BRAC University Journal, Vol. II, No. 2, 2005, pp. 57-64

A NEW APPROACH ON THE TORSION AND CURVATURE TENSORS IN SUPERMANIFOLDS

Md. Showkat Ali
Department of Mathematics, University of Dhaka
Dhaka-1000, Bangladesh

ABSTRACT

In this paper geometric space, super space, supermanifold, superconnection, the torsion and the curvature tensors are studied and some new notions of local forms of the torsion and curvature tensors are established, for deriving certain constraints in classical case called the super Bianchi identities.

1. INTRODUCTION

Ringed spaces give a way to describe essential features of geometry, expressed in the language of sheaf theory [7]. Using such methods, an alternative but equivalent definition of a manifold can be given and similar ideas will be used to define the notion of a supermanifold. Throughout this paper all rings are assumed to have an identity, and R will denote a commutative ring. Further, we assume that for any ring morphism $\phi : R \to S$ where S is a ring, $\phi(1_R) = 1_S$.

Definition 1.1 A *ringed space* over R is a pair (R, O_X) where X is a topological space and O_X is a sheaf of R-algebras over X.

Often we say X is a ringed space and call O_X the structure sheaf. It can be shown that the direct limit of a system of R-algebras is again an R-algebra. Hence the stalk $O_{X,\ x}$ [8] is an R-algebra $\forall x \in X$.

Example 1. Let X be any topological space and set O_X =Ù$_X$, the constant sheaf of integers. Then (X, O_X) is a ringed space over Ù$_X$.

Since the conception of supersymmetry, supergeometry has come to play an increasingly important role in theoretical physics [4] and is an essential part of almost every attempt to go beyond the standard model in particle physics [5]. The aim of this paper is to give a definition of a supermanifold and some associated basic theory on the torsion and curvature tensors in supermanifold for deriving certain constraints in classical case which is the super Bianchi identities. We use the definition similar to that given in [1], an equivalent definition is given in [3]. Throughout, we denote Ù/2Ù by Ù$_2$ = $\{\tilde{0}, \tilde{1}\}$.

2. GEOMETRIC SPACES AND MANIFOLDS

Definition 2.1 A subring $L \le R$ is called *local* if it has a unique maximal ideal.

Definition 2.2 Let $L \le R$ and $M \le S$ be local. A morphism $\phi : L \to M$ is called a local morphism of local rings if $\phi(L) \subset M$.

Definition 2.3 A ringed space over R is called *geometric space* over R if all the stalks $O_{X,\ x}$ are local rings.

Definition 2.4 Let (M, O_M) be a geometric space over a ring R which, we refer to as a model space. Another geometric space (X, O_X) over R is locally isomorphic to (M, O_M) if $\forall x \in X$ $\exists$ open $U \subseteq X$ with $x \in U$ and an open $V \subseteq M$ such that

$$(U, O_X|_U) \cong (V, O_M|_V),$$

as geometric spaces.

Definition 2.5 Let M be a class of model spaces and let X be a second countable Hausdorff space. A geometric space (X, O_X) is a *manifold* of type M over R if X has an open covering U such that, $\forall U \in$ U (U, $O_X|_U$) is locally isomorphic to (M, O_M) for some (M, O_M) $\in$ M.

Definition 2.6 A morphism of manifolds is a morphism of the underlying geometric spaces.

Example 2. Let $n \in$ Í and set M = {(Ñ^n, C^∞)}, where C^∞ is the sheaf of smooth functions over Ñ^n. Substituting this into the definition above 2.5, we obtain the notion of smooth real manifold of dimension n. Similarly, for $n \in$ Í and M = {(C^n , holomorphic functions)}, we obtain the notion of smooth complex manifolds of dimension n.

3. Ù$_2$- GRADED GROUPS AND RINGS

Definition 3.1 An Abelian group A is called ***Ù$_2$*** *-graded* if it is the direct sum of two subgroups $A = A_{\tilde{0}} \oplus A_{\tilde{1}}$. We call elements of $A_{\tilde{0}} \oplus A_{\tilde{1}}$ homogeneous, elements of $A_{\tilde{0}}$ even and elements of $A_{\tilde{1}}$ odd.

Definition 3.2 The parity function ~ : $A_{\tilde{0}} \cup A_{\tilde{1}} \rightarrow$ Ù$_2$ is defined by

$$x \mapsto \tilde{x} = \begin{cases} \tilde{0}, x \in A_{\tilde{0}} \\ \tilde{1}, x \in A_{\tilde{1}} \end{cases}$$

If A, B are **Ù$_2$**-graded Abelian groups, then the Abelian group *Hom(A,B)* is the direct sum,

$$Hom\ (A,B) = Hom\ (A_{\tilde{0}}, B_{\tilde{0}}) \oplus Hom\ (A_{\tilde{0}}, B_{\tilde{1}}) \oplus \text{Hom}\ (A_{\tilde{1}}, B_{\tilde{0}}) \oplus Hom\ (A_{\tilde{1}}, B_{\tilde{1}})$$

which we rewrite in the following way,

$$Hom\ (A,B) = Hom\ (A,B)_{\tilde{0}} \oplus Hom\ (A,B)_{\tilde{1}}$$

where *Hom* $(A,B)_{\tilde{0}} =$ *Hom* $(A_{\tilde{0}}, B_{\tilde{0}}) \oplus$ *Hom* $(A_{\tilde{1}}, B_{\tilde{1}})$ even homomorphisms, and *Hom* $(A,B)_{\tilde{1}} =$ *Hom* $(A_{\tilde{0}}, B_{\tilde{1}}) \oplus$ Hom $(A_{\tilde{1}}, B_{\tilde{0}})$ odd homomorphisms. Thus *Hom(A,B)* is naturally **Ù$_2$**-graded. For $\phi \in$ *Hom* $(A,B)_{\tilde{0}}$ and for $\tilde{\phi}(x) = \tilde{x} + \tilde{1}$.

Definition 3.3 A ring R is called **Ù$_2$ -**graded if,

i) The underlying Abelian group is **Ù$_2$**-graded.
ii) The set of homogeneous elements is closed under multiplication.
iii) The parity function satisfies $\widetilde{ab} = \tilde{a} + \tilde{b}\ \ \forall a,b \in R_{\tilde{0}} \cup R_{\tilde{1}}$.

4. SUPERSPACES AND SUPERMANIFOLDS

Definition 4.1 An R-superalgebra is a Ù$_2$-graded ring S together with an even ring morphism $\alpha : R \rightarrow Z(S)$. When S is supercommutative we say that S is a supercommutative R-superalgebra.

Definition 4.2 A ringed superspace over R is a pair (X, O$_X$) where X is a topological space and O$_X$ is a sheaf of R-superalgebras. A morphism of ringed superspaces over R is a pair (ϕ, ψ) where $\phi : X \rightarrow Y$ is a continuous mapping and ψ : O$_Y$ $\rightarrow$ ϕ_* O$_X$ is even.

Definition 4.3 Let M be a class of model superspaces and let X be a second countable Hausdorff space. A superspace (X, O$_X$) is a *supermanifold* of type M over R if X has an open covering W such that,

$\forall W \in$ W (W, O$_X|_W$) is locally isomorphic to (M, O$_M$),

for some (M, O$_M$) $\in$ M.

Example 3. If we set M = {(Ñ^p, O$_{p\ q}$)} = Ñ$^{p|q}$, for some p, q $\in$ Í , we obtain the notion of a smooth real supermanifold of dimension p|q. In particular Ñ$^{p|q}$ is a smooth supermanifold.

5. DIFFERENTIATION IN SUPERDOMAIN

As with algebra, many of the principles of calculus can be carried over to the Ù$_2$-graded case.

Definition 5.1 Let R denote a superalgebra. A linear operator $D \in End(R)$ is called a *derivation* of R if $\forall r_1, r_2 \in R$ the following holds,

$$D(r_1 r_2) = D(r_1) r_2 + (-1)^{\tilde{r}_1 \tilde{D}} r_1 D(r_2)$$

and we denote the set of all derivations of R by *Der(R).*

Let U be a superdomain in Ñ$^{m|n}$ and $(x, \eta) = (x_1, ..., x_m, \eta_1, ..., \eta_n)$ be a local coordinate system.

Definition 5.2 We define even partial derivatives

$$\frac{\partial}{\partial x_i} : C^\infty(\text{U}) \rightarrow C^\infty(\text{U})$$

such that

$$\frac{\partial}{\partial x_i}(f_{\nu_1...\nu_n}\eta_1^{\nu_1}...\eta_n^{\nu_n}) = \frac{\partial f_{\nu_1...\nu_n}}{\partial x_i}\eta_1^{\nu_1}...\eta_n^{\nu_n}$$

where $f_{\nu_1...\nu_n} \in C^\infty(\mathrm{U})$.

Definition 5.3 We define odd partial derivatives

$$\frac{\partial}{\partial x_i} : C^\infty(\mathrm{U}) \to C^\infty(\mathrm{U})$$

such that

$$\frac{\partial}{\partial \eta_i}(f_{\nu_1...\nu_n}\eta_1^{\nu_1}...\eta_2^{\nu_n}) = \nu_i(-1)^{\nu_1+...+\nu_{i-1}} f_{\nu_1...\nu_n}\eta_1^{\nu_1}...\eta_{i-1}^{\nu_{i-1}}\eta_{i+1}^{\nu_{i+1}}\eta_n^{\nu_n}$$

where $f_{\nu_1...\nu_n} \in C^\infty(\mathrm{U})$.

Proposition 1. Let $\frac{\partial}{\partial u}$ be an arbitrary homogeneous partial derivative. Then $\frac{\partial}{\partial u}$ satisfies the super-Leibnitz rule, $\frac{\partial}{\partial u}(fg) = \frac{\partial f}{\partial u}g + (-1)^{\tilde{\frac{\partial}{\partial u}}\tilde{f}} f\frac{\partial g}{\partial u}$ for $f, g \in C^\infty(\mathrm{U})_{\tilde{1}}$.

Definition 5.4 Let M denote a (p|q)-dimensional supermanifold. It can be shown [5] that Der(O_M) is a locally free sheaf of rank (p|q). We call this sheaf the tangent sheaf of M and denote it by TM. Also, in a superdomain U with local coordinates $(x_1,...,x_p,\eta_1,...,\eta_q)$ the sheaf TM is generated by its sections $(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_p},...,\frac{\partial}{\partial \eta_1},...,\frac{\partial}{\partial \eta_q})$. Further we define the tangent bundle of M to be the set of all sections of the sheaf TM..

Definition 5.5 Let M denote a (p|q)-dimensional supermanifold. We define the cotangent sheaf of M to be the sheaf of even morphisms TM* = *Hom* O_M (TM, O_M) and define the cotangent bundle denoted by Ω M to be the set of sections of TM*.

6. SUPERCONNECTIONS

As in the classical case, we now introduce a way to differentiate vectors and higher order objects in a suitable way. This leads to the following definition.

Definition 6.1 Let M be a supermanifold. A superconnection on M is an even map,

$$\nabla : \mathrm{TM} \otimes \mathrm{TM} \to \mathrm{TM}$$

satisfying the following conditions for $X_1, X_2, Y_1, Y_2 \in \mathrm{TM}$ and $f \in O_M$,

(i) TM linearity in the first variable,
$\nabla((X_1 + X_2) \otimes Y_1) = \nabla(X_1 \otimes Y_1) + \nabla(X_2 \otimes Y_1)$

(ii) TM linearity in the second variable,
$\nabla(X_1 \otimes (Y_1 \otimes Y_2)) = \nabla(X_1 \otimes Y_1) + \nabla(X_1 \otimes Y_2)$

(iii) O_M linearity in the first variable,
$$\nabla(fX_1 \otimes Y_1) = f\nabla(X_1 \otimes Y_1)$$

(iv) *and O_M satisfies the super-Leibnitz rule in the second variable,*
$$\nabla(X_1 \otimes fY_1) = X(f)Y + (-1)^{\tilde{f}\tilde{X}_1}\nabla(X_1 \otimes Y_1).$$

As in the classical case ∇ is not a tensor, however, as we shall now see, important tensors may be constructed from it in sections 7 and 8.

7. THE TORSION TENSOR

Definition 7.1 Let ∇ be a superconnection and define

$$T^\nabla : TM \otimes TM \to TM$$
$$X \otimes Y \mapsto \nabla_X Y - (-1)^{\tilde{Y}\tilde{X}}\nabla_Y X - [X,Y]$$

We call T^∇ the torsion of ∇. Often we simply denote T^∇ by *T* when there is no confusion about which superconnection we are considering.

There is an alternative way of defining the torsion of a superconnection in the abstract index formalism. An in depth discussion on this approach can be found in [6].

Definition 7.2 In the abstract index formalism the torsion of a superconnection is defined in the following way,

$$[\nabla_A, \nabla_B]f = -T_{AB}^C \nabla_C f$$

where $f \in O_M$.

Proposition 2. Let ∇ be a superconnection. Its torsion T is a tensor.

Proof. First notice that,

$$\begin{aligned} T(X\otimes Y) &= \nabla_X Y-(-1)^{\widetilde{Y}\widetilde{X}}\nabla_Y X-[X,Y] \\ &= \nabla_X Y-(-1)^{\widetilde{Y}\widetilde{X}}\nabla_Y X-(-1)^{\widetilde{Y}\widetilde{X}}[Y,X] \\ &= -(-1)^{\widetilde{Y}\widetilde{X}}(\nabla_Y X-(-1)^{\widetilde{Y}\widetilde{X}}\nabla_X Y-[Y,X]) \\ &= (-1)^{\widetilde{Y}\widetilde{X}}\,T(Y\otimes X) \end{aligned}$$

Now,

$$\begin{aligned} T(fX\otimes Y) &= \nabla_{fX} Y-(-1)^{\widetilde{Y}\widetilde{fX}}\nabla_Y fX-[fX,Y] \\ &= f\nabla_X Y-(-1)^{\widetilde{Y}\widetilde{fX}}Y(f)X-(-1)^{\widetilde{Y}\widetilde{fX}+\widetilde{Y}\widetilde{f}}f\nabla_Y X-f[X,Y]+(-1)^{\widetilde{Y}\widetilde{fX}}Y(f)X \\ &= f\nabla_X Y-(-1)^{\widetilde{Y}\widetilde{X}}f\nabla_Y X-f[X,Y]) \\ &= fT(X\otimes Y) \end{aligned}$$

Also,

$$\begin{aligned} T(X\otimes fY) &= -(-1)^{\widetilde{fY}\widetilde{X}}T(fY\otimes X) \\ &= -(-1)^{\widetilde{fY}\widetilde{X}}fT(Y\otimes X) \\ &= -(-1)^{\widetilde{fY}\widetilde{X}+\widetilde{Y}\widetilde{fX}}fT(X\otimes Y) \\ &= -(-1)^{\widetilde{X}\widetilde{f}}fT(X\otimes Y) \end{aligned}$$

Thus T is a tensor.

Note. The torsion can also be interpreted as a section of the bundle $TM\otimes\Lambda^2 TM^*$, i.e., $T\in\Gamma(M, TM\otimes\Lambda^2 TM^*)$.

8. THE CURVATURE TENSOR

Definition 8.1 Let ∇ be a superconnection and define

$$R^{\nabla}: TM\otimes TM\otimes TM\to TM$$
$$X\otimes Y\otimes Z\mapsto \nabla_X\nabla_Y Z-(-1)^{\widetilde{X}\widetilde{Y}}\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$$

We call R^{∇} the curvature of ∇. Often we simply denote R^{∇} by R when there is no confusion about which superconnection we are considering.

Again we may define the curvature of a superconnection ∇ in terms of abstract indices.

Definition 8.2 In the abstract index formalism the curvature of a superconnection is defined in the following way,

$$[\nabla_A,\nabla_B]V_C=-T^D_{AB}\nabla_D V_C-R^D_{ABC}V_D$$

where $V_C\in\Omega M$.

Proposition 3. Let ∇ be a superconnection and let $V^E\in TM$. Then

$$[\nabla_A,\nabla_B]V^E=-T^C_{AB}\nabla_C X^E+(-1)^{(\widetilde{C}+\widetilde{E})\widetilde{V}^C}R^E_{ABC}V^C.$$

Proof. Let $W_C \in \Omega M$, therefore the contraction $V^E W_E \in O_M$ and by definition,

$$[\nabla_A, \nabla_B](V^E W_E) = -T^C_{AB}\nabla_C(V^E W_E)$$
$$= -T^C_{AB}(\nabla_C V^E)W_E - (-1)^{(\tilde{A}+\tilde{B})\tilde{V}^E} V^E T^C_{AB}\nabla_C W_E$$

Also, by the Leibnitz rule,

$$[\nabla_A, \nabla_B](V^E W_E) = \{[\nabla_A, \nabla_B]V^E\}W_E + (-1)^{(\tilde{A}+\tilde{B})\tilde{V}^E} V^E[\nabla_A, \nabla_B]W_E$$
$$= \{[\nabla_A, \nabla_B]V^E\}W_E - (-1)^{(\tilde{A}+\tilde{B})\tilde{V}^E} V^E T^C_{AB}\nabla_C W_E - (-1)^{(\tilde{A}+\tilde{B})\tilde{V}^E} V^E R^C_{ABE} W_C$$

Now, equating these two expressions gives the required result.

Proposition 4. Let ∇ be a superconnection. Its curvature *R* is a tensor.

Proof. First notice that,

$$T(X\otimes Y\otimes Z) = \nabla_X\nabla_Y Z - (-1)^{\tilde{X}\tilde{Y}}\nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$
$$= \nabla_X\nabla_Y Z - (-1)^{\tilde{X}\tilde{Y}}\nabla_Y\nabla_X Z + (-1)^{\tilde{X}\tilde{Y}}\nabla_{[Y,X]}Z$$
$$= -(-1)^{\tilde{X}\tilde{Y}}(\nabla_X\nabla_Y Z - (-1)^{\tilde{X}\tilde{Y}}\nabla_X\nabla_Y Z - \nabla_{[Y,X]}Z)$$
$$= -(-1)^{\tilde{Y}\tilde{X}}\ R(Y\otimes X\otimes Z)$$

Now,

$$T(fX\otimes Y\otimes Z) = \nabla_{fX}\nabla_Y Z - (-1)^{\widetilde{fX}\tilde{Y}}\nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z$$
$$= f\nabla_X\nabla_Y Z - (-1)^{\widetilde{fX}\tilde{Y}}\nabla_Y(f\nabla_X Z) - f\nabla_{[X,Y]}Z + (-1)^{\widetilde{fX}\tilde{Y}}Y(f)\nabla_X Z$$
$$= f\nabla_X\nabla_Y Z - (-1)^{\widetilde{fX}\tilde{Y}}Y(f)\nabla_X Z - (-1)^{\widetilde{fX}\tilde{Y}+\tilde{f}\tilde{Y}} f\nabla_Y\nabla_X Z$$
$$- f\nabla_{[Y,X]}Z + (-1)^{\widetilde{fX}\tilde{Y}}Y(f)\nabla_X Z$$
$$= f(\nabla_X\nabla_Y Z - (-1)^{\tilde{X}\tilde{Y}}\nabla_Y\nabla_X Z - \nabla_{[Y,X]}Z)$$
$$= fR(X\otimes Y\otimes Z)$$

Therefore,

$$R(X\otimes fY\otimes Z) = -(-1)^{\tilde{X}\widetilde{fY}} R(fY\otimes X\otimes Z)$$
$$= -(-1)^{\tilde{X}\widetilde{fY}} fR(Y\otimes X\otimes Z)$$
$$= -(-1)^{\tilde{X}\widetilde{fY}+\tilde{X}\tilde{Y}} fR(X\otimes Y\otimes Z)$$
$$= -(-1)^{\tilde{f}\tilde{X}} fR(X\otimes Y\otimes Z)$$

Finally

$$T(X\otimes Y\otimes fZ) = \nabla_X\nabla_Y fZ - (-1)^{\tilde{X}\tilde{Y}}\nabla_Y\nabla_X fZ - \nabla_{[X,Y]}fZ$$
$$= \nabla_X(Y(f)Z + (-1)^{\tilde{f}\tilde{Y}} f\nabla_Y Z) - (-1)^{\tilde{X}\tilde{Y}}\nabla_Y(X(f)Z)$$
$$= f\nabla_X\nabla_Y Z - (-1)^{\widetilde{fX}\tilde{Y}}Y(f)\nabla_X Z - (-1)^{\widetilde{fX}\tilde{Y}+\tilde{f}\tilde{Y}} f\nabla_Y\nabla_X Z$$
$$-[X,Y](f)Z - (-1)^{\tilde{f}[\tilde{X},\tilde{Y}]} f\nabla_{[X,Y]}Z$$

$$= XY(f)Z + (-1)^{\widetilde{X}\widetilde{Y}(\widetilde{f})} Y(f)\nabla_X Z + (-1)^{\widetilde{f}\widetilde{Y}} X(f)\nabla_Y Z + (-1)^{\widetilde{f}\widetilde{Y}+\widetilde{f}\widetilde{X}} f\nabla_X \nabla_Y Z$$
$$- (-1)^{\widetilde{X}\widetilde{Y}} YX(f)Z - (-1)^{\widetilde{f}\widetilde{Y}} X(f)\nabla_Y Z - (-1)^{\widetilde{X}\widetilde{Y}+\widetilde{f}\widetilde{X}} Y(f)\nabla_X Z$$
$$- (-1)^{\widetilde{X}\widetilde{Y}+\widetilde{f}\widetilde{X}+\widetilde{f}\widetilde{Y}} f\nabla_Y \nabla_X Z - [X,Y](f)Z - (-1)^{\widetilde{f}\widetilde{X}+\widetilde{f}\widetilde{Y}} f\nabla_{[X,Y]} Z$$
$$= (-1)^{\widetilde{f}\widetilde{X}+\widetilde{f}\widetilde{Y}} f(\nabla_X \nabla_Y Z - (-1)^{\widetilde{X}\widetilde{Y}} \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z)$$
$$= (-1)^{\widetilde{f}\widetilde{X}+\widetilde{f}\widetilde{Y}} fR(X \otimes Y \otimes Z)$$

So R is a tensor.

Note. Like the torsion the curvature can also be viewed as a section of an appropriate bundle. In this case $R \in \Gamma(M, TM \otimes TM^* \otimes \Lambda^2 TM^*)$.

9. LOCAL FORM OF THE TORSION AND CURVATURE TENSORS

Proposition 5. In a local coordinate system (x_A) the torsion tensor T may be written as,

$$T(X \otimes Y) = (-1)^{\widetilde{Y}^B \widetilde{\partial}_A} X^A Y^B T^C_{AB} \partial_C$$

where,

$$T^C_{AB} = \Gamma^C_{AB} - (-1)^{\widetilde{\partial}_A \widetilde{\partial}_B} \Gamma^C_{BA}.$$

Proof. Let X, Y be vector fields, thus they can be written as, $X = X^A \partial_A$ and $Y = Y^B \partial_B$. So, as the torsion is a tensor,

$$T(X \otimes Y) = T(X^A \partial_A \otimes Y^B \partial_B)$$
$$= -(-1)^{\widetilde{Y}^B \widetilde{\partial}_A} X^A Y^B T(\partial_A \otimes \partial_B)$$

But,

$$T(\partial_A \otimes \partial_B) = \nabla_{\partial_A} \partial_B - (-1)^{\widetilde{\partial}^B \widetilde{\partial}_B} \nabla_{\partial_B} \partial_A - [\partial_A, \partial_B]$$
$$= (\Gamma^C_{AB} - (-1)^{\widetilde{\partial}_A \widetilde{\partial}_B} \Gamma^C_{BA}) \partial_C$$

Hence the proof is completed.

Proposition 6. In a local coordinate system (x_A) the curvature tensor R may be written as,

$$R(X \otimes Y \otimes Z) = (-1)^{(\widetilde{Y}^B + \widetilde{Z}^C)\widetilde{\partial}_A + \widetilde{Z}^C \widetilde{\partial}_B} X^A Y^B Z^C R^E_{ABC} \partial_E$$

where,

$$R^E_{ABC} = \partial_A(\Gamma^E_{BC}) - (-1)^{\widetilde{\partial}_A \widetilde{\partial}_B} \partial_B(\Gamma^C_{BA}) + (-1)^{(\widetilde{\partial}_B + \widetilde{\partial}_C + \widetilde{\partial}_G)(\widetilde{\partial}_E + \widetilde{\partial}_G)} (\Gamma^E_{AG} \Gamma^G_{BC} -$$
$$(-1)^{\widetilde{\partial}_A \widetilde{\partial}_B + (\widetilde{\partial}_A + \widetilde{\partial}_B)(\widetilde{\partial}_E + \widetilde{\partial}_G)} \Gamma^E_{BG} \Gamma^G_{AC})$$

Proof. Let X, Y and Z be three vector fields, thus they can be written as, $X = X^A \partial_A$, $Y = Y^B \partial_B$ and $Z = Z^C \partial_C$. So, as the curvature is a tensor,

$$R(X \otimes Y \otimes Z) = R(X^A \partial_A \otimes Y^B \partial_B \otimes Z^C \partial_C)$$
$$= (-1)^{(\widetilde{Y}^B + \widetilde{Z}^C)\widetilde{\partial}_A + \widetilde{Z}^C \widetilde{\partial}_B} X^A Y^B Z^C R(\partial_A \otimes \partial_B \otimes \partial_C)$$

But,

$$R(\partial_A \otimes \partial_B \otimes \partial_C) = \nabla_{\partial_A} \nabla_{\partial_B} \partial_C - (-1)^{\widetilde{\partial}_A \widetilde{\partial}_B} \nabla_{\partial_B} \nabla_{\partial_A} \partial_C - \nabla_{[\partial_A, \partial_B]} \partial_C$$
$$= \partial_A(\Gamma^E_{BC}) \partial_E - (-1)^{\widetilde{\partial}_A \widetilde{\partial}_B} \partial_B(\Gamma^E_{AC}) \partial_E + (-1)^{\widetilde{\partial}_A \widetilde{\Gamma}^G_{BC}} \Gamma^G_{BC} \Gamma^E_{AG} \partial_E$$
$$- (-1)^{\widetilde{\partial}_A \widetilde{\partial}_B + \widetilde{\partial}_B \widetilde{\Gamma}^G_{AC}} \Gamma^G_{AC} \Gamma^E_{BG} \partial_E$$

By reversing the order of the Γ's in the third and fourth terms we can write them as,

$$(-1)^{\widetilde{\Gamma}^G_{BC}\widetilde{\Gamma}^E_{AG}\widetilde{\partial}_A}(\Gamma^E_{AG}\Gamma^G_{BC} - (-1)^{(\widetilde{\partial}_A\widetilde{\partial}_B + \widetilde{\Gamma}^G_{BC}\widetilde{\Gamma}^E_{AG}\widetilde{\partial}_A + \widetilde{\Gamma}^G_{AC}\widetilde{\Gamma}^E_{BG}\widetilde{\partial}_B)}\Gamma^E_{BG}\Gamma^G_{AC})\partial_E$$

Now, using the identity $\widetilde{\Gamma}^C_{AB} = \widetilde{\partial}_A + \widetilde{\partial}_B + \widetilde{\partial}_C$, we get the required form.

10. THE SUPER-BIANCHI IDENTITIES

As in the classical case, it is possible to derive certain constraints that the curvature and torsion tensors of a superconnection must satisfy. In analogy with classical case, we call these the *super-Bianchi identities.*

Theorem 1. Let ∇ be a superconnection on a supermanifold M with torsion *T* and curvature *R*. The curvature tensor satisfies the following,

$$R^E_{[ABC]} = \nabla_{[A}T^E_{BC]} + T^D_{[AB}T^E_{|D|C]}$$

which we call the 1st *super-Bianchi identity*.

Proof. Using the abstract index definition of the curvature and torsion,

$$\nabla_A\nabla_B\nabla_C f - (-1)^{\widetilde{A}\widetilde{B}}\nabla_B\nabla_A\nabla_C f = T^E_{AB}\nabla_E\nabla_C f - R^E_{ABC}\nabla_C f.$$

Also, by definition,

$$\nabla_E\nabla_C f = (-1)^{\widetilde{E}\widetilde{C}}\nabla_C\nabla_E f - T^D_{EC}\nabla_D f.$$

Thus, by substituting this into the previous expression we obtain,

$$2\nabla_{[A}\nabla_{B]}\nabla_C f = -(-1)^{\widetilde{E}\widetilde{C}}T^E_{AB}\nabla_C\nabla_E f + T^E_{AB}T^D_{EC}\nabla_D f - R^E_{ABC}\nabla_E f.$$

Further,

$$\begin{aligned}(-1)^{\widetilde{E}\widetilde{C}}T^E_{AB}\nabla_C\nabla_E f &= (-1)^{\widetilde{A}\widetilde{B}\widetilde{C}}\nabla_C(T^E_{AB}\nabla_E f) - (-1)^{\widetilde{A}\widetilde{B}\widetilde{C}}(\nabla_C T^E_{AB})\nabla_E f \\ &= (-1)^{\widetilde{A}\widetilde{B}\widetilde{C}}\nabla_C(2\nabla_{[A}\nabla_{B]}\nabla_E f) - (-1)^{\widetilde{A}\widetilde{B}\widetilde{C}}(\nabla_C T^E_{AB})\nabla_E f\end{aligned}$$

Again, substituting this into the previous expression we obtain the following,

$$2\nabla_{[A}\nabla_{B]}\nabla_C f = 2(-1)^{\widetilde{A}\widetilde{B}\widetilde{C}}\nabla_C\nabla_{[A}\nabla_{B]}f + (-1)^{\widetilde{A}\widetilde{B}\widetilde{C}}(\nabla_C T^E_{AB})\nabla_E f + T^D_{AB}T^E_{DC}\nabla_E f - R^E_{ABC}\nabla_E f$$

Now, antisymmetrising in *A*, *B* and *C* gives the stated result.

Theorem 2. Let ∇ be a superconnection on a supermanifold M with torsion *T* and curvature *R*. The curvature tensor satisfies the following,

$$\nabla_{[A}R^E_{BC]D} = -T^E_{[AB}R^E_{|F|C]D}$$

which we call the 2nd *super-Bianchi identity*.

Proof. By definition,

$$\begin{aligned}2\nabla_{[A}\nabla_B\nabla_{C]} &= 2\nabla_{[A}\nabla_{[B}\nabla_{C]]}V^E \\ &= \nabla_{[A}\{-T^D_{BC]}\nabla_D V^E + (-1)^{(\widetilde{D}+\widetilde{E})\widetilde{V}^D}R^E_{BC]D}V^D\} \\ &= -\nabla_{[A}T^D_{BC]}(\nabla_D V^E) + (-1)^{(\widetilde{D}+\widetilde{E})\widetilde{V}^D}(\nabla_{[A}R^E_{BC]D})V^D - \frac{1}{3}\{(-1)^{\widetilde{C}\widetilde{D}}T^D_{AB}\nabla_C \\ &\quad + (-1)^{\widetilde{A}\widetilde{B}+\widetilde{A}\widetilde{C}+\widetilde{A}\widetilde{D}}T^D_{BC}\nabla_A + (-1)^{\widetilde{A}\widetilde{C}+\widetilde{B}\widetilde{C}+\widetilde{B}\widetilde{D}}T^D_{CA}\nabla_B\}\nabla_D V^E \\ &\quad + \frac{1}{3}(-1)^{(\widetilde{D}+\widetilde{E})\widetilde{V}^D}\{(-1)^{\widetilde{C}\widetilde{D}+\widetilde{C}\widetilde{E}}R^E_{ABD}\nabla_C + (-1)^{\widetilde{A}\widetilde{B}+\widetilde{A}\widetilde{C}+\widetilde{A}\widetilde{B}+\widetilde{A}\widetilde{E}}R^E_{BCD}\nabla_A \\ &\quad + (-1)^{\widetilde{A}\widetilde{C}+\widetilde{B}\widetilde{C}+\widetilde{B}\widetilde{D}+\widetilde{D}\widetilde{E}}R^E_{CAD}\nabla_D\}V^D\end{aligned}$$

Also

$$2\nabla_{[A}\nabla_{B}\nabla_{C]} = 2\nabla_{[[A}\nabla_{B]}\nabla_{C]}V^{E}$$

$$= -T^{D}_{[AB}\nabla_{|D|}\nabla_{C]}V^{E} - R^{D}_{[ABC]}\nabla_{D}V^{E} + \frac{1}{3}(-1)^{(\tilde{D}+\tilde{E})\tilde{V}^{D}}\{(-1)^{(\tilde{C}\tilde{D}+\tilde{C}\tilde{E})}R^{E}_{ABD}\nabla_{C}$$

$$+ (-1)^{\tilde{A}\tilde{B}+\tilde{A}\tilde{C}+\tilde{A}\tilde{B}+\tilde{A}\tilde{E}}R^{E}_{BCD}\nabla_{A} + (-1)^{\tilde{A}\tilde{C}+\tilde{B}\tilde{C}+\tilde{B}\tilde{D}+\tilde{D}\tilde{E}}R^{E}_{CAD}\nabla_{D}\}V^{D}$$

Using the 1st Bianchi identity and the definition of the curvature we find that,

$$2\nabla_{[A}\nabla_{B}\nabla_{C]}V^{E} = T^{D}_{[AB}T^{F}_{|D|C]}\nabla_{F}V^{E} - (-1)^{\tilde{V}^{F}(\tilde{E}+\tilde{F})}T^{D}_{[AB}R^{E}_{|D|C]F}V^{F} - \frac{1}{3}\{(-1)^{\tilde{C}\tilde{D}}T^{D}_{AB}\nabla_{C}$$

$$+ (-1)^{\tilde{A}\tilde{D}+\tilde{A}\tilde{B}+\tilde{A}\tilde{C}}T^{D}_{BC}\nabla_{A} + (-1)^{\tilde{B}\tilde{D}+\tilde{A}\tilde{C}+\tilde{B}\tilde{C}}T^{D}_{CA}\nabla_{B}\}\nabla_{D}V^{E} - (\nabla_{[A}T^{D}_{BC]})\nabla_{D}V^{E}$$

$$- T^{D}_{[AB}T^{F}_{|D|C]}\nabla_{F}V^{E} + \frac{1}{3}(-1)^{(\tilde{D}+\tilde{E})\tilde{V}^{D}}\{(-1)^{\tilde{C}\tilde{D}+\tilde{C}\tilde{E}}R^{E}_{ABD}\nabla_{C}$$

$$+ (-1)^{\tilde{A}\tilde{B}+\tilde{A}\tilde{C}+\tilde{A}\tilde{B}+\tilde{A}\tilde{E}}R^{E}_{BCD}\nabla_{A} + (-1)^{\tilde{A}\tilde{C}+\tilde{B}\tilde{C}+\tilde{B}\tilde{D}+\tilde{D}\tilde{E}}R^{E}_{CAD}\nabla_{D}\}V^{D}$$

Now, equating the two expressions for $2\nabla_{[A}\nabla_{B}\nabla_{C]}V^{E}$ we get the stated result.

Finally we conclude that the 1st and 2nd super-Bianchi identities are the two constraints which are new approach in supermanifolds.

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