STUDY ON CHOHOMOLOGY OF SUPERMANIFOLDS

Khondokar M. Ahmed
Department of Mathematics
University of Dhaka
Dhaka -1000, Bangladesh

ABSTRACT

In this present paper de Rham cohomology of graded manifolds, cohomology of graded differential forms and cohomology of DeWitt supermanifolds are studied. The structure of a G-supermanifold in general is not acyclic. So, the cohomology is defined via the complex of graded differential forms. This situation is investigated together with the graded Dolbeault cohomology of complex G-supermanifolds. A theorem is established that the structure sheaf of any DeWitt G-supermanifold is acyclic.

Key words: Manifolds, chomology, supermanifolds.

I. INTRODUCTION

The aim of this paper is to unfold a basic cohomology theory for supermanifolds. This cohomology does not embody only trivial extensions of results valid for differentiable manifolds. For instance the natural analogue of the de Rham theorem does not holds in general and also in the case of complex super-manifolds there is generally no analogue of the Dolbeault theorem. As a result the sturecture sheaf of a supermanifolds does not need to be cohomology trivial [2]. The fact that the cohomology of the complex of global graded differential forms on a G-supermanifold (M,A) depends on the G-supermanifold structure (M,A) so that super de Rham cohomology is a fine invariant of the supermanifold structure [3].

II. de Rham COHOMOLOGY OF GRADED MANIFOLDS

Graded manifolds are not very interesting as far as their cohomology is concerned. In the real case, the structure sheaf of a graded manifold (X,A) is fine and therefore A and all sheaves Ω^k A of graded differential forms are acyclic. This implies that the cohomology of the complex $\Omega^\bullet$ A (X) coincides with the de Rham cohomology of X. In the complex analytic case, a similar argument allows one to prove a Dolbeault-type theorem.

The complex of sheaves A$^\bullet$ is exact and moreover, it is a resolution of the constant sheaf Ñ on X; i.e., the sequence of sheaves of Ñ-modules

$$0 \to \tilde{N} \to A \xrightarrow{d} \Omega^1 A \xrightarrow{d}$$
$$\Omega^2 A \xrightarrow{d} \cdots \qquad (2.1)$$

is exact. This 'graded Poincaré Lemma' is most easily proved by working in local coordinates and proceeding on the analogy of the usual Poincaré Lemma. By defining the de Rham cohomology of (X,A), denoted by $H^\bullet_{DR}(X, A)$, as the cohomology of the complex of graded vector spaces A$^\bullet(X)$, from (2.1) and using the ordinary de Rham theorem, we obtain the following result (cf. [8] Theorem 4.7.).

Proposition 2.1. *There is a canonical isomorphism* $H^k_{DR}(X, A) ' H^k_{DR}(X)$ *for all* $k \geq 0$.

Here $H^k_{DR}(X)$ denotes the usual de Rham cohomology of X.

III. COHOMOLOGY OF GRADED DIFFERENTIAL FORMS

Let (M,A) be a G-supermanifold. The sheaves $\Omega^k \otimes_{\tilde{N}} B_L$ of smooth B_L-valued differential forms on M provide a resolution of the constant sheaf B_L on M, in the sense that the differential complex of sheaves of graded-commutative B_L-algebras $\Omega^\bullet \otimes_{\tilde{N}} B_L$ (with $\Omega^0 \otimes_{\tilde{N}} B_L ' C^\infty_L$) is a resolution of the constant sheaf B_L, i.e. the sequence

$$0 \to B_L \to C^\infty{}_L \xrightarrow{\ d\ } \Omega^1 \otimes_{\tilde{N}} B_L \xrightarrow{\ d\ }$$

$$\Omega^2 \otimes_{\tilde{N}} B_L \to \ \cdots \qquad (3.1)$$

is exact. The cohomology associated with this complex via the global section functor $\Gamma(\cdot, M)$, i.e. the cohomology of the complex $\Omega^\bullet \otimes_{\tilde{N}} B_L$, is denoted by $H^\bullet{}_{DR}(M, B_L)$ and is called the B_L-valued de Rham cohomology of M. Since B_L is a finite dimensional real vector space, the universal coefficient theorem [6] entails the isomorphism

$$H^\bullet{}_{DR}(M, B_L) \simeq H^\bullet{}_{DR}(M) \otimes_{\tilde{N}} B_L. \qquad (3.2)$$

By virtue of the de Rham theorem, equation (3.2) can be equivalently written as

$$H^\bullet{}_{DR}(M, B_L) \simeq H^\bullet(M, B_L). \qquad (3.3)$$

By $H^\bullet(M, \cdot)$ we designate interchangeable the 1ech or sheaf cohomology functor, which coincide since the base space is paracompact.

In order to gain information not on the topological or smooth structure of M, but rather on its G-supermanifold structure, we therefore need to define a new cohomology, obtained via a resolution of B_L, different from the differential complex (3.1). We consider the sheaves $\Omega^k{}_A$ of graded differential forms. The following result is a generalization of the usual Poincaré lemma (cf. [5]).

Proposition 3.1. *Given a G supermanifold (M, A), the differential complex of sheaves of graded B_L-algebras on M*

$$0 \to B_L \to A \xrightarrow{\ d\ } \Omega^1{}_A \xrightarrow{\ d\ }$$

$$\Omega^2{}_A \to \ \cdots \qquad (3.4)$$

is a resolution of B_L.

Proof. Since the claim to be proved is a local matter, we may assume that $(M, A) = (B_L^{m,n}, G)$; moreover, it is enough to show that, if U is an open ball around the origin in $B_L^{m,n}$ then any closed graded differential k-form $\lambda \in \Omega^k{}_G(U)$ is exact; i.e., there exists a graded differential $(k-1)$-form $\eta \in \Omega^{k-1}{}_G(U)$ such that $\lambda = d\eta$. Given coordinates $(z^1, \cdots, z^{m+n})$ in U, let

$$\omega = dz^{A_k} \wedge \cdots \wedge dz^{A_1} \omega_{A_1 \cdots A_k} \in \Omega^k{}_{H^\infty}(U) \quad \text{be}$$

an H^∞ graded differential k-form on U ($k > 0$); let us set

$$\hat{K}\omega(z) = (-1)^k dz^{A_{k-1}} \wedge \cdots \wedge$$

$$dz^{A_1} z^B \int_0^t t^{k-1} \omega_{BA_1 \cdots A_{k-1}}(tz) dt.$$

We have an isomorphism [2] $\Omega^k{}_G(U) \simeq \Omega^k{}_{H^\infty} \otimes_{\tilde{N}} B_L$; it is therefore possible to introduce a homotopy operator

$$K : \Omega^k{}_G(U) \to \Omega^{k-1}{}_G(U), \text{ defined by}$$

$$K(\omega \otimes a) = \hat{K}\omega \otimes a.$$

One can indeed verify easily that $dK\lambda + Kd\lambda = \lambda$ for any section $\lambda \in G^k(U)$, so that, if $d\lambda = 0$ then $\lambda = d(K\lambda)$. The case $k = 0$ has been left out. However, if $f \in G(U)$, by writing f as $f = \sum_i f \otimes a_i$ with $f_i \in H^\infty(U)$ and $a_i \in B_L$, the condition $df = 0$ implies directly that f is a constant in B_L.

Definition 3.2. *Given a G-supermanifold (M, A), the cohomology of the complex*

$$A(M) \xrightarrow{\ d\ } \Omega^1{}_A(M) \xrightarrow{\ d\ }$$

$$\Omega^2{}_A(M) \to \cdots \qquad (3.5)$$

denoted by $H^\bullet{}_{SDR}(M, A)$, is called the super de Rham cohomology of (M, A).

The operation of taking the SDR cohomology of a G-supermanifold is functorial. Indeed, given a G-morphism $(f, \phi) : (M, A) \to (N, B)$, it is easily proved that the morphism $\Omega^\bullet{}_B(N) \to f_*\Omega^\bullet{}_A(M)$ induced by ϕ commutes with the exterior differential and therefore yields a morphism of graded B_L-modules $\phi^\sharp : H^\bullet{}_{SDR}(N, B) \to H^\bullet{}_{SDR}(M, A)$. It should be noticed that the functor $H^\bullet{}_{SDR}(\cdot)$ does not fulfill the Eilenberg-Steenrod [10] axiomatics for cohomology (if it did, it would coincide with the B_L-valued de Rham cohomology functor) since it does not satisfy the excision axiom. Moreover, the functor $H^\bullet{}_{SDR}(\cdot)$ does not give rise to

topological invariants. On the other hand, it is easily varified that the graded B_L-modules

$H^k{}_{SDR}(M,\mathsf{A})$ are invariants associated with the G-supermanifold structure of M. Indeed, if $(f,\phi):(M,\mathsf{A})\to(N,\mathsf{B})$ is a G-isomorphism, it is easily proved that $\phi^\sharp:H^\bullet{}_{SDR}(N,\mathsf{B})\to H^\bullet{}_{SDR}(M,\mathsf{A})$. is an isomorphism.

The most natural thing to do to gain insight into the geometric significance of the groups $H^k{}_{SDR}(M,\mathsf{A})$ - which, as a matter of fact, are graded B_L-modules - is to compare them with the cohomology groups $H^k(M,B_L)$, which have a natural structure of graded B_L-modules as well. The morphisms $\Omega^k{}_\mathsf{A}(M)\to\Omega^k$

$(M)\otimes_{\tilde{\mathsf{N}}}B_L$ induced by the morphism $\delta:\mathsf{A}\to\mathsf{C}^\infty{}_L$ give rise to a morphism of differential complexes, which induces in cohomology a morphism of graded B_L-modules

$\cdot^k:H^k{}_{SDR}(M,\mathsf{A})\to$

$H^k{}_{DR}(M,B_L)\quad\forall k\geq 0.$ \hfill (3.6)

In degree zero, $\cdot^0$ is an isomorphism, in that one has manifestly

$H^0{}_{SDR}(M,\mathsf{A})'(B_L)^C{}'H^0{}_{DR}(M,B_L),$

where C is the number of connected components of M, which we assume to be finite. In degree higher than zero, we have, as a straightforward application of the abstract de Rham theorem, the following result.

Proposition 3.3. *Let (M,A) be a G-supermanifold and fix an integer $q\geq 1$. If $H^k(M,\Omega^p{}_\mathsf{A})=0$ for $0\leq p\leq q-1$ and $1\leq k\leq q$, there are isomorphisms*

$H^k{}_{SDR}(M,\mathsf{A})'H^k(M,B_L)$

for $1\leq k\leq q$.

From equation (3.3), still working under the hypotheses of Proposition 3.3., we obtain isomorphisms

$H^k{}_{SDR}(M,\mathsf{A})'H^k(M,B_L)$

for $0\leq k\leq q.$.

Proposition 3.3. provides a usefull tool for investigating the cohomological properties of the structure sheaf of a G- supermanifold. For instance, it suffices to exhibit a G-supermanifold (M,A) such that $H^1{}_{SDR}(M,\mathsf{A})\neq H^1{}_{DR}(M,B_L)$ to deduce that, in general, the sheaf A can not be expected to be acyclic.

IV. COHOMOLOGY OF DEWITT SUPERMANIFOLDS

Considering in M the fine topology we study the cohomology of a De Witt supermanifold (M,A); this is advantageous because in this way M is paracompact. Thus we continue to confuse the sheaf and 1ech cohomologies with coefficients in sheaves on M.

We need the following Lemma, which is obtained from a result given in [19] by strengthening certain hypotheses.

Lemma 4.1. *Let X and Y be topological spaces, with Y locally euclidean and F a sheaf of abelian groups on X; let we assume that all groups $H^k(X,\mathsf{F})$ are finitely generated. Then for all $n\geq 0$ there is an exact sequence of abelian groups*

$0\to\bigoplus_{j+k=n}H^j(X,\mathsf{F})\otimes_{\grave{\mathsf{U}}}H^k(Y,\grave{\mathsf{U}})\to$

$H^n(X\times Y,\pi^{-1}\mathsf{F})\to$

$\to\bigoplus_{j+k=n+1}Tor[H^j(X,\mathsf{F}),H^k(X,\grave{\mathsf{U}})]\to 0$

where $Tor[\cdot,\cdot]$ defines the torsion product [6], [7] and $\pi:X\times Y\to X$ is the canonical projection.

Proposition 4.2. *The G-supermanifold $(B_L{}^{m,n},\mathsf{G})$ is cohomologically trivial:*

$H^k(B_L{}^{m,n},\mathsf{G})=0\qquad\forall k>0$ \hfill (4.1)

Proof. In view of the definitions of the sheaves GH and G, one has an isomorphism

$\mathsf{G}'(\sigma^{m,n})^{-1}(\mathsf{C}^\infty{}_{\tilde{\mathsf{N}}}{}^m\otimes\wedge\tilde{\mathsf{N}}^n\otimes B_L).$

Therefore, applying Lemma 4.1. with the following identifications:

$X=\tilde{\mathsf{N}}^m,\quad Y=\mathbf{n}^{m,n}{}_L,$

$\mathsf{F}=\mathsf{C}^\infty{}_{\tilde{\mathsf{N}}}{}^m\otimes\wedge\tilde{\mathsf{N}}^n\otimes B_L,$

We obtain (since $H^k(\mathbf{n}^{m,n}{}_L, \grave{\mathbf{U}}) = 0$ for $k > 0$ and $H^0(\mathbf{n}^{m,n}{}_L, \grave{\mathbf{U}}) = \grave{\mathbf{U}}$)

$H^k(B_L{}^{m,n}, \mathsf{G})'$

$H^k(\tilde{\mathsf{N}}^m, \mathsf{C}^\infty{}_{\tilde{\mathsf{N}}}{}^m \otimes \wedge \tilde{\mathsf{N}}^n \otimes B_L)$.

Now, since the sheaf of rings $\mathsf{C}^\infty{}_{\tilde{\mathsf{N}}}{}^m$ is fine, the sheaf $\mathsf{C}^\infty{}_{\tilde{\mathsf{N}}}{}^m \otimes \wedge \tilde{\mathsf{N}}^n \otimes B_L$ of $\mathsf{C}^\infty{}_{\tilde{\mathsf{N}}}{}^m$-modules is soft and therefore is acyclic, which yields the sought result. $\square$

Coarse partitions of unity. DeWitt supermanifolds do not admit partitions of unity in a strict sense, that is to say, there cannot exist partitions of unity subordinated to any locally finite cover, since the structure sheaf of a DeWitt supermanifold is not soft and therefore is not even fine. However, any DeWitt supermanifold has a particular kind of partition of unity, that we call a coarse partition of unity.

Lemma 4.3. *Let* (M,A) *be a DeWitt supermanifold, with body* M_B *and projection* $\phi : M \to M_B$. *For any locally finite coarse cover* $\mathsf{U} = \{U_j\}$ *of* M *there exists a family* $\{g_j\}$ *of global sections of* A *such that*

(a) *Supp* $g_j \subset U_j$;

(b) $\sum_j g_j = 1$.

Corollary 4.4. *Let* C *be a locally finite coarse cover of M. Then* $\check{H}(\mathsf{U},\mathsf{F}) = 0$, $k > 0$, *for any sheaf* F *of* A*-modules.*

If we consider in M the coarse topology (let us denote the resulting space by M_{DW}, the sheaf A is apparently fine; however, this does not allow us to conclude that the sheaf cohomology of A is trivial, since M_{DW} is not paracompact. In any case, one can conclude that the 1ech cohomology $\check{H}^\bullet(M_{DW}, \mathsf{A})$ (or the cohomology $\check{H}^\bullet(M_{DW}, \mathsf{F})$, where F is any A-module) is trivial, since the direct limit over the covers involved in the definition of the 1ech cohomology can be taken on coarse covers.

Theorem 4.5. *The structure sheaf* A *of a DeWitt G-supermanifold* (M,A) *is acyclic.*

Proof. Any $p \in M_B$ has a system of neighbourhoods $\mathbf{m}$ such that for all $W \in \mathbf{m}$ the supermanifold $(\Phi^{-1}(W), \mathsf{A}_{|\Phi^{-1}(W)})$ is isomorphic to $(B_L{}^{m,n}, \mathsf{G})$; therefore, $\mathsf{A}_{|\Phi^{-1}(W)}$ is acyclic. We are then in the hypotheses of the inverse image in cohomology [2] and hence

$H^k(M, \mathsf{A})' H^k(M_B, \Phi_* \mathsf{A})$, $k \geq 0$.

is fine by Lemma 4.3., and hence acyclic, so that we achieve the thesis. $\square$

We notice that the same procedure that brought to Theorem 4.5. can be applied to the structure sheaves of an H^∞ or GH^∞ DeWitt supermanifolds, which are therefore acyclic as well.

Corollary 4.6. *Any locally free* A *module* F *is acyclic. Proof.* Let us at first assume that F trivializes on a coarse cover. Then, since $\Phi_* \mathsf{F}$ is a $\Phi_* \mathsf{A}$ -module, the same proof of the previous Proposition applies. Now we must prove that F actually trivializes on a coarse cover. Without any loss of generality we may assume that $(M,\mathsf{A}) = (B_L{}^{m,n}, \mathsf{G})$ and that F trivializes on subsets of $B_L{}^{m,n}$ which are diffeomorphic to open balls. Let U be one of these subsets; then $\mathsf{F}(U)'\mathsf{G}^{p/q}(U)$. In view of the definition of the sheaf G, if V is any other set of this kind such that $\Phi^{-1}\Phi(U) = \Phi^{-1}\Phi(V) = W$, then $\mathsf{F}(U)'$ $\mathsf{F}(V)$, so that one has $\mathsf{F}_{|W} = \mathsf{G}^{p|q}{}_{|W}$. $\square$

For instance, the sheaf of derivations $\mathsf{D}er\mathsf{A}$ and sheaves $\Omega^k\mathsf{A}$ of graded differential forms on (M,A) are acyclic.

SDR cohomology of DeWitt super- manifolds. The previous results have an immediate consequence in connection with the super de Rham cohomology of DeWitt supermanifolds.

Proposition 4.7. [9] *The super de Rham cohomology of a DeWitt supermanifold* (M,A) *is isomorphic with the* B_L *-valued de Rham cohomology of the body manifold* M_B:

$H^\bullet_{SDR}(M) \simeq H^\bullet_{DR}(M_B) \otimes_{\tilde{N}} B_L$. (4.2)

Proof. We have already seen that the sheaves of graded differential forms Ω^k A are acyclic, $H^k(M, \Omega^p$ A$)=0$ for all $k > 0$ and $p \geq 0$. Accordingly, Proposition 1.2. [9] implies

$H^\bullet_{SDR}(M) \simeq H^\bullet(M, B_B)$. (4.3)

On the other hand, M is a fibration over M_B with a contractible fibre, so that $H^\bullet_{DR}(M) \simeq H^\bullet_{DR}(M_B)$ and equation (4.3) is equivlent to equation (4.2) [2]. □

Dolbeault theorem. Let (M,B) be an (m, n)-dimensional complex G-super- manifold. We have that Ω^p B is the sheaf of holomorphic graded p-forms on(M,B), while $\Omega^{p,q}$ I is the sheaf of graded differential forms of type (p, q). Here I is the complexification of the sheaf A, i.e. I $=$A$\otimes_{\tilde{N}}$Â.

Proposition 4.8. *Let* (M,B) *be a complex DeWitt G-supermanifold. There are iso- morphisms of graded* C_L *-modules*

$H^{p,q}_{\bar{\partial}}(M, \text{B}) \simeq H^q(M, \Omega^p$ B).

Cohomology of G^∞ **DeWitt super- manifolds.** Theorem 4.5., which states the acyclicity of the structure sheaf of a DeWitt G-supermanifold can be shown to hold true also in the case of the sheaf A$^\infty$ of G^∞ functions on a DeWitt super- manifold.

Theorem 4.9. *The structure sheaf of a* G^∞ *DeWitt supermanifold is acyclic.*

Proof. Working as in Lemma 4.3., one can construct a coarse G^∞ partition of unity on M, so that the sheaf $\Phi_* A^\infty$ is fine and therefore is acyclic. Let us now consider for a while the G^∞ DeWitt super- manifold $(B_L^{m,n}, \text{G}^\infty)$. Lemma 4.1. implies

$H^k(B_L^{m,n}, (\sigma^{m,n})^{-1}(\sigma^{m,n})_* \text{G}^\infty) \simeq$
$H^k(\tilde{N}^m(\sigma^{m,n})_* \text{G}^\infty) = 0$

for all $k > 0$. Since $(\sigma^{m,n})^{-1}(\sigma^{m,n})_*$ G$^\infty$

$\simeq$ G$^\infty$ by the very definition of the sheaf G$^\infty$, the result is proved for the super- manifold $(B_L^{m,n},$ G$^\infty)$. From (cf. [1]

Proposition 4.11.) the result for a generic G^∞ DeWitt supermanifold now follows. □

REFERENCES

[1] K. M. Ahmed: A note on sheaves and cohomology on manifold, Ganit: *J. Bangladesh Math. Soc.*, **22**, pp 41-50. (2002)

[2] C. Bartocci and U. Bruzzo: Cohomology of supermanifolds, *J. Math. phys.* **28**, pp 2363-2368. (1987)

[3] C. Bartocci and U. Bruzzo: Cohomology of the structure sheaf of real and complex supermanifolds, *J. Math. phys* **29**, pp 1789-1795. (1988)

[4] G.E. Bredon: *sheaf theory,* New York: McGraw-Hill. (1967)

[5] U. Bruzzo: Supermanifolds, super-manifold cohomology, and super vector bundles, in : *Differential geometric methods in theoretical physics,* K. Bleuler and M. Werner eds., *NATO ASI Ser. C* Kluwer, Dordrecht, **250**, pp 417-440. (1988)

[6] R. Godement: *The´orie des faisceaux,* Hermann, Paris. (1964)

[7] P. J. Hilton and U. Stammbach: *A course in homological algebra,* New York: Springer-Verlag. (1971)

[8] B. Kostant: Graded manifolds, graded Lie theory, and prequantization, in: *Differential geometric methods in mathematical physics*, K. Bleuler and A. Reetz eds., *Lecture Notes Math.* Springer-Verlag, Berlin, **570**, pp 177-306. (1977)

[9] J. M. Rabin: Supermanifold cohomology and the Wess-Zumino term of the covariant superstring action, *Commun. Math. Phys.* **108**, pp 375-389. (1987)

[10] E. H. Spanier: *Algebraic topology,* New York: McGraw-Hill. (1966)