A Quantum $\mathcal{Z}$-Transform

By
Md. Sajibur Rahman Sarker
19341017

A thesis submitted to the Department of Computer Science and Engineering
in partial fulfillment of the requirements for the degree of
B.Sc. in Computer Science

Department of Computer Science and Engineering
Brac University
December 2019

Declaration

It is hereby declared that

1. The thesis submitted is my own original work while completing degree at Brac University.
2. The thesis does not contain material previously published or written by a third party, except where this is appropriately cited through full and accurate referencing.
3. The thesis does not contain material which has been accepted, or submitted, for any other degree or diploma at a university or other institution.
4. We have acknowledged all main sources of help.

Student's Full Name & Signature:

Md. Sajibur Rahman Sarker
19341017

Approval

The thesis titled "A Quantum $\mathscr{Z}$-Transform" submitted by Md. Sajibur Rahman Sarker (ID: 19341017) of Fall 2019 has been accepted as satisfactory in partial fulfillment of the requirement for the degree of Bachelor of Science in Computer Science on December, 2019.

Advisor:

Mahbub Alam Majumdar
Professor at Brac University
Dhaka, Bangladesh

Co-advisor:

Sathyawageeswar Subramanian
Ramanujan Scholar at University of Cambridge
Cambridgeshire, United Kingdom

Abstract

We investigate a quantum analog of the classical $\mathcal{Z}$-transform with the aim of making it implementable on quantum computers, potentially offering a speedup over the classical method. Unlike the discrete Fourier transform, which is limited to frequency analysis, the $\mathcal{Z}$-transform allows for versatile exploration of properties within the complex plane. Since the quantum Fourier transform underpins Shor's factoring algorithm and serves as a subroutine in many other quantum algorithms, a quantum $\mathcal{Z}$-transform promises broad applicability in quantum simulation, quantum machine learning, and quantum signal processing. This is especially relevant because $\mathcal{Z}$-transforms generalize Fourier transforms in certain aspects. Given that quantum computers are particularly adept at performing unitary operations, we discretize the classical definition of the $\mathcal{Z}$-transform and unitarize its matrix formulation to make it amenable for quantum computation. Our approach involves introducing a discrete $\mathcal{Z}$-transform, mapping the input sequence to a discrete set of values to represent them as quantum states, and redefining the $\mathcal{Z}$-transform as a finite summation to effectively handle the infinite summation of the classical definition. We then develop a matrix formulation for our redefined discrete $\mathcal{Z}$-transform and extend our approach by unitarizing this matrix formulation through block-encoding, constructing unitary operators that meet the criteria for efficient quantum operations using standard quantum gates and subroutines. Our approach establishes the groundwork by fulfilling the mathematical foundations for the potential discovery of a quantum $\mathcal{Z}$-transform and opens avenues for further exploration and implementation in quantum computing.

Keywords: quantum $\mathcal{Z}$-transform, quantum Fourier transform, discrete $\mathcal{Z}$-transform, quantum algorithms, quantum subroutines, quantum computing.

Table of Contents

List of Figures

Chapter 1

Introduction

In science and engineering, it is important to distinguish between processes that are discrete and processes that are continuous. At its most fundamental level, quantum mechanics is a mathematical framework describing discrete processes. However, given its fundamental importance in understanding the natural world, it is necessary to translate the mathematics of what is essentially a discrete theory to describe processes that are not necessarily discrete. In applied mathematics, tools have been developed to achieve this, even without any motivation from the natural sciences, and one such tool is the $\mathcal{Z}$-transform [6], [11], [14].

While the $\mathcal{Z}$-transform is commonly attributed to electrical engineers Ragazzini and Zadeh, its roots trace back to DeMoivre in the 1700s, who utilized the analogous concept of a *generating function* [13], [47]. To comprehend the $\mathcal{Z}$-transform's purpose, we consider discrete difference equations [24] of the form:

$$x_n = \alpha_n + \beta_n + \gamma, \quad n \in \mathbb{N}$$

Which can be transformed from a discrete n-domain to a continuous complex z-domain simply by multiplying throughout by an arbitrary complex variable z:

$$\sum_{n=0}^{\infty} x_n z_n = \sum_{n=0}^{\infty} \alpha_n z_n + \sum_{n=0}^{\infty} \beta_n z_n + \gamma \sum_{n=0}^{\infty} z_n,$$

Ultimately, this leads to an equation of the form $X(z) = A(z)+B(z)+\Gamma(z)$, to which an extensive array of techniques from complex analysis [1] can be applied. The $\mathcal{Z}$-transform can thus be seen as the discrete-time equivalent of the Laplace transform [50], making it highly applicable in signal processing and related disciplines.

1.1 Research Motivation

The $\mathcal{Z}$-transform shares remarkable similarities with the Fourier transform [25], differing primarily in that the Fourier transform uses sine waves or circles, while the $\mathcal{Z}$-transform utilizes exponential curves. Given that the Fourier transform has been adapted for quantum computation, known as the quantum Fourier transform, it is plausible to anticipate that the $\mathcal{Z}$-transform could also be adapted for quantum implementation, leading to the development of a quantum $\mathcal{Z}$-transform.

1.2 Problem statement

The $\mathcal{Z}$-transform and the Fourier transform are both linear transforms [25]. The Fourier transform, akin to the $\mathcal{Z}$-transform, can be interpreted as a change of basis in the frequency domain. Furthermore, both transforms are capable of expediting intermediate operations and are reversible. These parallels suggest a favorable outcome.

When applying the discrete $\mathcal{Z}$-transform, as we formulated in section (4.1), to the sequence $[1, 2, 3]$, it produces $[1+2+3, \frac{1}{2}+\frac{1}{2}+\frac{3}{8}, \frac{1}{3}+\frac{2}{9}+\frac{1}{9}] = [6, \frac{11}{8}, \frac{2}{3}]$. This resulting sequence illustrates a series of sums involving progressively decreasing terms, presented within distinct fractional representations. However, the primary challenge lies in the classical definition of the $\mathcal{Z}$-transform (2.1), which is expressed as an infinite summation. This cannot be implemented on a quantum computer, as infinite summations do not operate on finite sets of variables. Therefore, to realize a quantum implementation, the $\mathcal{Z}$-transform must be redefined as a finite summation.

While quantum computers excel at performing unitary operations and are particularly adept at matrix multiplication, a problem arises when representing the $\mathcal{Z}$-transform as a matrix: the columns and rows, when considered as vectors, fail to preserve their lengths.

To illustrate how length preservation fails in vectors under the discrete $\mathcal{Z}$-transform (4.1), consider the mapping of $[1, 0]$ to $[1, \frac{1}{2}]$. The initial vector $[1, 0]$ has a length of 1, while the transformed vector $[1, \frac{1}{2}]$ has a length of $\sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2}$. This discrepancy clearly indicates that the length is not conserved after the transformation, thus failing to satisfy the unitarity criterion essential for quantum implementation. Therefore, it is imperative to explore methods to ensure the transformation is unitary.

1.3 Contributions

When we commenced this research, there were no documented attempts in the existing literature specifically aimed at developing an algorithm or subroutine for a quantum $\mathcal{Z}$-transform that could be implemented on quantum computers. Thus, our work in formalizing the problem, sorting out the mathematical requirements, laying the groundwork for the algorithmic design, and formulating a strategic plan represents pioneering efforts in this research direction, particularly in the discovery of a quantum $\mathcal{Z}$-transform.

Our ongoing research has yielded the following findings to date:

- We introduced a discrete $\mathcal{Z}$-transform by discretizing the classical $\mathcal{Z}$-transform and redefining it as a finite summation.
- We provided a matrix formulation of the discrete $\mathcal{Z}$-transform, a critical step in developing a quantum algorithm for the $\mathcal{Z}$-transform.

- We unitarized the discrete $\mathcal{Z}$-transform by block-encoding its matrix formulation, enabling the construction of unitary operators that adhere to the preconditions for efficient quantum operations using standard quantum gates and subroutines.

As a result, the advancements achieved in this thesis have provided an optimal framework for developing the quantum $\mathcal{Z}$-transform and established a robust foundation for future exploration and implementation in quantum computing.

1.4 Preliminaries

To fully appreciate the discussions, arguments, and findings presented in this thesis, it is essential for the reader to have a solid foundation in several key areas. An undergraduate-level understanding of quantum computation [29], [33] is necessary, providing familiarity with the fundamental concepts and principles that drive this cutting-edge field. A background in complex analysis [1] and contour integral [12] techniques is crucial for grasping the mathematical machinery employed throughout the research. Additionally, a solid grounding in quantum mechanics [3], [42] will help in understanding the fundamental physical concepts that underpin the theoretical framework of this thesis. Knowledge of number theory [4], [31] basics is also beneficial, given its relevance to certain computational aspects explored herein. Furthermore, proficiency in linear algebra [5] is indispensable, as it forms the backbone of many mathematical formulations used in the thesis. Lastly, foundational concepts of computer science [27] will enable a comprehensive understanding of the computational dimensions of this research.

1.5 Thesis Overview

In Chapter 1, we introduce the central problem this thesis aims to address, providing a comprehensive context for the ensuing discussion. Chapter 2 follows with a succinct review of the classical $\mathcal{Z}$-transform, detailing its fundamental properties and laying the groundwork for the quantum analogs explored later. Chapter 3 discusses the quantum analogs of the $\mathcal{Z}$-transform properties, illustrating a crucial connection between Fock Space and Coherent states.

In Chapter 4, we address the inherent challenges of achieving a quantum $\mathcal{Z}$-transform. To manage the infinite summation, we redefined the classical $\mathcal{Z}$-transform to create a discretized version. Based on this new definition,, we developed a matrix formulation for the discrete $\mathcal{Z}$-transform. Subsequently, we unitarized this matrix formulation for a finite number of variables, paving the way for efficient quantum operations with the discrete $\mathcal{Z}$-transform. Building on these insights, Chapter 5 draws parallels with the quantum Fourier transform, a successful subroutine, to guide the development of our targeted quantum subroutine.

Finally, Chapter 6 offers a critical analysis, synthesizing the connections and insights gleaned from the preceding chapters. We reflect on the implications of our findings and propose a future research direction to build upon the advancements achieved in this thesis.

Chapter 2

Exploring the Classical $\mathcal{Z}$-transform

2.1 Definition of the classical $\mathcal{Z}$-transform

The classical $\mathcal{Z}$-transform [6], [25], [28], symbolized by $F(z)$, is defined through a bilateral infinite power series to facilitate a transformation on a complex sequence denoted by $f[n]$,

$$F(z) = \sum_{n=-\infty}^{\infty} f[n]z^{-n} \tag{2.1}$$

Where n represents an integer and z denotes a complex number, in general.

In contrast, the unilateral $\mathcal{Z}$-transform of the complex sequence $f[n]$ is distinct from its bilateral counterpart due to the difference in the lower summation limit, as indicated below,

$$F^{+}(z) = \sum_{n=0}^{\infty} f[n]z^{-n} \tag{2.2}$$

With z as a complex variable, $F^{+}(z)$ refers to the unilateral $\mathcal{Z}$-transform for the complex sequence $f[n]$, which is specified exclusively for $n \geq 0$.

Throughout our discussion in this chapter, we shall refer to the classical $\mathcal{Z}$-transform or, more succinctly, the $\mathcal{Z}$-transform as the bilateral $\mathcal{Z}$-transform, except where noted otherwise.

It is useful to view the classical $\mathcal{Z}$-transform as an operator, symbolized by $\mathcal{Z}\{.\}$, that transforms a sequence into a function.

$$\mathcal{Z}\{f[n]\} = \sum_{n=-\infty}^{\infty} f[n]z^{-n} = F(z) \tag{2.3}$$

The existence of the $\mathcal{Z}$-transform is confined to those z values ensuring the convergence of the series delineated in equation (2.1). Such values demarcate the Region of Convergence (ROC) for $F(z)$, defining its domain, with the range comprised of the elements constituting $F(z)$.

2.2 Condition for the existence of $\mathcal{Z}$-transform

It is essential to recognize that the $\mathcal{Z}$-transform does not apply to every sequence. Given its nature as a power series, it requires that the sequence $f[n]z^{-n}$ be absolutely summable to ensure convergence [35], [43]. Mathematically,

$$|\sum_{n=-\infty}^{\infty} f[n]z^{-n}| < \infty \tag{2.4}$$

Equivalently,

$$\sum_{n=-\infty}^{\infty} |f[n]z^{-n}| < \infty \tag{2.5}$$

Now let us substitute $z = re^{j\omega}$,

$$\sum_{n=-\infty}^{\infty} |f[n](re^{j\omega})^{-n}| < \infty \tag{2.6}$$

Which is equivalent to

$$\sum_{n=-\infty}^{\infty} |f[n]r^{-n}||e^{-j\omega n}| < \infty \tag{2.7}$$

But $e^{-j\omega n} = 1$, so we can write,

$$\sum_{n=-\infty}^{\infty} |f[n]r^{-n}| < \infty \tag{2.8}$$

This is the condition for the existence of $\mathcal{Z}$-transform.

2.3 Region of Convergence

It is vital to precisely delineate the region of convergence (ROC) [18], [28] since it demarcates the boundaries within which the $\mathcal{Z}$-transform remains valid. This region, known as the ROC, is comprised of the z-values for which the series, as outlined in the equation (2.1), converges.

$$\text{ROC} = \left\{ z : \left| \sum_{n=-\infty}^{\infty} f[n]z^{-n} \right| < \infty \right\} \tag{2.9}$$

The ROC stated in equation (2.9) must satisfy the condition for the existence of the $\mathcal{Z}$-transform as described in section (2.2).

2.4 Properties of the $\mathcal{Z}$-transform

The properties [6], [14], [25], [28] of the $\mathcal{Z}$-transform significantly facilitate the process of determining the z-domain analog of a given time domain function. The notation we adopt is as follows,

$$f[n] \overset{\mathcal{Z}}{\longleftrightarrow} F(z), \qquad ROC = R$$

Where $F(z)$ corresponds to the $\mathcal{Z}$-transform of $f[n]$, with its region of convergence denoted by R.

2.4.1 Scaling in the z-domain

If $f[n] \overset{\mathcal{Z}}{\longleftrightarrow} F(z)$ with $ROC = R$, then $z_0^n f[n] \overset{\mathcal{Z}}{\longleftrightarrow} F\left(z_0^{-1}z\right)$ with $ROC = z_0|R|$; where $ROC = z_0|R|$ is the scaled version of R and z_0 is a constant [6], [14], [25], [28].

Proof:

Consider the sequence denoted by $z_0^n f[n]$. Utilizing the $\mathcal{Z}$-transform definition allows us to represent the sequence $z_0^n f[n]$ in the subsequent manner,

$$\mathcal{Z}\left\{z_0^n f[n]\right\} = \sum_{n=-\infty}^{\infty} z_0^n f[n] z^{-n} \tag{2.10}$$

By reorganizing the terms present on the right-hand side of equation (2.10), we obtain,

$$\mathcal{Z}\left\{z_0^n f[n]\right\} = \sum_{n=-\infty}^{\infty} f[n](z_0^n z^{-n}) \tag{2.11}$$

Since the term $(z_0^n z^{-n}) = (z_0 z^{-1})^n$ in equation (2.11), we can write

$$\mathcal{Z}\left\{z_0^n f[n]\right\} = \sum_{n=-\infty}^{\infty} f[n](z_0 z^{-1})^n \tag{2.12}$$

Equivalently, since $(z_0 z^{-1})^n = (z_0^{-1} z)^{-n}$, equation (2.12) becomes

$$\mathcal{Z}\left\{z_0^n f[n]\right\} = \sum_{n=-\infty}^{\infty} f[n](z_0^{-1} z)^{-n} \tag{2.13}$$

It is evident that the right side of equation (2.13) constitutes the $\mathcal{Z}$-transform of the sequence $z_0^{-1}z$, thereby establishing a mapping to the function $F(z_0^{-1}z)$.

$$\mathcal{Z}\left\{z_0^n f[n]\right\} = \sum_{n=-\infty}^{\infty} f[n](z_0^{-1} z)^{-n} = F(z_0^{-1} z) \tag{2.14}$$

ROC:

For scaling in the z-domain, the region of convergence (ROC) is specified as follows,

$$r_1 < |z_0^{-1} z| < r_2 \tag{2.15}$$

Equivalently,

$$|z_0| r_1 < |z| < |z_0| r_2 \tag{2.16}$$

2.4.2 Time shifting

If $f[n] \overset{\mathcal{Z}}{\longleftrightarrow} F(z)$ with $ROC = R$, then $f[n-k] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-k}F(z)$; with $ROC = R$, except for $z = 0$ (if $k > 0$) or $z = \infty$ (if $k < 0$) [6], [14], [25], [28].

Proof:

Considering the sequence $f[n-k]$, we can apply the definition of the $\mathcal{Z}$-transform to express the sequence in the following form,

$$\mathcal{Z}\{f[n-k]\} = \sum_{n=-\infty}^{\infty} f[n-k]z^{-n} \tag{2.17}$$

Letting m represent $n-k$, we can rewrite n as $m+k$, which in turn reformulates equation (2.17),

$$\mathcal{Z}\{f[n-k]\} = \sum_{n=-\infty}^{\infty} f[m]z^{-(m+k)} \tag{2.18}$$

Equivalently we can write,

$$\mathcal{Z}\{f[n-k]\} = \sum_{n=-\infty}^{\infty} f[m]z^{-m}z^{-k} \tag{2.19}$$

Modifying the right-hand side terms to meet our particular needs,

$$\mathcal{Z}\{f[n-k]\} = z^{-k} \sum_{n=-\infty}^{\infty} f[m]z^{-m} \tag{2.20}$$

In light of the $\mathcal{Z}$-transform's definition, it is evident that, excluding z^{-k}, the remaining terms on the right-hand side equate to the function $F(z)$. Therefore,

$$\mathcal{Z}\{f[n-k]\} = z^{-k}F(z) \tag{2.21}$$

The term z^{-k} affects the poles and zeros at $z=0$ and $z=-\infty$.

2.4.3 First difference

If $f[n] \overset{\mathcal{Z}}{\longleftrightarrow} F(z)$ with $ROC = R$, then $f[n] - f[n-1] \overset{\mathcal{Z}}{\longleftrightarrow} (1-z^{-1})F(z)$ with $ROC = R$ [6], [14], [25], [28].

Proof:

We have a sequence $f[n] - f[n-1]$. Let us plug this sequence into the $\mathcal{Z}$-transform operator, and since the $\mathcal{Z}$-transform operator is linear, we can write

$$\mathcal{Z}\{f[n] - f[n-1]\} = \mathcal{Z}\{f[n]\} - \mathcal{Z}\{f[n-1]\} \tag{2.22}$$

The first $\mathcal{Z}$-transform operator $\mathcal{Z}\{f[n]\}$, as delineated on the right-hand side of equation (2.22), applies the standard definition to transform the sequence $f[n]$ into the function $F(z)$,

$$\mathcal{Z}\{f[n]\} = \sum_{n=-\infty}^{\infty} f[n]z^{-n} = F(z) \tag{2.23}$$

However, the second $\mathcal{Z}$-transform operator $\mathcal{Z}\{f[n-1]\}$ on the right hand side of equation (2.22) maps the sequence $f[n-1]$ into a function $F(z)$ with a factor z^{-1} following the Time Shifting property which we proved in the subsection (2.4.2). So we have,

$$\mathcal{Z}\{f[n-1]\} = z^{-1}F(z) \tag{2.24}$$

Substituting the outcomes of equations (2.23) and (2.24) into equation (2.22), we obtain,

$$\mathcal{Z}\{f[n] - f[n-1]\} = F(z) - z^{-1}F(z) \tag{2.25}$$

Finally, factoring out the common function, we have,

$$\mathcal{Z}\{f[n] - f[n-1]\} = (1 - z^{-1})F(z) \tag{2.26}$$

2.4.4 Convolution

If $f[n] \overset{\mathcal{Z}}{\longleftrightarrow} F(z)$ with $ROC = R_1$, and $g[n] \overset{\mathcal{Z}}{\longleftrightarrow} F(z)$ with $ROC = R_2$, then $f[n] * g[n] \overset{\mathcal{Z}}{\longleftrightarrow} F(z)G(z)$ with $ROC \supseteq R_1 \cap R_2$ [6], [14], [25], [28].

Proof:

Convolution serves as a mathematical procedure that combines two distinct functions, resulting in a third function. Let us consider $r(n)$ as the convolution result of $f[n]$ and $g[n]$; thus, we define $r(n)$ as

$$r[n] = f[n] * g[n] = \sum_{k=-\infty}^{\infty} f[k] * g[n-k] \tag{2.27}$$

Let us now take the $\mathcal{Z}$-transform of $r[n]$,

$$\mathcal{Z}\{r[n]\} = \sum_{n=-\infty}^{\infty} r[n]z^{-n} \tag{2.28}$$

According to the definition of convolution as stated in equation (2.27),

$$\mathcal{Z}\{f[n] * g[n]\} = \sum_{n=-\infty}^{\infty} \{f[n] * g[n]\} z^{-n} \tag{2.29}$$

We can further use the convolution sum from equation (2.27) to write as follows

$$\mathcal{Z}\{f[n] * g[n]\} = \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} f[k] * g[n-k] \right\} z^{-n} \tag{2.30}$$

Now let $m = n - k$, then $n = m + k$; therefore we obtain,

$$\mathcal{Z}\{f[n] * g[n]\} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f[k]g[m]z^{-(m+k)} \tag{2.31}$$

Equivalently we can state that,

$$\mathcal{Z}\{f[n]*g[n]\} = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} f[k]g[m]z^{-m}z^{-k} \tag{2.32}$$

Interchanging the order of summation, we can re-write,

$$\mathcal{Z}\{f[n]*g[n]\} = \sum_{k=-\infty}^{\infty} f[k]z^{-k}\sum_{n=-\infty}^{\infty} g[m]z^{-m} \tag{2.33}$$

As per the $\mathcal{Z}$-transform definition, we deduce,

$$\mathcal{Z}\{f[n]*g[n]\} = F(z)G(z) \tag{2.34}$$

ROC:

The region of convergence (ROC) for the convolution is at a minimum, the intersection of the ROCs for $F(z)$ and $G(z)$.

2.4.5 Accumulation

If $f[n] \xleftrightarrow{\mathcal{Z}} F(z)$ with $ROC = R$, then $\sum_{n=-\infty}^{\infty} f[k] \xleftrightarrow{\mathcal{Z}} F(z)\frac{1}{1-z^{-1}}$ with $ROC \supseteq R \cap \{(|z|>1)\}$ [6], [14], [25], [28].

Proof:

We can write the accumulation of $f[n]$ as its convolution with $u[n]$ as per the definition provided in equation (2.27),

$$f[n]*u[n] = \sum_{k=-\infty}^{\infty} f[k]u[n-k] \tag{2.35}$$

Herein, the unit step function $u[n]$ is specified as follows,

$$u[n] = \begin{cases} 0, & n<0 \\ 1, & n\geq 0 \end{cases}$$

Therefore, equation (2.35) turns out to be

$$f[n]*u[n] = \sum_{k=-\infty}^{n} f[k] \tag{2.36}$$

By employing the convolution property, we are able to represent,

$$\mathcal{Z}\left\{\sum_{n=-\infty}^{n} f[k]\right\} = \mathcal{Z}\{f[n]*u[n]\} \tag{2.37}$$

Given that $\mathcal{Z}\{u[n]\} = \frac{1}{1-z^{-1}}$, we can reformulate equation (2.37) as follows,

$$\mathcal{Z}\left\{\sum_{n=-\infty}^{n} f[k]\right\} = F(z)\frac{1}{1-z^{-1}} \tag{2.38}$$

2.4.6 Differentiation in the z-domain

If $f[n] \overset{\mathcal{Z}}{\longleftrightarrow} F(z)$ with $ROC = R$, then $nf[n] \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{dF(z)}{dz}$ with $ROC = R$ [6], [14], [25], [28].

Proof:

We have a sequence $nf[n]$ and we apply the $\mathcal{Z}$-transform on it,

$$\mathcal{Z}\{nf[n]\} = \sum_{n=-\infty}^{\infty} nf[n]z^{-n} \tag{2.39}$$

Since $z^{-1}z = 1$, we can write the above equation as

$$\mathcal{Z}\{nf[n]\} = z\sum_{n=-\infty}^{\infty} nf[n]z^{-n}z^{-1} \tag{2.40}$$

Equivalently we can write,

$$\mathcal{Z}\{nf[n]\} = z\sum_{n=-\infty}^{\infty} nf[n]z^{-n-1} \tag{2.41}$$

Differently stated,

$$\mathcal{Z}\{nf[n]\} = -z\sum_{n=-\infty}^{\infty} f[n](-nz^{-n-1}) \tag{2.42}$$

Since $\frac{d}{dz}(z^{-n}) = -nz^{-n-1}$, we can write from equation (2.42),

$$\mathcal{Z}\{nf[n]\} = -z\sum_{n=-\infty}^{\infty} f[n]\frac{d}{dz}(z^{-n}) \tag{2.43}$$

Rearranging the constant terms for our particular need,

$$\mathcal{Z}\{nf[n]\} = -z\frac{d}{dz}\left\{\sum_{n=-\infty}^{\infty} f[n]z^{-n}\right\} \tag{2.44}$$

In light of the $\mathcal{Z}$-transform definition, it is possible to express,

$$\mathcal{Z}\{nf[n]\} = -z\frac{dF(z)}{dz} \tag{2.45}$$

2.4.7 Time Expansion

If $f[n] \overset{\mathcal{Z}}{\longleftrightarrow} F(z)$ with $ROC = R$, then $f_{(k)}[n] \overset{\mathcal{Z}}{\longleftrightarrow} F(z^k)$ with $ROC = R^{\frac{1}{k}}$, [6], [14], [25], [28] where $f_{(k)}[n]$ is defined as

$$f_{(k)}[n] = \begin{cases} f[n/k], & n \text{ is a multiple of } k \\ 0, & \text{otherwise} \end{cases}$$

Proof:

In employing the $\mathcal{Z}$-transform definition on the sequence $f_{(k)}[n]$,

$$\mathcal{Z}\left\{f_{(k)}[n]\right\} = \sum_{n=-\infty}^{\infty} f_{(k)}[n]z^{-n} \tag{2.46}$$

Employing the given definition for the sequence $f_{(k)}[n]$, we may express,

$$\mathcal{Z}\left\{f_{(k)}[n]\right\} = \sum_{n=-\infty}^{\infty} f[n/k]\ z^{-n} \tag{2.47}$$

Assume $n/k = r$, from which it follows that $n = rk$; accordingly, equation (2.47) evolves into,

$$\mathcal{Z}\left\{f_{(k)}[n]\right\} = \sum_{n=-\infty}^{\infty} f[r]z^{-rk} \tag{2.48}$$

Differently stated for our particular need,

$$\mathcal{Z}\left\{f_{(k)}[n]\right\} = \sum_{n=-\infty}^{\infty} f[r](z^k)^{-r} \tag{2.49}$$

By the definition of $\mathcal{Z}$-transform, this turns out to be

$$\mathcal{Z}\left\{f_{(k)}[n]\right\} = F(z^k) \tag{2.50}$$

ROC:

$$r_1 < |z^k| < r_2 \tag{2.51}$$

Equivalently,

$$r_1^{\frac{1}{k}} < |z| < r_2^{\frac{1}{k}} \tag{2.52}$$

2.5 The Inverse $\mathcal{Z}$-transform

According to the Cauchy integral formula [12], [25], for any closed contour ∂C_z that circumscribes the origin in a counterclockwise manner, it follows that

$$\frac{1}{2\pi i}\oint_{\partial C_z} \mathrm{d}z\ z^{k-1} = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \tag{2.53}$$

The $\mathcal{Z}$-transform, denoted as $F(z)$, of a complex sequence $f[n]$ is precisely defined through

$$F(z) = \sum_{n=-\infty}^{\infty} f[n]z^{-n} \tag{2.54}$$

When both sides of equation (2.54) are multiplied by $\frac{1}{2\pi i}z^{k-1}$, this yields

$$\frac{1}{2\pi i}F(z)z^{k-1} = \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty} f[n]z^{-n}z^{k-1} \tag{2.55}$$

Integrating over a contour ∂C_z that encloses the origin counterclockwise and resides within the region of convergence (ROC) of $F(z)$ yields,

$$\frac{1}{2\pi i}\oint_{\partial C_z} \mathrm{d}z\, F(z)z^{k-1} = \frac{1}{2\pi i}\oint_{\partial C_z} \mathrm{d}z \sum_{n=-\infty}^{\infty} f[n]z^{-n+k-1} \tag{2.56}$$

Given the convergent nature of the series, the reordering of integration and summation is justified on the right-hand side of equation (2.56), which leads to

$$\frac{1}{2\pi i}\oint_{\partial C_z} \mathrm{d}z\, F(z)\, z^{k-1} = \sum_{n=-\infty}^{\infty} f[n]\, \frac{1}{2\pi i}\oint_{\partial C_z} \mathrm{d}z\, z^{-n+k-1} \tag{2.57}$$

The application of the Cauchy integral formula to the integral on the right-hand side of equation (2.57) provides us with,

$$\frac{1}{2\pi i}\oint_{\partial C_z} \mathrm{d}z\, z^{-n+k-1} = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases} \tag{2.58}$$

By the virtue of the approach specified in equation (2.58), we find that equation (2.57) evolves into

$$\frac{1}{2\pi i}\oint_{\partial C_z} \mathrm{d}z\, F(z)\, z^{k-1} = f[k] \tag{2.59}$$

As a result, the formula representing the inverse $\mathcal{Z}$-transform is encapsulated by the integral below,

$$f[n] = \frac{1}{2\pi \mathrm{i}}\oint_{\partial C_z} \mathrm{d}z\, z^{n-1}F(z) \tag{2.60}$$

For this integral, we consider a path defined by a counterclockwise enclosed circular contour ∂C_z, with its center at the origin and radius r, located within the region of convergence (ROC) of $F(z)$.

Chapter 3

A Bosonic Quantum $\mathcal{Z}$-transform

3.1 Fock Space

In quantum mechanics, Fock space [2], [22] serves as the state space for a variable number of elementary particles. Given the two fundamental categories of these particles—bosons and fermions—there emerge correspondingly distinct Fock spaces: bosonic and fermionic. Our work will specifically engage with the bosonic Fock space.

Within the bosonic Fock space, we identify the creation operator with $a^\dagger$ and the annihilation operator with a. The quantum states in this space, referred to as number states, are defined by their occupation numbers within the ket notation. The vacuum state, indicating no particles, is denoted by $|0\rangle$, whereas the states containing particles, known as non-vacuum states, are represented by $|n\rangle$, where n is a positive integer i.e. $n > 0$.

The action of the creation operator $a^\dagger$, applied n times to a vacuum state, results in the creation of n particles, indicated as,

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n |0\rangle$$

The action of the annihilation operator a on the vacuum state leads to the total annihilation of the state, yielding an outcome of zero.

$$a|0\rangle = 0$$

The non-vacuum states, represented as a set of eigenstates $|n\rangle$, form a basis, allowing any state to be expressed as their appropriate linear combination.

When applied to non-vacuum states, the creation operator acts as follows,

$$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

When applied to non-vacuum states, the annihilation operator acts as follows,

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$

Within the framework of a free scalar field, it is imperative to track unique excitation numbers, denoted by n_i. The creation operator $a_i^\dagger$ and the annihilation operator a_i interact with n_i in well-defined manners, detailed as follows,

The effect of creation operator $a_i^\dagger$ on the non-vacuum state manifests as

$$a_i^\dagger |n_1, n_2, \cdots, n_i, \cdots, n_j\rangle = \sqrt{n_i + 1}\, |n_1, n_2, \cdots, n_i + 1, \cdots, n_j\rangle$$

The effect of annihilation operator a_i on the non-vacuum state manifests as

$$a_i |n_1, n_2, \cdots, n_i, \cdots, n_j\rangle = \sqrt{n_i}\, |n_1, n_2, \cdots, n_i - 1, \cdots, n_j\rangle$$

We introduce the number operator as $\hat{n} := a^\dagger a$, which adheres to the corresponding eigenequation,

$$\hat{n} |n_1, n_2, \cdots, n_i, \cdots, n_j\rangle = n_i |n_1, n_2, \cdots, n_i, \cdots, n_j\rangle$$

The eigenstates of the number operator $\hat{n}$ form a basis for the entirety of Hilbert space, recognized as the Fock basis [22]; the space constructed from this basis is known as Fock space.

3.2 Coherent States

Coherent states [19], [48] are characterized as the displaced ground states of a harmonic oscillator [8], and they can be described in a non-normalized form as follows,

$$|(z)\rangle = e^{za^\dagger} |0\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \tag{3.1}$$

These coherent states serve as eigenkets for the boson annihilation operator a, in accordance with,

$$a|(z)\rangle = z|(z)\rangle \tag{3.2}$$

The inclusion of the complex variable z within the round brackets of $|(z)\rangle$ signifies that the states in the selected non-normalized standardization vary analytically with z, independent of the complex conjugate z^*.

We denote the adjoint states corresponding to $|(z)\rangle$ by $\langle(z^*)|$, which, upon relabeling $z^* \to z$, can be described explicitly as follows,

$$\langle(z)| = \langle 0| e^{za} = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \langle n| \tag{3.3}$$

These represent the eigenbras associated with the boson creation operator $a^\dagger$,

$$\langle(z)|\, a^\dagger = z \,\langle(z)| \tag{3.4}$$

The states $\langle(z)|$ are analytically dependent on the complex variable z.

Now, for auxiliary purposes, we introduce the notation $|(z)\rangle_\mu$ for non-normalizable eigenkets possessed by $a^\dagger$, where the subscript μ in $|(z)\rangle_\mu$ means the state belongs to $a^\dagger$.

To establish a one-to-one correspondence between the transformation from the number state representation $|n\rangle$ to the coherent state representation $|(z)\rangle$ with the mathematical $\mathcal{Z}$-transform, we employ the properties of the eigenkets $|(z)\rangle_\mu$ of the creation operator, particularly the completeness relation involving $|(z)\rangle_\mu$ and $\langle(z)|$ as

$$\oint_{\partial C} \mathrm{d}z \left\{|(z)\rangle_\mu\right\} \langle(z)| = I \tag{3.5}$$

Equivalently stated,

$$\oint_{\partial C} \mathrm{d}z\, |(z)\rangle \left\{{}_\mu\langle(z)|\right\} = I \tag{3.6}$$

Here, ∂C indicates a contour encircling the coordinate origin in the complex plane, and I stands for the identity operator.

The definition of the eigenkets $|(z)\rangle_\mu$ for the creation operator $a^\dagger$ is given by,

$$|(z)\rangle_\mu = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n!}} \delta^{(n)}(z)\, |n\rangle = e^{-a^\dagger \frac{\partial}{\partial z}}\, \delta(z)\, |0\rangle \tag{3.7}$$

The eigenbras ${}_\mu\langle(z)|$, associated with the annihilation operator a, are delineated as follows,

$$ {}_\mu\langle(z)| = \sum_{n=0}^{\infty} \langle n|\, \frac{(-1)^n}{\sqrt{n!}} \delta^{(n)}(z) = \langle 0|\, e^{-a \frac{\partial}{\partial z}}\, \delta(z) \tag{3.8}$$

Proceeding from $|(z)\rangle_\mu$ of $a^\dagger$, the eigenvalue equation unfolds as follows,

$$a^\dagger |(z)\rangle_\mu = z|(z)\rangle_\mu \tag{3.9}$$

Arising from ${}_\mu\langle(z)|$ of a is the eigenvalue equation, delineated as follows,

$$ {}_\mu\langle(z)|a = z_\mu\langle(z)| \tag{3.10}$$

Within equations (3.7) and (3.8), $\delta(z)$ represents the Dirac delta function, depicted through its contour integral representation as follows,

$$\delta(z) = \left.\frac{1}{\mathrm{i}2\pi z}\right|_{\partial C} \tag{3.11}$$

The n-th derivative of $\delta(z)$, represented by $\delta^{(n)}(z)$, is presented as follows,

$$\delta^{(n)}(z) = \left.\frac{(-1)^n n!}{\mathrm{i}2\pi z^{n+1}}\right|_{\partial C} = \frac{(-1)^n n!}{z^n}\delta(z) \tag{3.12}$$

The orthogonality relation is characterized by,

$$\langle(z')\,|(z)\rangle_\mu = \delta\,(z - z') \tag{3.13}$$

It is essential to underscore that, in alignment with the treatment in equation (3.3), we deviate from Dirac's customary notation where $\langle\psi|$ is the adjoint of $|\psi\rangle$. This deviation is crucial for maintaining the correct dependency of $\langle(z)|$ and $_\mu\langle(z)|$ on z rather than on the complex conjugate z^*.

3.3 Properties of Coherent States

The coherent states exhibit a range of noteworthy properties [10], [19], [32], [34], [46], some of which are relevant to our current work and will be discussed herein.

3.3.1 Coherent States in Fock State Basis

The basis of Fock states allows for the representation of coherent states [10], [19], [48]. The non-normalized coherent state $|(z)\rangle$ is delineated in equation (3.1) as follows

$$|(z)\rangle = e^{za^\dagger}\,|0\rangle \tag{3.14}$$

where $a^\dagger$ is the creation operator, z is a complex number, and $|0\rangle$ is the vacuum state. We can expand the exponential function by employing its Taylor series, which leads to

$$e^{za^\dagger} = \sum_{n=0}^{\infty} \frac{(za^\dagger)^n}{n!} \tag{3.15}$$

By applying this to the vacuum state $|0\rangle$, we produce

$$|(z)\rangle = \left(\sum_{n=0}^{\infty} \frac{(za^\dagger)^n}{n!}\right)|0\rangle \tag{3.16}$$

When the creation operator is raised to the n-th power and applied to the vacuum state, it produces

$$(a^\dagger)^n\,|0\rangle = \sqrt{n!}\,|n\rangle \tag{3.17}$$

The coherent state is then described by the following mathematical expression,

$$|(z)\rangle = \sum_{n=0}^{\infty} \frac{(za^\dagger)^n}{n!}\,|0\rangle = \sum_{n=0}^{\infty} \frac{z^n(a^\dagger)^n}{n!}\,|0\rangle = \sum_{n=0}^{\infty} \frac{z^n}{n!}\sqrt{n!}\,|n\rangle \tag{3.18}$$

By simplifying the terms, we can achieve

$$|(z)\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\,|n\rangle \tag{3.19}$$

This expression describes the non-normalized coherent state as an infinite summation over the Fock states $|n\rangle$, with each state weighted by the coefficient $\frac{z^n}{\sqrt{n!}}$.

3.3.2 Non-Orthogonality of Coherent States

Let $|(w)\rangle$ and $|(z)\rangle$ represent two coherent states, defined accordingly.

$$|(w)\rangle = \sum_{n=0}^{\infty} \frac{w^n}{\sqrt{n!}} |n\rangle \,, \quad |(z)\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \tag{3.20}$$

The inner product of these states is determined in the following manner,

$$\begin{aligned}
\langle (w)|(z)\rangle &= \left(\sum_{m=0}^{\infty} \frac{(\overline{w})^m}{\sqrt{m!}} \langle m| \right) \left(\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \right) \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\overline{w})^m}{\sqrt{m!}} \frac{z^n}{\sqrt{n!}} \delta_{mn} \\
&= \sum_{n=0}^{\infty} \frac{(\overline{w}z)^n}{n!} \\
&= e^{\overline{w}z}
\end{aligned}$$

The inner product $\langle (w)|(z)\rangle = e^{\overline{w}z}$ indicates that the coherent states $|(w)\rangle$ and $|(z)\rangle$ are orthogonal only if $\overline{w}z = 0$, illustrating the typically non-orthogonal nature of coherent states [10], [19], [48].

3.3.3 Non-Uniqueness of Coherent State Decompositions

Coherent states do not form a standard basis like the eigenstates of the number operator; they form an overcomplete set instead. This overcompleteness [7], [37], [38], [41] indicates that while every vector in the Hilbert space is representable through superposition of vectors from this set, the vectors within the set are not linearly independent, and there is more than one way to represent each vector in the space using the set. This surplus of vectors relative to a minimal spanning set allows multiple representations for the same state.

The integral over coherent states yields the identity operator, indicating their ability to span the space

$$\int \frac{d^2z}{\pi} |(z)\rangle\langle (z)| = I,$$

where I is the identity operator, and d^2z represents integration over the complex plane. This is not just a spanning set but an overcomplete one; and the combination of overcompleteness and non-orthogonality means that any decomposition using coherent states isn't unique.

Let us consider the quantum state $|\psi\rangle$. This state can potentially be expressed as a superposition of coherent states.

$$|\psi\rangle = \int \frac{d^2z}{\pi} f(z)|(z)\rangle$$

where $f(z)$ is some weighting function over z.

Considering the overcompleteness, we infer the existence of an additional function, $g(z)$, such that

$$|\psi\rangle = \int \frac{d^2z}{\pi} g(z)|(z)\rangle.$$

Although $f(z)$ and $g(z)$ both map to the same state $|\psi\rangle$, they do not necessarily have to be identical functions.

$$|\psi\rangle = \int \frac{d^2z}{\pi} f(z)|(z)\rangle = \int \frac{d^2z}{\pi} g(z)|(z)\rangle.$$

This lack of uniqueness arises owing to the non-orthogonal nature of coherent states $|(z)\rangle$.

$$\langle(w)|(z)\rangle = e^{\overline{w}z} \neq 0 \text{ for } w \neq z.$$

To effectively illustrate the concept of non-uniqueness, consider the modification of $f(z)$ by incorporating an additional function $h(z)$ such that

$$\int \frac{d^2z}{\pi} h(z)|(z)\rangle = 0.$$

This function $h(z)$ can be constructed because of the non-orthogonality of coherent states, and the infinite dimensionality of the function space. By setting $g(z) = f(z) + h(z)$, we find that both $f(z)$ and $g(z)$ equally and effectively continue to represent $|\psi\rangle$, even though they differ as functions.

This case highlights the non-uniqueness in the decomposition of quantum states into coherent states because of their overcompleteness. The existence of a possible function $h(z)$ indicates that the coefficients in this decomposition are not uniquely fixed by the state $|\psi\rangle$ itself.

3.4 The $\tilde{\mathcal{Z}}$-transform as a representation transform from $|n\rangle$ to $|(z)\rangle$

The $\mathcal{Z}$-transform strikingly resembles the discrete counterpart of the Laplace transform,

$$F(s) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-st} f(t) \tag{3.21}$$

With $f(t)$ being a real or complex function; the corresponding unilateral $\mathcal{Z}$-transform of the sequence $f[n]$ for a complex variable z is characterized by

$$\mathcal{Z}\{f[n]\} \equiv F(z) = \sum_{n=0}^{\infty} \frac{f[n]}{z^n} \tag{3.22}$$

The interaction between $f[n]$ and $F(z)$ is expressed through

$$f[n] \overset{\mathcal{Z}}{\longleftrightarrow} F(z) \tag{3.23}$$

The definition of the inverse $\mathcal{Z}$-transform is established as:

$$f[n] = \frac{1}{2\pi \mathrm{i}} \oint_{\partial C_z} \mathrm{d}z \; z^{n-1} F(z) \tag{3.24}$$

For this integral, the path taken is a closed circular contour ∂C_z, oriented counter-clockwise, featuring a radius of r, and is centered at the origin. Through the variable transformation $z \to z^{'} = \frac{1}{z}$, we arrive at the following expression:

$$\tilde{F}(z) \equiv F\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} f[n] z^n \tag{3.25}$$

Transforming $\tilde{F}(z)$ into $f[n]$ is defined as the inverse $\tilde{\mathcal{Z}}$-transform.

$$\begin{aligned} f[n] &= \frac{1}{\mathrm{i}2\pi} \oint_{\partial C''} \mathrm{d}z \left(\frac{1}{z}\right)^{n-1} F\left(\frac{1}{z}\right)\left(-\frac{1}{z^2}\right) \\ &= \frac{1}{\mathrm{i}2\pi} \oint_{\partial C_z'} \mathrm{d}z \frac{1}{z^{n+1}} \tilde{F}(z) \end{aligned} \tag{3.26}$$

We refer to the transformation from $f[n]$ to $\tilde{F}(z)$ as the $\tilde{\mathcal{Z}}$-transform [23], which is fundamentally similar to the $\mathcal{Z}$-transform, with the relationship between $f[n]$ and $\tilde{F}(z)$ being expressed as follows:

$$f[n] \stackrel{\tilde{\mathcal{Z}}}{\longleftrightarrow} \tilde{F}(z) \tag{3.27}$$

We now turn our attention to the $\tilde{\mathcal{Z}}$-transform as it applies to the Fock-state representation of states, in accordance with the approach described by Hong Yi (2004) [23]. Starting with any vector $|f\rangle$, we can generate the sequence:

$$f[n] = \frac{\langle n|f\rangle}{\sqrt{n!}}, \quad n = 0, 1, 2, \ldots \tag{3.28}$$

This, up to the factors of $1/\sqrt{n!}$, corresponds to the Fock-state representation denoted by $\langle n|f\rangle$. For the states to be normalized,

$$\sum_{n=0}^{\infty} n!(f[n])^* f[n] = \sum_{n=0}^{\infty} \langle f|n\rangle\langle n|f\rangle = \langle f|f\rangle = 1 \tag{3.29}$$

The definitions of $\tilde{F}(z)$ and $\langle (z)|$ are now set forth as follows:

$$\tilde{F}(z) = \langle (z)|f\rangle \tag{3.30}$$

$$\langle (z)| \equiv \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \langle n| \tag{3.31}$$

Through the use of the completeness relation

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = I \tag{3.32}$$

We acquire the explicit representation of $\tilde{F}(z)$, which is:

$$\begin{aligned}\tilde{F}(z) = \langle (z)| \sum_{n=0}^{\infty} |n\rangle \langle n|f\rangle &= \sum_{n=0}^{\infty} \frac{z^n \langle n|f\rangle}{\sqrt{n!}} \\ = \sum_{n=0}^{\infty} z^n f[n] \end{aligned} \tag{3.33}$$

This is consistent with the standard definition provided by equation (3.25) for the $\tilde{\mathcal{Z}}$-transform; therefore, we refer to the $\tilde{\mathcal{Z}}$-transform as a bosonic quantum $\mathcal{Z}$-transform. Conversely, by applying equations (3.28), (3.30), (3.31), along with the completeness relation in (3.6), the inversion is derived as follows,

$$\begin{aligned} f[n] &= \frac{1}{\sqrt{n!}} \langle n| \oint_{\partial C} \mathrm{d}z \left\{ |(z)\rangle_\mu \right\} \langle (z)|f\rangle \\ &= \frac{1}{\sqrt{n!}} \oint_{\partial C} \mathrm{d}z \frac{(-1)^n}{\sqrt{n!}} \delta^{(n)}(z) \tilde{F}(z) \\ &= \frac{1}{\mathrm{i}2\pi} \oint_{\partial C} \mathrm{d}z \frac{\tilde{F}(z)}{z^{n+1}} \end{aligned} \tag{3.34}$$

The function $\tilde{F}(z)$ is equivalent to the Bargmann representation of the state $|f\rangle$. [8] This implies that the bosonic quantum $\mathcal{Z}$-transform is related to the transition from the Fock-state representation to the Bargmann representation of a state. The benefit of viewing this transition as a type of bosonic quantum $\mathcal{Z}$-transform [23] or the $\tilde{\mathcal{Z}}$-transform is that it allows one to overtake some well-established rules for these transforms and describe them using the specific language of quantum mechanics.

3.5 Properties of Bosonic Quantum $\mathcal{Z}$-transform

The mathematical properties of the bosonic quantum $\mathcal{Z}$-transform or the $\tilde{\mathcal{Z}}$-transform, as noted by Hong and Yi (2004) [23], find their parallels within the realm of quantum mechanics.

3.5.1 Scaling in the z-domain

The scaling property [23] of the $\tilde{\mathcal{Z}}$-transform is illustrated by the relationship $z_0^n f[n] \xrightarrow{\tilde{\mathcal{Z}}} \tilde{F}(z_0 z)$. To explore its implications in the realm of quantum mechanics, we must define an operator, denoted as $\hat{P}$, in the following manner

$$\hat{P} \equiv \oint_{\partial C} \mathrm{d}z \left\{ |(z)\rangle_\mu \right\} \langle (z_0 z)| \tag{3.35}$$

Carrying out the contour integral by employing equation

$$f^{(n)}(0) = (-1)^n \oint_{\partial C} \mathrm{d}z f(z) \delta^{(n)}(z) \tag{3.36}$$

and by utilizing equations (3.9), (3.10), (3.11), (3.12), (3.30), and (3.31), we derive

$$\begin{aligned}\hat{P} &= \oint_{\partial C} \mathrm{d}z \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n!}} \delta^{(n)}(z)|n\rangle \sum_{m=0}^{\infty} \langle m| \frac{(z_0 z)^m}{\sqrt{m!}} \\ &= \sum_{n=0}^{\infty} z_0^n |n\rangle\langle n|\end{aligned} \tag{3.37}$$

It is thereby deduced that

$$\langle n|\, \hat{P} = z_0^n \,\langle n| \tag{3.38}$$

Making use of the orthogonality relation, we determine

$$\langle (z)|\, \hat{P} = \langle (z_0 z)| \tag{3.39}$$

When equations (3.38) and (3.39) are considered in tandem, we establish that

$$z_0^n f[n] \xrightarrow{\tilde{\mathcal{Z}}} \tilde{F}(z_0 z) \Rightarrow \frac{z_0^n}{\sqrt{n!}} \langle n|f\rangle = \frac{1}{\sqrt{n!}} \langle n|\{\hat{P}|f\rangle\} \xleftrightarrow{\tilde{\mathcal{Z}}} \langle (z)|\{\hat{P}|f\rangle\} \quad = \langle (z_0 z)|f\rangle. \tag{3.40}$$

3.5.2 Time Shifting

The time shifting property [23] associated with the $\tilde{\mathcal{Z}}$-transform is expressed as $f[n-k]u[n-k] \xleftrightarrow{\tilde{\mathcal{Z}}} z^k \tilde{F}(z)$. Here, the unit step function, denoted by $u[n-k]$, is defined such that

$$u[n-k] = \begin{cases} 0, & n < k \\ 1, & n \geq k \end{cases} \tag{3.41}$$

Consequently, by shifting the sequence $f[n]$ to the right by k steps, we obtain the sequence $f[n-k]u[n-k]$, with the initial k elements being 0. In light of the equations (3.28), (3.30), and (3.31), we can derive its counterpart in the realm of quantum mechanics,

$$\left.\begin{array}{rr} \frac{1}{\sqrt{(n-k)!}} \langle n-k|f\rangle, & n \geq k \\ 0, & n < k \end{array}\right\} \tag{3.42}$$

Therefore, for $n \geq k$, we obtain

$$\frac{1}{\sqrt{(n-k)!}} \langle n-k|f\rangle = \frac{1}{\sqrt{n!}} \left\langle n \left| \left\{ a^{\dagger k} |f\rangle \right\} \right. \longleftrightarrow \left\langle (z) \left| \left\{ a^{\dagger k} |f\rangle \right\} \right. \right. \quad = z^k \langle (z)|f\rangle. \tag{3.43}$$

3.5.3 First Difference

The first difference [23] of a sequence, denoted as $f[n]$, is characterized by

$$\nabla f[n] = f[n] - f[n-1]u[n-1] \tag{3.44}$$

The corresponding $\tilde{\mathcal{Z}}$-transform is given by

$$\nabla f[n] \overset{\tilde{\mathcal{Z}}}{\longleftrightarrow} (1-z)\tilde{F}(z) \tag{3.45}$$

Utilizing the identification $f[n] = \frac{1}{\sqrt{n!}}\langle n|f\rangle$ for $n = 0, 1, 2, \ldots$, we can formulate the quantum-mechanical representation of equation (3.44) accordingly,

$$\frac{1}{\sqrt{n!}}\langle n|f\rangle - \frac{1}{\sqrt{(n-1)!}}\langle n-1|f\rangle \longleftrightarrow (1-z)\langle(z)|f\rangle \tag{3.46}$$

As such, by employing the completeness relation $\sum_{n=0}^{\infty}|n\rangle\langle n| = I$, we can delineate the quantum mechanical expression for the first difference property,

$$\frac{1}{\sqrt{n!}}\left\langle n \left| \left\{ \left(1-a^{\dagger}\right) |f\rangle \right\} \right. \right. \longleftrightarrow \left\langle (z) \left| \left\{ \left(1-a^{\dagger}\right) |f\rangle \right\} \right. \right. \tag{3.47}$$

3.5.4 Convolution

The definition below formalizes the convolution [23] between two sequences, identified as $f[n]$ and $g[n]$,

$$f[n] * g[n] = \sum_{n_1=0}^{n} f\left[n_1\right] g\left[n-n_1\right] \tag{3.48}$$

The associated $\tilde{\mathcal{Z}}$-transform is delineated as follows,

$$f[n] * g[n] \overset{\tilde{\mathcal{Z}}}{\longleftrightarrow} \tilde{F}(z)\tilde{G}(z) \tag{3.49}$$

We present the two-mode coherent state and two-mode number state $|n_1, n_2\rangle$, highlighting their adherence to a completeness relation,

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{n} |n_1, n_2\rangle\langle n_1, n_2| = \sum_{n=0}^{\infty}\sum_{n_1=0}^{n} |n_1, n-n_1\rangle\langle n_1, n-n_1| = I \tag{3.50}$$

In light of the identifications given in equations (3.28), (3.30), and (3.31), we find the following,

$$\langle(z_1 = z, z_2 = z)\,|f, g\rangle = \tilde{F}(z)\tilde{G}(z) \tag{3.51}$$

In reference to equation (3.50), we can determine that

$$\begin{aligned}\tilde{F}(z)\tilde{G}(z) &= \langle(z_1 = z, z_2 = z)| \sum_{n=0}^{\infty}\sum_{n_1=0}^{n} \langle n_1, n-n_1| \langle n_1, n-n_1|f,g\rangle \\ &= \sum_{n=0}^{\infty}\sum_{n_1=0}^{n} \frac{z^n}{\sqrt{n_1!\,(n-n_1)!}} \langle n_1|f\rangle \langle n-n_1|g\rangle \\ &= \sum_{n=0}^{\infty} z^n \left(\sum_{n_1=0}^{n} f\left[n_1\right] g\left[n-n_1\right]\right) \\ &= \sum_{n=0}^{\infty} z^n f[n] * g[n]\end{aligned} \tag{3.52}$$

We verify this through calculation,

$$\frac{1}{\mathrm{i}2\pi}\oint_{\partial C} \mathrm{d}z \frac{1}{z^{n+1}} \langle(z_1 = z, z_2 = z)| = \sum_{n_1=0}^{n} \frac{z^n}{\sqrt{n_1!\,(n-n_1)!}} \langle n_1, n-n_1| \tag{3.53}$$

We can characterize the inverse transform of Equation (3.52) as

$$\begin{aligned}\sum_{n_1=0}^{n} f_{n_1} g_{n-n_1} &= \sum_{n_1=0}^{n} \frac{\langle n_1, n-n_1|}{\sqrt{n_1!\,(n-n_1)!}} |f,g\rangle \\ &= \frac{1}{\mathrm{i}2\pi}\oint_{\partial C} \mathrm{d}z \frac{1}{z^{n+1}} \langle(z_1 = z, z_2 = z)\,|f,g\rangle \\ &= \frac{1}{\mathrm{i}2\pi}\oint_{\partial C} \mathrm{d}z \frac{1}{z^{n+1}} \tilde{F}(z)\tilde{G}(z)\end{aligned} \tag{3.54}$$

Accordingly, the quantum mechanical formulation for the $\tilde{\mathcal{Z}}$-transform concerning the convolution of two sequences $f_n * g_n$ is

$$\sum_{n_1=0}^{n} \frac{z^n}{\sqrt{n_1!\,(n-n_1)!}} \langle n_1, n-n_1|f,g\rangle \longleftrightarrow \langle(z_1 = z, z_2 = z)\,|f,g\rangle \tag{3.55}$$

3.5.5 Accumulation

The property of accumulation [23] inherent to the $\tilde{\mathcal{Z}}$-transformation is characterized by

$$\sum_{j=0}^{n} f[j] \overset{\tilde{\mathcal{Z}}}{\longleftrightarrow} [1/1-z]\bar{F}(z) \tag{3.56}$$

In pursuit of its quantum mechanical equivalence, we introduce the operator $\hat{Q}$,

$$\hat{Q} \equiv \oint_{\partial C} \mathrm{d}z \frac{1}{1-z} \{|(z)\rangle_\mu\} \langle(z)| \tag{3.57}$$

By employing equations (3.9), (3.10), (3.11), (3.12), (3.30), and (3.31) to perform this contour integral, we achieve

$$\begin{aligned}\hat{Q} &= \int_{\partial C} \mathrm{d}z \sum_{m=0}^{\infty} \frac{(-1)^m}{\sqrt{m!}} \delta^{(m)}(z)|m\rangle \sum_{n=0}^{\infty} \langle n| \frac{z^n}{\sqrt{n!}} \frac{1}{1-z} \\ &= \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{1}{\sqrt{m!}\sqrt{n!}} |m\rangle\langle n| C_n^m n!(m-n)! \\ &= \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{\sqrt{m!}}{\sqrt{n!}} |m\rangle\langle n| \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{\sqrt{m!}}{\sqrt{n!}} |m\rangle\langle n|\end{aligned} \tag{3.58}$$

In light of this,

$$\frac{1}{\sqrt{n!}} \langle n| \hat{Q} = \sum_{j=0}^{n} \frac{1}{\sqrt{j!}} \langle j| \tag{3.59}$$

Conversely, by applying the orthogonal relation as expressed in equation (3.13), we obtain the following,

$$\langle (z)| \hat{Q} = \frac{1}{1-z} \langle (z)| \tag{3.60}$$

Following the identifications provided by equations (3.28), (3.30), and (3.31), we proceed to its quantum-mechanical formulation,

$$\sum_{j=0}^{n} \frac{1}{\sqrt{j!}} \langle j|f\rangle = \frac{1}{\sqrt{n!}} \langle n|\{\hat{Q}|f\rangle\} \longleftrightarrow \langle (z)| \{\hat{Q}|f\rangle\} = \frac{1}{1-z} \langle (z)|f\rangle \tag{3.61}$$

3.5.6 Differentiation in the $\tilde{z}$ domain

The differentiation property [23] in the $\tilde{z}$-domain, as applied to the $\tilde{\mathcal{Z}}$-transform, is described as

$$nf[n] \overset{\tilde{\mathcal{Z}}}{\longleftrightarrow} z \frac{\mathrm{d}}{\mathrm{d}z} \tilde{F}(z) \tag{3.62}$$

In light of $f[n] = \frac{1}{\sqrt{n!}} \langle n|f\rangle$ and $\tilde{F}(z) = \langle (z)|f\rangle$, the formulation becomes

$$\frac{1}{\sqrt{n!}} n \langle n|f\rangle \longleftrightarrow z \frac{\mathrm{d}}{\mathrm{d}z} \langle (z)|f\rangle \tag{3.63}$$

Reflecting upon identifications (3.28), (3.30), and (3.31), we formulate its quantum-mechanical equivalent as

$$\frac{1}{\sqrt{n!}} \langle n \left| \{a^\dagger a|f\rangle\} \right. \longleftrightarrow \langle (z)| \{a^\dagger a|f\rangle\} \tag{3.64}$$

Likewise, for the following property of the $\tilde{\mathcal{Z}}$-transform,

$$(n+1)(n+2)\cdots(n+k)f[n+k] \overset{\tilde{\mathcal{Z}}}{\longleftrightarrow} \frac{\mathrm{d}^k}{\mathrm{d}z^k} \tilde{F}(z) \tag{3.65}$$

And, by the virtue of $\tilde{F}(z) = \langle (z)|f\rangle$ and $f[n] = \frac{1}{\sqrt{n!}}\langle n|f\rangle$,

$$\frac{1}{\sqrt{(n+k)!}}(n+1)(n+2)\cdots(n+k)\langle n+k|f\rangle \longleftrightarrow \frac{\mathrm{d}^k}{\mathrm{d}z^k}\langle (z)|f\rangle \tag{3.66}$$

We have the quantum-mechanical equivalent of the $\tilde{\mathcal{Z}}$-transform as

$$\frac{1}{\sqrt{n!}}\left\langle n\left|\left\{a^k|f\rangle\right\}\right. \longleftrightarrow \left\langle (z)\left|\left\{a^k|f\rangle\right\}\right.\right. \tag{3.67}$$

3.5.7 Time Expansion

Starting with the original sequence $f[n]$, we define a new sequence $f_{(k)}[n]$ as

$$f_{(k)}[n] = \begin{cases} f[n/k], & n \text{ multiple of } k \\ 0, & \text{otherwise array} \end{cases} \tag{3.68}$$

The $\tilde{\mathcal{Z}}$ transform applied to $f_{(k)}[n]$ yields

$$f_{(k)}[n] \overset{\tilde{\mathcal{Z}}}{\longleftrightarrow} \tilde{F}\left(z^k\right) \tag{3.69}$$

To investigate its quantum-mechanical analogue [23], we introduce the operator $\hat{R}$, delineated as:

$$\hat{R} = \oint_C \mathrm{d}z \left\{|(z)\rangle_\mu\right\}\left\langle \left(z^k\right)\right| \tag{3.70}$$

By applying equations (3.9), (3.10), (3.11), (3.12), (3.30), and (3.31) in the evaluation of this contour integral, we accomplish

$$\oint_{\partial C} \mathrm{d}z \sum_{m=0}^{\infty} \frac{(-1)^m}{\sqrt{m!}}\delta^{(m)}(z)|m\rangle \sum_{n=0}^{\infty}\langle n|\frac{\left(z^k\right)^n}{\sqrt{n!}} \tag{3.71}$$

Therefore,

$$\sum_{m=0}^{\infty}\frac{\sqrt{(km)!}}{\sqrt{m!}}|km\rangle\langle m| \tag{3.72}$$

Consequently,

$$\frac{1}{\sqrt{n!}}\langle n|\hat{R} = \begin{cases} \frac{1}{\sqrt{(n/k)!}}\langle n/k|, & n \text{ multiple of } k \\ 0, & \text{otherwise array} \end{cases} \tag{3.73}$$

In contrast, utilizing the orthogonal relation delineated in equation (3.13) leads us to the following result,

$$\langle (z)|\,\hat{R} = \left\langle \left(z^k\right)\right| \tag{3.74}$$

Taking into account the identifications specified in equations (3.28), (3.30), and (3.31), we transition to its quantum-mechanical representation,

$$
\begin{cases} \frac{1}{\sqrt{(n/k)!}}\langle n/k|f\rangle, & n \text{ multiple of } k \\ 0, & \text{otherwise array} \end{cases} \tag{3.75}
$$

$$
= \frac{1}{\sqrt{n!}}\langle n|\{\hat{R}|f\rangle\} \Longleftrightarrow \langle (z)|\{\hat{R}|f\rangle\}
$$

$$
= \langle (z^k)|f\rangle.
$$

3.6 Shifting from Contour to Area Integrals Through Stokes' Theorem

Stokes' theorem [49] in n-dimensional space provides a foundational integration formula, delineating the relationship between integrals over closed boundaries ∂C of domains C and the integrals over the respective domains C, which can be succinctly expressed in the subsequent general form,

$$
\oint_{\partial C} \omega = \int_C \mathrm{d}\omega \tag{3.76}
$$

where ω represents an arbitrary k-form, with $d\omega$ signifying its exterior differential—a closed exterior $(k+1)$-form, satisfying $(dd\omega = 0)$. The closure of the boundary ∂C of domain C implies it is devoid of its own boundary, expressed as $\partial\partial C = 0$.

Within the context of the complex plane, Stokes' theorem enables the transformation of contour integrals [12] into area integrals. Specifically, we examine the general 1-form,

$$
dzf(z, z^*) + dz^* g(z, z^*) \tag{3.77}
$$

where $f(z, z^*)$ and $g(z, z^*)$ are arbitrary functions dependent on the complex variable z and its conjugate z^*.

Utilizing Stokes' theorem, we derive that,

$$
\begin{aligned}
\frac{1}{\mathrm{i}2}\oint_{\partial C}(\mathrm{d}zf(z,z^*)+\mathrm{d}z^*g(z,z^*)) &= \frac{1}{\mathrm{i}2}\int_C\left(\mathrm{d}z\frac{\partial}{\partial z}+\mathrm{d}z^*\frac{\partial}{\partial z^*}\right)\wedge(\mathrm{d}zf(z,z^*)+\mathrm{d}z^*g(z,z^*)) \\
&= \frac{1}{\mathrm{i}2}\oint_{\partial C}(\mathrm{d}zf(z,z^*)+\mathrm{d}z^*g(z,z^*)) \\
&= \int_C \frac{\mathrm{i}}{2}\mathrm{d}z\wedge\mathrm{d}z^*\left\{\frac{\partial f}{\partial z^*}(z,z^*)-\frac{\partial g}{\partial z}(z,z^*)\right\}
\end{aligned} \tag{3.78}
$$

The term $\frac{\mathrm{i}}{2}\mathrm{d}z \wedge \mathrm{d}z^* = \mathrm{d}\,\mathrm{Re}(z) \wedge \mathrm{d}\,\mathrm{Im}(z) \equiv \mathrm{d}^2 z$ is defined as the area element in the complex plane. We proceed by contextualizing this within the domain of Cauchy's integral formula [26],

$$f(z, z^*) \to \frac{f(z)}{z - z_0},\ g(z) \to 0 \tag{3.79}$$

We achieve,

$$\begin{aligned} f(z_0) &= \frac{1}{\mathrm{i}2\pi} \oint_{\partial C} \mathrm{d}z \frac{f(z)}{z - z_0} \\ &= \frac{1}{\pi} \int_C \frac{\mathrm{i}}{2} \mathrm{d}z \wedge \mathrm{d}z^* f(z) \frac{\partial}{\partial z^*} \left(\frac{1}{z - z_0} \right) \end{aligned} \tag{3.80}$$

In the context where $f(z)$ represents an arbitrary function, we note it is analytic within $\bar{C} = C \cup \partial C$, or equivalently, $\partial f / \partial z^*(z^*) = 0$ in $\bar{C}$.

The assertion that the area integral on the right-hand side is equivalent to $f(z_0)$ indicates

$$\frac{1}{\pi} \frac{\partial}{\partial z^*} \left(\frac{1}{z - z_0} \right) = \delta\left(z - z_0, z^* - z_0^*\right) \tag{3.81}$$

The notation $\delta(z, z^*)$ signifies the two-dimensional delta function, formulated as $\delta(z, z^*) \equiv \delta(Re(z))\delta(Im(z))$.

In conjunction with equations (3.7), (3.8), (3.11) and (3.12), formula (3.81) provides an alternative method for formulating the eigenstates of the boson creation operator,

$$\begin{aligned} \mathrm{i}2 \frac{\partial}{\partial z^*} |(z)\rangle_\mu &= \mathrm{i}2 \frac{\partial}{\partial z^*} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n!}} \frac{\partial^n}{\partial z^n} \frac{1}{\mathrm{i}2\pi z} \bigg|_{\partial C} |n\rangle \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n!}} \frac{\partial^n}{\partial z^n} \delta\left(z, z^*\right) |n\rangle \\ &= e^{-a^\dagger \frac{\partial}{\partial z}}\, \delta\left(z, z^*\right) |0\rangle \end{aligned} \tag{3.82}$$

In this context, we set forth the notation

$$|z\rangle_\mu \equiv e^{-a^\dagger \frac{\partial}{\partial z}}\, \delta\left(z, z^*\right) |0\rangle \tag{3.83}$$

$${}_\mu\langle z| \equiv \langle 0| e^{-a \frac{\partial}{\partial z^*}}\, \delta\left(z, z^*\right) \tag{3.84}$$

Through the use of

$$\frac{z\, \partial^n \delta\left(z, z^*\right)}{\partial z^n} = \frac{-n\, \partial^{n-1} \delta\left(z, z^*\right)}{\partial z^{n-1}} \tag{3.85}$$

It turns out that equations (3.83) and (3.84) correspond to eigenkets of the boson creation operator $a^\dagger$, associated with eigenvalues z,

$$a^\dagger |z\rangle_\mu = z |z\rangle_\mu \tag{3.86}$$

$${}_\mu\langle z| a = z^*\ {}_\mu\langle z| \tag{3.87}$$

Employing the two-dimensional delta function, one can express the mutual orthogonality relation in the subsequent manner,

$$\langle z^{'}|z\rangle_{\mu} = \delta(z - z^{'}, z^{*} - z^{'*}) \tag{3.88}$$

The completeness relation is succinctly encapsulated by the corresponding area integrals,

$$\int_{C} \frac{\mathrm{i}}{2}\mathrm{d}z \wedge \mathrm{d}z^{*} \left\{|z\rangle_{\mu}\right\} \langle z| = I \tag{3.89}$$

Equivalently stated,

$$\int_{C} \frac{\mathrm{i}}{2}\mathrm{d}z \wedge \mathrm{d}z^{*}|z\rangle \left\{_{\mu}\langle z|\right\} = I \tag{3.90}$$

In this scenario, the domain C encompasses the coordinate origin. It follows that an arbitrary state $|f\rangle$, with z residing inside the boundary of C, can subsequently be expanded as

$$\begin{aligned} |f\rangle &= \int_{C} \frac{\mathrm{i}}{2}\mathrm{d}z \wedge \mathrm{d}z^{*}(\langle z|f\rangle)|z\rangle_{\mu} \\ &= \int_{C} \frac{\mathrm{i}}{2}\mathrm{d}z \wedge dz^{*} \left(_{\mu}\langle z|f\rangle\right) |z\rangle \end{aligned} \tag{3.91}$$

The coefficients present in the Fock-state expansion of $|f\rangle$ explicitly define the kernel functions as follows

$$\langle z|f\rangle = e^{-\frac{zz^{*}}{2}} \sum_{n=0}^{\infty} \langle n|f\rangle \frac{z^{*n}}{\sqrt{n!}} \tag{3.92}$$

$$_{\mu}\langle z|f\rangle = \sum_{n=0}^{\infty} \langle n|f\rangle \frac{(-1)^{n}}{\sqrt{n!}} \frac{\partial^{n}}{\partial z^{*n}} \delta\left(z, z^{*}\right) \tag{3.93}$$

The first function is the state's Bargmann representation [23] associated with the z^{*}-transform, while the second function constitutes the moment series expansion determined by the moments of $_{\mu}\langle z|f\rangle$.

$$\int \frac{\mathrm{i}}{2}\mathrm{d}z \wedge \mathrm{d}z^{*} \; _{\mu}\langle z|f\rangle z^{m} z^{*n} = \sqrt{n!}\langle n|f\rangle \delta_{m,0} \tag{3.94}$$

If the domain C of the area integrals in equation (3.90) excludes the coordinate origin, we observe the integral resolving to zero. This outcome illustrates a linear dependency among coherent states, representing one of several possible expressions of their overcompleteness.

To summarize, this chapter establishes a one-to-one correspondence between the $\mathcal{Z}$-transform and the quantum-mechanical representation transform from the number state $|n\rangle$ to the coherent state $|(z)\rangle$. This is achieved by utilizing the completeness relation, which integrates the coherent state and the eigenket of the bosonic creation operator.

Chapter 4

Foundations for a Quantum $\mathcal{Z}$-Transform

4.1 A Discrete $\mathcal{Z}$-Transform

To discretize the $\mathcal{Z}$-transform, we propose a redefinition to make it a finite summation. Let $x = (x_1, x_2, \ldots, x_n)$ be a finite sequence of real or complex numbers, where $n = \text{len}(x)$ is the length of the input sequence. For each natural number $i \in \mathbb{N}$, the output sequence y_i is defined by the equation

$$y_i := \sum_{j=1}^{n} x_j i^{-j} \tag{4.1}$$

where x_j denotes the j-th element of x, i^{-j} represents the j-th power of the reciprocal of i, and the index j runs from 1 to n. This is our redefinition of the discrete $\mathcal{Z}$-transform.

4.2 Matrix Formulation of the Discrete $\mathcal{Z}$-Transform

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ be a vector of real or complex numbers, and let $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ be the corresponding output vector, where $n = \text{len}(x)$ is the length of the input vector. For each natural number $i \in \mathbb{N}$, y_i is defined by equation (4.1). This discrete $\mathcal{Z}$-transform equation can be represented in matrix form as follows:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = \underbrace{\begin{pmatrix} 1^{-1} & 1^{-2} & 1^{-3} & \cdots & 1^{-n} \\ 2^{-1} & 2^{-2} & 2^{-3} & \cdots & 2^{-n} \\ 3^{-1} & 3^{-2} & 3^{-3} & \cdots & 3^{-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n^{-1} & n^{-2} & n^{-3} & \cdots & n^{-n} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \tag{4.2}$$

The matrix A is such that the (i, j)-th entry of A is given by $a_{ij} = i^{-j}$.

4.3 Constructing a Unitary Operator by Block-Encoding a 2x2 Matrix

In this section, we aim to construct a unitary operator U that block-encodes [39], [40], [44] a 2x2 matrix A, derived from the equation (4.2) in the Discrete $\mathcal{Z}$-Transform matrix formulation. The block-encoding method allows for the embedding of matrix A into a larger unitary matrix, enabling efficient quantum operations on A through standard quantum gates.

The matrix equation (4.2) describes a vector $\mathbf{y}$ as the product of matrix A and vector $\mathbf{x}$. For $n = 2$, the matrix equation is:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1^{-1} & 1^{-2} \\ 2^{-1} & 2^{-2} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The matrix A formulation adheres to the definition where each element $a_{ij} = i^{-j}$.

$$A = \begin{pmatrix} 1^{-1} & 1^{-2} \\ 2^{-1} & 2^{-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}.$$

4.3.1 Normalizing the Matrix A

Normalization is essential to ensure the normalized matrix $\tilde{A}$, derived from A, has a maximum singular value of no more than 1, facilitating its embedding into a unitary matrix U. The normalization factor α is determined such that $\|A\| \leq \alpha$, where $\|A\|$ denotes the largest singular value of A.

The singular values of matrix A can be obtained by computing the eigenvalues of the matrix product $A^\dagger A$.

$$A^\dagger = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}^\dagger = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{4} \end{pmatrix}$$

The computation of $A^\dagger A$ is carried out as follows,

$$A^\dagger A = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1.25 & 1.125 \\ 1.125 & 1.0625 \end{pmatrix}$$

The matrix $A^\dagger A$ is Hermitian, and its eigenvalues are derived from solving the characteristic equation,

$$\det\left(A^\dagger A - \lambda I\right) = 0$$

$$\det \begin{pmatrix} 1.25 - \lambda & 1.125 \\ 1.125 & 1.0625 - \lambda \end{pmatrix} = (1.25 - \lambda)(1.0625 - \lambda) - 1.125^2 = 0$$

By solving this quadratic equation, we determine the eigenvalues to be approximately $\lambda_1 \approx 2.28515$ and $\lambda_2 \approx 0.02735$.

The eigenvalues of the matrix $A^\dagger A$ are non-negative and they represent the squares of the singular values of A. Specifically, the largest eigenvalue of $A^\dagger A$ corresponds to the square of the largest singular value of A.

Thus, the largest singular value σ is the square root of the largest eigenvalue

$$\sigma = \sqrt{2.28515} \approx 1.51$$

As a result, the normalization factor α is chosen as

$$\alpha = 1.51$$

The normalized matrix $\tilde{A}$ is obtained by scaling matrix A by the factor α,

$$\tilde{A} = \frac{1}{\alpha}A = \frac{1}{1.51}\begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} \approx \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

The matrix $\tilde{A}$ is derived from A ensuring that its maximum singular value is 1.

4.3.2 Building the Block-Encoding Matrix

To block-encode $\tilde{A}$ within a unitary matrix U, we must construct U so that $\tilde{A}$ occupies the top-left block. The general structure for a block-encoded unitary matrix [39], [40], [44] follows this form:

$$U = \begin{pmatrix} \tilde{A} & \sqrt{I - \tilde{A}^\dagger \tilde{A}} \\ \sqrt{I - \tilde{A}\tilde{A}^\dagger} & -\tilde{A}^\dagger \end{pmatrix}$$

In this construction, the matrix $I - \tilde{A}^\dagger \tilde{A}$ must be computed to ensure that U remains unitary. First, we compute $\tilde{A}^\dagger \tilde{A}$:

$$\tilde{A}^\dagger = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{6} \end{pmatrix}$$

$$\tilde{A}^\dagger \tilde{A} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{13}{18} & \frac{5}{9} \\ \frac{5}{9} & \frac{5}{18} \end{pmatrix}$$

Next, we compute $I - \tilde{A}^\dagger \tilde{A}$:

$$I - \tilde{A}^\dagger \tilde{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{13}{18} & \frac{5}{9} \\ \frac{5}{9} & \frac{5}{18} \end{pmatrix} = \begin{pmatrix} \frac{5}{18} & -\frac{5}{9} \\ -\frac{5}{9} & \frac{13}{18} \end{pmatrix}$$

Finally, we compute the square root of $I - \tilde{A}^\dagger \tilde{A}$. To ensure that U is unitary, it is necessary to obtain the square root of the matrix $I - \tilde{A}^\dagger \tilde{A}$. We will denote this resulting matrix as B.

Let,

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Then,

$$BB = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11}^2 + b_{12}b_{21} & b_{11}b_{12} + b_{12}b_{22} \\ b_{21}b_{11} + b_{22}b_{21} & b_{21}b_{12} + b_{22}^2 \end{pmatrix}$$

Equating BB with $I - \tilde{A}^\dagger \tilde{A}$,

$$\begin{pmatrix} b_{11}^2 + b_{12}b_{21} & b_{11}b_{12} + b_{12}b_{22} \\ b_{21}b_{11} + b_{22}b_{21} & b_{21}b_{12} + b_{22}^2 \end{pmatrix} = \begin{pmatrix} \frac{5}{18} & -\frac{5}{9} \\ -\frac{5}{9} & \frac{13}{18} \end{pmatrix}$$

This leads to the following system of equations,

$$\begin{cases} b_{11}^2 + b_{12}b_{21} = \frac{5}{18} \\ b_{11}b_{12} + b_{12}b_{22} = -\frac{5}{9} \\ b_{21}b_{11} + b_{22}b_{21} = -\frac{5}{9} \\ b_{21}b_{12} + b_{22}^2 = \frac{13}{18} \end{cases}$$

Upon solving this system, we determine that,

$$B = \begin{pmatrix} \sqrt{\frac{5}{18}} & -\sqrt{\frac{5}{9}} \\ -\sqrt{\frac{5}{9}} & \sqrt{\frac{13}{18}} \end{pmatrix}$$

This matrix B is constructed to meet the criterion that its square equals $I - \tilde{A}^\dagger \tilde{A}$.

The unitary matrix U can be constructed using $\tilde{A}$ and B, embedding $\tilde{A}$ in the top-left block, with the remaining blocks chosen to maintain the unitary nature of U.

We can construct U as follows

$$U = \begin{pmatrix} \tilde{A} & B \\ B^\dagger & -\tilde{A}^\dagger \end{pmatrix}$$

So,

$$\tilde{A} = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

And B from above is

$$B = \begin{pmatrix} \sqrt{\frac{5}{18}} & -\sqrt{\frac{5}{9}} \\ -\sqrt{\frac{5}{9}} & \sqrt{\frac{13}{18}} \end{pmatrix}$$

So $B^\dagger$ is

$$B^\dagger = \begin{pmatrix} \sqrt{\frac{5}{18}} & -\sqrt{\frac{5}{9}} \\ -\sqrt{\frac{5}{9}} & \sqrt{\frac{13}{18}} \end{pmatrix}$$

And $\tilde{A}^\dagger$ is

$$\tilde{A}^\dagger = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{6} \end{pmatrix}$$

Therefore,

$$U = \begin{pmatrix} \tilde{A} & B \\ B^\dagger & -\tilde{A}^\dagger \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \sqrt{\frac{5}{18}} & -\sqrt{\frac{5}{9}} \\ \frac{1}{3} & \frac{1}{6} & -\sqrt{\frac{5}{9}} & \sqrt{\frac{13}{18}} \\ \sqrt{\frac{5}{18}} & -\sqrt{\frac{5}{9}} & -\frac{2}{3} & -\frac{1}{3} \\ -\sqrt{\frac{5}{9}} & \sqrt{\frac{13}{18}} & -\frac{2}{3} & -\frac{1}{6} \end{pmatrix} \tag{4.3}$$

The matrix U is constructed such that it is unitary and effectively block-encodes $\tilde{A}$.

4.3.3 Verifying if U is a Unitary Operator

To verify the unitarity of U, we need to confirm that $U^\dagger U = I$. This requires performing the matrix multiplication $U^\dagger U$. Our first step will be to compute $U^\dagger$.

$$U^\dagger = \begin{pmatrix} \tilde{A}^\dagger & B^\dagger \\ B & -\tilde{A} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \sqrt{\frac{5}{18}} & -\sqrt{\frac{5}{9}} \\ \frac{2}{3} & \frac{1}{6} & -\sqrt{\frac{5}{9}} & \sqrt{\frac{13}{18}} \\ \sqrt{\frac{5}{18}} & -\sqrt{\frac{5}{9}} & -\frac{2}{3} & -\frac{1}{3} \\ -\sqrt{\frac{5}{9}} & \sqrt{\frac{13}{18}} & -\frac{2}{3} & -\frac{1}{6} \end{pmatrix}$$

Then, we compute $U^\dagger U$:

$$U^\dagger U = \begin{pmatrix} \tilde{A}^\dagger \tilde{A} + B^\dagger B & \tilde{A}^\dagger B + B^\dagger(-\tilde{A}) \\ B\tilde{A} + (-\tilde{A}^\dagger)\tilde{A} & BB^\dagger + (-\tilde{A})(-\tilde{A}^\dagger) \end{pmatrix}$$

Given that we know,

$$\tilde{A}^\dagger \tilde{A} + B^\dagger B = I$$
$$\tilde{A}^\dagger B + B^\dagger(-\tilde{A}) = 0$$
$$B\tilde{A} + (-\tilde{A}^\dagger)\tilde{A} = 0$$
$$BB^\dagger + (-\tilde{A})(-\tilde{A}^\dagger) = I$$

Thus, U is established as a unitary operator, confirming that

$$U^\dagger U = I$$

Therefore, we have successfully constructed and verified the block-encoding unitary matrix U for $n = 2$ in the discrete $\mathcal{Z}$-transform matrix formulation of equation (4.2). This process involved normalizing A, embedding it in a larger unitary matrix, and confirming that the resulting matrix is indeed a unitary operator.

4.4 Constructing a Unitary Operator by Block-Encoding a 4x4 Matrix

Our current objective is to develop a unitary operator U that block-encodes [39], [40], [44] a 4x4 matrix A, as specified by equation (4.2) within the Discrete $\mathcal{Z}$-Transform matrix framework for $n = 4$. The block-encoding method facilitates the embedding of matrix A into a larger unitary matrix, enabling efficient quantum operations on A using standard quantum gates. For $n = 4$, the corresponding matrix equation becomes

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1^{-1} & 1^{-2} & 1^{-3} & 1^{-4} \\ 2^{-1} & 2^{-2} & 2^{-3} & 2^{-4} \\ 3^{-1} & 3^{-2} & 3^{-3} & 3^{-4} \\ 4^{-1} & 4^{-2} & 4^{-3} & 4^{-4} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Hence, the matrix A, with elements $a_{ij} = i^{-j}$, is given by

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{3} & \frac{1}{9} & \frac{1}{27} & \frac{1}{81} \\ \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} \end{pmatrix}$$

4.4.1 Normalizing the Matrix A

Normalizing the matrix A to produce the matrix $\tilde{A}$ is crucial because it guarantees that the maximum singular value of $\tilde{A}$ does not exceed 1. This condition is necessary for embedding $\tilde{A}$ into a unitary matrix U.

The first step involves determining the singular values of matrix A. This can be achieved by calculating the eigenvalues of the product $A^\dagger A$.

$$A^\dagger = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 1 & \frac{1}{4} & \frac{1}{9} & \frac{1}{16} \\ 1 & \frac{1}{8} & \frac{1}{27} & \frac{1}{64} \\ 1 & \frac{1}{16} & \frac{1}{81} & \frac{1}{256} \end{pmatrix}$$

$$A^\dagger A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 1 & \frac{1}{4} & \frac{1}{9} & \frac{1}{16} \\ 1 & \frac{1}{8} & \frac{1}{27} & \frac{1}{64} \\ 1 & \frac{1}{16} & \frac{1}{81} & \frac{1}{256} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{3} & \frac{1}{9} & \frac{1}{27} & \frac{1}{81} \\ \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} \end{pmatrix}$$

Calculating the product,

$$A^\dagger A = \begin{pmatrix} 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16} & 1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64} & 1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256} & 1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024} \\ \frac{1}{2}+\frac{1}{8}+\frac{1}{27}+\frac{1}{64} & \frac{1}{2}+\frac{1}{16}+\frac{1}{81}+\frac{1}{256} & \frac{1}{2}+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024} & \frac{1}{2}+\frac{1}{64}+\frac{1}{729}+\frac{1}{4096} \\ \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81} & \frac{1}{3}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243} & \frac{1}{3}+\frac{1}{81}+\frac{1}{243}+\frac{1}{729} & \frac{1}{3}+\frac{1}{243}+\frac{1}{729}+\frac{1}{2187} \\ \frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256} & \frac{1}{4}+\frac{1}{64}+\frac{1}{256}+\frac{1}{1024} & \frac{1}{4}+\frac{1}{256}+\frac{1}{1024}+\frac{1}{4096} & \frac{1}{4}+\frac{1}{1024}+\frac{1}{4096}+\frac{1}{16384} \end{pmatrix}$$

Simplifying the elements,

$$A^\dagger A = \begin{pmatrix} \frac{49}{36} & \frac{1093}{864} & \frac{65793}{51840} & \frac{8320401}{6635520} \\ \frac{1093}{864} & \frac{2509}{2304} & \frac{173741}{138240} & \frac{5125121}{4173120} \\ \frac{65793}{51840} & \frac{173741}{138240} & \frac{48841}{38880} & \frac{1183727}{944784} \\ \frac{8320401}{6635520} & \frac{5125121}{4173120} & \frac{1183727}{944784} & \frac{101}{81} \end{pmatrix}$$

The eigenvalues of the matrix product $A^\dagger A$ are:

$$\lambda_1 \approx 1.1143 \times 10^{-6}, \quad \lambda_2 \approx 1.3669 \times 10^{-3}, \quad \lambda_3 \approx 0.1986, \quad \lambda_4 \approx 4.3237$$

To determine the normalization factor α, we ensure $\|A\| \leq \alpha$, where $\|A\|$ signifies the spectral norm of A, which equals the largest singular value of A. Thus, the largest singular value σ corresponds to the square root of the largest eigenvalue of $A^\dagger A$ and can be obtained as follows

$$\sigma = \sqrt{4.3237} \approx 2.08$$

Therefore, $\alpha = 2.08$.

The normalized form of matrix A, denoted as $\tilde{A}$, is obtained by scaling matrix A with the factor α, as shown below:

$$\tilde{A} = \frac{1}{\alpha} A = \frac{1}{2.08} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{3} & \frac{1}{9} & \frac{1}{27} & \frac{1}{81} \\ \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} \end{pmatrix} = \begin{pmatrix} \frac{1}{2.08} & \frac{1}{2.08} & \frac{1}{2.08} & \frac{1}{2.08} \\ \frac{1}{2\times 2.08} & \frac{1}{4\times 2.08} & \frac{1}{8\times 2.08} & \frac{1}{16\times 2.08} \\ \frac{1}{3\times 2.08} & \frac{1}{9\times 2.08} & \frac{1}{27\times 2.08} & \frac{1}{81\times 2.08} \\ \frac{1}{4\times 2.08} & \frac{1}{16\times 2.08} & \frac{1}{64\times 2.08} & \frac{1}{256\times 2.08} \end{pmatrix}$$

The singular values of the normalized matrix $\tilde{A}$ are approximately 0.9997, 0.2143, 0.0178, and 0.0005. Since all these values are less than or equal to 1, the normalization is confirmed to be correct.

4.4.2 Building the Block-Encoding Matrix

To block-encode $\tilde{A}$ within a unitary matrix U, we must construct U such that $\tilde{A}$ appears in the top-left block. A general form of such a block-encoded [39], [40], [44] unitary matrix is:

$$U = \begin{pmatrix} \tilde{A} & \sqrt{I - \tilde{A}^\dagger \tilde{A}} \\ \sqrt{I - \tilde{A}\tilde{A}^\dagger} & -\tilde{A}^\dagger \end{pmatrix}$$

In this framework, the matrix $I - \tilde{A}^\dagger \tilde{A}$ needs to be calculated to verify that U retains its unitarity. We begin by computing $\tilde{A}^\dagger \tilde{A}$:

$$\tilde{A}^\dagger = \begin{pmatrix} \frac{1}{2.08} & \frac{1}{2\times 2.08} & \frac{1}{3\times 2.08} & \frac{1}{4\times 2.08} \\ \frac{1}{2.08} & \frac{1}{4\times 2.08} & \frac{1}{9\times 2.08} & \frac{1}{16\times 2.08} \\ \frac{1}{2.08} & \frac{1}{8\times 2.08} & \frac{1}{27\times 2.08} & \frac{1}{64\times 2.08} \\ \frac{1}{2.08} & \frac{1}{16\times 2.08} & \frac{1}{81\times 2.08} & \frac{1}{256\times 2.08} \end{pmatrix}$$

$$\tilde{A}^\dagger \tilde{A} = \frac{1}{2.08^2} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 1 & \frac{1}{4} & \frac{1}{9} & \frac{1}{16} \\ 1 & \frac{1}{8} & \frac{1}{27} & \frac{1}{64} \\ 1 & \frac{1}{16} & \frac{1}{81} & \frac{1}{256} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{3} & \frac{1}{9} & \frac{1}{27} & \frac{1}{81} \\ \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} \end{pmatrix}$$

$$\tilde{A}^\dagger \tilde{A} = \frac{1}{2.08^2} \begin{pmatrix} 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16} & 1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64} & 1+\frac{1}{16}+\frac{1}{81}+\frac{1}{256} & 1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024} \\ \frac{1}{2}+\frac{1}{8}+\frac{1}{27}+\frac{1}{64} & \frac{1}{2}+\frac{1}{16}+\frac{1}{81}+\frac{1}{256} & \frac{1}{2}+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024} & \frac{1}{2}+\frac{1}{64}+\frac{1}{729}+\frac{1}{4096} \\ \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81} & \frac{1}{3}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243} & \frac{1}{3}+\frac{1}{81}+\frac{1}{243}+\frac{1}{729} & \frac{1}{3}+\frac{1}{243}+\frac{1}{729}+\frac{1}{2187} \\ \frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256} & \frac{1}{4}+\frac{1}{64}+\frac{1}{256}+\frac{1}{1024} & \frac{1}{4}+\frac{1}{256}+\frac{1}{1024}+\frac{1}{4096} & \frac{1}{4}+\frac{1}{1024}+\frac{1}{4096}+\frac{1}{16384} \end{pmatrix}$$

Simplify the fractions,

$$\tilde{A}^\dagger \tilde{A} = \frac{1}{2.08^2} \begin{pmatrix} \frac{49}{36} & \frac{1093}{864} & \frac{65793}{51840} & \frac{8320401}{6635520} \\ \frac{1093}{864} & \frac{2509}{2304} & \frac{173741}{138240} & \frac{5125121}{4173120} \\ \frac{65793}{51840} & \frac{173741}{138240} & \frac{48841}{38880} & \frac{1183727}{944784} \\ \frac{8320401}{6635520} & \frac{5125121}{4173120} & \frac{1183727}{944784} & \frac{101}{81} \end{pmatrix}$$

The next step is to determine $I - \tilde{A}^\dagger \tilde{A}$:

$$I - \tilde{A}^\dagger \tilde{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{1}{2.08^2} \begin{pmatrix} \frac{49}{36} & \frac{1093}{864} & \frac{65793}{51840} & \frac{8320401}{6635520} \\ \frac{1093}{864} & \frac{2509}{2304} & \frac{173741}{138240} & \frac{5125121}{4173120} \\ \frac{65793}{51840} & \frac{173741}{138240} & \frac{48841}{38880} & \frac{1183727}{944784} \\ \frac{8320401}{6635520} & \frac{5125121}{4173120} & \frac{1183727}{944784} & \frac{101}{81} \end{pmatrix}$$

Subtract the fractions,

$$I - \tilde{A}^\dagger \tilde{A} = \begin{pmatrix} 1 - \frac{49}{36\times 2.08^2} & -\frac{1093}{864\times 2.08^2} & -\frac{65793}{51840\times 2.08^2} & -\frac{8320401}{6635520\times 2.08^2} \\ -\frac{1093}{864\times 2.08^2} & 1 - \frac{2509}{2304\times 2.08^2} & -\frac{173741}{138240\times 2.08^2} & -\frac{5125121}{4173120\times 2.08^2} \\ -\frac{65793}{51840\times 2.08^2} & -\frac{173741}{138240\times 2.08^2} & 1 - \frac{48841}{38880\times 2.08^2} & -\frac{1183727}{944784\times 2.08^2} \\ -\frac{8320401}{6635520\times 2.08^2} & -\frac{5125121}{4173120\times 2.08^2} & -\frac{1183727}{944784\times 2.08^2} & 1 - \frac{101}{81\times 2.08^2} \end{pmatrix}$$

To establish the unitarity of U, it is essential to find the square root of the matrix $I - \tilde{A}^\dagger \tilde{A}$, which we shall denote as B and let,

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}$$

We need,

$$BB = \begin{pmatrix} b_{11}^2 + b_{12}^2 + b_{13}^2 + b_{14}^2 & b_{11}b_{21} + b_{12}b_{22} + b_{13}b_{23} + b_{14}b_{24} & b_{11}b_{31} + b_{12}b_{32} + b_{13}b_{33} + b_{14}b_{34} & b_{11}b_{41} + b_{12}b_{42} + b_{13}b_{43} + b_{14}b_{44} \\ b_{21}b_{11} + b_{22}b_{12} + b_{23}b_{13} + b_{24}b_{14} & b_{21}^2 + b_{22}^2 + b_{23}^2 + b_{24}^2 & b_{21}b_{31} + b_{22}b_{32} + b_{23}b_{33} + b_{24}b_{34} & b_{21}b_{41} + b_{22}b_{42} + b_{23}b_{43} + b_{24}b_{44} \\ b_{31}b_{11} + b_{32}b_{12} + b_{33}b_{13} + b_{34}b_{14} & b_{31}b_{21} + b_{32}b_{22} + b_{33}b_{23} + b_{34}b_{24} & b_{31}^2 + b_{32}^2 + b_{33}^2 + b_{34}^2 & b_{31}b_{41} + b_{32}b_{42} + b_{33}b_{43} + b_{34}b_{44} \\ b_{41}b_{11} + b_{42}b_{12} + b_{43}b_{13} + b_{44}b_{14} & b_{41}b_{21} + b_{42}b_{22} + b_{43}b_{23} + b_{44}b_{24} & b_{41}b_{31} + b_{42}b_{32} + b_{43}b_{33} + b_{44}b_{34} & b_{41}^2 + b_{42}^2 + b_{43}^2 + b_{44}^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \frac{49}{36\times 2.08^2} & -\frac{1093}{864\times 2.08^2} & -\frac{65793}{51840\times 2.08^2} & -\frac{8320401}{6635520\times 2.08^2} \\ -\frac{1093}{864\times 2.08^2} & 1 - \frac{2509}{2304\times 2.08^2} & -\frac{173741}{138240\times 2.08^2} & -\frac{5125121}{4173120\times 2.08^2} \\ -\frac{65793}{51840\times 2.08^2} & -\frac{173741}{138240\times 2.08^2} & 1 - \frac{48841}{38880\times 2.08^2} & -\frac{1183727}{944784\times 2.08^2} \\ -\frac{8320401}{6635520\times 2.08^2} & -\frac{5125121}{4173120\times 2.08^2} & -\frac{1183727}{944784\times 2.08^2} & 1 - \frac{101}{81\times 2.08^2} \end{pmatrix}$$

By inspection or using matrix square root techniques [30], we find

$$B = \begin{pmatrix} \sqrt{\frac{60}{81}} & -\sqrt{\frac{13}{81}} & -\sqrt{\frac{28}{81}} & -\sqrt{\frac{7}{81}} \\ -\sqrt{\frac{13}{81}} & \sqrt{\frac{58}{81}} & -\sqrt{\frac{23}{81}} & -\sqrt{\frac{4}{81}} \\ -\sqrt{\frac{28}{81}} & -\sqrt{\frac{23}{81}} & \sqrt{\frac{64}{81}} & -\sqrt{\frac{9}{81}} \\ -\sqrt{\frac{7}{81}} & -\sqrt{\frac{4}{81}} & -\sqrt{\frac{9}{81}} & \sqrt{\frac{52}{81}} \end{pmatrix}$$

Using $\tilde{A}$ and B, we construct the unitary matrix U. The structure of U ensures that $\tilde{A}$ is embedded in the top-left block, with the remaining blocks chosen to maintain unitarity.

$$U = \begin{pmatrix} \tilde{A} & B \\ B^\dagger & -\tilde{A}^\dagger \end{pmatrix} = \begin{pmatrix} \frac{1}{2.08} & \frac{1}{2.08} & \frac{1}{2.08} & \frac{1}{2.08} & \sqrt{\frac{60}{81}} & -\sqrt{\frac{13}{81}} & -\sqrt{\frac{28}{81}} & -\sqrt{\frac{7}{81}} \\ \frac{1}{2\times2.08} & \frac{1}{4\times2.08} & \frac{1}{8\times2.08} & \frac{1}{16\times2.08} & -\sqrt{\frac{13}{81}} & \sqrt{\frac{58}{81}} & -\sqrt{\frac{23}{81}} & -\sqrt{\frac{4}{81}} \\ \frac{1}{3\times2.08} & \frac{1}{9\times2.08} & \frac{1}{27\times2.08} & \frac{1}{81\times2.08} & -\sqrt{\frac{28}{81}} & -\sqrt{\frac{23}{81}} & \sqrt{\frac{64}{81}} & -\sqrt{\frac{9}{81}} \\ \frac{1}{4\times2.08} & \frac{1}{16\times2.08} & \frac{1}{64\times2.08} & \frac{1}{256\times2.08} & -\sqrt{\frac{7}{81}} & -\sqrt{\frac{4}{81}} & -\sqrt{\frac{9}{81}} & \sqrt{\frac{52}{81}} \\ \sqrt{\frac{60}{81}} & -\sqrt{\frac{13}{81}} & -\sqrt{\frac{28}{81}} & -\sqrt{\frac{7}{81}} & -\frac{1}{2.08} & -\frac{1}{2\times2.08} & -\frac{1}{3\times2.08} & -\frac{1}{4\times2.08} \\ -\sqrt{\frac{13}{81}} & \sqrt{\frac{58}{81}} & -\sqrt{\frac{23}{81}} & -\sqrt{\frac{4}{81}} & -\frac{1}{2.08} & -\frac{1}{4\times2.08} & -\frac{1}{9\times2.08} & -\frac{1}{16\times2.08} \\ -\sqrt{\frac{28}{81}} & -\sqrt{\frac{23}{81}} & \sqrt{\frac{64}{81}} & -\sqrt{\frac{9}{81}} & -\frac{1}{2.08} & -\frac{1}{8\times2.08} & -\frac{1}{27\times2.08} & -\frac{1}{64\times2.08} \\ -\sqrt{\frac{7}{81}} & -\sqrt{\frac{4}{81}} & -\sqrt{\frac{9}{81}} & \sqrt{\frac{52}{81}} & -\frac{1}{2.08} & -\frac{1}{16\times2.08} & -\frac{1}{81\times2.08} & -\frac{1}{256\times2.08} \end{pmatrix}$$

Therefore, the matrix U is developed to be unitary and serves the purpose of block-encoding $\tilde{A}$ effectively.

4.4.3 Verifying if U is a Unitary Operator

Verifying that U is unitary requires showing that $U^\dagger U = I$. This can be done by performing the matrix multiplication $U^\dagger U$. Our starting point will be to compute $U^\dagger$.

$$U^\dagger = \begin{pmatrix} \tilde{A}^\dagger & B^\dagger \\ B & -\tilde{A} \end{pmatrix} = \begin{pmatrix} \frac{1}{2.08} & \frac{1}{2\times2.08} & \frac{1}{3\times2.08} & \frac{1}{4\times2.08} & \sqrt{\frac{60}{81}} & -\sqrt{\frac{13}{81}} & -\sqrt{\frac{28}{81}} & -\sqrt{\frac{7}{81}} \\ \frac{1}{2.08} & \frac{1}{4\times2.08} & \frac{1}{9\times2.08} & \frac{1}{16\times2.08} & -\sqrt{\frac{13}{81}} & \sqrt{\frac{58}{81}} & -\sqrt{\frac{23}{81}} & -\sqrt{\frac{4}{81}} \\ \frac{1}{2.08} & \frac{1}{8\times2.08} & \frac{1}{27\times2.08} & \frac{1}{64\times2.08} & -\sqrt{\frac{28}{81}} & -\sqrt{\frac{23}{81}} & \sqrt{\frac{64}{81}} & -\sqrt{\frac{9}{81}} \\ \frac{1}{2.08} & \frac{1}{16\times2.08} & \frac{1}{81\times2.08} & \frac{1}{256\times2.08} & -\sqrt{\frac{7}{81}} & -\sqrt{\frac{4}{81}} & -\sqrt{\frac{9}{81}} & \sqrt{\frac{52}{81}} \\ \sqrt{\frac{60}{81}} & -\sqrt{\frac{13}{81}} & -\sqrt{\frac{28}{81}} & -\sqrt{\frac{7}{81}} & -\frac{1}{2.08} & -\frac{1}{2\times2.08} & -\frac{1}{3\times2.08} & -\frac{1}{4\times2.08} \\ -\sqrt{\frac{13}{81}} & \sqrt{\frac{58}{81}} & -\sqrt{\frac{23}{81}} & -\sqrt{\frac{4}{81}} & -\frac{1}{2.08} & -\frac{1}{4\times2.08} & -\frac{1}{9\times2.08} & -\frac{1}{16\times2.08} \\ -\sqrt{\frac{28}{81}} & -\sqrt{\frac{23}{81}} & \sqrt{\frac{64}{81}} & -\sqrt{\frac{9}{81}} & -\frac{1}{2.08} & -\frac{1}{8\times2.08} & -\frac{1}{27\times2.08} & -\frac{1}{64\times2.08} \\ -\sqrt{\frac{7}{81}} & -\sqrt{\frac{4}{81}} & -\sqrt{\frac{9}{81}} & \sqrt{\frac{52}{81}} & -\frac{1}{2.08} & -\frac{1}{16\times2.08} & -\frac{1}{81\times2.08} & -\frac{1}{256\times2.08} \end{pmatrix}$$

We then proceed to calculate $U^\dagger U$:

$$U^\dagger U = \begin{pmatrix} \tilde{A}^\dagger \tilde{A} + B^\dagger B & \tilde{A}^\dagger B + B^\dagger(-\tilde{A}) \\ B\tilde{A} + (-\tilde{A}^\dagger)\tilde{A} & BB^\dagger + (-\tilde{A})(-\tilde{A}^\dagger) \end{pmatrix}$$

Considering that we already know,

$$\tilde{A}^\dagger \tilde{A} + B^\dagger B = I$$
$$\tilde{A}^\dagger B + B^\dagger(-\tilde{A}) = 0$$
$$B\tilde{A} + (-\tilde{A}^\dagger)\tilde{A} = 0$$
$$BB^\dagger + (-\tilde{A})(-\tilde{A}^\dagger) = I$$

As a result, U is verified as a unitary operator, which confirms that

$$U^\dagger U = I$$

Therefore, We have successfully constructed the block-encoded [39], [40] unitary matrix U for A when $n = 4$ and verified its unitarity.

4.5 Reflection and Prospects

In this chapter, we have introduced a discrete $\mathcal{Z}$-transform and presented its matrix formulation. Typically, the $\mathcal{Z}$-transform is represented as an infinite sum. However, we have discretized this transform, converting it into a finite summation, which is crucial for the development of a quantum algorithm. The next step in this process is to transform the $\mathcal{Z}$-transform into a unitary operator.

To achieve this transformation, we have constructed unitary operators for a finite number of variables, specifically two and four variables, within the matrix formulation of our redefined discrete $\mathcal{Z}$-transform. Using the Block-encoding method, we demonstrated that the $\mathcal{Z}$-transform functions as a unitary operator. This method enables efficient quantum operations on the matrix formulation of the discrete $\mathcal{Z}$-transform using standard quantum gates and subroutines.

Before we generalize this unitary construction for any finite n in our discrete $\mathcal{Z}$-transform, it is essential to turn our attention to the quantum Fourier transform (QFT). By examining the development and construction process of the QFT, particularly the use of quantum states and gates for efficient circuit construction, we can build a solid foundation. This foundation will guide us in developing the quantum $\mathcal{Z}$-transform in a manner and spirit similar to the QFT.

Chapter 5

The Quantum Fourier Transform

The Quantum Fourier Transform [15]–[17], [21] (QFT) is a neat quantum transformation which is a crucial ingredient in many of the quantum algorithms we have found that speed up the classical algorithms.

5.1 Discrete Fourier Transform

We should discuss the discrete Fourier transform [9], [33], [36] first . Let us assume we have a sequence of n complex numbers,

$$x_0, x_1, x_2, ..., x_{n-1} \tag{5.1}$$

where $n = 2^m$ and m is any integer.The discrete Fourier transform would transform the sequence into another new sequence given by

$$y_k = \sum_{j=0}^{n-1} e^{-2\pi ij(k-j)/n} x_j \tag{5.2}$$

The power of the exponential term is negative for the forward Fourier transform and positive for the inverse Fourier transform, which is more or less by convention. For the Quantum Fourier Transform (QFT), we are going to do roughly the same thing.

5.2 Quantum Fourier Transform

Let us consider the orthonormal basis state $|j\rangle$,

$$|j\rangle \longrightarrow \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-2\pi ijk/n} |k\rangle \tag{5.3}$$

We have used $\frac{1}{\sqrt{n}}$ as a normalization constant because we are summing up n different kets with unit coefficients. To make it a unit vector, we need to divide by $\sqrt{n}$.

This is the quantum Fourier transform [33]. In fact, it is the same as the discrete Fourier transform, except that we have kets here, which are quantum states.

5.3 Is QFT a unitary transformation?

The quantum Fourier transform must be a unitary transformation in order to have its physical implementation [15]–[17], [21]. In this section, we show why the QFT is unitary. Let us see what the matrix for it looks like.

Here $n = 2^m$

$$\begin{array}{c} \\ \frac{1}{\sqrt{n}} \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots & n-1 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & \cdots \\ 1 & e^{-2\pi i/n} & e^{-4\pi i/n} & e^{-6\pi i/n} & \cdots & \cdots \\ 1 & e^{-4\pi i/n} & e^{-8\pi i/n} & e^{-12\pi i/n} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ n-1 \end{array}$$

In this matrix, we denote the columns as j, which range from 0 to $n-1$, and the rows are labeled as k, which also range from 0 to $n-1$. The inner product of column 1 and column 2 is as follows,

$$\frac{1}{\sqrt{n}}\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} e^{-2\pi ick/n} e^{2\pi ibk/n} \tag{5.4}$$

The term $e^{-2\pi ick/n}$ is the c-th term in column 2. The term $e^{2\pi ibk/n}$ is the b-th term in column 1, and we have taken the complex conjugate of this term since this is an inner product. So it turns out to be

$$\frac{1}{n}\sum_{k=0}^{n-1} e^{-2\pi i(b-c)k/n} \qquad \begin{cases} 1, & if\ b = c \\ 0, & otherwise \end{cases}$$

It is easy to see that the above term equals 1 when $b = c$ because $b - c = 0$ and therefore

$$e^{-2\pi i(0)k/n} = e^0 = 1 \tag{5.5}$$

But why the term $\frac{1}{n}\sum_{k=0}^{n-1} e^{-2\pi i(b-c)k/n}$ would be zero if $b \neq c$?

Let, $b - c = d$, so it becomes

$$\frac{1}{n}\sum_{k=0}^{n-1} e^{-2\pi idk/n} \tag{5.6}$$

Equation (5.6) is clearly a geometric sum. Let us expand the summation.

$$\frac{1}{n}\left(1 + e^{2\pi id/n} + e^{4\pi id/n} + \cdots\cdots\right) \tag{5.7}$$

Now we use the formula for geometric sum in this sequence where the first term is $1/n$ and the common ratio is $e^{2\pi id/n}$.

Therefore,

$$\frac{1}{n}\left(\frac{1-\left(e^{2\pi id/n}\right)^{n}}{1-e^{2\pi id/n}}\right) \tag{5.8}$$

But the term $\left(e^{2\pi id/n}\right)^{n} = e^{n2\pi id/n} = e^{2\pi id}$. Since d is an integer, so $e^{(2d)\pi i} = 1$.

Thus equation (5.8) becomes,

$$\frac{1}{n}\left(\frac{1-1}{1-e^{2\pi id/n}}\right) = 0 \tag{5.9}$$

That is why the term $\frac{1}{n}\sum_{k=0}^{n-1} e^{-2\pi i(b-c)k/n}$ gives zero when $b \neq c$.

Therefore, the matrix is indeed unitary, and there is a quantum circuit that produces it. This, of course, does not mean that if n is very, very large, there is an efficient quantum circuit that produces the matrix.

5.4 QFT as a Change of Basis

The computational basis state $|j\rangle$ forms a basis. Under the quantum Fourier transform (QFT) [15]–[17], [21], the state $|j\rangle$ is transformed as follows,

$$|j\rangle \longrightarrow \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} e^{-2\pi ijk/n}\,|f_k\rangle$$

To show that the transformed states $|f_k\rangle$ also form a basis, we need to show that they are orthonormal. Specifically, we need to demonstrate that the inner product $\langle f_k|f_k\rangle = 1$ and the inner product $\langle f_k|f_l\rangle = 0$ for $k \neq l$.

First, we compute the inner product $\langle f_k|f_k\rangle$,

$$\langle f_k|f_k\rangle = \left(\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1} e^{2\pi ijk/n}\langle j|\right)\left(\frac{1}{\sqrt{n}}\sum_{j'=0}^{n-1} e^{-2\pi ijk/n}\,|j'\rangle\right)$$

Simplifying the expression,

$$\langle f_k|f_k\rangle = \frac{1}{n}\sum_{j=0}^{n-1}\sum_{j'=0}^{n-1} e^{2\pi ijk/n}e^{-2\pi ijk/n}\langle j|j'\rangle$$

Using $\langle j|j'\rangle = \delta_{jj'}$,

$$\langle f_k|f_k\rangle = \frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi ijk/n}e^{-2\pi ijk/n}$$

$$\langle f_k|f_k\rangle = \frac{1}{n}\sum_{j=0}^{n-1} 1 = 1$$

Next, we compute the inner product $\langle f_k|f_l\rangle$,

$$\langle f_k|f_l\rangle = \left(\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1} e^{2\pi ijk/n}\langle j|\right)\left(\frac{1}{\sqrt{n}}\sum_{j'=0}^{n-1} e^{-2\pi ijl/n}\,|j'\rangle\right)$$

Simplifying the expression,

$$\langle f_k|f_l\rangle = \frac{1}{n}\sum_{j=0}^{n-1}\sum_{j'=0}^{n-1} e^{2\pi ijk/n}e^{-2\pi ijl/n}\langle j|j'\rangle$$

Using $\langle j|j'\rangle = \delta_{jj'}$,

$$\langle f_k|f_l\rangle = \frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi ijk/n}e^{-2\pi ijl/n}$$

$$\langle f_k|f_l\rangle = \frac{1}{n}\sum_{j=0}^{n-1} e^{2\pi ij(k-l)/n}$$

For $k \neq l$, the sum $\sum_{j=0}^{n-1} e^{2\pi ij(k-l)/n}$ is a geometric series with the common ratio $e^{2\pi i(k-l)/n} \neq 1$. The sum of this geometric series is zero,

$$\sum_{j=0}^{n-1} e^{2\pi ij(k-l)/n} = 0$$

Thus,

$$\langle f_k|f_l\rangle = \frac{1}{n}\cdot 0 = 0$$

Since we have shown that $\langle f_k|f_k\rangle = 1$ and $\langle f_k|f_l\rangle = 0$ for $k \neq l$, the transformed states $|f_k\rangle$ are orthonormal. Therefore, the transformed states $|f_k\rangle$ form a new orthonormal basis, proving that the QFT is indeed a change of basis.

5.4.1 QFT for $m = 1$ as a Hadamard

QFT for 1 qubit i.e., m =1, we have $n = 2^m = 2^1 = 2$. According to the matrix from section (5.3),

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & e^{-2\pi i/2} \end{pmatrix} \tag{5.10}$$

But $e^{-2\pi i/2} = e^{-\pi i}$ = -1, so it becomes

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{5.11}$$

Now it looks like a Hadamard transform, and this is quantum Fourier transform for $m = 1$.

5.4.2 Quantum Fourier Transform for powers of two

The Quantum Fourier Transform [15]–[17], [21] is defined as

$$|j\rangle \longrightarrow \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-2\pi ijk/n} |k\rangle \tag{5.12}$$

Let m be a power of 2.

We can write out j and k in terms of bits. This would be a binary representation of j and k.

$$j = b_{n-1}b_{n-2}b_{n-3}\cdots b_0 \tag{5.13}$$

$$k = b'_{n-1}b'_{n-2}b'_{n-3}\cdots b'_0 \tag{5.14}$$

Now we want to write equation (5.12) in terms of b's and b''s,

$$|b_{n-1}b_{n-2}\cdots b_0\rangle \longrightarrow \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-2\pi i\left(b_{n-1}2^{n-1}+b_{n-2}2^{n-2}+\cdots+b_0\right)\left(b'_{n-1}2^{n-1}+b'_{n-2}2^{n-2}+\cdots+b'_0\right)/2^n} |k\rangle$$

Here is the magic part comes in. We can multiply the whole thing out in the exponent, and it turns out that only half of these terms actually matter. That is because of the roots of unity. We have the term,

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-2\pi ib_jb'_k2^{j+k}/2^n} \tag{5.15}$$

But this term results 1 only when the sum of j and k is bigger than or equal to n.

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-2\pi ib_jb'_k2^{j+k}/2^n} = 1, \quad if\ j+k \geq n \tag{5.16}$$

5.4.3 Two Qubit Quantum Fourier Transform

For 2 qubits: $n = 2^m = 2^2 = 4$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{-2\pi i/4} & e^{-4\pi i/4} & e^{-6\pi i/4} \\ 1 & e^{-4\pi i/4} & e^{-8\pi i/4} & e^{-12\pi i/4} \\ 1 & e^{-6\pi i/4} & e^{-12\pi i/4} & e^{-18\pi i/4} \end{pmatrix} \tag{5.17}$$

Let, $e^{-2\pi i/4} = -i$. Then, the matrix becomes,

$$\frac{1}{\sqrt{4}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & 1 & -i \end{pmatrix} \tag{5.18}$$

Now we want to build the matrix out of gates. If the columns are labeled as j and rows are labeled as k, then the jk entry of the matrix is going to be

$$e^{-2\pi ijk/4} \tag{5.19}$$

Equivalently,

$$e^{-2\pi i(j_0+2j_1)(k_0+2k_1)/4} \tag{5.20}$$

Multiplying the whole thing out in the exponent of (5.20), we have

$$\underbrace{e^{-2\pi ij_0k_0/4}}_{1st\ term} \cdot \underbrace{e^{-2\pi ij_1k_0/2}}_{2nd\ term} \cdot \underbrace{e^{-2\pi ij_0k_1/2}}_{3rd\ term} \tag{5.21}$$

We do not get a j_1k_1 term because $2j_1 * 2k_1$ is a multiple of 4. The j_i's are represented by qubits. So we can think of (5.21) as three potential qubits.

Let us try implementing the 2nd term $e^{-2\pi ij_1k_0/2}$ from (5.21),

$$|j_1\rangle \longrightarrow \frac{1}{\sqrt{2}} \sum_{k=0}^{n-1} e^{-2\pi ij_1k_0/2} |k_0\rangle \tag{5.22}$$

We have used $\frac{1}{\sqrt{2}}$ as a normalization factor to make this a unitary. From equation (5.22) we observe,

$$|0\rangle \longrightarrow \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \tag{5.23}$$

$$|1\rangle \longrightarrow \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \tag{5.24}$$

We can recognize this gate, this is just a Hadamard gate [20].

$$\begin{matrix} & |0\rangle & |1\rangle & \\ \frac{1}{\sqrt{2}} & \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & & \begin{matrix} |0\rangle \\ |1\rangle \end{matrix} \end{matrix}$$

So the Hadamard gate implements the 2nd term $e^{-2\pi i j_1 k_0/2}$ and similarly the Hadamard gate also implements the 3rd term $e^{-2\pi i j_0 k_1/2}$ because this is just the same thing with j_0 and k_1.

Let us try drawing the circuit. Since j_1 was Hadamarded from k_0 and k_1 was Hadamarded from j_0. We have,

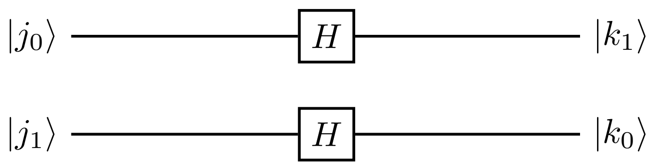

Figure 5.1: Two qubits Hadamarded

This might be the first attempt of the circuit, but it does not work. We need to implement the first term $e^{-2\pi i j_0 k_0/4}$ over here to make this circuit work. So what does bringing the term $e^{-2\pi i j_0 k_0/4}$ do? It means we want $|j_0\rangle$ to interact with $|k_0\rangle$.

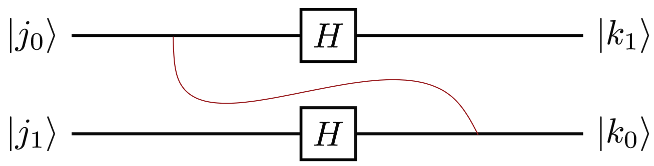

Figure 5.2: Plan to establish interaction between $|j_0\rangle$ and $|k_0\rangle$

So we want to draw a circuit that connects $|j_0\rangle$ with $|k_0\rangle$. If we move the top Hadamard later than the bottom Hadamard, this would do the trick.

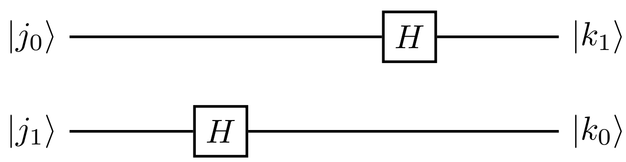

Figure 5.3: Top H moved later than the bottom H

Now what does the first term $e^{-2\pi i j_0 k_0/4}$ mean in equation (5.21)?

$$|j_0 k_0\rangle \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} |j_0 k_0\rangle \tag{5.25}$$

For the state $|j_0k_0\rangle$, nothing happens, or rather we multiply by 1, if j_0=0 or k_0=0. So that is 1, 1, 1 in the diagonal; otherwise, we multiply by $e^{-2}\pi i/4$, which we recognize as $-i$. That is just a controlled-$S^\dagger$ gate.

The controlled S gate has an i in the last index, so the conjugate transpose of the controlled S gate i.e., $S^\dagger$ gives a $-i$ in the last index.

$$|j_0k_0\rangle \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} |j_0k_0\rangle \tag{5.26}$$

When the first qubit is 1, we multiply the phase by $-i$ if the second qubit is also a 1. This is the matrix for controlled $S^\dagger$, and this gate is symmetric in j_0 and k_0.

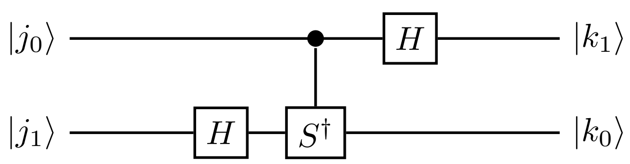

Figure 5.4: A controlled $S^\dagger$ between two Hadamards

In figure (5.4), we put the smallest bit of j last and the largest bit of j first. Let us try writing this out in terms of matrices.

The first thing we do is apply a Hadamard gate on j_1, which is the first qubit. Then we use the matrix for the controlled $S^\dagger$ gate. Finally, we apply a Hadamard gate on the second qubit, which is j_0.

We write them out in the standard way we write binary numbers.

$$\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}}_{H\ on\ j_0} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}}_{Controlled-S^\dagger} \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}}_{H\ on\ j_1} \tag{5.27}$$

Now we multiply two matrices from the right and it results,

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -i & 0 & i \end{pmatrix} \tag{5.28}$$

And multiplying this two matrices in (5.28) gives the following matrix,

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{pmatrix} \tag{5.29}$$

These bits are not in the correct order, so the order needs to be changed. We want to swap the bits of k, and to do that, we need to multiply the above matrix by a SWAP gate.

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{SWAP\ gate}\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \\ 1 & i & -1 & -i \end{pmatrix} \tag{5.30}$$

This SWAP gate will swap the 2nd and 3rd rows of the matrix (5.30),

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \tag{5.31}$$

Finally, the circuit will work properly with a SWAP gate reversing the bits of k,

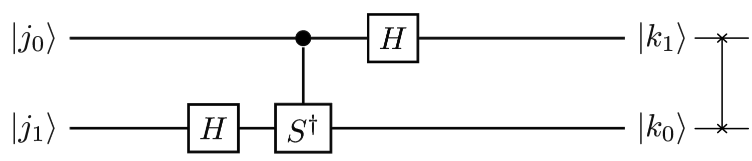

Figure 5.5: A two qubit QFT circuit

5.5 n-Qubit Quantum Fourier Transform

We can now generalize this to n-qubit quantum Fourier transform [15]–[17], [21]. Let us recall,

$$|j\rangle \longrightarrow \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} e^{-2\pi i\left(b_{n-1}2^{n-1}+b_{n-2}2^{n-2}+\cdots+b_0.1\right)\left(b'_{n-1}2^{n-1}+b'_{n-2}2^{n-2}+\cdots+b'_0.1\right)/2^n}\,|k\rangle \tag{5.32}$$

where b_s's are the bits of j and b''s are the bits of k. We have the term,

$$e^{-2\pi i b_s b'_{n-s-1}2^{n-1}/2^n} \tag{5.33}$$

If b_s and b'_{n-s-1} both are 1, then this results -1 and a zero otherwise. So this is clearly a Hadamard.

Let us start with $b_0 b_1 \cdots b_{n-1}$ and we need to Hadamard them all. Although in figure (5.6), we called them $j_0 j_1 \cdots j_{n-1}$, we are switching variables here. Let us consider a more specific case when $n = 4$ before we try to generalize it for any n.

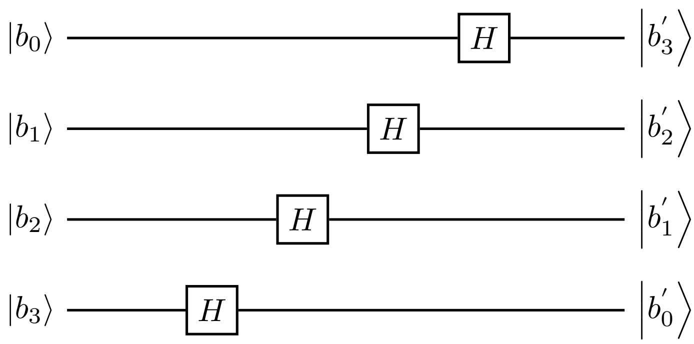

Figure 5.6: Four qubits Hadamarded

In figure (5.6), we put the smallest bits of b last and largest bits of b first.

Now, we need to put the gates in as they appear in the exponent of equation (5.32), which perform all the other transformations.

Multiplying the entire expression in the exponent of equation (5.32), we obtain one term from $(b_{n-1}2^{n-1} + b_{n-2}2^{n-2} + \cdots + b_0.1)$ and one term from $\left(b'_{n-1}2^{n-1} + b'_{n-2}2^{n-2} + \cdots + b'_0.1\right)$, resulting in the following gate,

$$e^{-2\pi i b_s b'_t 2^s 2^t/2^2} \tag{5.34}$$

$$e^{-2\pi i b_s b'_t 2^{s+t-n}} \tag{5.35}$$

This gives the matrix,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-2\pi i 2^{s+t-n}} \end{pmatrix} \tag{5.36}$$

If $s+t=n-1$, it comes from the term $e^{-2\pi i b_s b'_{n-s-1}/2^n}$, which transforms into a Hadamard gate in our circuit.

In the qubits $|b_s\rangle$ and $\left|b'_t\right\rangle$ of figure (5.6), if $s+t<n-1$, then there is room here, so we can put a new gate in.

It does not matter exactly where you put the new gate, as long as you place it after the Hadamard gate on b_s and before the Hadamard gate on b'_t.

Let us call this gate to be,

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{-2\pi i/k} \end{pmatrix} \tag{5.37}$$

which is taking $|0\rangle \longrightarrow |0\rangle$ and $|1\rangle \longrightarrow e^{-2\pi i/2^k}\,|1\rangle$.

Let us put the gates in now using R_k and later explain why we put them in.

1. We put R_2 gate between all adjacent qubits. We place R_2 between the Hadamards of $|b_3\rangle$ and $|b_2\rangle$, $|b_2\rangle$ and $|b_1\rangle$, $|b_1\rangle$ and $|b_0\rangle$.

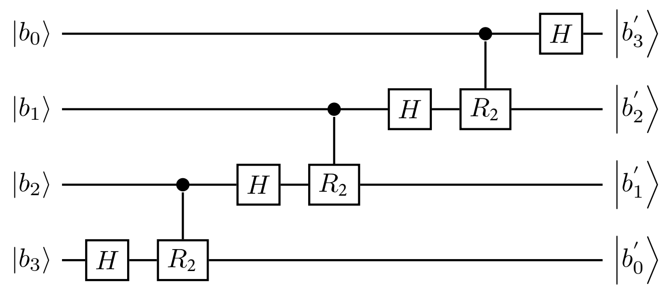

Figure 5.7: Placement of R_2 gate

2. We put R_3 between all pairs of qubits that are two away. We place R_3 between the Hadamards of $|b_3\rangle$ and $|b_1\rangle$, $|b_2\rangle$ and $|b_0\rangle$.

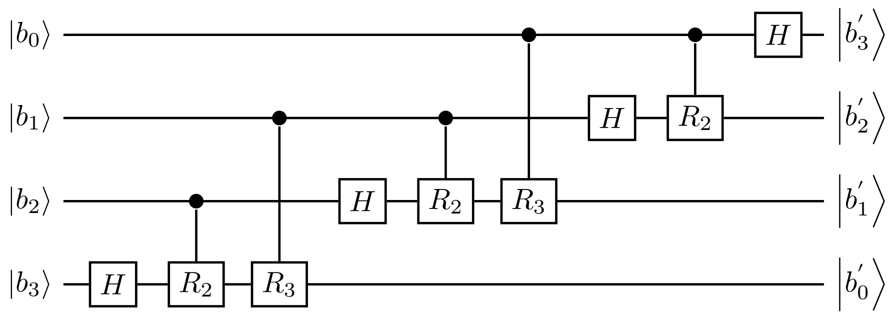

Figure 5.8: Placement of R_3 gate

3. And we put R_4 between all pairs of qubits which are three away from each other. We place R_4 between the Hadamards of $|b_3\rangle$ and $|b_0\rangle$.

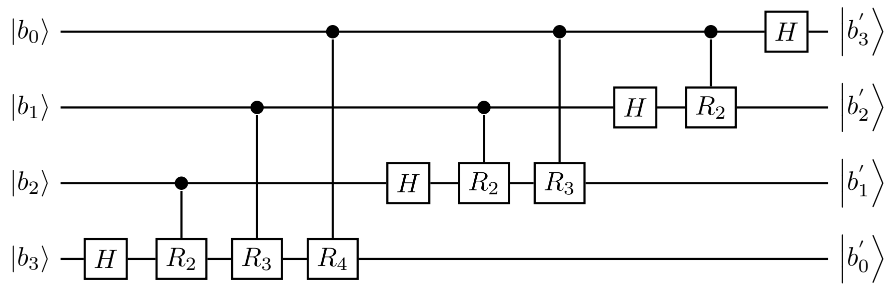

Figure 5.9: Placement of R_4 gate

Generally, we will put R_t between all pairs of qubits that are $(t+1)$ away. And $R_1, R_2, R_3 \cdots = R_k$, which is a controlled R_k gate.

Here, $\left|b_0'\right\rangle$ is interacting with $|b_1\rangle$ via R_3 gate.

But why did we choose R_3 from R_k? We want R_3 from (5.37) because it depends on $(n-s-t)$. In this case, $n=4$ and if $s=1$ in $|b_s\rangle$ and $t=0$ in $\left|b_t'\right\rangle$, then the R_k gate we want to help interact between $|b_1\rangle$ and $\left|b_0'\right\rangle$is $(n-s-t)=(4-1-0)=3$ i.e. R_3.

The sum of t and s depends on the distance between the wires. It turns out that we have R_2 between the adjacent wires, R_3 between the pairs of wires that are two apart, and R_4 between the pairs of wires that are three apart, etc.

Therefore, this circuit performs a 4-qubit quantum Fourier transform, and, of course, we have to swap the bits at the very end to get them in the right order.

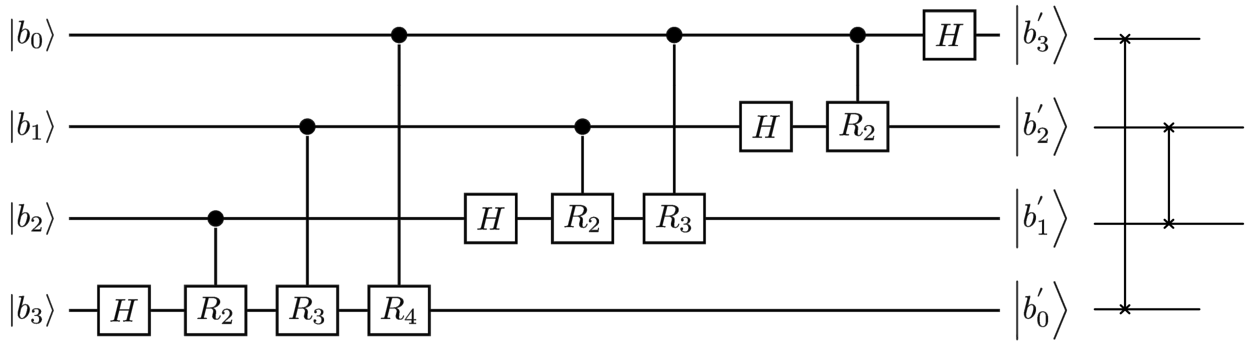

Figure 5.10: A four qubit QFT circuit

Building upon our earlier discussion of the design principles for a four-qubit Quantum Fourier Transform (QFT), it becomes apparent that these principles can be extended [45] to design a QFT circuit for any number of qubits n. The process does not require much imagination, as the underlying logic and structure remain consistent across different scales.

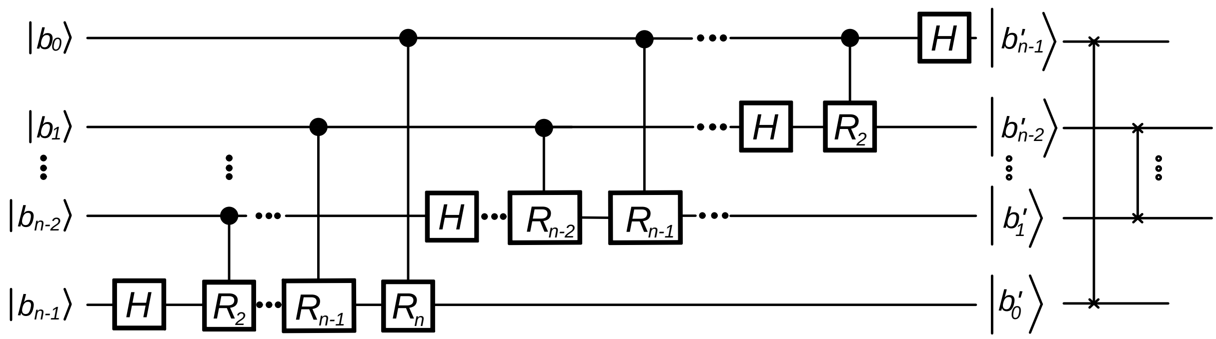

Figure 5.11: n-Qubit QFT Circuit

5.6 The Quantum Fourier Transform Versus a Discrete $\mathcal{Z}$-Transform

The Quantum Fourier Transform (QFT) and the Discrete $\mathcal{Z}$-transform (DZT) are both employed to expedite intermediate operations. They exhibit the following notable similarities:

5.6.1 Linearity of QFT versus DZT

To demonstrate the linearity of a matrix transformation T that maps from one vector space V to another vector space W, we need to establish that T adheres to two fundamental properties: additivity and homogeneity.

Additivity requires that for any vectors v_1 and v_2 in V, the transformation of their sum is equal to the sum of their individual transformations. Formally, we express this property as:

$$T(v_1 + v_2) = T(v_1) + T(v_2)$$

This equation states that applying the transformation T to the vector sum $v_1 + v_2$ is equivalent to applying T to v_1 and v_2 separately and then summing the results.

Homogeneity requires that for any scalar $\alpha \in \mathbb{F}$ (where $\mathbb{F}$ is the field over which the vector space is defined) and any vector $v \in V$, the transformation of a scalar multiple of the vector is equal to the scalar multiple of the transformation of the vector. Formally, we express this property as:

$$T(\alpha v) = \alpha T(v)$$

This equation states that applying the transformation T to the scaled vector αv is equivalent to scaling the result of applying T to v by the same scalar α.

By confirming both additivity and homogeneity, we establish the linearity of the matrix transformation $T : V \to W$.

Linearity of QFT

To establish the linearity of the quantum Fourier transform (QFT) represented by the matrix F, we need to prove that F satisfies both the additivity and homogeneity properties as defined earlier.

The QFT matrix F as in equation (5.10) for a two-variable transform is given by

$$F = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{e^{-2\pi i/2}}{\sqrt{2}} \end{pmatrix}$$

Consider a scalar c and two vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$,

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Additivity:

To prove additivity, we need to show:

$$F(\vec{x} + \vec{y}) = F(\vec{x}) + F(\vec{y})$$

Compute $F(\vec{x} + \vec{y})$,

$$\vec{x} + \vec{y} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$

$$F(\vec{x}+\vec{y}) = F\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{e^{-2\pi i/2}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}(x_1+y_1) + \frac{1}{\sqrt{2}}(x_2+y_2) \\ \frac{1}{\sqrt{2}}(x_1+y_1) + \frac{e^{-2\pi i/2}}{\sqrt{2}}(x_2+y_2) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}(x_1+x_2+y_1+y_2) \\ \frac{1}{\sqrt{2}}(x_1+y_1) + \frac{e^{-2\pi i/2}}{\sqrt{2}}(x_2+y_2) \end{pmatrix}$$

Compute $F(\vec{x}) + F(\vec{y})$,

$$F(\vec{x}) = F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{e^{-2\pi i/2}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2 \\ \frac{1}{\sqrt{2}}x_1 + \frac{e^{-2\pi i/2}}{\sqrt{2}}x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(x_1+x_2) \\ \frac{1}{\sqrt{2}}x_1 + \frac{e^{-2\pi i/2}}{\sqrt{2}}x_2 \end{pmatrix}$$

$$F(\vec{y}) = F\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{e^{-2\pi i/2}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}y_1 + \frac{1}{\sqrt{2}}y_2 \\ \frac{1}{\sqrt{2}}y_1 + \frac{e^{-2\pi i/2}}{\sqrt{2}}y_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(y_1+y_2) \\ \frac{1}{\sqrt{2}}y_1 + \frac{e^{-2\pi i/2}}{\sqrt{2}}y_2 \end{pmatrix}$$

$$F(\vec{x}) + F(\vec{y}) = \begin{pmatrix} \frac{1}{\sqrt{2}}(x_1+x_2) \\ \frac{1}{\sqrt{2}}x_1 + \frac{e^{-2\pi i/2}}{\sqrt{2}}x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}}(y_1+y_2) \\ \frac{1}{\sqrt{2}}y_1 + \frac{e^{-2\pi i/2}}{\sqrt{2}}y_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}(x_1+x_2) + \frac{1}{\sqrt{2}}(y_1+y_2) \\ \frac{1}{\sqrt{2}}x_1 + \frac{e^{-2\pi i/2}}{\sqrt{2}}x_2 + \frac{1}{\sqrt{2}}y_1 + \frac{e^{-2\pi i/2}}{\sqrt{2}}y_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}(x_1+y_1+x_2+y_2) \\ \frac{1}{\sqrt{2}}(x_1+y_1) + \frac{e^{-2\pi i/2}}{\sqrt{2}}(x_2+y_2) \end{pmatrix}$$

Since both $F(\vec{x}+\vec{y})$ and $F(\vec{x}) + F(\vec{y})$ yield the same result, additivity is satisfied.

Homogeneity

To prove homogeneity, we need to show:

$$F(c\vec{x}) = cF(\vec{x})$$

Compute $F(c\vec{x})$,

$$c\vec{x} = c\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix}$$

$$F(c\vec{x}) = F\begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{e^{-2\pi i/2}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}(cx_1) + \frac{1}{\sqrt{2}}(cx_2) \\ \frac{1}{\sqrt{2}}(cx_1) + \frac{e^{-2\pi i/2}}{\sqrt{2}}(cx_2) \end{pmatrix}$$

$$= \begin{pmatrix} c\left(\frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2\right) \\ c\left(\frac{1}{\sqrt{2}}x_1 + \frac{e^{-2\pi i/2}}{\sqrt{2}}x_2\right) \end{pmatrix}$$

Compute $cF(\vec{x})$,

$$F(\vec{x}) = F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{e^{-2\pi i/2}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2 \\ \frac{1}{\sqrt{2}}x_1 + \frac{e^{-2\pi i/2}}{\sqrt{2}}x_2 \end{pmatrix}$$

$$cF(\vec{x}) = c\begin{pmatrix} \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2 \\ \frac{1}{\sqrt{2}}x_1 + \frac{e^{-2\pi i/2}}{\sqrt{2}}x_2 \end{pmatrix}$$

$$= \begin{pmatrix} c\left(\frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2\right) \\ c\left(\frac{1}{\sqrt{2}}x_1 + \frac{e^{-2\pi i/2}}{\sqrt{2}}x_2\right) \end{pmatrix}$$

Since both $F(c\vec{x})$ and $cF(\vec{x})$ yield the same result, homogeneity is satisfied.

The matrix F representing the quantum Fourier transform for a two-variable system satisfies both additivity and homogeneity. Therefore, F is a linear transformation.

Linearity of DZT

To establish the linearity of our redefined Discrete $\mathcal{Z}$-transform (DFT) represented by the block-encoded matrix U as in equation (4.3), we need to prove that U satisfies both the additivity and homogeneity properties.

The DFT matrix for a two-variable transform, block-encoded into a larger matrix U, is given by

$$U = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \sqrt{\frac{5}{18}} & -\sqrt{\frac{5}{9}} \\ \frac{1}{3} & \frac{1}{6} & -\sqrt{\frac{5}{9}} & \sqrt{\frac{13}{18}} \\ \sqrt{\frac{5}{18}} & -\sqrt{\frac{5}{9}} & -\frac{2}{3} & -\frac{1}{3} \\ -\sqrt{\frac{5}{9}} & \sqrt{\frac{13}{18}} & -\frac{2}{3} & -\frac{1}{6} \end{pmatrix}$$

Additivity

We need to show that for any vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in V:

$$U(\mathbf{v}_1 + \mathbf{v}_2) = U(\mathbf{v}_1) + U(\mathbf{v}_2)$$

Let $\mathbf{v}_1 = \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \\ v_{24} \end{pmatrix}$. Then the sum of these vectors is:

$$\mathbf{v}_1 + \mathbf{v}_2 = \begin{pmatrix} v_{11} + v_{21} \\ v_{12} + v_{22} \\ v_{13} + v_{23} \\ v_{14} + v_{24} \end{pmatrix}$$

Applying the transformation U to this sum,

$$U(\mathbf{v}_1 + \mathbf{v}_2) = U \begin{pmatrix} v_{11} + v_{21} \\ v_{12} + v_{22} \\ v_{13} + v_{23} \\ v_{14} + v_{24} \end{pmatrix}$$

Using matrix multiplication,

$$U(\mathbf{v}_1 + \mathbf{v}_2) = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \sqrt{\frac{5}{18}} & -\sqrt{\frac{5}{9}} \\ \frac{1}{3} & \frac{1}{6} & -\sqrt{\frac{5}{9}} & \sqrt{\frac{13}{18}} \\ \sqrt{\frac{5}{18}} & -\sqrt{\frac{5}{9}} & -\frac{2}{3} & -\frac{1}{3} \\ -\sqrt{\frac{5}{9}} & \sqrt{\frac{13}{18}} & -\frac{2}{3} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} v_{11} + v_{21} \\ v_{12} + v_{22} \\ v_{13} + v_{23} \\ v_{14} + v_{24} \end{pmatrix}$$

This expands to

$$U(\mathbf{v}_1 + \mathbf{v}_2) = \begin{pmatrix} \frac{2}{3}(v_{11} + v_{21}) + \frac{2}{3}(v_{12} + v_{22}) + \sqrt{\frac{5}{18}}(v_{13} + v_{23}) - \sqrt{\frac{5}{9}}(v_{14} + v_{24}) \\ \frac{1}{3}(v_{11} + v_{21}) + \frac{1}{6}(v_{12} + v_{22}) - \sqrt{\frac{5}{9}}(v_{13} + v_{23}) + \sqrt{\frac{13}{18}}(v_{14} + v_{24}) \\ \sqrt{\frac{5}{18}}(v_{11} + v_{21}) - \sqrt{\frac{5}{9}}(v_{12} + v_{22}) - \frac{2}{3}(v_{13} + v_{23}) - \frac{1}{3}(v_{14} + v_{24}) \\ -\sqrt{\frac{5}{9}}(v_{11} + v_{21}) + \sqrt{\frac{13}{18}}(v_{12} + v_{22}) - \frac{2}{3}(v_{13} + v_{23}) - \frac{1}{6}(v_{14} + v_{24}) \end{pmatrix}$$

Separating this into two matrix-vector multiplications,

$$U(\mathbf{v}_1 + \mathbf{v}_2) = \begin{pmatrix} \frac{2}{3}v_{11} + \frac{2}{3}v_{12} + \sqrt{\frac{5}{18}}v_{13} - \sqrt{\frac{5}{9}}v_{14} \\ \frac{1}{3}v_{11} + \frac{1}{6}v_{12} - \sqrt{\frac{5}{9}}v_{13} + \sqrt{\frac{13}{18}}v_{14} \\ \sqrt{\frac{5}{18}}v_{11} - \sqrt{\frac{5}{9}}v_{12} - \frac{2}{3}v_{13} - \frac{1}{3}v_{14} \\ -\sqrt{\frac{5}{9}}v_{11} + \sqrt{\frac{13}{18}}v_{12} - \frac{2}{3}v_{13} - \frac{1}{6}v_{14} \end{pmatrix} + \begin{pmatrix} \frac{2}{3}v_{21} + \frac{2}{3}v_{22} + \sqrt{\frac{5}{18}}v_{23} - \sqrt{\frac{5}{9}}v_{24} \\ \frac{1}{3}v_{21} + \frac{1}{6}v_{22} - \sqrt{\frac{5}{9}}v_{23} + \sqrt{\frac{13}{18}}v_{24} \\ \sqrt{\frac{5}{18}}v_{21} - \sqrt{\frac{5}{9}}v_{22} - \frac{2}{3}v_{23} - \frac{1}{3}v_{24} \\ -\sqrt{\frac{5}{9}}v_{21} + \sqrt{\frac{13}{18}}v_{22} - \frac{2}{3}v_{23} - \frac{1}{6}v_{24} \end{pmatrix}$$

Which simplifies to

$$U(\mathbf{v}_1 + \mathbf{v}_2) = U(\mathbf{v}_1) + U(\mathbf{v}_2)$$

This confirms the additivity property of the matrix transformation U.

Homogeneity

We need to show that for any scalar $\alpha \in \mathbb{F}$ and any vector $\mathbf{v} \in V$:

$$U(\alpha\mathbf{v}) = \alpha U(\mathbf{v})$$

Let $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$. Then the scaled vector is:

$$\alpha\mathbf{v} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha v_3 \\ \alpha v_4 \end{pmatrix}$$

Applying the transformation U to this scaled vector,

$$U(\alpha\mathbf{v}) = U \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha v_3 \\ \alpha v_4 \end{pmatrix}$$

Using matrix multiplication,

$$U(\alpha\mathbf{v}) = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \sqrt{\frac{5}{18}} & -\sqrt{\frac{5}{9}} \\ \frac{1}{3} & \frac{1}{6} & -\sqrt{\frac{5}{9}} & \sqrt{\frac{13}{18}} \\ \sqrt{\frac{5}{18}} & -\sqrt{\frac{5}{9}} & -\frac{2}{3} & -\frac{1}{3} \\ -\sqrt{\frac{5}{9}} & \sqrt{\frac{13}{18}} & -\frac{2}{3} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha v_3 \\ \alpha v_4 \end{pmatrix}$$

This expands to

$$U(\alpha\mathbf{v}) = \begin{pmatrix} \alpha\left(\frac{2}{3}v_1 + \frac{2}{3}v_2 + \sqrt{\frac{5}{18}}v_3 - \sqrt{\frac{5}{9}}v_4\right) \\ \alpha\left(\frac{1}{3}v_1 + \frac{1}{6}v_2 - \sqrt{\frac{5}{9}}v_3 + \sqrt{\frac{13}{18}}v_4\right) \\ \alpha\left(\sqrt{\frac{5}{18}}v_1 - \sqrt{\frac{5}{9}}v_2 - \frac{2}{3}v_3 - \frac{1}{3}v_4\right) \\ \alpha\left(-\sqrt{\frac{5}{9}}v_1 + \sqrt{\frac{13}{18}}v_2 - \frac{2}{3}v_3 - \frac{1}{6}v_4\right) \end{pmatrix}$$

Factoring out α,

$$U(\alpha\mathbf{v}) = \alpha \begin{pmatrix} \frac{2}{3}v_1 + \frac{2}{3}v_2 + \sqrt{\frac{5}{18}}v_3 - \sqrt{\frac{5}{9}}v_4 \\ \frac{1}{3}v_1 + \frac{1}{6}v_2 - \sqrt{\frac{5}{9}}v_3 + \sqrt{\frac{13}{18}}v_4 \\ \sqrt{\frac{5}{18}}v_1 - \sqrt{\frac{5}{9}}v_2 - \frac{2}{3}v_3 - \frac{1}{3}v_4 \\ -\sqrt{\frac{5}{9}}v_1 + \sqrt{\frac{13}{18}}v_2 - \frac{2}{3}v_3 - \frac{1}{6}v_4 \end{pmatrix}$$

Recognizing the transformed vector $U(\mathbf{v})$,

$$U(\alpha\mathbf{v}) = \alpha U(\mathbf{v})$$

This confirms the homogeneity property of the matrix transformation U.

By confirming both additivity and homogeneity, we establish that the block-encoded matrix U representing the redefined Discrete $\mathcal{Z}$-transform (DFT) is linear.

5.6.2 Reversibility of QFT versus DZT

Unitarity is a fundamental concept in quantum mechanics that ensures a transformation is reversible. Specifically, a unitary transformation preserves the inner product in Hilbert space, which guarantees that no information is lost during the process, making reversibility a key feature.

We have shown that the Quantum Fourier Transform (QFT) is unitary, meaning it maintains orthonormality and can be inverted by its conjugate transpose. This establishes the QFT as a reversible transformation.

Our research extends these principles to the redefined Discrete $\mathcal{Z}$-Transform (DZT). Using block-encoding techniques, we constructed a matrix representation of the DZT and demonstrated its unitarity. This confirms that the DZT, like the QFT, is reversible.

These findings are significant for quantum computing, where reversible transformations are essential. We provided evidence and a theoretical basis that confirm the unitary nature of both the QFT and our redefined DZT, thus establishing their reversibility.

5.7 Basis Transformation

Basis transformations are fundamental to various algorithms in quantum computing. Notably, the Quantum Fourier Transform (QFT), discussed earlier in this chapter, the Discrete $\mathcal{Z}$-transform (DZT) covered in Chapter 4, and the bosonic quantum $\mathcal{Z}$-transform explored in Chapter 3, all serve as changes of basis.

The Quantum Fourier Transform is a linear transformation on qubits that maps computational basis states to another set of basis states. To validate these transformed states as a legitimate basis, their orthonormality is established. Specifically, the inner product of a transformed basis state with itself is equal 1, and the inner product with any other transformed basis state is equal 0. These conditions ensure that the transformed basis states are orthogonal and normalized, which is crucial for preserving the inner product structure of the Hilbert space in quantum computing.

Similarly, the Discrete $\mathcal{Z}$-transform, when block encoded as a unitary matrix, inherently satisfies the orthogonality and normalization properties, much like the QFT. This ensures its suitability as a basis transformation in the potential gate-based model of a quantum $\mathcal{Z}$-transform. By maintaining unitarity, the DZT preserves the orthonormality of the transformed basis vectors, ensuring that the structure of the Hilbert space is maintained. This capability to transform and preserve the basis vectors makes the DZT a valid change of basis amenable to quantum computations.

Additionally, the bosonic quantum $\mathcal{Z}$-transform extends the classical $\mathcal{Z}$-transform into the quantum domain, connecting number states (Fock states) and coherent states. Both can form bases in the Hilbert space of a quantum system. To establish the bosonic quantum $\mathcal{Z}$-transform as a change of basis, we analyze the orthogonality and completeness of the number states and coherent states.

Number states are inherently orthonormal, as indicated by

$$\langle n|m\rangle = \delta_{nm}$$

where δ_{nm} is the Kronecker delta, equating to 1 if $n = m$ and 0 otherwise. Coherent states, while not orthonormal in the same sense as number states, form an overcomplete basis. This overcompleteness allows any state in the Hilbert space to be represented as a linear combination of coherent states. The connection between number states and coherent states established through the quantum $\tilde{\mathcal{Z}}$-transform demonstrates it as a valid change of basis in the quantum context.

In summary, the QFT, discrete $\mathcal{Z}$-transform, and bosonic quantum $\mathcal{Z}$-transform each exhibit crucial properties for basis transformations in quantum mechanics. The QFT directly preserves the orthonormality of basis states, the DZT ensures orthonormality through its unitary block-encoded matrices, and the quantum $\tilde{\mathcal{Z}}$-transform connects number states and coherent states, leveraging the overcomplete nature of the latter. These properties set them up in an optimal condition for developing quantum algorithms.

Chapter 6

Conclusion

In this chapter, we construct a conceptual map encompassing all major ideas we have explored, positioning our investigation within a broader context. We present a critical analysis of the problem and demonstrate how our work establishes a foundation for future research in discovering a desired quantum $\mathcal{Z}$-transform.

6.1 Critical Analysis

In Chapter 1, we present the problem statement, summarize the key contributions, and offer a concise overview of the thesis, effectively setting the stage for our research. Chapter 2 provides an in-depth look at the classical $\mathcal{Z}$-transform, covering its definition, existence criteria, and key properties with proofs, preparing the reader for the later discussion of quantum analogs.

In Chapter 3, we explore the quantum analogs of classical $\mathcal{Z}$-transform properties, establishing a connection between number states and coherent states. Traditionally, the $\mathcal{Z}$-transform is not considered a basis transformation. However, since both number states and coherent states form bases, we treat this connection as a basis transformation.

Importantly, we discuss the $\tilde{\mathcal{Z}}$-transform as a basis transformation within a bosonic quantum system. The persistence of classical $\mathcal{Z}$-transform properties in a quantum system, especially a bosonic one, is significant as it opens possibilities for demonstrating the $\mathcal{Z}$-transform as a basis transformation in other quantum systems. As a result, in developing a quantum subroutine, our focus will shift from bosonic to qubit quantum systems to enable implementation on quantum computers.

In this context, it is crucial to note that coherent states are overcomplete[38], meaning they do not form the usual orthonormal basis. This overcompleteness leads to an infinite number of representations, allowing a single ket vector to be decomposed in various ways using the same set of vectors. Therefore, the basis of coherent states does not provide a unique decomposition, enabling multiple possible decompositions. Understanding this property is essential as it influences how we approach the implementation of the $\mathcal{Z}$-transform in quantum computing.

In Chapter 4, we begin by discretizing the classical $\mathcal{Z}$-transform. The traditional definition of the $\mathcal{Z}$-transform involves an infinite summation, which remains infinite even when the $\tilde{\mathcal{Z}}$-transform acts as a basis transformation in a bosonic quantum system to derive its quantum properties. However, this infinite nature complicates its implementation on quantum computers, thus necessitating discretization. This discretization process is not straightforward and involves redefining the $\mathcal{Z}$-transform to handle the infinite summation effectively.

We also developed a matrix formulation for our discrete $\mathcal{Z}$-transform. Initially, the matrix derived from this discrete definition is not unitary. Ensuring the unitarity of our discrete $\mathcal{Z}$-transform is crucial for developing a quantum algorithm. To address this, we must convert the matrix formulation into a unitary operator.

Using the block-encoding method, we successfully converted two-variable and four-variable discrete $\mathcal{Z}$-transform matrices into unitary operators. This process involved normalizing the matrices by dividing them by their largest singular value, ensuring the normalized matrices had singular values less than or equal to one. We then verified that the matrix formed by subtracting the conjugate transpose of the normalized matrix multiplied by itself from the identity matrix was positive semi-definite. This verification allowed us to decompose the matrix using the matrix square root technique. We employed eigenvalue decomposition to find the square roots of the eigenvalues and constructed the matrix B accordingly. Finally, we structured the unitary matrix U to embed the normalized matrix while maintaining unitarity.

To summarize, the discrete definition and the successful unitarization of the discrete $\mathcal{Z}$-transform matrix through the block-encoding method fulfill the preconditions for efficient quantum operations on the matrix formulation of the discrete $\mathcal{Z}$-transform using standard quantum gates and subroutines. This work lays a solid foundation for further exploration and implementation in quantum computing.

In Chapter 5, we begin our discussion by examining the foundational principles underlying the discovery of the quantum Fourier transform to glean insights for the development of a quantum $\mathcal{Z}$-transform, owing to the considerable similarities between them. The discrete Fourier transform (DFT), intrinsically discrete, is unitary because it conforms to the orthogonality property in its matrix representation and preserves vector lengths post-transformation. These characteristics make the DFT ideal for quantum computation, ensuring its unitarity and applicability on quantum computers. The quantum Fourier transform (QFT) is essentially the discrete Fourier transform, but instead of transforming the values of the variables themselves as the DFT does, it considers the variables as input/output vectors representing the probability amplitudes of the quantum states in QFT. Drawing inspiration from the parallels between the discrete Fourier transform and the quantum Fourier transform, we establish the analogous properties of our redefined discrete $\mathcal{Z}$-transform and the quantum Fourier transform: both are linear, involve basis changes in the frequency domain, are reversible, and enhance the efficiency of intermediary operations. This makes our discrete $\mathcal{Z}$-transform amenable to quantum computation in a manner and spirit similar to the quantum Fourier transform.

6.2 Future Direction

Achieving a quantum implementation of the $\mathcal{Z}$-transform, akin to the development of the quantum Fourier transform, requires two principal modifications. First, the classical $\mathcal{Z}$-transform must be discretized and redefined as a finite summation. This milestone has been accomplished in our research. Second, it is necessary to ensure that the matrix formulation of this redefined $\mathcal{Z}$-transform is unitary. We have successfully demonstrated this by converting two-variable and four-variable discrete $\mathcal{Z}$-transform matrices into unitary operators. We are also prepared to extend this to any finite number of variables for our discrete $\mathcal{Z}$-transform.

The discretization of the $\mathcal{Z}$-transform maps the signal to a discrete set of values, which can be naturally represented as quantum states. The unitary matrix from block-encoding allows us to manipulate these quantum states while preserving their norm, an essential property for quantum operations. As a result, the progress we have made thus far in our research has provided the optimal setup for developing the intended quantum $\mathcal{Z}$-transform.

Due to the pioneering nature of this work and the inherent time constraints of an undergraduate thesis, we could not explore all our innovative ideas, especially given the project's ambitious scope that exceeded typical undergraduate expectations. However, in future research, using our newfound insights, we intend to continue developing and implementing this on a gate-based model of quantum computation with lower circuit depth and complexity. To that end, we plan to use quantum phase estimation to extract frequency components and apply the quantum Fourier transform to convert the time-domain signal into the frequency domain efficiently. By measuring the quantum states post-transformation, we will be able to obtain the $\mathcal{Z}$-transform coefficients. This approach, starting with block-encoding, will allow us to efficiently implement the $\mathcal{Z}$-transform on a quantum computer, harnessing quantum parallelism and interference for potentially significant speedups over classical methods.

Bibliography

[1] B. Riemann, "Foundations for a general theory of functions of a complex variable," *Mathematische Annalen*, 1851. [Online]. Available: https://link.springer.com/article/10.1007/BF02418511.

[2] V. A. Fock, "Konfigurationsraum und zweite quantelung," *Zeitschrift für Physik*, 1932.

[3] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*. Princeton University Press, 1932. [Online]. Available: https://archive.org/details/mathematicalfoun0613vonn.

[4] G. Hardy and E. Wright, *An Introduction to the Theory of Numbers*. Oxford University Press, 1938. [Online]. Available: https://global.oup.com/academic/product/an-introduction-to-the-theory-of-numbers-9780199219865.

[5] P. R. Halmos, *Finite-Dimensional Vector Spaces*. Princeton University Press, 1958.

[6] L. A. Zadeh and C. A. Desoer, *Theory and Applications of the Z-Transform Method*. Prentice-Hall, 1963.

[7] K. E. Cahill, "Overcompleteness of coherent states," *Physical Review*, vol. 138, B1566, 1965. [Online]. Available: https://doi.org/10.1103/PhysRev.138.B1566.

[8] R. Glauber, "Optique et electronique quantiques — quantum optics and electronics," in *Quantum Optics and Electronics*, C. DeWitt, A. Blandin, and C. Cohen-Tannoudji, Eds., New York: Gordon and Breach, 1965.

[9] H. Kung and C. E. Leiserson, "Discrete and fast fourier transform made clear," *arXiv preprint arXiv:1908.07154*, 1978. [Online]. Available: https://arxiv.org/abs/1908.07154.

[10] J. R. Klauder and B.-S. Skagerstam, *Coherent States: Applications in Physics and Mathematical Physics*. World Scientific, 1985. [Online]. Available: https://www.springer.com/gp/book/9789971966016.

[11] A. V. Oppenheim and R. W. Schafer, *Discrete-Time Signal Processing*. Prentice Hall, 1989.

[12] G. Stephenson and P. M. Radmore, *Advanced Mathematical Methods for Engineering and Science Students*. Cambridge: Cambridge University Press, 1990, ISBN: 978-0521368598.

[13] H. S. Wilf, *Lectures on Generating Functions*. American Mathematical Society, 1994. [Online]. Available: https://www.ams.org/books/mbk/033/.

[14] A. V. Oppenheim, A. S. Willsky, and S. H. Nawab, *Signals and Systems*. Prentice Hall, 1997.

[15] P. W. Shor, "Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer," *SIAM J. Comput.*, vol. 26, no. 5, pp. 1484–1509, Oct. 1997, ISSN: 0097-5397. DOI: 10.1137/S0097539795293172. [Online]. Available: http://dx.doi.org/10.1137/S0097539795293172.

[16] R. Jozsa, "Quantum fourier transforms," *arXiv preprint quant-ph/0301093*, 1998. [Online]. Available: https://arxiv.org/abs/quant-ph/0301093.

[17] M. Mosca and A. Ekert, "Generic quantum fourier transforms," *arXiv preprint quant-ph/0304064*, 1999. [Online]. Available: https://arxiv.org/abs/quant-ph/0304064.

[18] B. P. Lathi, *Signal Processing and Linear Systems*. Oxford University Press, 2000, ISBN: 978-0195219173.

[19] F. Bagarello, "Introduction to coherent states and quantum information theory," *arXiv preprint arXiv:0112090*, 2001. [Online]. Available: https://arxiv.org/abs/0112090.

[20] D. Coppersmith, "An approximate fourier transform useful for quantum factoring," Feb. 2002.

[21] A. Kitaev, "Exact quantum fourier transforms and discrete logarithm algorithms," *arXiv preprint quant-ph/0301093*, 2003. [Online]. Available: https://arxiv.org/abs/quant-ph/0301093.

[22] S. Carroll, "Spacetime and geometry - an introduction to general relativity," 2004.

[23] F. Hong-Yi, F. Liang, and A. Wünsche, "Quantum mechanical version of z-transform related to eigenkets of boson creation operator," *Communications in Theoretical Physics*, vol. 42, no. 5, pp. 675–680, Nov. 2004. DOI: 10.1088/0253-6102/42/5/675. [Online]. Available: https://doi.org/10.1088%2F0253-6102%2F42%2F5%2F675.

[24] S. Elaydi, *An Introduction to Difference Equations* (Undergraduate Texts in Mathematics). Springer, 2005. DOI: 10.1007/0-387-27602-8. [Online]. Available: https://doi.org/10.1007/0-387-27602-8.

[25] L. Debnath and D. Bhatta, *Integral Transforms and Their Applications*, 2nd. Boca Raton: Chapman & Hall/CRC, 2006, ISBN: 9781584885757.

[26] J. J. Kelly, *Graduate Mathematical Physics*. Wiley, 2006, ISBN: 978-0470048067.

[27] D. C. Kozen, *Introduction to the Theory of Computation*. Springer, 2006, ISBN: 978-1846282973.

[28] R. El Attar, "Lecture notes on z-transform," 2007.

[29] P. Kaye, R. Laflamme, and M. Mosca, *An Introduction to Quantum Computing*. Oxford University Press, 2007, ISBN: 978-0198570493.

[30] N. J. Higham, *Functions of Matrices: Theory and Computation*. SIAM, 2008, ISBN: 978-0898716467.

[31] W. Stein, *Elementary Number Theory: Primes, Congruences, and Secrets*. Springer, 2008, ISBN: 978-0387855240.

[32] C. Schröder, C. Lüders, and D. Wong, "Generalized coherent states, reproducing kernels, and quantum support vector machines," *Journal of Machine Learning Research*, vol. 10, pp. 2193–2228, 2009. [Online]. Available: http://jmlr.org/papers/v10/schroder09a.html.

[33] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 10th Anniversary Edition. Cambridge University Press, 2010, ISBN: 978-1107002173.

[34] H.-Y. Fan and H.-L. Lu, "Classical optical transforms studied in the context of quantum optics via the route of developing dirac's symbolic method," *International Journal of Modern Physics B*, 2012.

[35] W. Wasylkiwskyj, *Signals and Transforms in Linear Systems Analysis*. Springer Nature, 2013, ISBN: 978-3642384246.

[36] K. Kido, "Discrete fourier transform," in *Digital Fourier Analysis: Fundamentals*, Springer, 2015, ch. 4, pp. 127–155. DOI: 10.1007/978-1-4614-9260-3_4. [Online]. Available: https://doi.org/10.1007/978-1-4614-9260-3_4.

[37] F. Parisio, "Coherent-state overcompleteness, path integrals, and weak values," *Journal of Mathematical Physics*, vol. 57, p. 062 105, 2016. [Online]. Available: https://doi.org/10.1063/1.527142.

[38] F. Parisio, "Coherent-state overcompleteness, path integrals, and weak values," *Journal of Mathematical Physics*, vol. 57, no. 3, p. 032 101, 2016. DOI: 10.1063/1.4943014. eprint: https://doi.org/10.1063/1.4943014. [Online]. Available: https://doi.org/10.1063/1.4943014.

[39] I. Kerenidis and A. Prakash, "Quantum algorithms for matrix inversion and solving linear systems," *arXiv preprint arXiv:1704.04992*, 2017. [Online]. Available: https://arxiv.org/abs/1704.04992.

[40] A. M. van Apeldoorn and A. Gilyén, "Quantum singular value transformation and beyond: Exponential improvements for quantum matrix arithmetics," *arXiv preprint arXiv:1806.01838*, 2018. [Online]. Available: https://arxiv.org/abs/1806.01838.

[41] "Coherent states in quantum optics: An oriented overview," *arXiv preprint arXiv:1810.06473*, 2018. [Online]. Available: https://arxiv.org/abs/1810.06473.

[42] D. J. Griffiths and D. F. Schroeter, *Introduction to Quantum Mechanics*, 3rd ed. Cambridge University Press, 2018, ISBN: 978-1107189638.

[43] B. Guo and J. Zhu, *Signals and Systems*. Walter de Gruyter GmbH, 2018, ISBN: 9783110611604.

[44] M. E. Newman and C. H. Lee, "Block-encoding technique in quantum algorithms," *arXiv preprint arXiv:1806.01838*, 2018. [Online]. Available: https://arxiv.org/abs/1806.01838.

[45] C. Shao, "Generalization of quantum fourier transformation," *Journal of Quantum Information Processing*, vol. 7, no. 3, pp. 123–130, 2018, Accessed: 2024-07-18. DOI: 10.1007/s11128-018-1234-5.

[46] R. D. Somma, “Quantum circuit synthesis for generalized coherent states,” *arXiv preprint arXiv:1811.08479*, 2018. [Online]. Available: https://arxiv.org/abs/1811.08479.

[47] R. Chattamvelli and R. Shanmugam, *Generating Functions in Engineering and the Applied Sciences*. Springer Cham, 2019. DOI: 10.1007/978-3-031-79410-0. [Online]. Available: https://doi.org/10.1007/978-3-031-79410-0.

[48] O. Rosas-Ortiz, “Coherent and squeezed states: Introductory review of basic notions, properties, and generalizations,” in *Integrability, Supersymmetry and Coherent States: A Volume in Honour of Professor Véronique Hussin*, Ş. Kuru, J. Negro, and L. M. Nieto, Eds. Cham: Springer International Publishing, 2019, pp. 187–230, ISBN: 978-3-030-20087-9. DOI: 10.1007/978-3-030-20087-9_7. [Online]. Available: https://doi.org/10.1007/978-3-030-20087-9_7.

[49] I. L. Shapiro, “Theorems of green, stokes, and gauss,” in *A Primer in Tensor Analysis and Relativity*, Springer, 2019, pp. 123–143. DOI: 10.1007/978-3-030-26895-4_9. [Online]. Available: https://doi.org/10.1007/978-3-030-26895-4_9.

[50] P. LibreTexts, *Laplace transforms and their applications in engineering*, Accessed: 2024-07-18. [Online]. Available: https://phys.libretexts.org/Bookshelves/Mathematical_Physics_and_Pedagogy/Book%3A_Mathematical_Methods_in_Physics_(Wilkins)/14%3A_Laplace_Transforms.