with $|\delta \mathbf{p}| \ll p_F$. Additionally, we introduce a relativity parameter $x = \frac{p_F}{m}$ which depends on the number densities.

$$x \approx 0.28 \left(\frac{n}{n_0} \right)^{\frac{1}{3}} \quad (3.28)$$

Usually, $n_0 = 0.16 \text{ fm}^{-3}$. Therefore, one can arrive at $E_p \approx m(1 + \mathbf{p}^2/2m^2)$ and $\mu \approx m(1 + p_F^2/2m^2)$. From these two approximations-

$$p_0 \approx \frac{\mathbf{p}^2 - p_F^2}{2m} \approx x \hat{\mathbf{n}} \cdot \delta \mathbf{p} \quad (3.29)$$

From eq. 3.27, it can be inferred that p_0 is much smaller than \mathbf{p} . Hence, $(p - k)^2 \approx -(\mathbf{p} - \mathbf{k})^2$. With these approximations:

$$2E_k q \approx (\mathbf{p} - \mathbf{k})^2 [(p_0 + \mu)\gamma_0 - \mathbf{p} \cdot \boldsymbol{\gamma}] \quad (3.30)$$

These approximations will allow for a significant reduction of calculation complexity. Once eq. 3.30 has been substituted into the expression $s + q$ -

$$s + q \approx \frac{(\mathbf{p} - \mathbf{k})^2}{2m} [m + (p_0 + \mu)\gamma_0 - \boldsymbol{\gamma} \cdot \mathbf{p}] \quad (3.31)$$

Nevertheless, said approximations, in no way, changes the structure of the self-energy. As shown before, the self-energy decomposes into a scalar and vector parts. This form holds true even after the approximation, yet the expression has been hugely simplified and an analytical evaluation can be carried out.

3.4 Calculation results

Continuing from section 3.3, we can proceed to further simplify the expression for $\Sigma(p)$. From eq. 3.23, it is evident that $\Sigma(p)$ consists of an integral times

the factor $(s + \not{q})n_F(k_0)$. With the help of eq. 3.31,

$$\begin{aligned}\Sigma(p) &\approx [m + (p_0 + \mu) - \boldsymbol{\gamma} \cdot \mathbf{p}][\sigma_+(p) + \sigma_-(p)] \\ &= [m + (p_0 + \mu) - \boldsymbol{\gamma} \cdot \mathbf{p}]\sigma(p)\end{aligned}\quad (3.32)$$

where,

$$\sigma_{\pm}(p) = \pm \frac{3g_A^2}{8mf_\pi^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{pk}} \frac{(\mathbf{p} - \mathbf{k})^2}{p_0 - k_0 \mp \omega_{pk} + i\eta} n_F(k_0) \quad (3.33)$$

is the reduced self-energy. In terms of this newly defined quantity, the Lorentz components of $\Sigma(p)$ are

$$\begin{aligned}\Sigma_s(p) &\approx m\sigma(p) \\ \Sigma_0(p) &\approx (p_0 + \mu)\sigma(p) \\ \Sigma_v(p) &\approx -\sigma(p)\end{aligned}\quad (3.34)$$

3.4.1 Evaluating $\sigma(p)$

$$\sigma_+(p) = \frac{3g_A^2}{8mf_\pi^2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{2\omega_{pk}} \frac{(\mathbf{p} - \mathbf{k})^2}{p_0 - k_0 - \omega_{pk} + i\eta} n_F(k_0)$$

$$d^3\mathbf{k} = -2\pi\mathbf{k}^2 d|\mathbf{k}| dx \quad x = \cos\theta$$

Then,

$$\sigma_+(p) = -\frac{3g_A^2}{32m\pi^2 f_\pi^2} \int_0^\infty \frac{d|\mathbf{k}|\mathbf{k}^2}{2\omega_{pk}} n_F(k_0) \int_{-1}^1 \frac{(\mathbf{p} - \mathbf{k}^2) dx}{\omega_{pk} - (p_0 - k_0) - i\eta}$$

Since, $\omega_{pk}^2 = \mathbf{p}^2 + \mathbf{k}^2 - 2|\mathbf{p}||\mathbf{k}|x + m_\pi^2$

$$\frac{dx}{\omega_{pk}} = -\frac{d\omega_{pk}}{|\mathbf{p}||\mathbf{k}|}$$

$$\sigma_+(p) = \frac{3g_A^2}{32m\pi^2 f_\pi^2} \int_0^\infty \frac{d|\mathbf{k}|\mathbf{k}^2}{2|\mathbf{p}||\mathbf{k}|} n_F(k_0) \int \frac{d\omega_{pk}(\mathbf{p} - \mathbf{k}^2)}{\omega_{pk} - (p_0 - k_0) - i\eta}$$

Using Sokhotski-Plemelj theorem, which states that:

$$\lim_{\epsilon \rightarrow 0^+} \int_a^b \frac{f(x)}{x \pm i\epsilon} dx = \mp i\pi f(0) + \mathcal{P} \int_a^b \frac{f(x)}{x} dx$$

$$\lim_{\eta \rightarrow 0^+} \int \frac{d\omega_{pk}(\mathbf{p}-\mathbf{k}^2)}{\omega_{pk} - (p_0 - k_0) - i\eta} = i\pi (\mathbf{p}-\mathbf{k}^2) \Big|_{\omega_{pk}=p_0-k_0} + \mathcal{P} \int \frac{d\omega_{pk}(\mathbf{p}-\mathbf{k}^2)}{\omega_{pk} - (p_0 - k_0)}$$

$$\begin{aligned} \text{Re}[\sigma_+(p)] &= \frac{3g_A^2}{32m\pi^2 f_\pi^2} \int_0^\infty \frac{d|\mathbf{k}|\mathbf{k}^2}{2|\mathbf{p}||\mathbf{k}|} n_F(k_0) \mathcal{P} \int \frac{d\omega_{pk}(\mathbf{p}-\mathbf{k}^2)}{\omega_{pk} - (p_0 - k_0)} \\ &= -\frac{3g_A^2}{32m\pi^2 f_\pi^2} \int_0^\infty \frac{d|\mathbf{k}|\mathbf{k}^2}{2|\mathbf{p}||\mathbf{k}|} n_F(k_0) \mathcal{P} \int_{-1}^1 \frac{dx |\mathbf{p}||\mathbf{k}| (\mathbf{p}-\mathbf{k})^2}{\omega_{pk}^2 - \omega_{pk}(p_0 - k_0)} \end{aligned}$$

$$\begin{aligned} \omega_{pk}^2 - \omega_{pk}(p_0 - k_0) &= \mathbf{p}^2 + \mathbf{k}^2 - 2|\mathbf{p}||\mathbf{k}|x + m_\pi^2 - \omega_{pk}(p_0 - k_0) \\ &= -2|\mathbf{p}||\mathbf{k}| \left[x - \frac{1}{2|\mathbf{p}||\mathbf{k}|} (\mathbf{p}^2 + \mathbf{k}^2 + m_\pi^2 - (p_0 - k_0)^2) \right] \\ &= -2|\mathbf{p}||\mathbf{k}| (x - x_0) \end{aligned}$$

$$\text{Re}[\sigma_+(p)] = \frac{3g_A^2}{32m\pi^2 f_\pi^2} \int_0^\infty \frac{d|\mathbf{k}|\mathbf{k}^2}{2|\mathbf{p}||\mathbf{k}|} n_F(k_0) \mathcal{P} \frac{1}{2} \int_{-1}^1 \frac{dx (\mathbf{p}-\mathbf{k})^2}{x - x_0}$$

Hence,

$$\begin{aligned} \text{Re}[\sigma(p)] &= \text{Re}[\sigma_+(p) + \sigma_-(p)] \\ &= \frac{3g_A^2}{32m\pi^2 f_\pi^2} \int_0^\infty \frac{d|\mathbf{k}|\mathbf{k}^2}{2|\mathbf{p}||\mathbf{k}|} n_F(k_0) \mathcal{P} \int_{-1}^1 \frac{dx (\mathbf{p}-\mathbf{k})^2}{x - x_0} \end{aligned} \quad (3.35)$$

Eq. 3.35 is the full extent to which analytic calculations can be performed on the self-energy. Any subsequent results need to be obtained through numerical implementations.

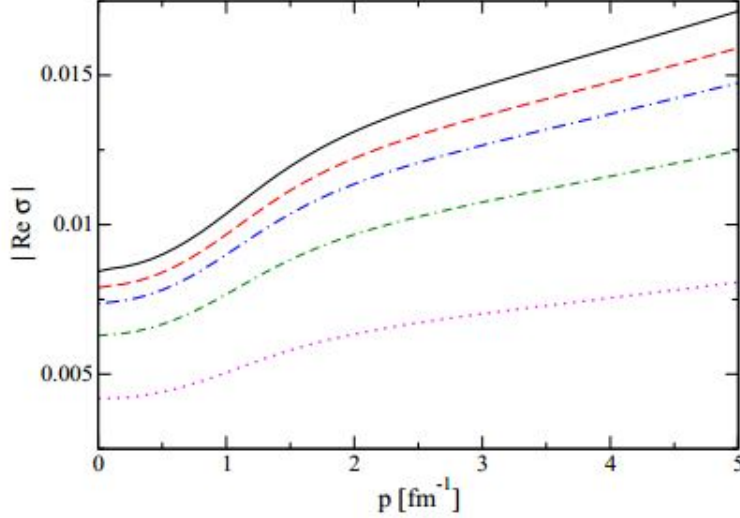


Figure 3.5: The result for $|\text{Re } \sigma(p)|$ vs p , at $T = 0$, as obtained in [2]

Fig. 3.5 shows the actual dependence of $|\text{Re } \sigma(p)|$ on the momentum transfer. This result is obtained through solving eq. 3.35 iteratively until convergence is obtained. To achieve the correct result through this process the Schwinger-Dyson equation

$$S^{-1}(p) = S_0^{-1}(p) - \Sigma(p) \quad (3.36)$$

is used to determine the excitation spectrum $p_0^* = E_p^* - \mu^*$ by solving $\det S^{-1}$. The starred quantities are the renormalized equivalents. In practice, the process is first carried out using the free variables to obtain the first iteration (pink dotted line in fig. 3.5a). The process is repeated, using the renormalized quantities, until convergence is reached (black solid line). For small densities, the number of iterations required to achieve convergence is small. But for higher densities, the number of iterations needed would be larger in comparison. [2].

However, the plot obtained for the purposes of this thesis (fig. 3.6) is a very rudimentary one- one done without the use of the computational stages mentioned above. As a consequence, the result obtained has been far from accurate. It is manifest from the comparison of the two graphs that the actual variation of $|\text{Re } \sigma(p)|$ was not obtained from the basic numerical implements

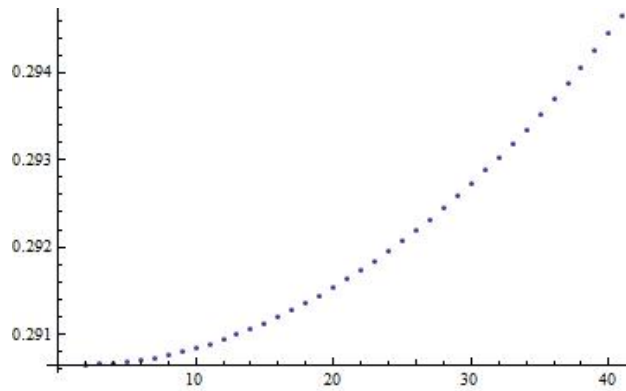


Figure 3.6: The result for $|\text{Re } \sigma(p)|$ vs p , at $T = 0$, obtained without implementing the numerical methods of [2].

used. Although, for both graphs, the reduced self-energy increases with increase in momentum transfer, the nature of the increase is vastly different. Also, fig. 3.5b shows the reduced self-energy to be zero at zero momentum transfer which is a major discrepancy.

Chapter 4

Conclusion

In this work, as a literature review of [2], the one-pion exchange contribution to the nucleon self-energy has been calculated. The exposition of the thesis has been mostly analytical. Starting off with chiral symmetries, we have been able to write the effective Lagrangian $\mathcal{L}_{\pi N}^{(1)}$ which, in turn, generated the leading order Feynman diagram needed for the calculation of $\Sigma(p)$. Using the computational conventions used in thermal field theory (chapter 2), we have been able to show that $\Sigma(p)$ decomposes into vacuum and finite temperature parts: the latter of which further breaks down into scalar and vector components. Further approximations lead to a reduced self-energy $\sigma(p)$. The $|\text{Re } \sigma| - p$ plot generated shows a source of a great degree of variation once it has been compared with the results in the reference material.

This, however, needs to be subject to more improvement. As a starting point, better codes need to be written in order to be on equal footing with the results obtained in [2]. Thus, the behaviour of nuclear systems where $T > 0$ can be studied.

Extension of this work can be brought about by studying the ρ -meson contribution to the nucleon self-energy. The calculations can be performed in a similar manner. Another interesting point of investigation can be directed at conditions where the Lorentz components self-energy that were discarded become valid and contribute in the process. Examples of such cases that are of current research interest are astrophysical objects like pulsars which have very strong magnetic fields ($\sim 100 \text{ G}$). Symmetry considerations, in particular time-reversal symmetry, were able to vastly reduce the complexity of the cal-

culations that were performed. However, presence of external fields (strong \mathbf{B} fields) would mean that time-reversal is no longer a symmetry of the system under consideration, introducing additional components of $\Sigma(p)$.

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