On the Seiberg-Witten Invariants of Smooth 4-Manifolds

by

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Declaration

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Abstract

This thesis reviews Seiberg-Witten Gauge theory and the Seiberg-Witten invariants of smooth 4D manifolds. After reviewing some preliminaries on Clifford Algebras, Spin-bundles, Dirac Operators, We go into discussing a system of mildly nonlinear partial differential equations on a $U(1)$ bundle which are commonly known as Seiberg-Witten equations. We discuss its properties, consider their solution space and then quotient it by the equivalence due to gauge transformations. The moduli space that we get after moding on the space of solutions has some nicer properties as compared to Donaldson's. In the last chapter, we briefly talk about the Witten conjecture which makes a connection between the Seiberg-Witten Invariants and the Donaldson invariants. Many physicists argue that using S-duality, SW theory and Donaldson theory can be viewed as the two extreme cases (one $N \to \infty$, and the other $N \to 0$) of a common theory, but S-duality is not yet mathematically understood fully rigorously. Even with seminal progresses regarding proving this conjecture which is widely believed to be true by many professional physicists- it still remains to be proven true in the general sense. This thesis acts as a review of these ideas as an introduction to Seiberg-Witten theory.

Keywords: Invariants; Seiberg-Witten Invariants; Principal $U(1)$-bundles; Complex Line Bundle; Clifford algebras; Spinor Bundles; Dirac Operator; Seiberg-Witten Gauge theory; Monopoles; Moduli Space; Witten Conjecture; S-duality.

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Chapter 1

Clifford Algebras and Spin Groups

In this chapter, we review some basic results and preliminaries on Clifford Algebras, their complexifications, splittings, the representations of the complexified Clifford Algebras, Pin and Spin groups, Spin-bundles and $Spin^c$-bundles, connection, and curvature. Our discussions follow [7], [6], [12], [5]

1.1 Clifford Algebras

Definition (Clifford Algebras). The Clifford algebra $C(V)$ of a real inner product space $(V,(,))$ is the algebra generated by the elements of V, subject to the relations

$$e\cdot e'+e'\cdot e=-2(e,e') \tag{1.1}$$

It may also be defined as the algebra which is the quotient of the Tensor algebra $T(V)$ associated with V by the two-sided ideal $\mathscr{I}(V)$ generated by all elements of the form

$$v\otimes v+||v||^2 1 \tag{1.2}$$

The multiplicaiton of in the Clifford algebra is called Clifford multiplication.
Given an orthornormal basis $\{e_i\}$ of V,

$$e_{i_1}^{\varepsilon_1}\cdots e_{i_n}^{\varepsilon_n} \tag{1.3}$$

where $\varepsilon_i=0$ or 1 is a basis of $C(V)$ as a vector space.

The $\mathbf{Z}_2$-grading on $T(V)$ descends to a $\mathbf{Z}_2$-grading on $C(V)$ as $C_0(V)\oplus C_1(V)$, where $C_0(V)$ is the image of the quotient map $T(V)\to T(V)/\mathscr{I}(V)=C(V)$ restricted to $T_0(V)$, and similarly for $C_1(V)$, the quotient map restricted to $T_1(V)$. In the language of super-mathematics, this $C(V)$ is a superalgebra, of which $C_0(V)$ is a subalgebra, and $C_1(V)$ is a $C_0(V)$ module over this algebra.

Defintion (Complex Spinor Representation). If V is even dimensional, then there is a unique irreducible representation of $C(V)$ on a complex inner product space S such that the elements of V act as skew-hermitian (or anti-hermitian) operators. This is called the complex spinor representation and it is a 2^m-dimensional representation of $C(V)$. This representation has a spectral decomposition $S=S^+\oplus S^-$ with respect to the action of the volume form of $C(V)$.

Example 1. $C(\mathbf{R}^1)$, the Clifford algebra associated with the 1D real vector space is supposed to be

$$C(\mathbb{R}) = T(\mathbb{R}^1)/\mathscr{I}(\mathbb{R}^1) = \frac{\mathbb{R}\oplus\mathbb{R}\oplus(\mathbb{R}\otimes\mathbb{R})\oplus\dots}{\mathscr{I}(\mathbb{R}^1)} \cong \frac{\mathbb{R}[R]}{\langle(x^2+1)\rangle} \cong \mathbb{C} \tag{1.4}$$

Example 2. $C(\mathbb{R}^2)$, the Clifford algebra associated with the 2D real vector space is supposed to be

$$C(\mathbb{R}^2) = T(\mathbb{R}^2)/\mathscr{I}(\mathbb{R}^2) = \frac{\mathbb{R}\oplus\mathbb{R}^2\oplus(\mathbb{R}^2\otimes\mathbb{R}^2)\oplus\dots}{\langle(x^2+1),(y^2+1),(xy+yx)\rangle} \cong \mathbb{H} \tag{1.5}$$

It's easier to see that $C(\mathbb{R}^2) \cong \mathbb{H}$ by considering the vector space structure of $C(\mathbb{R}^2)$. Let $\{e_1, e_2\}$ be an ONB of $\mathbb{R}^2$. Then $\{1, e_1, e_2, e_1 \cdot e_2\}$ will be a basis for $C(\mathbb{R}^2)$, and we can have an isomorphism of algebras between $C(\mathbb{R}^2)$ and $\mathbb{H}$ by mapping $1 \mapsto 1$, $e_1 \mapsto i$, $e_2 \mapsto j$, and $e_1 \cdot e_2 \mapsto k$.

Result 1. For any inner product space V, we have an isomorphism of algebras

$$C(V) \cong C_0(V \oplus \mathbb{R}) \tag{1.6}$$

Let e be a unit vector in $\mathbb{R}$. Then the map $\phi : C(V) \to C_0(V \oplus \mathbb{R})$ which takes a sum of two homogeneous elements $v_0 + v_1$ ($v_0 \in C_0(V)$ and $v_1 \in C_1(V)$) and turns it into: $v_0 + v_1 e$ is an isomorphism of algebras.

Definition (Clifford Algebra of the Tangent Bundle of a Riemannian Manifold). Given a Riemannian manifold M, at each point $p \in M$, we have $(T_pM.(,))$ to be an inner product space. Then we can consider a vector bundle on M which has the Clifford algebra $C(T_pM)$ as its fibers at each $p \in M$, which we call the Clifford algebra of the tangent bundle of X, and denote by $C(TM)$.

Definition (Spinor Bundle on a Riemannian Manifold). Let M be a hermitian vector bundle W of complex rank 2^m with a map $\# : TM \to End(W)$ saisfying $\#(v) + \#(v^*) = 0$ and $\#(v) \cdot \#(v^*) = -|v|^2 Id$ for all $v \in TM$. A spin connection on W is a connection compatible with the Levi-Civita connection ∇^{LC}, that is, a connection

$$\nabla : \mathscr{C}^\infty(W) \to \mathscr{C}^\infty(T^*M \otimes W) \tag{1.7}$$

such that given any two vector fields u and v on M and any sections s of W, we have

$$\nabla_u(v \cdot s) = \nabla_u^{LC}(v) \cdot s + v \cdot \nabla_u(s) \tag{1.8}$$

1.2 $Pin(V)$ and $Spin(V)$ groups

Let $C^\times(V)$ be the multiplicative group of units of the Clifford algebra $C(V)$.

Definition ($Pin(V)$). The group $Pin(V)$ is defined as the subgroup of $C^\times(V)$ generated by elements $v \in V$ with $||v||^2 = 1$. The given generators of $Pin(V)$ are units, since the square of any of them is -1 in the Clifford algebra.

Definition ($Spin(V)$). The group $Spin(V)$ is defined as the intersection of $Pin(V)$ with $C_0(V)$, in other words, the kernel of the group homomorphism $Pin(V) \to \mathbb{Z}_2$, induced by the splitting $C_0(V) \oplus C_1(V)$. One can also define it to be the universal covering group of $SO(V)$.

Example. $Pin(1) \equiv Pin(\mathbb{R})$ is basically the subgroup of $\mathbb{C}^\times$ generated by $\pm i$. So it is a cyclic group of order 4. $Spin(1)$ then would be a cyclic group of order two $\{+1, = 1\}$ inside $\mathbb{R}$.
Example. $Pin(2) \cong S^1 \times S^1$, $Spin(2) \cong S^1$. $Spin(3) \cong S^3$. $Spin(4) \cong SU(2) \times SU(2)$.

Remark. $Pin(V)$ contains a vector space basis for $C(V)$, and $Spin(V)$ contains a vector space basis for $C_0(V)$.
If $e_1, \ldots, e_n$ is an ONB for V, then every product $e_{i_1} \cdots e_{i_k}$ is an element of $Pin(V)$. Taking $k = 1$ here lets us have $e_1, \cdots, e_n$ is contained in $C(V)$. A similar argument holds for $Spin(n)$.

Proposition. Two (real or complex) representations of the algebra $C_0(V)$ whose restrictions to $Spin(V)$ are isomorphic representations are in fact isomorphic representations of the algebra.
Suppose two modules A and A' for $C_0(V)$ admit a linear isomorphism ϕ which commutes with the $Spin(V)$ actions. Then ϕ commutes with the actions of an $\mathbb{R}$-basis of $C_0(V)$ (since Spin(V)) contains an orthonormal basis of $C_0(V)$) and hence commutes with the $Cl_0(V)$ actions. This implies ϕ is an isomorphism of $C_0(V)$-modules.

1.3 Splitting and Complexification of Clifford Algebras

Let V be an oriented real inner product space and $C(V)$ be the Clifford algebra associated with V. We consider the **complexification** of the Clifford algebra as the complex algebra $C(V) \otimes_{\mathbb{R}} \mathbb{C}$.
Let $e_1, \ldots, e_n$ be an oriented orthonormal basis for V. Then we define

$$\omega_{\mathbb{C}} = i^{[\frac{n+1}{2}]} e_1 \cdots e_n \implies \omega_{\mathbb{C}}^2 = 1 \tag{1.9}$$

$\omega_{\mathbb{C}}$ is independent of the choice of an oriented orthonormal basis. Since it is basis independent, and squared to 1, the action on $C(V) \otimes_{\mathbb{R}} \mathbb{C}$ is an idempotent, so it has an eigenvalue of ± 1. We then decompose $C(V) \otimes_{\mathbb{R}} \mathbb{C}$ into $(C(V) \otimes_{\mathbb{R}} \mathbb{C})^+$ and $(C(V) \otimes_{\mathbb{R}} \mathbb{C})^-$ based on the eigenspaces.

Remark. If $dim(V)$ is odd, then $\omega_{\mathbb{C}}$ is in the center of $C(V) \otimes_{\mathbb{R}} \mathbb{C}$ and that $(C(V) \otimes_{\mathbb{R}} \mathbb{C})^\pm$ are subalgebras which annihilates each other- whereas if $dim(V)$ is even, then it is in the center of $C_0(V) \otimes_{\mathbb{R}} \mathbb{C}$ but anti-commutes with the elements in $C_1(V) \otimes_{\mathbb{R}} \mathbb{C}$.

Lemma. If $dim(V)$ is odd, then the algebras $(C(V) \otimes_{\mathbb{R}} \mathbb{C})^\pm$ are both isomorphic to $(C_0(V) \otimes_{\mathbb{R}} \mathbb{C})$.
Multiplication by $\omega_{\mathbb{C}}$ interchanges $(C_0(V) \otimes_{\mathbb{R}} \mathbb{C})$ and $(C_1(V) \otimes_{\mathbb{R}} \mathbb{C})$. Therefore, we have the following composition

$$C_0(V) \otimes \mathbb{C} \to C(V) \otimes \mathbb{C} \to (C(V) \otimes \mathbb{C})^\pm \tag{1.10}$$

where the first map is just an inclusion, and the second map is the projection over any of the $\pm$ subalgebras, and this is an isomorphism of algebras.

Example. $C(\mathbb{R}^1) \cong \mathbb{C}$, so $C(\mathbb{R}^1) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. The decomposition of $C(\mathbb{R}^1) \otimes_{\mathbb{R}} \mathbb{C}$ into $(C(\mathbb{R}^1) \otimes_{\mathbb{R}} \mathbb{C})^+$ and $(C(\mathbb{R}^1) \otimes_{\mathbb{R}} \mathbb{C})^-$ then corresponds to the usual decomposition

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \tag{1.11}$$

Example. $C(\mathbb{R}^2) \cong \mathbb{H}$, and so $C(\mathbb{R}^2) \otimes_{\mathbb{R}} \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}$. Define a map from $\mathbb{H}$ to $\mathbb{C}[2]$- the algebra of 2×2 complex matrices by

$$(\alpha + j\beta) \mapsto \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \tag{1.12}$$

Wriiting element of $\mathbb{H}$ as $x + jy$ where $x, y \in \mathbb{C}$, this matrix basically gives us the action of $\alpha + j\beta$ by left multiplication on $\mathbb{H}$ viewed as $\mathbb{C}^2$. Extending scalars, we get a homomorphism of complex algebras $\mathbb{H} \otimes \mathbb{C} \to \mathbb{C}[2]$. In fact, this extension is an isomorphism and gives us an identification of $C(\mathbb{R}^2) \otimes \mathbb{C}$ with the matrix algebra $\mathbb{C}[2]$.
Note. These computations are enough to determine the structure of the complexifications of all the Clifford algebras by induction.
Example. $C(\mathbb{R}^3) \cong \mathbb{H} \oplus \mathbb{H}$ which is already splitted. Thus, in this case, the splitting complexification of the complex algebra is induced from the splitting in the real algebra.

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \tag{1.13}$$

Lemma. If the dimension of V is congruent to $3 mod 4$, then $C(V)$ splits as an orthogonal sum of two algebras, $C(V) = C(V)^+ \oplus C(V)^-$ induces the above splitting on the complexified algebras.
$dimV \equiv 3 \text{ mod } 4 \implies \omega_{\mathbb{C}} = (-1)^{\frac{n+1}{4}} e_1 \cdots e_n$ and is therefore contained in the real algebra, and so its ± 1 eigensppaces are real subspaces.

Lemma.

$$C(V \oplus \mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C} \cong (C(V) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (C(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C}) \tag{1.14}$$

Let $v_1, v_2, \cdots, v_n$ be an ONB for V. Let e_1, e_2 be the standard basis for $\mathbb{R}^2$. Then we define a map,

$$V \oplus \mathbb{R}^2 \to (C(V) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (C(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C}) \tag{1.15}$$

by sending v_j to $iv_j \otimes e_1 e_2$ for all $1 \leq j \leq n$ and by sending e_r to $1 \otimes e_r$. This map satisfies the condition to extend to an algebra homomorphism

$$C(V) \to (C(V) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (C(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C}) \tag{1.16}$$

and by the extension of scalars to a map from $C(V) \otimes_{\mathbb{R}} \mathbb{C}$ to this tensor product.

Some Corollaries to this Lemma.

- If V is even dimensional, then $C(V) \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the matrix algebra $\mathbb{C}[2^n]$.

- If V is odd dimensional, then $C(V)\otimes_{\mathbb{R}}\mathbb{C}$ is isomorphic to the direct sum of two copies of $\mathbb{C}[2^n]$ as an algebra.
- If V is even dimensional, then $(C_0(V)\otimes_{\mathbb{R}}\mathbb{C})^+\cong\mathbb{C}[2^{n-1}]$
- If V is even dimensional, then $C(V)$ has a unique irreducible finite dimensional complex representation $S_{\mathbb{C}}(V)$ upto isomorphism. Any such representation has dimension = 2^n. The action of $C(V)\otimes\mathbb{C}$ on $S_{\mathbb{C}}(V)$ induce an isomorphism

$$C(V)\otimes\mathbb{C}\to End_{\mathbb{C}}(S_{\mathbb{C}}(V))=S_{\mathbb{C}}(V)\otimes S_{\mathbb{C}}(V)^* \tag{1.17}$$

- If V is odd dimensional, then $C(V)$ has exactly two irreducible, finite dimensional, complex representations upto isomorphism. These induce isomorphic representations of $C_0(V)$ by restriction. Any such representation has dimension 2^n, and the action of Clifford multiplication induces a map

$$C_0(V)\otimes\mathbb{C}\to End_{\mathbb{C}}(S_{\mathbb{C}}(V))=S_{\mathbb{C}}(V)\otimes S_{\mathbb{C}}(V)^* \tag{1.18}$$

The proof of the last two lemmas involve a theorem in Representation theory called Wedderburn's theorem, which tells us that $\mathbb{C}[n]$ has a unique irreducible, finite dimensional, complex representation $S^n_{\mathbb{C}}$ up to isomorphism, and that furthermore, the map

$$\mathbb{C}[n]\to End(S^n_{\mathbb{C}}) \tag{1.19}$$

for this representation is an isomorphism of algebras. The result for even dimensional V's follows from this.

Some important results.

- Suppose V is even dimensional. Let $S_{\mathbb{C}}(V)$ be an irreducible (complex) representation of $C(V)\otimes\mathbb{C}$. Then, $S_{\mathbb{C}}(V)$ decomposes into $S_{\mathbb{C}^{\pm}}(V)$ under the action of $\omega_{\mathbb{C}}$.
- This is a decomposition of modules over $C_0(V)\otimes\mathbb{C}$ whereas the action of $C_1(V)\otimes\mathbb{C}$ interchanges $S^{\pm}_{\mathbb{C}}(V)$.
- Clifford multiplicaiton induces the following isomorphisms

$$(C_0(V)\otimes\mathbb{C})^+\cong End_{\mathbb{C}}(S^+_{\mathbb{C}}(V)) \tag{1.20}$$

$$(C_0(V)\otimes\mathbb{C})^-\cong End_{\mathbb{C}}(S^-_{\mathbb{C}}(V)) \tag{1.21}$$

$$(C_1(V)\otimes\mathbb{C})^-\cong Hom_{\mathbb{C}}(S^+_{\mathbb{C}}(V),S^-_{\mathbb{C}}(V)) \tag{1.22}$$

$$(C_1(V)\otimes\mathbb{C})^+\cong Hom_{\mathbb{C}}(S^-_{\mathbb{C}}(V),S^+_{\mathbb{C}}(V)) \tag{1.23}$$

- $S^{\pm}_{\mathbb{C}}(V)$ are the only two inequivalent irreducible representations of $C_0(V)\otimes\mathbb{C}$ up to isomorphism.

Most of these results follow directly from the fact that the Clifford multiplication induces an isomorphism

$$C(V) \cong End(S_{\mathbb{C}}(V)). \tag{1.24}$$

- There is a unique complex representation of $Spin(V)$ upto isomorphism induced from any irreducible finite dimensional representation of $C(V)$. This representation is call the **complex spin representation**, and is denoted by

$$\Delta_{\mathbb{C}}\colon Spin(V) \to Aut_{\mathbb{C}}(S_{\mathbb{C}}(V)) \tag{1.25}$$

This follows from the previous results and we also use the fact that $Spin(n) \subseteq C_0(V)$.

1.4 The Complex Spin Representation

There is a unique complex representation of $Spin(V)$ up to isomorphism induced from the irreducible complex finite dimensional representation of $C(V)$- which is called the **complex spin representation** and is denoted by $\Delta_{\mathbb{C}}\colon Spin(V) \times S_{\mathbb{C}}(V) \to S_{\mathbb{C}}(V)$.

Some Results that follow.

- If V is even dimensional, then this representation $\Delta_{\mathbb{C}}$ decomposes into two inequivalent irreducible representations of $Spin(V)$

$$\Delta_{\mathbb{C}}^{+} : Spin(V) \times_{\mathbb{C}}^{+} (V) \to S_{\mathbb{C}}^{+}(V) \tag{1.26}$$

$$\Delta_{\mathbb{C}}^{-} : Spin(V) \times S_{\mathbb{C}}^{-}(V) \to S_{\mathbb{C}}^{-}(V) \tag{1.27}$$

- If $dim(V) = 2n$, then these two representations $\Delta_{\mathbb{C}}^{+}$ and $\Delta_{\mathbb{C}}^{-}$ are 2^{n-1} dimensional.

Example. $C(\mathbb{R}) \cong \mathbb{H}$, and hence $C(\mathbb{R}) \otimes \mathbb{C}$ is isomorphic to $\mathbb{C}[2]$. Hence the spin representation is basically the map

$$\Delta_{\mathbb{C}}\colon Spin(\mathbb{R}^2) \to Aut(\mathbb{C}^2). \tag{1.28}$$

This representation decomposes as a sum of two 1D complex representations

$$\Delta_{\mathbb{C}}^{\pm}\colon Spin(\mathbb{R}^2) \to Aut(S_{\mathbb{C}}^{\pm}(\mathbb{R}^2)) \tag{1.29}$$

But, $Spin(\mathbb{R}^2) \cong S^1$ embedded in the standard way in $\mathbb{C} \subset \mathbb{H}$. Under the embedding $\mathbb{H} \subset \mathbb{C}[2]$, this circle S^1 is embedded as

$$\alpha \in S^1 \to \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}. \tag{1.30}$$

Since in this case, $\omega_{\mathbb{C}} = ie_1e_2$, the element e_1e_2 acts on $S_{\mathbb{C}}^{+}\mathbb{R}^2$ by $-i$ and thus $S_{\mathbb{C}}^{+}\mathbb{R}^2$ has the action conjugate to the standard action of S^1 on $\mathbb{C}$. Similarly, $S_{\mathbb{C}}^{-}\mathbb{R}^2$ has the standard action.

Example. $Spin(\mathbb{R}^3) \cong SU(2)$ as both of them are double covers of $SO(3)$. Th spin representation $\Delta_{\mathbb{C}}$ is the standard representation of $SU(2)$ on $\mathbb{C}^2$.

Example. $Spin(\mathbb{R}^4) \cong SU(2) \times SU(2)$. The spin representation $\Delta_{\mathbb{C}}^{+}$ is the projection of $Spin(4)$ onto the first factor, followed by the standard representation of $SU(2)$ on $\mathbb{C}$, whereas the spin representation $\Delta_{\mathbb{C}}^{-}$ is the projection of $Spin(4)$ onto the second factor, followed by the standard representation of $SU(2)$ on $\mathbb{C}$.

1.5 *Spin* and *Spin*^{c}-structures

A Riemannian manifold M equipped with a spinor bundle as defined above is said to have a spin structure. Not all manifolds admit a spinor bundle, and the existence of such a bundle is equivalent to a $Spin^c$-structure on M, which we will discuss shortly. But before that, we need the notion of $Spin^c$ groups.

Definition 1. ($Spin^c$ group) $Spin^c(V)$ is the multiplicative group of units of $C(V) \otimes_{\mathbb{R}} \mathbb{C}$ generated by $Spin(V)$ and the unit circle of complex scalars.

Definition 2. ($Spin^c$ group) The group $Spin^c(n) := (Spin(n) \times U(1))/\mathscr{Z}_2$ is an extension

$$1 \to \mathscr{Z}_2 \to Spin^c(n) \to SO(n) \times U(1) \to 1. \tag{1.31}$$

This yields the following long exact sequence in sheaf cohomology:

$$\dots \to H^1(M; Spin^c(n)) \to H^1(M; SO(n)) \oplus H^1(M; U(1)) \to H^2(M; \mathscr{Z}_{\in}) \to \dots \tag{1.32}$$

Here, $H^1(M;G)$ represents the equivalence classes of principal G-bundles over M.

The connecting homomorphism of the sequence $H^1(M;SO(n)) \oplus H^1(M;U(1)) \to H^2(M;\mathscr{Z}_{\in})$ is given by

$$\delta : (P_{SO(n)}, P_{U(1)}) \mapsto w_2(P_{SO(n)}) + \bar{c}_1(P_{U(1)}), \tag{1.33}$$

where $\bar{c}_1(P_{U(1)})$ is the reduction mod 2 of the first Chern class of the principal bundle $P_U(1)$ and w_2 is the second Stiefel-Whitney class.

1.5.1 Spin Bundles

Let $(V, (,))$ be a real inner product space with $\dim V = n \geq 2$. Suppose $P \to M$ is a principal $SO(n)$-bundle. We want this bundle P to lift to a principal $Spin(n)$-bundle $\tilde{P} \to M$ on M, meaning that we want to find a principal $Spin(n)$-bundle $\tilde{P} \to M$ whose quotient by the center $\{-1, 1\}$ of $Spin(n)$ is isomorphic as an $SO(n)$-bundle on M to P. This is a standard problem in obstruction theory, the soltution to which is quite well-known.

Result. The $SO(V)$-bundle $P \to M$ lifts to a $Spin(V)$-bundle if and only if the Second Stiefel-Whitney class $w_2(P) \in H^2(M; \mathbb{Z}_2)$ is equal to zero.

Definition. (Spin Structure for a Principal $SO(n)$-bundle) If a lifting to a principal $Spin(V)$-bundle $\tilde{P} \to M$ exists for a principal $SO(V)$-bundle $P \to M$, then that lifting is called a Spin structure for the principal $SO(V)$-bundle P on M.

Definition. (Spin Structure for a Manifold) In the special case when P happens to be the tangent or cotangent bundles TM or T^*M of a Riemannian manifold, then the lifting, if it exists, is called a spin structure for the manifold.
Definition. (Associated complex vector Bundle) Suppose that $P \to M$ is a principal $SO(n)$- bundle with a spin structure $\tilde{P} \to M$. Then there is an associated complex spin bundle

$$\tilde{P} \times_{Spin(n)} S_{\mathbb{C}}(\mathbb{R}^n) \tag{1.34}$$

induced by the representation $\Delta_{\mathbb{C}} : Spin(n) \to Aut(S_{\mathbb{C}}(\mathbb{R}^n))$, which we denote by $S_{\mathbb{C}}(\tilde{P})$ For even n, the decomposition of $\Delta_{\mathbb{C}}$ into $\Delta_{\mathbb{C}}^+ \oplus \Delta_{\mathbb{C}}^-$ corresponds to a decomposition of the associated complex vector bundle

$$S_{\mathbb{C}}(\tilde{P}) = S_{\mathbb{C}}^+(\tilde{P}) \oplus S_{\mathbb{C}}^-(\tilde{P}) \tag{1.35}$$

$$S_{\mathbb{C}}^{\pm}(\tilde{P}) = \tilde{P} \times S_{\mathbb{C}}^{\pm}(\mathbb{R}^n) \tag{1.36}$$

These $S_{\mathbb{C}}^{\pm}(\tilde{P})$ are called the plus and minus spin bundles associated with $\tilde{P}$.

Some results.

- These plus and minus spin bundles are complex vector bundles of complex dimensions $2^{\frac{n}{2}-1}$
- $Spin(n)$ is compact.
- These bundles $S^{\pm}_{\mathbb{C}}(\tilde{P})$ carry hermitian inner products unique up to isomorphism.
- We can choose the metric to be invariant under the action of $Pin(\mathbb{R}^n)$ since the bundles are induced by an action of the Clifford Algebra on $S_{\mathbb{C}}(\mathbb{R}^n) \implies$ Clifford multiplication by a unit vector in $\mathbb{R}^n \subset C(\mathbb{R}^n)$ is an isometry of $S_{\mathbb{C}}(\mathbb{R}^n)$.

From here on, we will implicitly assume that we are working with the metric that is invariant under the action of $Pin(\mathbb{R}^n)$ as per the result.

1.5.2 $Spin^c$ Bundles

We ask an analogous question as before. Given a principal $Spin^c(n)$-bundle $\tilde{P} \to M$ on a smooth manifold M, does a lifting of $\tilde{P}$ to a principal $SO(n)$-bundle $P \to M$ exist?

Definition (Determinant Line Bundle associated with a Principal $Spin^c(n)$- bundle) We can have a map $Spin^c(n) \to SO(n)$ where we divide out the map $Spin(n) \to SO(n)$ (double covering) by the center of $Spin(n)$. The homomorphism $Spin^c(V) \to S^1$ given by dividing out by $Spin(n)$ determines a complex line bundle $\mathscr{L} \to M$- which is associated with any principal $Spin^c(n)$- bundle. This is called the determinant line bundle of the $Spin^c(n)$-bundle.

Definition ($Spin^c$-structure). A $Spin^c$-structure on an oriented n-dimensional Riemannian manifold M. is a lift of the bundle Fr of oriented orthonormal frames to a principal $Spin^c(n)$-bundle.

If the Principal $Spin^c$-bundle lifts to a $SO(n)$-bundle, then the determinant line bundle $\mathscr{L}$ has a first Chern class $c_1(\mathscr{L})$ which agrees mod 2 with the second Stiefel-Whitney class. Conversely, given any line bundle $\mathscr{L} \to M$ whose first Chern-class satisfies this mod 2 equation, there is a $Spin^c(n)$-lifting of P with determinant line bundle isomorphic to $\mathscr{L}$.

Proposition. Let M be an oriented $4-$manifold and let $P \to M$ be the frame bundle of the tangent bundle. Then there is a lifting $\tilde{P}$ of P to a $Spin^c(4)$-bundle.

One of the great advantages of $Spin^c$-structures in studying 4-manifolds is that every oriented $4-$manifold possesses one.

Clifford Bundle and Their Actions on the Spin bundles. Let P be a $SO(n)-$bundle. We thus can form the associated complex spin bundles to a $Spin$ or $Spin^c$-bundle $\tilde{P}$ lifting P. We can also form bundles of complexified Clifford Algebras associated to P.
Since $SO(n)$ acts on the Clifford algebra $C(\mathbb{R}^n)$, we can associate to $P \to M$ a bundle without the need of a spin structure.

$$C(P) = P \times_{SO(n)} C(\mathbb{R}^n) \tag{1.37}$$

which is a locally trivial bundle of Clifford algebras. The complex version

$$C(P) \otimes \mathbb{C} = P \times_{SO(n)} (C(\mathbb{R}^n) \otimes \mathbb{C}) \tag{1.38}$$

a bundle of complexified Clifford algebras. These bundles decompose as

$$C(P) = C_0(P) \oplus C_1(P) \tag{1.39}$$

$$C(P) \otimes \mathbb{C} = (C(P) \otimes \mathbb{C})^+ \oplus (C(P) \otimes \mathbb{C})^- \tag{1.40}$$

In the presence of a spin (or $Spin^c$) structure $\tilde{P}$ on P, these Clifford bundles then act on the complex spin bundles. Let $S_{\mathbb{C}}(\tilde{P}) \to M$ be the associated complex spin bundle.

1.6 Connections and Curvature

Let $P \to M$ be a smooth principal G-bundle over a smooth manifold. At each point $p \in P$, we have the vertical tangent space $T^v P_p$. This is the subspace of the tangent space of P which is tangent to the fiber of the projection mapping.
Defintion (Connections on Principal Bundle) A connection on a principal bundle is a distribution $\{H_p\}_{p \in P}$, i.e. a smoothly varying family of linear subspaces of the tangent bundle TP which is everywhere complementary to the vertical distribution and which is invariant under the action of the group G.
The condition that the distribution be complementary to the vertical distribution simply means that under the project.ion mapping each linear subspace of the distribution projects isomorphically onto the tangent space to M at the image point. This condition is expressed by calling the distribution the horizontal distribution

Definition. (Connection one-form) Given a connection on $P \to M$ there is an associated 1-form ω on P with values in the adjoint bundle $ad(P)$ of P, i.e. the vector bundle associated to P and the adjoint action of G on its Lie algebra g. This one-form is called the connection one-form. Its values at any $p \in P$ is a linear map ω_p defined as follows:

$$\omega_p \colon TP_p \to T^v P_p \cong g \tag{1.41}$$

where the first map is the projection with kernel H_p, and the second map is the inverse of the isomorphism induced by the action of G at p. The equivariance property of the distribution translated into the condition that ω transforms by the adjoint action:

$$(\forall h \in G)(\forall p \in P)(\forall \tau \in TP_p)\omega_p h(\tau \cdot h) = h^{-1}\omega_p(\tau)h. \tag{1.42}$$

Also, the restriction of ω_p to the fiber of the projection is identified with the left invariant Mauer-Cartan form on G. This is the form on G with values in g whose value at any $\tau \in TG_h$ is equal to $h^{-1}\tau \in$ g. It is the unique form on G which is invariant under left multiplication by any element in the group and which is the the identity map at the identity of the group.

These two properties characterize the connection one-forms. Given a connection one-form, one can recover the horizontal distribution as the kernels of the one-form.

Definition. (Induced connection on a Vector Bundle) Given a connection on a principal bundle P, there is an induced connection on any vector bundle $E = P \times_G V$ coming from a linear representation of G on a vector space V. The natural way to view this connection is as a covariant derivative as follows.

$$\nabla\colon \Omega^0(M;E) \to \Omega^1(M;E) \tag{1.43}$$

which is

- linear over the scalars
- a derivation over the scalar-valued functions with respect to the usual d, i.e. for any section $\sigma \in \Omega^0(M;E)$ and any scalar-valued function f on M, we have

$$(f \cdot \sigma) = f \cdot \nabla(\sigma) + df \otimes \sigma \tag{1.44}$$

1.6.1 Formulae for the Connection One-form and the Covariant Derivative in a Local Trivialization

Let $P \to M$ be a principal G-bundle with a connection $\omega \in \Omega^1(P;g)$. We fix a local trivialization of $P|_U$ for some open subset $U \subset M$. We can view this trivialization as a section $\sigma_0 : U \to P|_U$. We then have the form $\sigma_0 * (\omega) \in \Omega^1(U;g)$. This is the connection one-form with respect to the given trivialization.

If $G \subset GL(n,\mathbb{R})$ so that $g \subset M_{nxn}(\mathbb{R})$, then $\sigma_)^*(\omega)$ is a matrix valued one-form $(\omega_{i,j})$ whose value at every point belongs to the subspace g. For example, if $G = SO(n)$, then $(\omega_{i,j})$ is a skew-symmetric matrix-valued one form; i.e.,

$$\omega_{i,j} = -\omega_{j,i} \tag{1.45}$$

Suppose that $\sigma : U \to P|_U$ is another section. Then there is a smooth map $h : U \to G$ such that $\sigma(u) = \sigma_0(u)h(u)$. It follows that

$$\sigma^*(\omega)(u) = h(u)^-1\sigma_0^*(\omega)(u)h(u) + h(u)^{-1}dh(u). \tag{1.46}$$

In other words, if $\sigma(u) = (u,h(u))$ with respect to the original trivialization and if τ is a tangent vector to U at u, then

$$\omega(\frac{\partial\sigma}{\partial\tau}) = h(u)^-1\tilde{\omega}(\tau)h(u)h(u)^-1\frac{\partial h}{\partial\tau}(u) \tag{1.47}$$

1.6.2 The Curvature of a Connection

For one-forms ω on P with values in g, we denote by $\frac{1}{2}\omega\wedge\omega$ the two-form whose value on a pair of tangent vectors (τ_1,τ_2) and a point p is given by

$$[\omega_p(\tau_1),\omega_p(\tau_2)]. \tag{1.48}$$

Lemma. Let ω be a connection one-form on a principal G-bundle over M. Then the two-form

$$d\omega + \frac{1}{2}\omega \wedge \omega \tag{1.49}$$

is a two-form on P which is induced via the pullaback from a two-form on M with values in adP.

Definition (Curvature form of the connection). The two form on M- the existence of which is guaranteed by the previous lemma, is called the curvature form of the connection.

For a covariant derivative ∇ in a vector bundle E associated to a principal G-bundle P over M, and a representation of G on a vector space V, we have the operator

$$\nabla \circ \nabla :^0 (M;E) \to \Omega^2(M;E) \tag{1.50}$$

is linear over the functions, and hence is a section of $\Omega^2(M;End(E))$. This section is the image of the curvature of the connection under the map $g \to End(V)$ induced by the action of G on V. Using a local trivialization, we can write that

$$\nabla \circ \nabla(e_i) = \nabla(\sum_j (\omega_{j,i} \otimes e_j)) \tag{1.51}$$

$$\nabla \circ \nabla(e_i) = \sum_j d\omega_{j,i} \otimes e_j - \sum_j \omega_{j,i} \wedge \nabla(e_j) \tag{1.52}$$

$$\nabla \circ \nabla(e_i) = \sum_j d\omega_{j,i} \otimes e_j - \sum_k \omega_{k,i} \wedge (\sum_j \omega_{j,k} \otimes e_j) \tag{1.53}$$

$$\nabla \circ \nabla(e_i) = \sum_j d\omega_{j,i} \otimes e_j + (\sum_j \sum_k \omega_{j,k} \wedge \omega_{k,i}) \otimes e_j \tag{1.54}$$

$$\nabla \circ \nabla(e_i) = \sum_j (d\omega_{j,i} + \frac{1}{2}(\omega \wedge \omega)_{j,i}) \otimes e_j \tag{1.55}$$

$$\nabla \circ \nabla(e_i) = \sum_j \Omega_{j,i} \otimes e_j \tag{1.56}$$

Which means that

$$\nabla_{e_r} \circ \nabla_{e_s} - \nabla_{e_s} \circ \nabla_{e_r} = \Omega(e_r, e_s) \tag{1.57}$$

as sections of endomorphism bundle of $E|U$.

The space of connection one-forms for a bundle There is more than one possible connection form on a principal bundle $\pi : P \to M$. For any connection one-form ω and for any one-form η on M with values in adP, the sum $\omega + \pi^*\eta$ is a connection one-for as well. A bit non-trivial fact is that one can obtain all connection one-forms on P like this... So the space of connection one-forms for P becomes an affine space associated to the vector space $\Omega^1(M;adP)$.

1.7 Action of the Group of Changes of Gauge

Definition (Change of Gauge) A *change of gauge* is nothing but a bundle automorphism of the principal bundle, i.e. gauge group $\mathscr{G} = Aut(P \to M)$.
We can consider the action of the group of automorphisms of $P \to M$ on the space of connection one-forms. If $\phi : P \to P$ is a diffeomorphism commuting with the action of G and with the projection to M, then ϕ is said to be an automorphism of P. These from a group: $Aut(P;M)$ or just simply $Aut(P)$. These bundle automorphisms can be interpreted as a function from $P \to G$ which satisfies $\psi(ph) = h^{-}1\psi(p)h$ for any $h \in G$.

If ω is a connection $one - form$ and ϕ is a bundle automorphism, then $\phi^*\omega$ is also a connection one form, where

$$\phi^*(\omega) = \phi^{-1}\omega\phi + \phi^{-1}d\phi \tag{1.58}$$

The effect of this action on Ω, the curvature of the connection is to conjugate it by ϕ, i.e. if Ω is the curvature from for the connection ω and Ω' is that for $\phi^*\omega$, then

$$\Omega' = \phi^{-1}\Omega\phi \tag{1.59}$$

So the noem-squared of the curvature is left invariant by the action of the group of changes of gauges.

Chapter 2

The Dirac Operator

The discussions here mostly follow [7], and
Let M be a Riemannian manifold; let $P \to M$ be the $SO(n)$-principal bundle associated to the tangent bundle; let $\tilde{P}$ be a lifting of this bundle to a *Spin*-bundle, or a *Spin*^{c}-bundle. Let $S_{\mathbb{C}}(\tilde{P})$ be the associated spin bundle, which is a complex vector bundle inherently.
In the case of a *Spin*^{c}-bundle, we also need to fix a $U(1)$-connection $\mathscr{A}$ on the determinant line bundle $\mathscr{L} \to M$. Let $\tilde{\nabla}$ be the spin connection induced by the Levi-Civita connection and the connection $\mathscr{A}$.
Definition The Dirac operator is defined as a map

$$\mathscr{D}^{\mathscr{A}} : C^\infty(S_{\mathbb{C}}(\tilde{P})) \to C^\infty(S_{\mathbb{C}}(\tilde{P})) \tag{2.1}$$

which is defined as follows.
Take $\{e_1, \dots, e_n\}$ to be an oriented orthonormal frame for TM_p. Then the operator can be locally defined as

$$\mathscr{D}^{\mathscr{A}}(\sigma)(p) = \sum_{i=1}^{n} e_i \cdot \tilde{\nabla}_{e_i}(\sigma)(p) \tag{2.2}$$

where the $\cdot$ is the Clifford multiplication. This is in the case of a *Spin*^{c}-bundle. For a *Spin*-bundle, there is no connection $\mathscr{A}$ and we then simply denote it by $\mathscr{D}$.

Lemma. The operators $\mathscr{D}^{\mathscr{A}}$ and $\mathscr{D}$ are independent of the choice of orthonormal frame $\{e_1, \dots, e_n\}$
Proof. Suppose that $\{e'_1, \dots, e'_n\}$ is another oriented orthonormal frame. Suppose that

$$e'_i = \sum_{j=1}^{n} B_{i,j} e_j \tag{2.3}$$

Then, B is an element of $SO(n)$. Let us consider

$$\sum_{i=1}^{n} e'_i \cdot \tilde{\nabla}_{e'_i}(\sigma) \tag{2.4}$$

Using the bilinearity of the Clifford multiplication, we can say that for any $\alpha \in S_{\mathbb{C}}(\tilde{P})$,

$$e'_i \cdot \alpha = \sum_{j=1}^{n} B_{i,j} e_j \cdot \alpha \tag{2.5}$$

Also, since $\tilde{\nabla}_e$ is linear in e, we see that

$$\tilde{\nabla}_{e'_i}(\sigma) = \sum_{j=1}^{n} B_{i,j} e_j \tilde{\nabla}_{e'_j}(\sigma) \tag{2.6}$$

Combining these, we have

$$\sum_{i=1}^{n} e'_i \cdot \tilde{\nabla}_{e'_i}(\sigma) = \sum_{i=1}^{n}(\sum_{j=1}^{n} B_{i,j} e_j \cdot (\sum_{j'=1}^{n} B_{i,j'} \tilde{\nabla}_{e_{j'}}(\sigma))) \tag{2.7}$$

$$\implies \sum_{i=1}^{n} e'_i \cdot \tilde{\nabla}_{e'_i}(\sigma) = \sum_{i,j,j'} B_{i,j} B_{i,j'} e_j \cdot \tilde{\nabla}_{e_{j'}}(\sigma) \tag{2.8}$$

Since B is an orthogonal matrix, it satisfies

$$\sum_{i} B_{i,j} B_{i,j'} = \delta_{j,j'} \tag{2.9}$$

Thus,

$$\implies \sum_{i=1}^{n} e'_i \cdot \tilde{\nabla}_{e'_i}(\sigma) = \sum_{j,j'} \delta_{j,j'} e_j \cdot \tilde{\nabla}_{e_{j'}}(\sigma) = \sum_{j=1}^{n} e_j \cdot \tilde{\nabla}_{e_j}(\sigma) \tag{2.10}$$

This completes the proof.

Dirac operator in a local trivialization of a $SO(n)$-bundle. Let us write out the expression for the Dirac Operator with respect to a local trivialization of the principal $SO(n)$-bundle $P \to M$ associated to the tangent bundle of a Riemannian manifold M with a spin structure $\tilde{P}$.
Let M be a Rimannian manifold with a principal $SO(n)$-bundle $P \to M$ associated to the tangent bundle TM, and a spin structure $\tilde{P}$. In a local trivialization, let $\tilde{\omega}_{ij}$ be the connection 1-form and let σ be a local section of $S_{\mathbb{C}}(\tilde{P})$ given by $\sigma(u) = (u, s(u))$ in the induced local trivialization. Let $\{e_1 \ldots, e_n\}$ be the orthonormal basis at $u \in X$ corresponding to the standard basis for $\mathbb{R}^n$ under trivialization. We then have,

$$\mathscr{D}(\sigma)(u) = \sum_i e_i \tilde{\nabla}_{e_i}(\sigma)(u) = (u_i \sum_i e_i(\frac{ds(u)}{de_i} + \frac{1}{2}\sum_{j<k} \tilde{\omega}_{k,j}(e_i)(e_j e_k) \cdot s(u))) \tag{2.11}$$

$$\mathscr{D}(\sigma)(u) = (u_i \sum_i e_i \cdot \frac{ds(u)}{de_i} + \frac{1}{2}\sum_i \sum_{j<k} \tilde{\omega}_{k,j}(e_i)(e_i e_j e_k) \cdot s(u)) \tag{2.12}$$

Here the $\cdot$ represents Clifford multiplication. From the last expression, we see that the operators $\mathscr{D}$ at the point p is of the form $\sum_{i=1}^{n} e_i \cdot \frac{\partial}{\partial e_i}$ plus a zeroth order term involving no derivative. In particular, we see that $\mathscr{D}$ is a linear first order operator.

For the case of a $Spin^c$-lifting, $\tilde{P}$ and a $U(1)$-connection $\mathscr{A}$ on the determinant line bundle $\mathscr{L} \to M$, the formula is as follows.

$$\mathscr{D}^{\mathscr{A}}(\sigma)(u) = \sum_i e_l \cdot \frac{ds(u)}{de_l} + \frac{1}{2}\sum_l (\mathscr{A}(e_l)e_l + \sum_{j<k} \tilde{\omega}_{k,j}(e_i)(e_j e_k)) \cdot s(u) \tag{2.13}$$

To know how $\mathscr{D}^{\mathscr{A}}$ changes with the connection $\mathscr{A}$, and we have the following lemmas.
Lemma. Let $\mathscr{A}$ and $\mathscr{A}' = \mathscr{A} + \alpha$ be two $U(1)$-connections on the determinant line bundle $\mathscr{L}$ of a $Spin^c$ structure $\tilde{P}$ for M. Then for any section ψ of $S_{\mathbb{C}}(\tilde{P})$, we have

$$\mathscr{D}^{\mathscr{A}'}\psi = \mathscr{D}^{\mathscr{A}}\psi + \frac{1}{2}\alpha \cdot \psi \tag{2.14}$$

Lemma. Let M be a closed manifold with a spin or $Spin^c$-structure $\tilde{P}$. Then the Dirac operator $\mathscr{D} : S_{\mathbb{C}}(\tilde{P}) \to S_{\mathbb{C}}(\tilde{P})$ is **formally self-adjoint** in the sense that

$$(\mathscr{D}(\sigma_1), \sigma_2)_{L^2} = (\sigma_1, \mathscr{D}(\sigma_2))_{L^2} \tag{2.15}$$

where $(\cdot,\cdot)_{L^2}$ is the L^2 inner product on sections of $S_{\mathbb{C}}(\tilde{P})$ induced from the pointwise Hermitian inner product on the fibers.
Proof. The Hermitian inner product on C^∞ sections of $S_{\mathbb{C}}(\tilde{P})$ is given by

$$(\mathscr{D}(\sigma_1), \sigma_2)_{L^2} = \int_M \langle \mathscr{D}(\sigma_1, \sigma_2) \rangle dVol_M \tag{2.16}$$

where the inner product on the RHS is the pointwise hermitian inner product on the fibers of the complex spin bundle. Let us fix a coordinate system at a point p so that the standard unit tangent vectors $\{e_1, \ldots, c_n\}$, we have $\nabla_{e_i} e_i = 0$ at that point. Computing at $p \in M$ we have

$$\langle \mathscr{D}(\sigma_1), \sigma_2 \rangle_p = \langle \sum_i e_i \tilde{\nabla}_{e_i}(\sigma_1), \sigma_2 \rangle_p \tag{2.17}$$

$$\implies \langle \mathscr{D}(\sigma_1), \sigma_2 \rangle_p = -\langle \sum_i \tilde{\nabla}_{e_i}(\sigma_1), e_i\sigma_2 \rangle_p \tag{2.18}$$

$$\implies \langle \mathscr{D}(\sigma_1), \sigma_2 \rangle_p = \langle \sum_i \sigma_1, \tilde{\nabla}_{e_i}(e_i\sigma_2) \rangle_p - \frac{\partial}{\partial e_i}\langle \sigma_1, e_i\sigma_2 \rangle_p \tag{2.19}$$

$$\implies \langle \mathscr{D}(\sigma_1), \sigma_2 \rangle_p = \langle \sum_i \sigma_1, e_i \tilde{\nabla}_{e_i}(\sigma_2) \rangle_p + \langle \sum_i \sigma_1, \tilde{\nabla}_{e_i}(e_i)\sigma_2 \rangle_p - \frac{\partial}{\partial e_i}\langle \sigma_1, e_i\sigma_2 \rangle_p \tag{2.20}$$

$$\implies \langle \mathscr{D}(\sigma_1), \sigma_2 \rangle_p = \langle \sum_i \sigma_1, e_i \tilde{\nabla}_{e_i}(\sigma_2) \rangle_p - \frac{\partial}{\partial e_i}\langle \sigma_1, e_i\sigma_2 \rangle_p \tag{2.21}$$

$$\implies \langle \mathscr{D}(\sigma_1), \sigma_2 \rangle_p = \langle \sigma_1, \mathscr{D}(\sigma_2) \rangle_p - \sum_i \frac{\partial}{\partial e_i}\langle \sigma_1, e_i\sigma_2 \rangle_p \tag{2.22}$$

Now let us define a complexified vector field V on M, i.e. a section of $TM \otimes \mathbb{C}$ by the condition

$$\langle V(p), W(p) \rangle = \langle \sigma_1, W \cdot \sigma_2(p) \rangle \tag{2.23}$$

for all vector fields W and all $p \in M$. Then the above equalities can be written as

$$\langle \mathscr{D}(\sigma_1), \sigma_2 \rangle_p = \langle \sigma_1, \mathscr{D}(\sigma_2) \rangle_p - div(V)_p \tag{2.24}$$

All the quantities defined in this expression are **global**- this expression holds at every point $p \in M$. By integrating over M, we have

$$\langle \mathscr{D}(\sigma_1), \sigma_2 \rangle_{L^2} = \langle \sigma_1, \mathscr{D}(\sigma_2) \rangle_{L^2} - \int_M div(V_) dVol_M \tag{2.25}$$

and we're done.

2.1 The symbols of $\mathscr{D}$ and $\mathscr{D}^{\mathscr{A}}$

Definition. (Symbol of a first order differential operator) Let D be a first order differential operator from sections of a bundle $E \to M$ to sections of $F \to M$. Then the symbol of D, denoted by $Symb(D)$ is a bundle map from π^*E to π^*F between the pullbacks of the bundles over the cotangent bundle T^*M of M.

The symbol of a differential operator is a bundle map which is a function only of its leading order term. As for the discussions on Dirac operators, the symbol(s) should be a function only of the first order term.
Let us fix a local coordinate $(x^1, x^2, \ldots, x^n)$ which are orthonomal at the a point p. Let us assume that an operator D can be expressed in the local coordinates by

$$D(\sigma) = \sum_I \alpha_I \frac{\partial^{|I|}}{\partial x^I} + (lower-order-terms) \tag{2.26}$$

where the leading sum ranges over multi-indices I of total length $|I| = n$, and α_I is a linear map from E to F.
The symbol of D on a cotangent vector $\zeta \in T_p^*M$ is defined as the linear map

$$Symb(D)(\zeta) = i^n \sum \alpha_I \zeta^I \tag{2.27}$$

where $\zeta = (\zeta_1, \ldots, \zeta_n)$ in the dual basis of T*M to $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$. Thus the symbol of $\mathscr{D}$ on $(T^*M)_p$ is given by

$$Symb(\mathscr{D})(\zeta) = i \sum_j \zeta_j \cdot () \tag{2.28}$$

where $\cdot$ is the Clifford multiplication.

For any non-zero cotangent vector ζ, Clifford multiplication by $i\zeta$ induces an isomorphism from the fiber of $S_{\mathbb{C}\tilde{P}}$ to itself. This means that **The Dirac operator is an elliptic first order linear differential operator**.

2.2 Index Theory of the Dirac Operator

Let M be a closed, oriented, Riemannian 4-manifold. Let $\tilde{P}$ be a spin structure for M. Since $\mathscr{D} : C^\infty(S^+_{\mathbb{C}}(\tilde{P})) \to C^\infty(S^-_{\mathbb{C}}(\tilde{P}))$ is an elliptic operator, it follows that **kernel of $\mathscr{D}$ is finite-dimensional**, and that the **image is a closed subspace of finite codimension**.

Definition. (Index) The index of $\mathscr{D}$ is the complex dimension of the kernel minus he complex dimension of the cokernel. [1]

The Atiyah-Singer Index theorem computes the index of the operator from this element in relative K- theory, In this case of the Dirac operator, the index is $\hat{A}(M)$- the so-called A-hat genus of M. For a closed oriented 4-manifold, we have

$$\hat{A}(M) = \int_M -\frac{p_1(M)}{24} = -\frac{\sigma(M)}{8} \tag{2.29}$$

where $\sigma(M)$ is the signature of M. In the case of a $Spin^c$-structure $\tilde{P}$ on M, the bundles in question change, and the index formula gives

$$\frac{c_1(det(\tilde{P}))^2 - \sigma(M)}{8} \tag{2.30}$$

as the index of the Dirac operator in this context.

A couple of consequences of this theory:

- the $\hat{A}$-genus of a spin manifold is an integer
- the index of a spin 4-manifold is divisible by 8. (Actually, it is divisible by 16).

Chapter 3

Seiberg-Witten Theory

The discussions in this chapter mostly follow [13] [11], [6], and [7].

Let M be a smooth Riemannian 4-manifold with a $spin^c$ structure s, its associated spinor bundles $\mathscr{W}^{\pm}$, and its determinant line bundle $\mathscr{L}$. The objects we will be looking at are pairs

$$(A, \psi) \tag{3.1}$$

where $\psi \in \Gamma(\mathscr{W}^+)$ or $S^+_{\mathbb{C}}(\tilde{P})$ is a self-dual spinor field, and $A \in Conn(\mathscr{L})$ is a $U(1)$-connection on the determinant line bundle $\mathscr{L} \to M$. The whole story can be summarised as follows: We will be looking at solutions (ϕ, A) to a couple of non-linear partial differential equations called the Seiberg-Witten equations, consider their space of solutions, from which we will form a moduli space where two solutions would be equivalent upto the action of the gauge group $\mathscr{G}(\mathscr{L})$.

3.1 The Seiberg Witten Equations

The SW-equations are:

$$\mathscr{D}^A \psi = 0 \tag{3.2}$$

$$F_A^+ = \sigma(\psi) = \psi \otimes \psi^* - \frac{|\psi|^2}{2} Id \tag{3.3}$$

If we have a moduli problem formulated in terms of differential equations, then we may consider a functional of which the solutions represent the absolute minima. An example of this is the case of the Yang-Mills functional, and the anti-self-dual equation for $SU(2)$-Donaldson gauge theory.

Seiberg Witten Action. In the case of Seiberg-Witten gauge theory, such a particular functional would be

$$S(A, \psi) = \int_M (|\mathscr{D}^A \psi|^2 + |F_A^+|^2 + \frac{\kappa}{4}|\psi|^2 + \frac{1}{8}|\psi|^4) dvol \tag{3.4}$$

The equations above are the Euler-Lagrange equations for this actional.

3.1.1 Explanation of the Equations

$\mathscr{D}^A$ is the Dirac operator induced by the connection A, so the first equation basically says that ϕ should be in the kernel of this operator. F_A^+ is the imaginary-valued curvature 2-form of A, and $F_A^+ = \frac{1}{2}(F_A + *F_A)$ is its self-dual part. As to the right hand side of the second equation, $\mathscr{W}^+$ has a hermitian metric, and this complex vector space can be identified with its dual via an anti-complex isomorphism. The image of a ϕ under this isomorphism is defined by $\phi*$ which lives in the dual of $\sqsupseteq^+$. So,

$$\phi \otimes \phi^* \in \mathscr{W}^+ \otimes (\mathscr{W}^+)^* \cong End_{\mathbf{C}}(\mathscr{W}^+) \tag{3.5}$$

3.1.2 Space of Configurations

The solutions of the SW equations are called (Seiberg-Witten) monopoles. We will produce a moduli space out of them from which we will construct the SW invriants of the $Spin^C$ structure. For that, we need to talk about the space of solutions first.

Let us denote the space of unitary L_2^2-connections on $\mathscr{L}$ by $\mathscr{A}_L^2(\mathscr{L})$. (For any vector bundle V over M, $L_k^2(V)$ means the space of L_k^2-sections of V.) Define $\mathscr{C}(\tilde{P})$ to be the space

$$\mathscr{C}(\tilde{P}) = \mathscr{A}_{L_2^2}(\mathscr{L}) \times L_2^2(S^+(\tilde{P})) \tag{3.6}$$

where $\mathscr{A}_{L_2^2}(\mathscr{L})$ is as we described. We have the choice to work with any strong or stronger norm; the moduli space will turn out to be the same consisting of C^∞ objects up to gauge equivalence.

The tangent space at any point to $\mathscr{C}(\tilde{\mathscr{P}})$ is naturally identified with

$$L_2^2((T^*M \otimes i\mathbb{R}) \oplus S^+\tilde{P}) \tag{3.7}$$

This space has a natural L^2-inner product associated with it.

Seiberg-Witten Functional We define the Seiberg-Witten functional as

$$F : \mathscr{C}(\tilde{P}) \to L_1^2((\Lambda_+^2 T^*X \otimes i\mathbb{R}) \oplus S^-(\tilde{P})) \tag{3.8}$$

by

$$F(A, \psi) = (F_{\mathscr{A}}^+ - \sigma(\psi), \mathscr{D}^{\mathscr{A}}(\psi)) \tag{3.9}$$

In this context, Seiberg-Witten equations can be written simply as

$$F(\mathscr{A}, \psi) = 0 \tag{3.10}$$

Lemma. The SW Functional F is a smooth mapping and its differential at (A, ψ) is given by the following linear map.

$$DF_{(A,\psi)} = \begin{pmatrix} P + d & -D\sigma_\psi \\ \cdot\frac{1}{2}\psi & \mathscr{D}^{\mathscr{A}} \end{pmatrix} \tag{3.11}$$

where $\cdot \frac{1}{2}\psi$ represents the Clifford multiplication map sending one-form α to the element $\frac{1}{2}\alpha \cdot \psi \in L^2_1(S^-\tilde{P})$. Furthermore,

$$D_{\sigma_\psi}(\eta) = \psi \otimes \eta^* + \eta \otimes \psi^* - \frac{\langle \eta, \psi \rangle + (\langle \eta, \psi \rangle)^*}{2} Id \tag{3.12}$$

This element is traceless, self-adjoint automorphism and hence is identified via Clifford multiplication with a purely imaginary two-form.
The proof of this lemma involves direct computation of the derivatives, and also uses the following ideas that F here is an affine mapping plus the quadratic mapping $\sigma(\psi)$. The affine map is continuous, hence smooth- and thus it follows from Sobolev multiplication theorem that $\sigma(\psi)$ is a smooth map.

3.2 The Group of Gauge Changes

The group of gauge changes is basically the group of **Bundle Automorphisms**. If we are working with a Principal $Spin^c$-bundle $\tilde{P} \to M$, we can choose these to be the automorphisms of the principal $Spin^c$-bundle which cover the identity on the frame bundle of the tangent bundle.

Result. Automorphisms of the principal $Spin^c$-bundle that cover the identity on the frame bundle of the tangent bundle can be given by maps from the manifold M to the center S^1 of $Spin^c(4)$.

Another derivative on the bundle automorphism is then controlled, so that one can have a suitable action, and the L^2_3 maps are taken in this context. Finally we denote the space of such mappings with the $L^2_3 - topology$ as $\mathscr{G}(\tilde{P})$. This is a **Hilbert manifold** whose tangent space at the identity is the L^2_3-functions on M with values in $i\mathbb{R} \subset \mathbb{C}$.

Result. $\mathscr{G}(\tilde{P})$ is an infinite dimensional abelian Lie group with respect to pointwise multiplication. Its Lie Algebra is $L^2_3(M; i\mathbb{R})$ (the space of L^2_3-sections of the trivial bundle $M \times i\mathbb{R}$ over M) with the trivial bracket.

3.2.1 Action of $\mathscr{G}(\tilde{P})$ on the Space of Configurations

Let $g \in \mathscr{G}(\tilde{P})$ be a group element. The action is given by

$$(\mathscr{A}, \psi) \cdot g = ((detg)^*\mathscr{A}, S^+(g^{-1})(\psi)) \tag{3.13}$$

Lemma. The above action of $\mathscr{G}(\tilde{P})$ on $\mathscr{C}(\tilde{P})$is a smooth right action.
The proof of this theorem relies on the Sobolev multiplication theorems.

Lemma. Let $g \in G$ and let $\psi \in S^+(\tilde{P})$. Then

$$\mathscr{D}^{(detg)^*\mathscr{A}}(S^+(g^{-1})(\psi)) = S^-(g^{-1})(\mathscr{D}^{\mathscr{A}}\psi) \tag{3.14}$$

Proof. If ∇' is the covariant derivative on $S^{\pm}(\tilde{P})$ determined by $(detg)^*\mathscr{A}$ and the Levi-Civita connection, and if ∇ is the covariant derivative on $S^{\pm}(\tilde{P})$ determined by $\mathscr{A}$ and the Levi-Civita connection, then

$$\nabla' = S^{\pm}(g)^*\nabla \tag{3.15}$$

It follows that

$$\nabla'(S^{\pm}(g^{-1})(\psi)) = S^{\pm}(g^{-1})(\nabla\psi) \tag{3.16}$$

The lemma follows from this since the Clifford Multiplication commutes with the automorphisms $S^{\pm}(g)$.

Result. The Seiberg-Witten equations are invariant under the action of $\mathscr{G}(\tilde{P})$.

3.3 The Quotient Space

We have an action of $\mathscr{G}(\tilde{P})$ on $\mathscr{C}(\tilde{P})$, and now we can talk about the quotient of the action. The point of view that one wants to adopt is, since the SE equations are invariant under the action of the gauge group $\mathscr{G}(\tilde{P})$ But, before we do that- we should wait and ask, "Can we reasonably take the quotient at all?" Let us first look at the various types of stabilizers that appear in this group action. Our following discussion assumes M is connected.

Lemma. The stabilizer in $\mathscr{G}(\tilde{P})$ of an element $(\mathscr{A},\psi) \in \mathscr{C}(\tilde{P})$ is **trivial** unless $\psi = 0$, in which case the stabilizer is the group consisting of the constant maps $M \to S^1$- a group naturally identified with S^1.

M is connected $\implies$ the stabilizers of any connection $\mathscr{A}$ is exactly the group of constant maps from M to S^1. This subgroup acts freely on ψ unless ψ is identically zero.

Definition. (Irreducible and Reducible) We say that an element $(\mathscr{A},\psi)$ is irreducible if $\psi \neq 0$, and otherwise, it is reducible. The (open) subset of irreducible configurations in $\mathscr{C}(\tilde{P})$ is denoted by $\mathscr{C}^*(\tilde{P})$.

For reducible solutions, i.e. for the case $\psi = 0$, the Seiberg-Witten equations become

$$F^+_{\mathscr{A}} = 0 \tag{3.17}$$

whose solutions must be all anti-self-dual connections on $\mathscr{L}$- which are also studied in Donaldson gauge theory. So somehow the Seiberg-Witten equations are also capturing something that the Donaldson gauge theory tries to do.

Theorem. If $b_2^+(M) \geq 1$, then for a generic Riemannian metric, the Seiberg-Witten moduli space $\mathscr{M}$ is either empty, or is a smooth manifold of dimension

$$dim\mathscr{M} = \frac{1}{4}(c_1(\tilde{P})^2 - 2\chi(M) - 3\cdot sgnM) \tag{3.18}$$

Theorem. If $b_2^+(M) \geq 2$, then for every two generic metrics g_0 and g_1, and for every generic path g_t connecting them, all corresponding moduli spaces $\mathscr{M}_t$ are smooth manifolds (maybe empty), and draw smooth cobordism between $\mathscr{M}_0$ and $\mathscr{M}_1$.

$$dim\mathscr{M} = \frac{1}{4}(c_1(\tilde{P})^2 - 2\chi(M) - 3\cdot sgn(M)) \tag{3.19}$$

Here $b_2^+(M)$ is the dimension of the maximal positive subspace for the intersection form on H_2, $sgn(M)$ is the "sign" of the manifold, depending on the orientation, $c_1(\tilde{P})$ is the first Chern class of the Principal $Spin(n)$-bundle (or $Spin^c(n)-$bundle) $\tilde{P}$, and $\chi(M)$ is the Euler characteristic of M.

For the proof that $\mathscr{M}$ is a manifold, Along with perturbing the metric, one of the two SW equations is also perturbed to

$$F_{\mathscr{A}}^{+} = \sigma(\psi) + i\eta^{+} \tag{3.20}$$

for some parameter $\eta^{+} \in \Gamma(\Lambda_{+}^{2})$. This perturbative approach is quite fruitful for applications of Seiberg-Witten theory in the cases of Symplectic manifolds, where a suitable eta^{+} is "grown to infinity". A proof that it is sufficient to perturb only the metric and not the equations is in the book by **T. Friedrich** titled **Dirac Operators in Riemannian Geoemtry**. **[TFr97]**

The main consequence of the above theorem seems to be that when $b_2^{+}(M) \geq 2$, the Seiberg-Witten moduli space $\mathscr{M}$ determines a well-defined bordism class inside $Conn(\mathscr{L}) \times \Gamma(S_{\mathbb{C}}^{+}(\tilde{P}))$ modded out by $\mathscr{G}$, which only depends on the manifold M, and the spin-structure $\tilde{P}$, but not on the Riemannian metric. Therefore, by evaluating various cohomology classes on $\mathscr{M}$, we can obtain numerical invariants of 4-manifolds M which would then only depend on the Chern class $c_1(\tilde{P})$.

Theorem. The moduli space $\mathscr{M}$ is compact.

To prove this, one needs to use Uhlenbeck's general compactness criterion alongwith some apriori pointwise bounds on the spinor fields (which arises due to the special nature of the SW equations) and hence on the self-dual part of the curvature of any solution. There are also some bounds the come from the assumption that the formal dimension of the moduli space is non-compact, which implies an L^2-bound on the curvature, and eventually compactness is established.

Theorem. The SW moduli space $\mathscr{M}$ is non-empty for at most finitely many $spin^c$-structures.
Proof. We first obtain a bound on the curvature $F_{\mathscr{A}}$ of any solution $(\mathscr{A}, \psi)$, then use the result that $[F_{\mathscr{A}}] = -2\pi i c_1(\tilde{P})$, restrict to positive-dimensional moduli spaces to conclude that $c_1(\tilde{P})$ must be condined to a finite subset of $H^2(M;Z)$.
Let us find the bound on the curvature.

Theorem (Integral Curvature Bound) If $(\mathscr{A}, \psi)$ is a solution of the SW equations, then we must have:

$$2\sqrt{2}||F_{\mathscr{A}}^{+}|| \leq ||Scal|| \tag{3.21}$$

where $||\cdot||$ is the L^2-norm $||\alpha||^2 = \int_M |\alpha|^2 dVol_M$, and "Scal" refers to scalar curvature.
Proof. The proof of the Integral Curvature bound uses the Lichnerowicz formula which is:

$$(\mathscr{D}^{\mathscr{A}})^{*}(\mathscr{D}^{\mathscr{A}})\psi = (\nabla^{\mathscr{A}})^{*}(\nabla^{\mathscr{A}})\psi + \frac{1}{4} scal \cdot \psi + \frac{1}{2} F_{\mathscr{A}}^{+}\psi \tag{3.22}$$

The solutions of SW equations $(\mathscr{A},\psi)$ would satisfy $\mathscr{D}^{\mathscr{A}}\psi = 0$ and $F^+_{\mathscr{A}} = \sigma(\psi)$. Pluggin these in the Lichnerowitz formula, we get:

$$0 = (\nabla^{\mathscr{A}})^*(\nabla^{\mathscr{A}})\psi + \frac{1}{4}scal \cdot \psi + \frac{1}{4}|\psi|^2 \cdot \psi \tag{3.23}$$

Taking the inner product with ψ on both sides, we get

$$0 = \langle(\nabla^{\mathscr{A}}) * (\nabla^{\mathscr{A}})\psi, \psi\rangle + \frac{1}{4}\langle scal \cdot \phi, \phi\rangle + \frac{1}{4}|\phi|^4. \tag{3.24}$$

Then integrating over M (using the Riemannian volume element $dVol_M$, using the fact that $(\nabla^{\mathscr{A}})^*$ is adjoint to $(\nabla^{\mathscr{A}})$, we get

$$0 = \int_M |\nabla^{\mathscr{A}}\psi|^2 dVol_M + \frac{1}{4}\int_M scal \cdot |\psi|^2 dVol_M + \frac{1}{4}\int_M |\psi|^4 dVol_M \tag{3.25}$$

We rearrange this equality by separating the scalar curvature term. and use the Cauchy-Schwarz inequality on the right (and let us drop with the subscripts M and the measure $dVol_M$ for a bit):

$$\frac{1}{4}\int|\psi|^4 \leq \int|\nabla^{\mathscr{A}}\psi|^2 + \frac{1}{4}\int|\psi|^4 = \frac{1}{4}\int(-scal)|\psi|^2 \leq \frac{1}{4}(\int(scal)^2)^{1/2}(\int|\psi|^4)^{1/2} \tag{3.26}$$

This means that

$$(\int|\psi|^4)^{1/2} \leq (\int(scal)^2)^{1/2} \tag{3.27}$$

But $|\sigma(\psi)| = \frac{1}{2\sqrt{2}}|\psi|^2 = F^+_{\mathscr{A}}$. So,

$$2\sqrt{2}(\int|F^+_{\mathscr{A}}|^2)^{1/2} \leq (\int(scal)^2)^{1/2} \tag{3.28}$$

which is exactly what we wanted to show in terms of L^2-norms.
Lemma. Let α be any closed 2-form on a 4-manifold. Then:

$$[\alpha]\cdot[\alpha] = ||\alpha^+||^2 - ||\alpha^-||^2. \tag{3.29}$$

Proof. We have to use the following:

$$\alpha^+ \wedge \alpha^- = 0, *\alpha^+ = \alpha^+, *\alpha^- = -\alpha^-, \beta \wedge *\beta = |\beta|^2. \tag{3.30}$$

$$[\alpha]\cdot[\alpha] = \int\alpha\wedge\alpha = \int((\alpha^+ + \alpha^-)\wedge(\alpha^+ + \alpha^-)) \tag{3.31}$$

$$\implies [\alpha]\cdot[\alpha] = \int(\alpha^+\wedge\alpha^+) + \int(\alpha^-\wedge\alpha^-) \tag{3.32}$$

$$\implies [\alpha]\cdot[\alpha] = \int(\alpha^+\wedge(*\alpha^+)) + \int(\alpha^-\wedge(-*\alpha^-)) \tag{3.33}$$

From which the statement follows automatically.

Applying this lemma to $F_{\mathscr{A}}$, with the fact that $[F_{\mathscr{A}}] = -2i\pi c_1(\mathscr{L}) \implies \frac{i}{2\pi}[F_{\mathscr{A}}] = c_1(\mathscr{L})$, we get:

$$4\pi^2 c_1^2 = ||F_A^+||^2 - ||F_A^-||^2 \leq ||F_A^+||^2 \leq \frac{1}{8}||Scal||^2. \tag{3.34}$$

We have an upper bound on $c_1(\mathscr{L}) \cdot c_1(\mathscr{L})$- and it only makes sense to look at these moduli spaces that are expected to be of positive dimension. Using the formula $dim\mathscr{M} - \frac{1}{4}(c_1^2 - 2\chi(M) - 3 \cdot sgn(M)$, it is concluded that

$$2\chi(M) - 3 \cdot sgn(M) \leq c_1 \cdot c_1 \leq \frac{1}{32\pi^2}||scal||^2 \tag{3.35}$$

Thus, only finitely many choices of $C_1(\mathscr{L})$ from the integral lattice $H^2(M;Z)$ have any chance to yield a non-empty moduli space.
Theorem. The moduli space M is always compact.

Proof. First we need to obtain a pointwise a priori bound on $|\psi|$:

Lemma (Pointwise Curvature Bound) If $(\mathscr{A}, \psi)$ is a solution of the SW equations, then either we have

$$|\psi|^2 \leq max_{x \in M}\{-scal(x)\} \tag{3.36}$$

or ψ is identically zero.

Proof. If $f \colon M \to \mathbb{R}$ has a local maximum at some $p \in M$, then it must have $(\Delta f)(p) \geq 0$, where $\Delta = -\sum \partial_{e_k}\partial_{e_k}$ is the Laplace operator. Because $\Delta f = -Tr(Hessian(f))$, and a maximum at p implies that all the eigenvalues of $Hessian(f)$ are non-positive, so it follows that $(\Delta f)(p) \geq 0$.
Then choosing an Orthonormal local frame $\{e_1, e_2, e_3, e_4\}$ in the tangent bundle TM, we compute.

$$\Delta(|\psi|^2) = -\sum \partial_{e_k}\partial_{e_k}\langle \psi, \psi \rangle_{\mathbb{R}} \tag{3.37}$$

$$\implies \Delta(|\psi|^2) = -\sum \partial_{e_k} 2\langle \nabla_{e_k}^{\mathscr{A}} \psi, \psi \rangle \tag{3.38}$$

$$\implies \Delta(|\psi|^2) = -\sum 2\langle \nabla_{e_k}^{\mathscr{A}} \nabla_{e_k}^{\mathscr{A}} \psi, \psi \rangle - \sum 2\langle \nabla_{e_k}^{\mathscr{A}} \psi, \nabla_{e_k}^{\mathscr{A}} \psi \rangle \tag{3.39}$$

Here we have used that ∇^A is compatible with the fiber-metric of $S_{\mathbb{C}}^+(\tilde{P})$. We arrange to

$$\Delta(|\psi|^2) + 2\sum |\nabla_{e_k}^{\mathscr{A}} \psi|^2 = -\sum 2\langle \nabla_{e_k}^{\mathscr{A}} \nabla_{e_k}^{\mathscr{A}} \psi, \psi \rangle \tag{3.40}$$

Assuming that $p \in M$ is the absolute maximum point of $\psi|^2$, then $\Delta(|\psi|^2)|_p \geq 0$. Therefore, at p, we will have

$$-\sum 2\langle \nabla_{e_k}^{\mathscr{A}} \nabla_{e_k}^{\mathscr{A}} \psi, \psi \rangle \geq 0 \tag{3.41}$$

On the other hand, by taking inner-product with test-spinor fields ϕ, integrating over M, and using the idea due to Uhlenbeck that on compact manifolds, divergences integrate to 0, we can arrive at the following result that if the chosen local frame $\{e_1, e_2, e_3, e_4\}$ in TM is such that at p, the Levi-Civita connection has $\nabla_{e_i} e_j|_p = 0$ (so called geodesic coordinates),

$$(\nabla^{\mathscr{A}})^*(\nabla^{\mathscr{A}})\psi = -\sum(\nabla^{\mathscr{A}}_{e_k})(\nabla^{\mathscr{A}}_{e_k})\psi \tag{3.42}$$

Implies that at the maximum point p, we have

$$2\langle(\nabla^{\mathscr{A}})^*(\nabla^{\mathscr{A}})\psi, \psi\rangle|_p \geq 0 \tag{3.43}$$

If we look again at the Lichnerowicz formula

$$(\mathscr{D}^{\mathscr{A}})^*(\mathscr{D}^{\mathscr{A}})\psi = (\nabla^{\mathscr{A}})^*(\nabla^{\mathscr{A}})\psi + \frac{1}{4} scal \cdot \psi + \frac{1}{2} F^+_{\mathscr{A}} \psi \tag{3.44}$$

applied to a SW solution $(\mathscr{A}, \psi)$ exactly as in the proof of the integral curvature bound theorem, we are led to

$$0 = \langle(\nabla^{\mathscr{A}})^*(\nabla^{\mathscr{A}})\psi, \psi\rangle + \frac{1}{4} scal \cdot |\psi|^2 + \frac{1}{4}|\psi|^4. \tag{3.45}$$

At the maximum point p, the first term is positive and that forces

$$\frac{1}{4} scal(p)|\psi(p)|^2 + \frac{1}{4}|\psi(p)|^4 \leq 0 \tag{3.46}$$

If ψ is non-zero somewhere, i.e. if it is not zero everywhere, then $\psi(p) \neq 0$, and we can cancel:

$$|\psi(p)|^2 \leq -scal(P) \tag{3.47}$$

Since $-scal(p) \leq max{-}scal(x)$ and $|\psi(x)| \leq |\psi(p)|$, the result follows.

After this pointwise bound on $|\psi|$ has been established, one uses a standard technique called ”elliptic bootstrapping” arguments to bound all the higher derivatives of both ψ and $\mathscr{A}$, deducing the compactness of the moduli space.

Result. $\mathscr{M}$ is orientable, and its orientations correspond to orientations in the vector space $H^1(M;\mathbb{R}) \otimes H^2_+(M;\mathbb{R})$.

Chapter 4

Seiberg-Witten Invariants

The discussions in this chapter mostly follow [7] [6] [11] [10].

We've seen that the Seiberg-Witten equations have some nice moduli spaces. There is something more. It can be further shoun that for simply connected M the natural ambient space of $\mathscr{M}$, which is the space of all connections-and-spinor fiels pairs modulo gauge-equivalence, has the homoptopy type of $\mathbb{CP}^{\infty}$- which leads to the result that the cohomology ring of the ambient space is $\mathbb{Z}[u]$ for a degree-2 class u. So, if $\mathscr{M}$ is even dimensional, we can evaluate the appropriate class $u \cup u \cup \cdots \cup u$ (repeated cup products with itself) on it and obtain a numerical invariant of M as the following.

$$\mathscr{SW}_M(\tilde{P}) = \int_{\mathscr{M}} u \cup u \cup \cdots \cup u \tag{4.1}$$

We call it the Seiberg-Witten invariant of the $Spin^C$-structure $\tilde{P}$. It can be shown that it will only depend on M, and the first Chern class $c_1(\tilde{P})$.

If the moduli space $\mathscr{M}$ is odd dimensional, then the best we can do is to define

$$\mathscr{SW}_M(\tilde{P}) = 0 \tag{4.2}$$

and no information is obtained. There is a result that says that $dim\mathscr{M}$ is odd if and only if $b_2^+(M)$ is even. So in case $b_2^+(M)$ is even, the SW invariants are blind.

What about non-simply-connected cases?
In the above discussion, we saw what to do with the simply connected cases, but that would be too restrictive while talking about $4-manifolds$, and specially because there are infinitely many diffeomorphism classes. In any case, the non-simply connected cases are also dealt with in a similar way. The moduli spaces in that case are either all even, or all odd dimensional depending on whether- not b_2^+- but

$$b_2^+(M) + b_1(M) + 1 \tag{4.3}$$

is even or odd. The cases $b_2^+(M) + b_1(M)$ = even is particularly uninteresting here since then $dim\mathscr{M}$ would be odd, the homology class of $\mathscr{M}$ become trivial and SW can tell us nothing.

4.1 Simple Type Conjecture (Open)

Statement. For any simply-connected 4-manifold with $b_2^+ \geq 2$, if the Seiberg-Witten moduli space is non-empty, then it must be zero-dimensional, and thus consists of finitely many isolated points.
If this were true, then the implications would be enough to merely count (with signs) their solutions.Symplectic 4-manifolds constitute a large class of 4-manifolds for which the above conjecture is true.
There is no known example of a simply-connected manifold with $b_2^+ \geq 2$ that has higher-dimensional moduli spaces. On the other hand, there are examples from non-simply-connected cases and of manifolds with $b_2^+ = 1$ that each have SW moduli spaces of arbitrarily high dimensions.

Definition. (Simple-type manifold) A 4-manifold for which only 0 dimensional SW moduli spaces appear is said to be of Seiberg-Witten type, or simple type.

From our earlier discussion, we saw that

$$dim\mathscr{M} = \frac{1}{4}(c_1(\tilde{P})^2 - 2\chi(M) - 3 \cdot sgn(M)) \tag{4.4}$$

So, it's interesting that 0-dimensional moduli spaces occur exactly for those $Spin^c$-structures for which

$$c_1\tilde{P} \cdot c_1\tilde{P} = 2\chi(M) + 3 \cdot sgn(M) \tag{4.5}$$

-and these are exactly the $Spin^C$ structures that arise from almost-complex structures. There is a result that says that if M admits an almost-complex structure, then $b_2^+(M) + b_1(M)$ must be odd- so it's good on that end as well.

If the above conjecture turns out to be correct, then one can think of the SW invariants as an invariant no only of the $Spin^C$-structure, but also of the almost-complex structures. In the following part of this chapter, we will try to see some results that would follow if this conjecture is true.

4.2 Main Results and Properties

Definition. (The invariants) The SW invariant is a map

$$\mathscr{SW}_M : \{Spin^c - structures - on - M\} \to \mathbb{Z} \tag{4.6}$$

with $\mathscr{SW}_M(\tilde{P})$ defined by counting the number of solutions of the SW-equations for $Spin^c$ structure $\tilde{P}$, considered up to gauge equivalence:

$$\mathscr{SW}_M : \#(\{(\mathscr{A}, \psi) | \mathscr{D}^{\mathscr{A}}\psi = 0, F_A^+ = \sigma(\psi)\}/\mathscr{G}(\tilde{\mathscr{P}}) \tag{4.7}$$

4.2.1 Sketch of the Seiberg-Witten Proof of Donaldson's Theorem

Definition. (Intersection form) Let M be a compact, oriented, simply-connected 4-manifold. The Poincare-duality isomorphism between homology and cohomology is equivalent to a bilinear form

$$Q : H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{Z}. \tag{4.8}$$

This is called the **intersection form** of the manifold.

Donaldson's Theorem. If M is a smooth 4-manifold with negative-definite intersection form, then in fact, its intersection from must be

$$Q_M = \oplus m[-1] \tag{4.9}$$

Proof. (Due to P. Kronheimer and N. Elkeis) Assume that M is a smooth 4-manifold with negative-definite intersection form. In other words, $b_2^+(M) = 0$ and thus $b_2(M) = b_2^-(M)$ and $sgn(M) = -b_2(M)$.
Let w be any characteristic element of M. Then we must have $w \cdot w = sgn(M)(mod 8)$ and hence

$$w \cdot w + b_2(M) = 0(mod 8) \tag{4.10}$$

Characteristic elements w correspond to $Spin^c$-structures $\tilde{P}$ with $c_1(\tilde{P} = w$. The virtual dimension of the corresponding Seiberg-Witten moduli space is

$$vdim\mathscr{M} = \frac{1}{4}(w \cdot w - 3 \cdot sgn(M) - 2 \cdot \chi(M)) \tag{4.11}$$

$$\implies vdim\mathscr{M} = \frac{1}{4}(w \cdot w - 3 \cdot sgn(M)) - 1 \tag{4.12}$$

A consequence is that the dimension of the moduli space is always odd.
Assume that there is some characteristic element w for which the virtual dimension $vdim(\mathscr{M})$ is non-negative- that is to say, at least 1. Then the moduli space is either empty or a singular manifold of the expected dimension. Since $b_2^+(M) = 0$, there are always reducible solutions in $\mathscr{M}$ which here cannot be empty.

The space $\mathscr{M}^0$ of Seiberg-Witten solutions modulo the action of

$$\mathscr{G} = g\colon M \to S^-1|g(p) = 1 \tag{4.13}$$

is a smooth manifold of dimension $vdim(\mathscr{M}) + 1$. It's dimension is even and at lease 2. The group S^1 acts on $\mathscr{M}^0$ with fixed points at the reducible solutions, and its quotient is $\mathscr{M}$.

Assume first that $H^1(M;\mathbb{R}) = 0$. Then there is a unique gauge class of reducible solutions. In other words, there is only one fixed point of the action of S^1 on $\mathscr{M}$.

For the manifold $\mathscr{M}^0$, a discussion in terms of the complex

$$0 \to T_{\mathscr{G})'\mathscr{L}}|_1 \xrightarrow{dg} T_{\Gamma(S_{\mathbb{C}}\tilde{P}^+)\times Conn(\mathscr{L})}|_{(A,\psi)} \xrightarrow{d_{\mathscr{SW}}} T_{\Gamma(S_{\mathbb{C}}\tilde{P}^-)\times i\Gamma(\Lambda^2_+)}|_{(0,0)} \to 0$$

leads to an identification of the tangent space of $\mathscr{M}^0$ with the first cohomology group of this complex,

$$T_{\mathscr{M}^0}|_{[A,\psi]} = \mathscr{H}^1_{(A,\psi)} \tag{4.14}$$

The only difference now is that the tangent spact is not the full space $i\Gamma(\mathbb{R}) = if\colon M \to i\mathbb{R}$, but its codimension 1 subspace of maps with $f(p) = 0$.
At every reducible solution $(0,A)$, the derivatives are

$$d_{\mathscr{SW}}|_{(0,A)}(i\nu,\psi) = (\mathscr{D}^{\mathscr{A}}\psi, id^{+}\nu), dg|_1(if) = (0, 2idf), \tag{4.15}$$

and thus the tangent space to $\mathscr{M}^0$ at [0,A] is merely $\mathscr{H}(0,A)^1$ = Ker $\mathscr{D}^{\mathscr{A}}$ since $H^1(M;\mathbb{R}$ was assumed to be trivial.
We think of $T_{\mathscr{M})}|_{[}(0,A)] = Ker\mathscr{D}^{\mathscr{A}}$ as an approximation to the manifold $\mathscr{M}^0$ around the point $[0,A]$. The proof then applies Kuranishi techniques which was created by **M. Kuranishi** in the paper *On the locally complete families of complex analytic structures* to study moduli spaces of complex varieties. Its use in gauge theoretic ideas originates with Atiyah, Hitchin and Singer's *Self duality in four dimensional Riemannian Geometry*.

The main proof also requires to use the following algebraic result proved in the paper *A characterization of $\mathbb{Z}^n$ lattice* by Elkies. See [4]

Theorem. ([2]) Let $Q\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ be a symmetric unimodular biinear form. If Q is neither $\oplus[-1]$ nor $\oplus[1]$, then there exists a characteristic value w such that $|w\cdot w| < rankQ$.
Theorem. If M is a smooth manifold with a negative-definite intersection form, then for every characteristic element w of M, we must have $w\cdot w \leq -b_2(M)$.

Corollary. If M is a smooth manifold with definite intersection form, then Q_M cannot be even.
Proof. Any even form would have $w = 0$ as characteristic element, and thus $w\cdot w = 0 > -b_2(M)$ which is contradictory.
Consequence. $E_8 \oplus E_8$ is out of the smooth realm. [3]

4.2.2 Seiberg-Witten and Donaldson Theory: Witten Conjecture

In a fundamental paper, Kronheimer and Mrowka described a relation that constrains the values of the Donaldson invariants for a manifold of *Donaldson finite type*- which is a rather technical hypothesis. The polynomial invariants are defined in terms of homology classes $\alpha \in H_2(M;\mathbb{Z})$ and a moduli space of Anti-self-dual $SU(2) - connections$ on a bundle E with instanton number $k = c_2(E)$. The moduli space in the Donaldson theory, $\mathscr{M}_k^{ASD}$ has a dimension $= 8k - 3(b_2^+ + 1)$.

Definition. M is of *Donaldson type* if the polynomial invariant satisfies

$$q_{d-4}(\alpha) = 4\langle \mu(\alpha) \wedge \nu^2, \mathscr{M}^{ASD} \rangle, \tag{4.16}$$

where $\alpha = (\alpha_1, \dots, \alpha_{d-4})$.

The following result is due to Kronheimer and Mrowka, and is true for manifolds of Donaldson Simple Type.

Theorem. Let M be a simply-connected manifold of Donaldson simple type with $b^{2+} > 1$ odd. Combine the Donaldson polynomial invariants q_d in the expression

$$q = \sum_d \frac{q_d}{d!} \tag{4.17}$$

Then this expression satisfies

$$q(\alpha) = exp(\frac{Q(\alpha)}{2}) \sum_k a_k e^{x_k \cdot \alpha} \tag{4.18}$$

Here Q is the intersection from of the four-manifols M and the classes

$$x_k \in H^2(M;\mathbb{Z}) \tag{4.19}$$

are called the Kronheimer-Mrowka basica classes. They are subject to constraints that the mod 2 reductions of each is the Stiefel-Whitney class $w_2(M)$, and the corresponsding coefficients are non-zero rationals.

Edward Witten has a still open conjecture about the relationship between the SW invariants and the Donaldson invariants which is stated below.

Conjecture. (Witten) In the expression

$$q = exp(\frac{Q}{2}) \sum_k a_k e^{x_k} \tag{4.20}$$

the basic classes $x_k \in H^2(M;\mathbb{Z})$ are exactly the Seiberg-Witten basic classes, namely those that satisfy

$$s_k^2 = c_1(\sqrt{L_k})^2 = \frac{2\chi + 3\sigma}{4}, \tag{4.21}$$

and correspond to a $Spin^c$-structure $\tilde{P}$ with non-trivial Seiberg-Witten invariant,

$$\mathscr{SW}_M(\tilde{P}) \neq 0. \tag{4.22}$$

Moreover, the corresponding a_k is exactly, upto a topological factor, the Seiberg-Witten invariant $\mathscr{SW}_M(\tilde{P})$, that is- we have

$$a_k = 2^{2+7\chi(M)+11\sigma(M)}\mathscr{SW}_M(\tilde{P}) \tag{4.23}$$

A related conjecture:

Conjecture. A Simply connected manifold X is Donaldson simple type if and only if it is of a Seiberg-Witten simple type, namely that if the only non-trivial SW invariants correspond to the choice of a $Spin^c$-structure such that $dim(\mathscr{M}) = 0$.

The physics-way to approach the conjectured duality between SW gauge theory and Donaldson gauge theory is through a twisted SUSY QFT, and by means of a term called "S-duality", which are not well understood mathemtically. The process is roughly as follows.

We start with a higher dimensional theory, from which we can get some other theory by dimensional reduction. What goes on is a compactification process- where physicists would say- this and this dimensions have been "curled up". But there are essentially a lot of ways of doing these compactifications- the process is not unique.
Starting from the same theory, different sort of compactifications and twists can lead to different types of theories. But physicists argue that since they come from the same theory in a higher dimension, there should be some kind of duality going on.

In String theory, people have found out that the Donaldson gauge theory and SW gauge theories come from a common theory ,but in different limits. Trying to argue that there would be some duality (perhaps also some obstructions) becuase they arise from the same theory by different twists or limits is essentially, the whole essence of many duality-arguments that physicists now-a-days make, and $S-duality$ also falls under this- at least this is what I could decipher.

Mathematicians are also trying to understand these problems and are trying to formulate these things in a rigorous way. Regarding the conjectures which are widely believed to be true but are still open, there have been a detailed strategy lined out by Pidstrigatch and Tyurin. The main idea of the strategy is to relate Donaldson and Seiberg-Witten theory with a "mixed theory" of non-abelian monopoles, designed in such a way that the Donaldson and the SW moduli spaces appear as singular submanifolds of the moduli space $\mathscr{M}$ of non-abelian monopoles. In this way, the larger moduli space describes a cobordism between the links of the two types of moduli, thus defining a relation between the invariants- which turns out to be the one prescribed by the Witten conjecture.

Even though the strategy has been clearly outlined, the actual construction of the cobordism presents a lot of analytical difficulty. Most of the difficulty lies in the lower strata of the moduli space of non-abelian monopoles. The compactness argument fails in the non-abelian case. Infact, the moduli space in the Donaldson gauge theory itself is non-compact, and a lot of analytical tools need to be employed for compactificaiton.

There are a series of papers by Feehan and Leness where they try to tackle the problem by a large display of technical skills. There is an Uhlenbeck compactification $\tilde{\mathscr{M}}$ of the moduli space of non-abelian monopoles, obtained by adding lower dimensional strata- where each strata may themselves contain reducibles. Also, like many invariants of this language, the Donaldson invarians are obtained by integrating some cohomology classes over the fundamental class of $\mathscr{M}$. If there is a non-trivial intersection with the links of the reducibles in the lower strata, then that is a problem, and one expects that an analogue to the *Kotschick-Morgan Conjecture* holds true.

Conjecture (Kotschick-Morgan) The *Wall-crossing terms* in the Donaldson gauge theory only depend on the homotopy type of teh manifold M. [**Kot89**]

A substantial part of Feehan and Leness is put into proving an algoue of the KM conjecture, which had not been worked out even in the *simpler* cases, and in the *simpler* context of ASD moduli space. [8] [9]

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