BRAC University Journal, Vol. IV, No. 2, 2007, pp. 87-92

AN ALGORITHM FOR RESTRICTED NORMAL FORM TO SOLVE DUAL TYPE NON-CANONICAL LINEAR FRACTIONAL PROGRAMMING PROBLEM

Mozzem Hossain
Department of Mathematics
Ghior Govt. College, Ghior, Manikgonj

and

Monjur Morshed
Department of Mathematics
Darul ihsan University, Dhaka, Bangladesh

and

Mohammed Forhad Uddin
Department of Mathematics
BUET, Dhaka, Bangladesh

ABSTRACT

In this paper, an algorithm is presented to solve non-canonical linear fractional programming (LFP) problem, considering the restricted normal form. It provides a new way to solve all types of LFP problems. When the LFP problem is only in canonical form, Forhad et al. [13] derived an algorithm considering the restricted normal form. In this paper, the algorithm of Forhad et al. [13] has been generalized to solve the LFP problem which is non-canonical form also. This algorithm required neither transformation nor the iterative calculations of simplex method. But it requires only algebraic elimination.

Keywords: Linear Fractional Programming (LFP), Canonical and Non-canonical form, Transformation, Simplex method.

I. INTRODUCTION

To solve LFP problems Charnes-Cooper's [2] developed a transformation technique which transform the LFP into two Linear Programming (LP) problems, Bitran-Novaes [1] developed an algorithm which transform the objective function of LFP problem into a linear objective function and solves a sequence of LP problems, so, it takes more time and labor. On the other hand, Swarup [10] developed an algorithm that has fewer steps than previous techniques but cannot avoid the iterative calculations of simplex method of Dantzig [3, 4]. To overcome the complexities of LP problems William et al. [9] suggested the restricted normal form.

Further, Forhad et al [12] modified Swarup [11] primal simplex type method for solving LFP problem based on primal simplex method [4] for solving LP problem, which extends the scope of the method. Swarup's [10] primal simplex type method can be applied only when the constraints set is in canonical form. Latter on, Swarup suggest to apply the dual simplex type method in the case where the set of constraints is not in canonical form. But Swarup's [11] dual simplex type method cannot be applied in the case where the dual feasible basis is not obtained. To over come the complexities of these methods, Forhad et al [12] suggested a modified approach to solve any type of LFP problem.

Moreover, when the LFP problem is only in canonical form, Forhad et al. [13] derived an algorithm considering the restricted normal form [9]. In this paper, the algorithm of Forhad et al. [13] has been generalized to solve the LFP problem which is non-canonical form also.

II. LINEAR FRACTIONAL PROGRAMMING (LFP) PROBLEMS

The LFP problem can be defined as follows:

(LFP) $Maximize \ \ F(x) = \dfrac{cx+\alpha}{dx+\beta}$ (1)

Subject to primary constraints

$x_1, x_2, x_3, ..., x_N \geq 0$ (2)

and simultaneously subject to $M = m_1 + m_2 + m_3$ additional constraints, m_1 of them are of the form

$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + ... + a_{iN}x_N \leq b_i,$ (3)

$i = 1, \ 2, ..., m_1$

m_2 of them are of the form

$a_{j1}x_1 + a_{j2}x_2 + a_{j3}x_3 + ... + a_{jN}x_N \geq b_j \geq 0,$ (4)

$j = m_1 + 1, \ ..., m_1 + m_2$

and m_3 of them are of the form

$$\left.\begin{array}{l} a_{k1}x_1 + a_{k2}x_2 + a_{k3}x_3 + ... + a_{kN}x_N = b_k \geq 0, \\ k = m_1 + m_2 + 1, \ ..., m_1 + m_2 + m_3 \end{array}\right\} \quad (5)$$

The various a_{ij} 's can have either sign, or be zero. The fact that b's must all be non-negative is the matter of convention only, since one can multiply any contrary inequality by -1. There is no particular significance in the number of the constraints m being less than, equal to, or greater than the number of unknowns N.

III. DIFFERENT TYPES OF METHODS FOR SOLVING LFPP

III.1 Swarup's dual simplex type method

Swarup's [10] primal simplex type method showed that the basic feasible solution will be optimal if $\Delta_j \leq 0$,

where $z^1 = c_B x_B + \alpha$

$z^2 = d_B x_B + \beta$

$z_j^1 = c_B a_j$

$z_j^2 = d_b a_j$

$\Delta_j = z^2(c_j - z_j^1) - z^1(d_j - z_j^2), \ j = 1,2,\ldots,n.$

The above observation presents the following interesting possibility, if one can start with some basic but not feasible solution to a given LFP problem with all $\Delta_j \leq 0$ and remove from this basic solution to another by changing one vector at a time in such a way that he keeps all $\Delta_j \leq 0$ provided no basic solution is to be repeated, an optimal solution to LFP problem will be obtained in a finite number of iterations. That is the fact that this algorithm maintains all $\Delta_j \leq 0$ at each iteration and is not concerned about the feasibility of the basic solution.

III.2 The modified approach of Swarup's primal simplex type method

Swarup[10] first developed a method for solving LFP problem. However, the method can be applied only when the system Ax = b is in a canonical form, that is, all constraints are less than or equal form (≤). The problem that is not in canonical form, one can solve by using Swarup's[11] dual simplex type method. Likewise, LP problem, dual simplex type method also cannot be applied in the case where the dual feasible basis is not obtained.

To overcome the above limitation of Swarup's [10&11] methods, Forhad et al. [13] suggested a modification based on Dantiz [3] two phase method for solving linear programming problems.

III.3 Numerical example:

Example 3.3.1

(LFP) Maximize $Z = \dfrac{x_2 - 5}{-x_1 - x_2 + 9}$

Subject to

$2x_1 + 5x_2 \geq 10$

$4x_1 + 3x_2 \leq 20$

$-x_1 + x_2 \leq 2$

$x_1, \ \ x_2 \geq 0$

Now, introducing surplus and slack variables s_1 s_2 and s_3 to 1st, 2nd and 3rd constraints respectively to make the LFP problem in the standard form as follows:

(LFPI) Maximize $Z = \dfrac{x_2 - 5}{-x_1 - x_2 + 9}$

Subject to

$2x_1 + 5x_2 - s_1 = 10$

$4x_1 + 3x_2 + s_2 = 20$

$-x_1 + x_2 + s_3 = 2$

$x_1, \ \ x_2, s_1, s_2, s_3 \geq 0$

Thus the initial basic solution
$s_1 = -10$, $s_2 = 20$, $s_3 = 2$ and $x_1 = x_2 = 0$

Since $s_1 = -10 < 0$, it fails the feasibility, that is, Swarup's [10] primal simplex type method fails to solve the LFP problem.

Now, we start, Swarup's [11] dual simplex type method to solve the above LFP problem.

Initial Table

c_B ↓	d_B ↓	c_j → d_j →	0 1 0 0 0 -1 -1 0 0 0
		x_{Bi}	x_1 x_2 s_1 s_2 s_3
0 0 0	0 0 0	s_1= -10 s_2= 20 s_3=2	-2 –5 1 0 0 4 3 0 1 0 -1 1 0 0 1
z^1=-5	z^2=9	Z=-5/9	
		$c_j-z_j^1$ $d_j-z_j^2$	0 1 0 0 0 -1 -1 0 0 0
		Δ_j →	-5 **[4]** 0 0 0

To obtain optimal solution it must be maintained that all $\Delta_j \le 0$ at each optimization stage. But in the initial table, it is observed that $\Delta_2 = 4 > 0$, which indicates the **failure of Swarup's [11] dual simplex type method.**

III. 4 The algorithm of this paper

To overcome the above limitation of Swarup's [10 & 11] methods, we suggest a modification on Forhad et al. [13].

Finally, the Example 3.3.1 is solved by our derived restricted normal form as follows:

PHASE 1:

(ALP) Minimize $L^* = w = 10 - 2x_1 - 5x_2 + s_1$ (6)

Subject to
$$\begin{aligned} w &= 10 - 2x_1 - 5x_2 + s_1 \\ s_2 &= 20 - 4x_1 - 3x_2 \\ s_3 &= 2 + x_1 - x_2 \\ x_1, x_2&, s_1, s_2, s_3, w \ge 0 \end{aligned} \quad (7)$$

Here, N= 6 and M=3; the left-hand variables are w, s_2 and s_3; the right- hand variables are x_1, x_2 and s_1. The objective function is written so as to depend only on the right-hand variables.

For any problem in restricted normal form, it can be instantly read off a feasible basic vector (although not necessarily the optimal feasible basic vector). Simply set all right -hand variables equals to zero, and the equations (7) then give the values of the left-hand variables for which the constraints are satisfied.

The idea of the simplex method is to proceed by series of exchanges. In each exchange, right-hand variables and a left-hand variables change the places. At each stage we maintain a problem in restricted normal form that is equivalent to the original problem.

It is convenient to record the information constant of the equations (6) and (7) in a so-called tableau, as follows:

Table: 1

		x_1	x_2	s_1
L^*	10	-2	-5	1
w	10	-2	$\boxed{-5}$	1
s_2	20	-4	-3	0
s_3	2	1	-1	0

Step1: The most negative L^* row entry is –5 so x_2 is the left hand variable.

Step2: There are three negative entry below it. The ratios are $10 \div |-5| = 2$, $20 \div |-3| = 6.67$, $2 \div |-1| = 2$, the minimum value is 2, so, w is the right hand variable and so w is replaced by x_2.

Step 3: Now, solving x_2 in favor of w, namely

$$\therefore \quad 5x_2 = 10 - 2x_1 - w + s_1$$

$$x_2 = 2 - \frac{2}{5}x_1 - \frac{1}{5}w + \frac{1}{5}s_1$$

Then substitute this value into the old objective function,

$$\begin{aligned} L^* &= 10 - 2x_1 - 10 + 2x_1 + w - s_1 + s_1 \\ &= w \end{aligned}$$

and into all other old left hand variables,

$$s_2 = 20 - 4x_1 - 3\left(2 - \frac{2}{5}x_1 - \frac{1}{5}w + \frac{1}{5}s_1\right)$$

$$= 14 - \frac{14}{5}x_1 + \frac{3}{5}w - \frac{3}{5}s_1$$

$$s_3 = 2 + x_1 - 2 + \frac{2}{5}x_1 + \frac{1}{5}w - \frac{1}{5}s_1$$

$$= \frac{7}{5}x_1 + \frac{1}{5}w - \frac{1}{5}s_1$$

In table form:
Table: 2

		x_1	w	s_1
L^*	0	0	1	0
x_2	2	$-\frac{2}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$
s_2	14	$-\frac{14}{5}$	$\frac{3}{5}$	$-\frac{3}{5}$
s_3	0	$\frac{7}{5}$	$\frac{1}{5}$	$-\frac{1}{5}$

Since all $c_j^* \geq 0$ and there is no artificial variable in the last table, it yields a primal feasible solution, this table gives another sub optimal point $s_1 = 0$, $s_2 = 14$, $s_3 = 0$ and $x_1 = 0$, $x_2 = 2$ with Min $L^* = 0$.

Now, the Phase 2 of the problem is as follows:

PHASE 2: Now the initial basic solution is $s_2 = 14, s_3 = 0, x_2 = 2$
and the original objective function becomes

$$\text{Maximize } z = \frac{x_2 - 5}{-x_1 - x_2 + 9}$$

$$= \frac{2 - \frac{2}{5}x_1 + \frac{1}{5}s_1 - 5}{-x_1 - 2 + \frac{2}{5}x_1 - \frac{1}{5}s_1 + 9}$$

$$= \frac{-3 - \frac{2}{5}x_1 + \frac{1}{5}s_1}{7 - \frac{3}{5}x_1 - \frac{1}{5}s_1}$$

In table form :
Table: 3

		x_1	s_1
c_j	-3	$-\frac{2}{5}$	$\frac{1}{5}$
d_j	7	$-\frac{3}{5}$	$-\frac{1}{5}$
x_2	2	$-\frac{2}{5}$	$\frac{1}{5}$
s_2	14	$-\frac{14}{5}$	$-\frac{3}{5}$
s_3	0	$\frac{7}{5}$	$\boxed{-\frac{1}{5}}$
	Δ_j	$-\frac{23}{5}$	$\frac{4}{5}$ ↑

Algorithm of the Restricted Normal form:

Here relative profit factor $c_j - Z_j^{1}$, relative cost factor $d_j - Z_j^{2}$ and the ratio ∇_j,
Where

$$Z^1 = c_B x_B + \alpha$$

$$Z^2 = d_B x_B + \beta$$

$$Z_j^{1} = c_B a_j$$

$$Z_j^{2} = d_B a_j \quad and \quad \nabla_j = z^2(c_j - Z_j^{1}) - Z^1(d_j - Z_j^{2})$$

Step I: To select the pivot column, consider

$$\Delta_j = z^2 (c_j - z_j^1) - z^1 (d_j - z_j^2)$$

Choose max $\Delta_j > 0$.

Here, $\Delta_1 = -\frac{23}{5}, \Delta_2 = \frac{4}{5}$

So, s_1 is the new left-hand variable.

Step II: To choose pivot element, the minimum ratio test need to apply. In our problem there are two negative entries, namely, $-\frac{3}{5}$ and $-\frac{1}{5}$.

The ratios are $\frac{14}{\left|-\frac{3}{5}\right|} = \frac{70}{3}$ and $\frac{0}{\left|-\frac{1}{5}\right|} = 0$, the minimum value is 0 and so right hand variable s_3 and s_3 is replaced by left hand variable s_1.

Step III: Now, by solving the pivot-row equation for the new left-hand variable s_1 in favor of the old, s_3 namely,

$$\frac{1}{5}s_1 = \frac{7}{5}x_1 - s_3$$

$$\therefore \quad s_1 = 7x_1 - 5s_3$$

then substitute this value into the original objective function

$$z = \frac{-3 - \frac{2}{5}x_1 + \frac{7}{5}x_1 - s_3}{7 - \frac{3}{5}x_1 - \frac{7}{5}x_1 + s_3}$$

$$= \frac{-3 + x_1 - s_3}{7 - 2x_1 + s_3}$$

and into all other the old left-hand variable rows,

$$x_2 = 2 - \frac{2}{5}x_1 + \frac{7}{5}x_1 - s_3$$

$$= 2 + x_1 - s_3$$

$$s_2 = 14 - \frac{14}{5}x_1 - \frac{3}{5}(7x_1 - 5s_3)$$

$$= 14 - 7x_1 + 3s_3$$

Step IV: Go back and repeat the first step, until all $\Delta_j \le 0$, signaling that no further improve is possible.

Thus after first iteration,
Table: 4

		x_1	s_3
c_j	-3	1	-1
d_j	7	-2	1
x_2	2	1	-1
s_2	14	$\boxed{-7}$	3
s_1	0	7	-5
	Δ_j	1 ↑	-4

Since all is not $\Delta_j \le 0$, it is needed to improve the result and repeat the above steps

Second iteration: Repeating step I, $\Delta_1 = 1$ and $\Delta_2 = -4$, that is first column is the pivot column, and using the minimum ratio test of step II, the ratio is $\frac{14}{|-7|} = 2$, the minimum value is 2 and so right hand variable x_1 and x_1 replaces by left hand variable s_2.

Now, by solving the pivot-row equation for the new left-hand variable x_1 in favor of the old s_2, namely,

$$7x_1 = 14 - s_2 + 3s_3$$

$$\therefore x_1 = 2 - \frac{1}{7}s_2 + \frac{3}{7}s_3$$

then substitute this value into the old objective function

$$z = \frac{-3 - \frac{1}{7}s_2 + \frac{3}{7}s_3 + 2 - s_3}{7 - 4 + \frac{2}{7}s_2 - \frac{6}{7}s_3 + s_3}$$

$$= \frac{-1 - \frac{1}{7}s_2 - \frac{4}{7}s_3}{3 + \frac{2}{7}s_2 + \frac{1}{7}s_3}$$

And into all other the old left-hand variable rows, in this case,

$$x_2 = 2 - \frac{1}{7}s_2 + \frac{3}{7}s_3 + 2$$

$$= 4 - \frac{1}{7}s_2 + \frac{3}{7}s_3$$

$$s_1 = -s_2 + 3s_3 + 14 - 5s_3$$

$$= 14 - s_2 - 2s_3$$

Hence after second iteration,
Table: 5

		s_2	s_3
c_j	-1	$-\frac{1}{7}$	$-\frac{4}{7}$
d_j	3	$\frac{2}{7}$	$\frac{1}{7}$
x_2	4	$-\frac{1}{7}$	$\frac{3}{7}$
x_1	2	$-\frac{1}{7}$	$\frac{3}{7}$
s_1	14	-1	-2
	Δ_j	$-\frac{1}{7}$	$-\frac{11}{7}$

Since all is $\Delta_j \leq 0$, signaling that no further improve is possible. Thus the solution of the example 3.3.1 is $x_1 = 2$, $x_2 = 4$ with $Z_{\max} = -\frac{1}{3}$

IV. CONCLUSION

It is observed that to solve LFP problems by using Swarup's [10] method, it is not needed any transformation but it requires the iterative calculations of simplex method of Dantzig [3, 4]. Forhad et al. [13] derived restricted normal form to solve LFP problem, which is only in canonical form. On the other hand, this paper presents an algorithm on restricted normal form to solve LFP problem, which is not in canonical form. Further, this algorithm requires neither transformation nor the iterative calculations of simplex method. It requires only algebraic elimination.

Finally, it is noted that to use Swarup's [10 & 11] method, it has to consider the slack or surplus variables in each table. For this reason, the number of variables is increased; and extra calculations are needed. But the algorithm described in this papers does not require to consider non-basic variables in each table, that is why, it is needed less calculations, saves time and labor. Hopefully the discussed algorithm helps to solve all types of LFP problems easily.

REFERENCES

1. Bitran, G.R. and Novaes, A.G.: Linear Programming with a Fractional Objective Function, Operations Research, Vol. 21, pp. 22-29, (1972).
2. Charnes, A., & Cooper, W.W.: Programming with Fractional Functionals, Naval Research Logistics Quarterly 9, pp. 181-186,(1962).
3. Dantzig, G.B.: Linear Programming and extension, Princeton University Press, Princeton, N. J., (1962).
4. Dantzig, G.B.: Inductive proof of the simplex method, IBM Journal of research and development, Vol.14, No. 5, (1960) .
5. Gillet B.E.: Introduction to Operations Research, McGraw Hill, Inc., New York, (1998).
6. Islam, M.A., & Nath. G.: Investigation on some Algorithm for solving Linear Fractional Programming Problems, Bangladesh Sci. Res., 14(1), pp. 1-11, (1996).
7. Lipschutz, S. & Poe, A.: Programming with FORTRAN including Structural Fortran, McGraw Hill, Inc., New York, (1998).
8. Mayo, E.W. & Cwiakala,M.: Programmng with FORTRAN 77, McGraw Hill, Inc., New York, (1995).
9. Press, H.W., Teukolsky, A.S., Vetterling, T.W. & Flannery, P.B.: Numerical Recipes in FORTRAN 77, Cambridge University Press, (1992).
10. Swarup, K.: Linear Fractional Programming, Operations Research, Vol. 13, No. 6, pp. 1029-1036.
11. Swarup, K.:, Some Aspects of Linear Fractional Functions Programming, Australian Journal of Statistics, V(3), pp. 90-104 (1965).
12. Farhad, U. M., Ahad,M.A. & Islam, M.A. Modified approach of Swarup's method, Ganit: J. Bangladesh Math. Soc. Vol.24, pp.89-98(2004).
13. Farhad, U. M., & obayedullah, M. A New Approach to Solve LFPP, Ganit: J. Bangladesh Math. Soc. Vol.25 pp.3-15(2005).