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DERIVATIONS OF GENERALIZED INVERSE USING CONTOUR INTEGRATION AND INTERPOLATION POLYNOMIAL

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ABSTRACT

This paper deals with a representation of Generalized inverse (g-inverse) by the contour integral formula that supports the four major properties of g-inverse. Here we have used Cauchy's integral formula. These are verified numerically. This paper also includes the derivation of g-inverse by using minimal polynomial. Here we express A^+ as a Lagrange-Sylvester interpolation polynomial in powers of A, A^*. Mathematica codes are used in these examples.

Key words: contour integral, minimal polynomial, inconsistent system, eigenvalue.

1. INTRODUCTION

The inverse of a nonsingular square matrix and its various properties are available in textbooks. It is stated that if a matrix A has an inverse, the matrix must be square and its determinant must be non-zero. Let us consider a system of linear equations $Ax=b$.

If A is an $n\times n$ non-singular matrix, the solution to the system in the equation $Ax=b$ exists and is unique and is given by
$x = A^{-1}b$

However, there are cases where A is not a square matrix (i.e. rectangular matrix) and also the cases where A is $n\times n$ singular matrix; i.e when the linear equations are inconsistent. In these cases there may still be solution to the system and a unified theory to treat all cases may be desirable. One such theory involves the use of **generalized inverse of matrices**. The generalized inverse is also referred to as **Pseudo-inverse, Moore-Penrose inverses or simply g-inverse** with possible subscripting of the letter g.

Moore [3] first published the work on generalized inverses. **Penrose** [4] defined uniquely determined generalized inverse matrix and investigated some of its properties.

2. DEFINITION

Generalized inverse (g- inverse)

Let A be $m\times n$ matrix of rank $R(A) = r \le \min(m, n)$. Then a *generalized inverse (g-inverse)* of A is an $n\times m$ matrix denoted by A^- such that $x = A^- b$ is a solution of both the consistent and inconsistent set of linear equations $Ax=b$.

A matrix A^- satisfying $AA^-A = A$ obviously coincides with A^{-1} when A^{-1} exists.

3. DIFFERENT CLASSES OF G- INVERSES

Let A be an $m\times n$ matrix over the complex field ***C***. Clearly, analogous results are obtainable when the matrices are defined over a real field.

Consider the following matrix equations:

$$\text{(i)} \quad AXA = A, \tag{1.1}$$

$$\text{(ii)} \quad XAX = X, \tag{1.2}$$

$$\text{(iii)} \quad (XA)^* = XA, \tag{1.3}$$

$$\text{(iv)} \quad (AX)^* = AX, \tag{1.4}$$

where` * `denotes the conjugate transpose.

X is a **g-inverse** if equation *(1.1)* is satisfied and we denote $X = A^-$.

(a) If *(1.1)* and *(1.2)* are satisfied then X is a **reflexive g-inverse** and we denote $X=A^r$.

(b) If (*1.1), (1.2)* & *(1.3)* are satisfied then X is a **left weak g-inverse** and we denote $X = A^w$.

(c) If (*1.1), (1.2)* & *(1.4)* are satisfied then X is **right weak g-inverse** and we denote $X = A^n$.

(d) If *(1.1), (1.2), (1.3)* & *(1.4)* are all satisfied then we call X is **Pseudo -inverse** or (**Moore & Penrose generalized inverse**) and we denote $X = A^+$**.** It is also known as **M-P g-inverse.**

4. SPECIAL REPRESENTATION OF GENERALIZED INVERSE BY CONTOUR INTEGRATION

Here we will discuss two theorems that represent the generalized inverse using contour integral formula. Numerical examples will support our proof.

Theorem 1 If A is any $m \times n$ matrix such that $(AA^*)^{-1}$ exists, then

$$A^+ = \frac{1}{2\pi i}\int_C A^*(AA^* - Iz)^{-1}\frac{1}{z}dz$$

where C is a closed contour containing non-zero eigenvalues of AA^* but not containing the zero eigenvalue of AA^* in or on C.

Proof Let $X = \frac{1}{2\pi i}\int_C A^*(AA^* - Iz)^{-1}\frac{1}{z}dz$

Then we have to show that X satisfies the following four conditions:

1) $AXA = A$
2) $XAX = X$
3) $(AX)^* = AX$
4) $(XA)^* = XA$

(1) $$AXA = \frac{1}{2\pi i}\int_C AA^*(AA^* - Iz)^{-1}A\frac{1}{z}dz$$

$$= \frac{1}{2\pi i}\int_C \{(AA^*)^{-1}\}^{-1}(AA^* - Iz)^{-1}A\frac{1}{z}dz$$

$$= \frac{1}{2\lambda i}\int \{(AA^* - Iz)(AA^*)^{-1}\}^{-1}A\frac{1}{z}dz$$

$$= \frac{1}{2\pi i}\int_C \{I - z(AA^*)^{-1}\}^{-1}A\frac{1}{z}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)}dz \text{ (say) where } z_0 = 0$$

and $f(z) = \{I - z(AA^*)^{-1}\}^{-1}A$

$= f(z_0)$

$= f(0)$

$= (I - 0)^{-1}A$

$= A$

$\therefore\ AXA = A$

(2) $$XAX = \frac{1}{2\pi i}\int_C A^*(AA^* - Iz)^{-1}\frac{1}{z}dzA\frac{1}{2\pi i}\int_C A^*(AA^* - Iz)^{-1}\frac{1}{z}dz$$

$$= \frac{1}{2\pi i}\int_C A^*(AA^* - Iz)^{-1}\frac{1}{z}dzI$$

$= XI$

$= X$

$\therefore\ XAX = X$

(3) $$AX = A\frac{1}{2\pi i}\int_C A^*(AA^* - Iz)^{-1}\frac{1}{z}dz = I$$

$\therefore\ (AX)^* = I^* = I = AX$

(4) $$XA = \frac{1}{2\pi i}\int_C A^*(AA^* - Iz)^{-1}\frac{1}{z}Adz$$

$$= \frac{1}{2\pi i}\int \frac{f(z)dz}{(z - z_0)} \text{ where } z_0 = 0$$

and $f(z) = A^*(AA^* - Iz)^{-1}A$

$= f(z_0)$ (using Cauchy's integral formula)[1]

$= A^*(AA^*)^{-1}A$ which is hermitian.

$\therefore\ (XA)^* = XA$.

---

[1] Suppose U is an open subset of the complex plane **C**, and $f: U \to$ **C** is a holomorphic function, and the disk $D = \{ z : |z - z_0| \leq r\}$ is completely contained in U. Let C be the circle forming the boundary of D. Then we have for every a in the interior of D:

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - a}dz$$

where the contour integral is to be taken counter-clockwise

Thus X satisfies the four conditions of M-P g-inverse. Hence $X = A^+$

So, we have $A^+ = \frac{1}{2\pi i}\int_C A^*(AA^* - Iz)^{-1}\frac{1}{z}dz$

Example Consider a complex matrix $A = \begin{pmatrix} 0 & -i \\ 1 & -2 \end{pmatrix}$

Then $A^* = \begin{pmatrix} 0 & 1 \\ i & -2 \end{pmatrix}$ and

$$AA^* = \begin{pmatrix} 1 & 2i \\ -2i & 5 \end{pmatrix}$$

Now det[AA^*] =1 ≠ 0. So ($AA^*)^{-1}$ exists and

$$(AA^*)^{-1} = \begin{pmatrix} 5 & -2i \\ 2i & 1 \end{pmatrix}$$

$$A^+ = \frac{1}{2\pi i}\int_C A^*(AA^* - Iz)^{-1}\frac{1}{z}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{A^*(AA^* - Iz)^{-1}}{z}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z - z_0} \text{ (say) where } z_0 = 0$$

and $f(z) = A^*(AA^* - Iz)^{-1}$

$= f(0)$

$= A^*(AA^*)^{-1}$

& $(AA^*)^{-1} = \begin{pmatrix} 5 & -2i \\ 2i & 1 \end{pmatrix}$

$$A^* = A^*(AA^*)^{-1}$$

$$= \begin{pmatrix} 0 & 1 \\ i & -2 \end{pmatrix}\begin{pmatrix} 5 & -2i \\ 2i & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2i & 1 \\ i & 0 \end{pmatrix}$$

Now we will verify the four conditions

(1) $AA^+A = A$

(2) $A^+AA^+ = A^+$

(3) $(AA^+)^* = AA^+$

(4) $(A^+A)^* = A^+A$

(1) $AA^+A = \begin{pmatrix} 0 & -i \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2i & 1 \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ 1 & -2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ 1 & -2 \end{pmatrix}$

$= \begin{pmatrix} 0 & -i \\ 1 & -2 \end{pmatrix}$

$= A$

(2) $A^+AA^+ = \begin{pmatrix} 2i & 1 \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2i & 1 \\ i & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2i & 1 \\ i & 0 \end{pmatrix}$

$= \begin{pmatrix} 2i & 1 \\ i & 0 \end{pmatrix}$

$=A$

(3) $AA^+ = \begin{pmatrix} 0 & -i \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2i & 1 \\ i & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore (AA^+)^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = AA^+$

(4) $A^+A = \begin{pmatrix} 2i & 1 \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ 1 & -2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore (A^+A)^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A^+A$

Hence all the four conditions are satisfied.

Theorem 2 The M-P g-inverse of a $m \times n$ matrix A of complex numbers is given by the formula

$$A^+ = \int_0^\infty e^{-A^*At}A^*dt$$

Proof Let $X = \int_0^\infty e^{-A^*At} A^* dt$,then we have to show that X satisfies the four condition of M-P g-inverse.

(1) $AXA = A\int_0^\infty e^{-A^*At} A^* A dt$

$= - A[e^{-\infty} - e^0]$

$= - A[\frac{1}{e^\infty} - I]$

$= A$

$\therefore AXA = A$

(2) $XAX = \int_0^\infty e^{-A^*At} A^* dt A \int_0^\infty e^{-A^*At} A^* dt$

$= -[e^{-\infty} - e^0] \int_0^\infty e^{-A^*At} A^* dt$

$= \int_0^\infty e^{-A^*At} A^* dt$

$= X$

$\therefore XAX = X$

(3) $AX = A \int_0^\infty e^{-A^*At} A^* dt$

$= -[e^{-\infty} - e^0]$

$= I$

$\therefore (AX)^* = AX$

(4) $XA = \int_0^\infty e^{-A^*At} A^* dt$

$= -[e^{-\infty} - e^0]$

$=I$

$\therefore (XA)^* = XA$

Hence X satisfies the four conditions of M-P g – inverse

So, $A^+ = \int_0^\infty e^{-A^*At} A^* dt$

5. AN INTERPOLATION POLYNOMIAL FOR THE M-P INVERSE

Here we express A^+ as a Lagrange-Sylvester interpolation polynomial in powers of A, A^*. For any complex square matrix A let $\sigma(A)$ denote the spectrum of A and $\psi(A)$ its *minimal polynomial* written as $\psi(\lambda) = \prod_{\mu\in\sigma(A)} (\lambda - \mu)^{\nu(\mu)}$, where the root $\mu \in \sigma(A)$ is *simple* if $\nu(\mu) = 1$ and *multiple* otherwise.

We intend to construct A^+ as the matric function $f(A)$ corresponding to the scalar function $f(\lambda) = \lambda^+$ and consider only the case where $\lambda = 0 \in \sigma(A)$ as otherwise A is nonsingular.

Corollary If $\lambda = 0$ is a simple root, this effort to construct A^+ this way lead only to the satisfaction of *(1.1)* and *(1.2)*.

We use therefore $A^+ = (A^*A)^+ A^*$ to construct A^+ by using the Lagrange-Sylvester interpolation polynomial to give explicit expressions associated with (A^*A), as all the roots in $\sigma(A^*A)$ are simple.

$$A^*A = \sum_{\lambda\in\sigma(A^*A)} \lambda \frac{\prod_{\lambda\neq\mu\in\sigma(A^*A)} (A^*A - \mu I)}{\prod_{\lambda\neq\mu\in\sigma(A^*A)} (\lambda - \mu)} \quad (1.5)$$

so that $A^+ = \sum_{\lambda\in\sigma(A^*A)} \lambda^+ \left(\frac{\prod_{\lambda\neq\mu\in\sigma(A^*A)} (A^*A - \mu I)}{\prod_{\lambda\neq\mu\in\sigma(A^*A)} (\lambda - \mu)}\right) A^*$

.

We call (1.5) the *Lagrange-Sylvester interpolation polynomial* for A^+.

Example Let $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$

Then $A^*A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

And $\psi\,(A^*A) = (A^*A)^2 - 2(A^*A)$ is the minimal polynomial.

Writing $\psi(\lambda) = \lambda(\lambda - 2)$

we have $(A^*A)^+ = \frac{1}{2}\,\frac{(A^*A)}{2} = \begin{pmatrix} 1/4 & -1/4 \\ -1/4 & 1/4 \end{pmatrix}$

and

$$A^+ = (A^*A)^+ A^* = \begin{pmatrix} 1/4 & -1/4 \\ -1/4 & 1/4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 0 \end{pmatrix}.$$

Example Let $A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

$$A^*A = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

The minimal polynomial of (A^*A) is $\psi(\lambda) = \lambda(\lambda-2)(\lambda-4)$

Therefore,

$$(A^*A)^+ = \frac{1}{2}\left(\frac{A^*A(A^*A - 4I)}{2(2-4)}\right) + \frac{1}{4}\left(\frac{A^*A(A^*A - 2I)}{4(4-2)}\right) = \frac{14}{32}(A^*A) - \frac{3}{32}\left(A^*A\right)^2 = \frac{1}{16}\begin{pmatrix} 5 & -1 & -3 & -1 \\ -1 & 5 & -1 & -3 \\ -3 & -1 & 5 & -1 \\ -1 & -3 & -1 & 5 \end{pmatrix}.$$

Hence

$$A^+ = \frac{1}{16}\begin{pmatrix} 5 & -1 & -3 & -1 \\ -1 & 5 & -1 & -3 \\ -3 & -1 & 5 & -1 \\ -1 & -3 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{8} & \frac{-3}{8} & \frac{-1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} & \frac{-3}{8} & \frac{-1}{8} \\ \frac{-1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{-3}{8} \\ \frac{-3}{8} & \frac{-1}{8} & \frac{1}{8} & \frac{3}{8} \end{pmatrix}.$$

Mathematica code

```
a = {{1,0,0,-1}, {-1,1,0,0}, {0,-1,1,0}, {0,0,-1,1}}
    {{1,0,0,-1}, {-1,1,0,0}, {0,-1,1,0}, {0,0,-1,1}}

b = Transpose[a]
    {{1,-1,0,0}, {0,1,-1,0}, {0,0,1,-1}, {-1,0,0,1}}

c = b.a
    {{2,-1,0,-1},{-1,2,-1,0},{0,-1,2,-1},{-1,0,-1,2}}

gi = PseudoInverse[c]
```

$\{\{\frac{5}{16}, -\frac{1}{16}, -\frac{3}{16}, -\frac{1}{16}\}, \{-\frac{1}{16}, \frac{5}{16}, -\frac{1}{16}, -\frac{3}{16}\}, \{-\frac{3}{16}, -\frac{1}{16}, \frac{5}{16}, -\frac{1}{16}\}, \{-\frac{1}{16}, -\frac{3}{16}, -\frac{1}{16}, \frac{5}{16}\}\}$

```
ginverse = gi.b
```

$\{\{\frac{3}{8}, -\frac{3}{8}, -\frac{1}{8}, \frac{1}{8}\}, \{\frac{1}{8}, \frac{3}{8}, -\frac{3}{8}, -\frac{1}{8}\}, \{-\frac{1}{8}, \frac{1}{8}, \frac{3}{8}, -\frac{3}{8}\}, \{-\frac{3}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8}\}\}$

```
% // MatrixForm
```

$$\begin{bmatrix} \frac{3}{8} & -\frac{3}{8} & -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} & -\frac{3}{8} & -\frac{1}{8} \\ -\frac{1}{8} & \frac{1}{8} & \frac{3}{8} & -\frac{3}{8} \\ -\frac{3}{8} & -\frac{1}{8} & \frac{1}{8} & \frac{3}{8} \end{bmatrix}$$

6. REFERENCES

[1] C. R. Rao and S. K. Mitra, Generalized inverse of matrices and its applications, Wiley, New York, 1971.

[2] Graybill F. A., Matrices with applications in Statistics. The Wadsworth Statistics/Probability Series, Wadsworth International Group Belmont, California, 1969.

[3] Moore E. H., 1920 on the reciprocal of the general Algebric matrix, Abst. Bull. Amer. Math. Soc. pp.304-305.

[4] Penrose R., 1955 A generalized inverses for matrices, Proc. Camb. Philo. Soc. 51. pp. 406-413.

[5] Rohde C.A., Some results on generalized inverses, SIAM Review vol. 8. No. 2.