An Examination of the Big Bang Singularity Using the Matrix Formulation of M-Theory

by

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Declaration

It is hereby declared that

1. The thesis submitted is my own original work while completing degree at BRAC University.

2. The thesis does not contain material previously published or written by a third party, except where this is appropriately cited through full and accurate referencing.

3. The thesis does not contain material which has been accepted, or submitted, for any other degree or diploma at a university or other institution.

4. I have acknowledged all main sources of help.

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Approval

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Abstract

M theory is a proposed quantum theory of everything which is connected to type IIA string theory via S duality. In its low energy limit, M theory is approximated by 11 dimensional supergravity.

In this thesis, we look at the BFSS formulation of M theory, known as Matrix theory. We discover that this formulation yields correct expressions for the velocity dependent scattering potential between supergravitons at one and two loops. Furthermore, it also gives us the correct M2 brane tension.

Another form of the BFSS conjecture at finite N is then used to show that a static potential does exist between supergravitons. This static potential falls rapidly at late times, in agreement with the flat spacetime calculation.

Keywords: M-theory, Matrix Theory, BFSS, String Theory

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Contents

1 Introduction

M-theory is a proposed theory of everything, one that unifies all versions of superstring theory. At a string conference in 1995, Edward Witten showed that the M theory is consistent with all versions of superstring theory - one could obtain type IIA superstring theory or $E_8 \times E_8$ heterotic string theory from M theory by transformations called S dualities. Witten's revelation produced a flurry of papers known as the *second superstring revolution*.

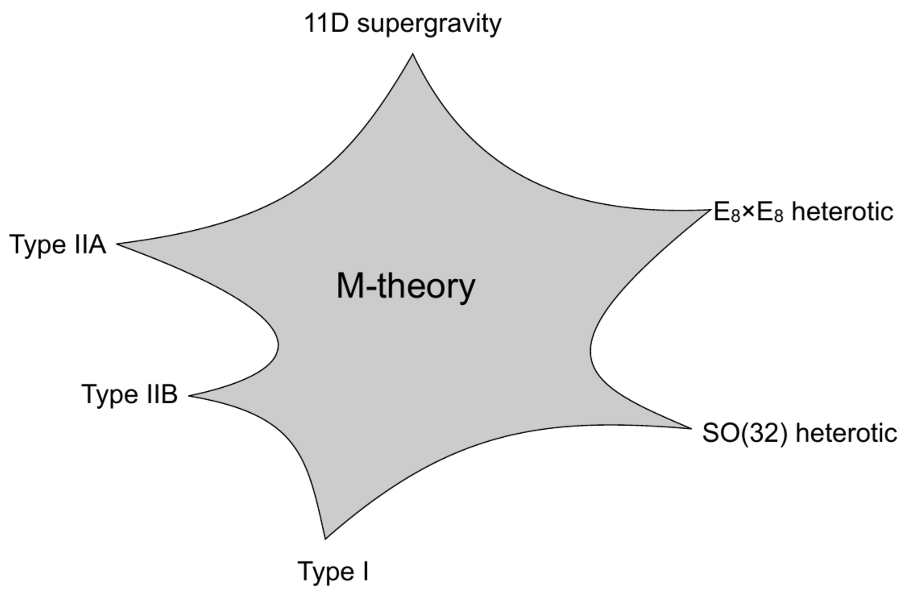

Figure 1: The five string theories and 11D supergravity are all united by M-theory

M-theory is a consistent quantum theory, but not much is known of it. We know for a fact that the low energy effective action for M-theory must be 11 dimensional supergravity. This is in contrast to the $9+1$ dimensions of superstring theory. Thus, M theory must contain objects called 2 branes and 5 branes, the latter of which is magnetically dual to the former. They are restricted to these dimensions by supersymmetry.

In this thesis, we will first explore some properties of D-branes. Even though D-branes are boundary conditions for strings to end, we will see that they are actually dynamic objects and are central to non-perturbative formulations of string theory. Then we will look at a particular formulation of M-theory, the BFSS matrix theory. We will establish correspondences between Matrix theory and M theory in the low energy limit (i.e. 11 dimensional supergravity).

Subsequently, we will introduce another formulation of Matrix theory, using Discrete Light Cone Quantization (DLCQ).Using DLCQ Matrix theory in a Type IIA linear dilaton background, we will probe the singularity of the big bang, which is inaccessible via the non-remormalizable theory of General Relativity. We will see that due to twisted boundary conditions on a Milne orbifold, there is a static potential between supergravitons that decays rapidly at late times and spacetime emerges.

2 String Theory and D-Branes

The contents of this section are based on Joseph Polchinski's String Theory textbook [1] and Adel Belali's paper [3].

2.1 T-Duality

2.1.1 Closed Strings

Consider a world sheet parametrized by the variables τ and σ. Now make the change of variables $z = \mathrm{e}^{\tau - i\sigma}$ and $\bar{z} = \mathrm{e}^{\tau + i\sigma}$. From the Polyakov action for a closed string, we get the equation of motion

$$\partial_z \partial_{\bar{z}} X^\mu(z, \bar{z}) = 0$$

This yields the expansion:

$$X^\mu = x^\mu - i\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}^{\infty}\Big(\frac{\alpha_m^\mu}{m}z^{-m} + \frac{\tilde{\alpha}_m^\mu}{m}\bar{z}^{-m}\Big) \tag{2.1}$$

Now, owing to translational invariance of the action, momentum is conserved, by Noether's theorem. The momentum of the string field is

$$p^\mu = \frac{1}{\sqrt{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)$$

If a dimension X^μ is non-compact, then to ensure the invariance of X^μ under $\sigma \to \sigma + 2\pi$ requires $\alpha_0^\mu = \tilde{\alpha}_0^\mu$.

On the contrary, if we compactify a certain dimension, say X^{25} on a circle of radius R, such that $X^{25} + 2\pi R \simeq X^{25}$, then $\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu) = mR$ for some integer m. Then under the transformation $\sigma \to \sigma + 2\pi, X^{25} \to X^{25} + 2\pi mR \simeq X^{25}$. Now, for the wavefunction $\sim \exp(ix^{25}p_{25})$ to remain invariant, the momentum is $p_{25} = \frac{n}{R}$. If we force X^{25} to be compact, we must then have

$$\begin{aligned} \alpha_0^{25} &= \sqrt{\frac{\alpha'}{2}}\Big(\frac{n}{R} + m\frac{R}{\alpha'}\Big) \\ \text{and } \tilde{\alpha}_0^{25} &= \sqrt{\frac{\alpha'}{2}}\Big(\frac{n}{R} - m\frac{R}{\alpha'}\Big), \end{aligned} \tag{2.2}$$

where m is known as the *winding number.*

We can split up X^{25} into a holomorphic (z-dependent) left moving and an antiholomorphic ($\bar{z}$ - dependent) parts. Now consider the two fields:

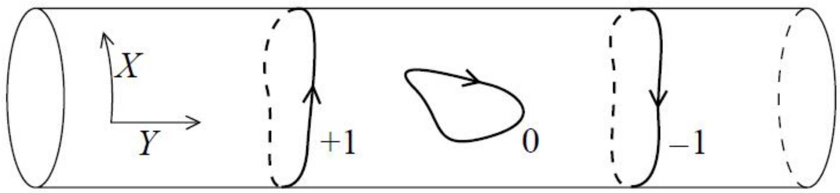

Figure 2: Oriented Strings with winding numbers m = +1,0 and -1

$$\begin{aligned} X^{25} &\equiv X^{25}(z) + X^{25}(\bar{z}), \\ \text{and } \hat{X}^{25} &\equiv X^{25}(z) - X^{25}(\bar{z}). \end{aligned} \tag{2.3}$$

Here,

$$\begin{aligned} X^{25}(z) &= x_L^{25} - i\frac{\alpha'}{2}p_L^{25} + i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{+\infty}\frac{\alpha_m^{25}}{mz^m}, \\ \text{and } X^{25}(\bar{z}) &= x_R^{25} - i\frac{\alpha'}{2}p_R^{25} + i\sqrt{\frac{\alpha'}{2}}\sum_{m=-\infty}^{+\infty}\frac{\tilde{\alpha}_m^{25}}{m\bar{z}^m}. \end{aligned} \tag{2.4}$$

We have $x^{25} = x_L^{25} + x_R^{25}$. p_L^{25} and p_R^{25} are the left and right handed momenta which are respectively given by:

$$p_L^{25} = \sqrt{\frac{2}{\alpha'}}\alpha_0^{25} = \frac{n}{R} + \frac{mR}{\alpha'},$$

and

$$p_R^{25} = \sqrt{\frac{2}{\alpha'}}\tilde{\alpha}_0^{25} = \frac{n}{R} - \frac{mR}{\alpha'}.$$

If we make the exchange $n \leftrightarrow m$ and $R \leftrightarrow \frac{\alpha'}{R}$ simultaneously, then the mass squared of the string state, given by $\alpha_0^2 + \tilde{\alpha}_0^2 +$ oscillator modes, which remains invariant. The momentum modes and the winding modes in this theory are then interchanged. This is known as *T-duality*: a theory compactified on a circle of radius R is equivalent to a theory compactified on a circle of radius $\frac{\alpha'}{R}$.

Furthermore, for the fields referred to in (2.3), the transformation $X^{25} \to \hat{X}^{25}$ leave the OPE, energy momentum tensor and correlation functions invariant. Therefore, T-duality is a symmetry of the perturbative string theory as well.

2.1.2 Open Strings and D-branes

For open strings, we must set booundary conditions for the ends of the string. We can have either Neumann (N) or Dirichlet (D) conditions. A Neumann condition is to set the σ- derivative at $\sigma = 0, \pi$ equal to 0. In so doing, we have $\alpha_n^\mu = \tilde{\alpha}_n^\mu$ for all n.

And therefore, we have

$$X^\mu = x^\mu - i\alpha' p^\mu \ln z\bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\alpha_m^\mu}{m}(z^{-m} + \bar{z}^{-m}). \tag{2.5}$$

Now we compactify X^{25} on a circle of radius R. As before, we have $p^{25} = \frac{n}{R}$. The T-dual field to $X^{25} is \hat{X}^{25}$ and given by:

$$\hat{X}^{25} = \hat{x}^{25} - i\alpha' p^{25} \ln \frac{z}{\bar{z}} + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0}^{\infty} \frac{\alpha_m^{25}}{m}(z^{-m} - \bar{z}^{-m}). \tag{2.6}$$

It can be shown that $\partial_\sigma X^\mu = i\partial_\tau \hat{X}^\mu$. This implies that $\partial_\tau \hat{X}^{25} = 0$ at $\sigma = 0, \pi$. Therefore, at any boundary, $\hat{X}^{25}$ is constant - this is what is known as a *Dirichlet* boundary condition. The difference between the values of $\hat{X}^{25}$ at the two boundaries is given by

$$\hat{X}^{25}(\pi) - \hat{X}^{25}(0) = 2\pi\alpha' p^{25} = 2\pi\alpha' \frac{n}{R} = 2\pi n \hat{R}. \tag{2.7}$$

Thus, the difference between the two boundaries is a multiple of $2\pi\hat{R}$, *i.e.* both values of the field are identified with each other. Hence, both ends of the string lie on the same $24 + 1$ dimensional hypersurface known as a *D-brane*.

Here, we have started with N boundary conditions and have derived D conditions in the T-dual space. We could equivalently have started with D conditions and have ended up with N conditions in the T dual space.

We have T-dualized in only one direction. Thus, our D-brane is a D 24-brane. In general, if we T-dualize k directions, then we have a D p-brane with $p = 25 - k$. If we did not T-dualize at all, we would be left with a D25-brane, which covers all of spacetime. The strings are ordinary open strings propagating in $25 + 1$ dimensions.

2.1.3 Fractional Momentum Modes

Open strings can carry Chan-Paton [4] factors at their endpoints. The states are labeled as $|ij\rangle$ with $i, j = 1, 2, \ldots N$ labels for the Chan Paton charges at the end of the string. For oriented strings, we associate the fundamental representation with the $\sigma = 0$ and and the antifundamental representation with the $\sigma = \pi$ end. This describes a gauge group $U(N)$.

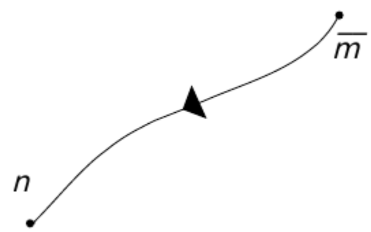

Figure 3: Chan Paton factors at the ends of an open string

The full quantum state is given by

$$|\phi, k, \lambda\rangle = \sum_{i,j=1}^{N} \lambda_{ij} |\phi, k, ij\rangle, \tag{2.8}$$

where ϕ is the Fock state space label and k is the momentum. The λ_{ij} are $U(N)$ matrices known as Chan Paton matrices.

If we include a Wilson line, corresponding to a background gauge field in the compactified direction,

$$A_{25} = \frac{1}{2\pi R}\mathrm{diag}(\theta_1, \ldots, \theta_N),$$

the $U(N)$ symmetry breaks to the diagonal subgroup $U(N) \to U(1)^N$ if the gauge invariant holonomy matrix or Wilson line

$$U = \exp\Big(i \oint dx^\mu A_\mu(x)\Big), \tag{2.9}$$

are all distinct.

The introduction of the gauge field shifts the momentum along the compactified direction such that

$$p^{25} = \frac{n}{R} + \frac{\theta_i - \theta_j}{2\pi R}. \tag{2.10}$$

To see how this works in the context of a $U(1)$ theory, consider the action of a point particle with charge q:

$$S = \int d\tau \Big(\frac{1}{2}\dot{X}^\mu \dot{X}_\mu + \frac{m^2}{2} - qA_\mu \dot{X}^\mu\Big) \equiv \int d\tau \mathcal{L}.$$

The gauge field $A_{25} = -\frac{\theta}{2\pi R} = -i\Lambda^{-1}\frac{\partial \Lambda}{\partial X^{25}}$ is pure gauge, where $\Lambda(X^{25}) = \exp(-\frac{i\theta X^{25}}{2\pi R})$. The term with the gauge field is simply $-iq\int dx^\mu A_\mu$. In the path integral weight $\exp(-S)$, the gauge field term picks up a factor equal to the holonomy matrix. For $\mu \neq 25$, the conjugate momentum Π^μ is given by

$$\Pi_\mu = i\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = i\dot{X}_\mu, \tag{2.11}$$

and

$$\Pi_{25} = i\frac{\partial \mathcal{L}}{\partial \dot{X}^{25}} = i\dot{X}_{25} - \frac{q\theta}{2\pi R} = \frac{n}{R}. \tag{2.12}$$

The last equality results from the periodicity of the wavefunction which contains a factor of $\exp(i\Pi_{25}X^{25})$. Thus, the momentum in the X_{25} direction is given by

$$i\dot{X}^{25} = \frac{n}{R} + \frac{q\theta}{2\pi R}. \tag{2.13}$$

This leads to "fractional winding numbers" in the dual theory $\widehat{X}^\mu$. The expression for the T-dual space corresponding to equation (2.7) is then:

$$\widehat{X}^{25}(\pi) - \widehat{X}^{25}(0) = (2\pi n + \theta_i - \theta_j)\widehat{R}. \tag{2.14}$$

The mass shell relation of the string is given by

$$M^2 = (p^{25})^2 + \frac{1}{\alpha'}(\mathcal{N} - 1) = \Big(\frac{(2\pi n + \theta_j - \theta_i)\hat{R}}{2\pi\alpha'}\Big)^2 + \frac{1}{\alpha'}(\mathcal{N} - 1). \tag{2.15}$$

In the equation above, $\mathcal{N}$ is the eigenvalue of the *level operator*, which is defined by $\sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n$, which is analogous to the number operator of the SHO.

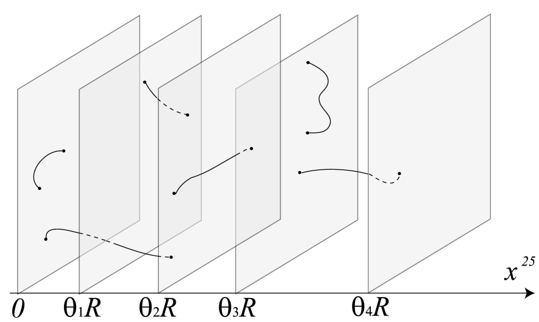

Figure 4: Including the Wilson line yields D-branes at positions $\theta_1 R \ldots \theta_4 R$

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2.2 D-brane Action

2.2.1 Motivation

Why do we consider D-branes to be dynamical objects?

Massles states of the string only occur when $\theta_i = \theta_j$, since work must be done to lengthen the string. Thus, in order to be massless, a state must have a Chan-Paton factor of $|ii\rangle$. We have, for a Dp-brane, the directions $a = 0, 1, \ldots, p$ parallel to the brane and the directions $\alpha = p+1, \ldots, D-1$ orthogonal to the brane, the massless modes of the open string

$$\alpha^a_{-1}\, |k; ii\rangle \tag{2.16a}$$

$$\text{and } \alpha^\alpha_{-1}\, |k; ii\rangle\,. \tag{2.16b}$$

The first set of excitations in 2.16 are vector bosons. In particular, they are photons that are the quanta of gauge fields A_a living on the brane.

The second set of states are scalar fields ϕ^α perpendicular to the brane. For Dp-branes, both translational and Lorentz symmetry are broken in the $D-p-1$ transverse directions. The string states fall into representations of Poincare$(1,p)\times SO(D-p-1)$.

2.2.2 D-brane action

The low-energy effective action is, with the gauge choice $X^a = \xi^a, a = 0, \ldots, p$ and setting $\phi^\alpha = \frac{X^\alpha}{2\pi\alpha'}, \alpha = p+1, \ldots D-1$,

$$S = -(2\pi\alpha')^2 T_p \int d^{p+1}\xi \Big(1 + \frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\partial_a\phi^\alpha\partial^a\phi^\alpha + \ldots\Big), \tag{2.17}$$

where T_p is the brane tension. The brane energy is tension $\times$ volume. Now suppose that there are no gauge field excitations on the brane and that $\phi^\alpha \equiv \phi^\alpha(t)$. The action 2.17 takes on the form

$$\begin{aligned} S &= \int dt \Big(- M_{\text{Dp-brane}} + T_p V_{\text{Dp-brane}} \times \frac{1}{2}(\dot{X}^\alpha(t))^2 \Big), \\ &= \int dt \Big(- M_{\text{Dp-brane}} + \frac{1}{2} M_{\text{Dp-brane}} (\dot{X}^\alpha(t))^2 \Big). \end{aligned} \tag{2.18}$$

Here, $V_{\text{Dp-brane}}$ and $M_{\text{Dp-brane}}$ are Dp-brane volume and mass respectively. The term $M_{\text{Dp-brane}}$ is the vacuum energy (no excitations) of the Dp-brane. This action is kinetic energy term minus potential and it motivates us to think of the scalar field as fluctuations of the Dp-brane itself.

At low energies and slowly varying fields, the effective action is given by the Dirac-Born-Infeld action, which was first postulated to get rid of the infinities in Maxwell theory,

$$S_{DBI} = -T_p \int d^{p+1}\xi \sqrt{-\det(\gamma_{ab} + 2\pi\alpha' F_{ab})}. \tag{2.19}$$

Here $\gamma_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\eta_{\mu\nu} = \eta_{ab} + \frac{\partial X^\alpha}{\partial \xi^a}\frac{\partial X^\beta}{\partial \xi^b}\eta_{\alpha\beta}$ and μ, ν are directions both parallel and orthogonal to the brane. In other words, γ_{ab} is the pullback of the target space metric onto the Dp brane. In matrix form, we can write the DBI action 2.19 as

$$\mathcal{L}_{DBI} = \sqrt{-\det\gamma}\sqrt{\det(\mathbb{1}+M)}, M = 2\pi\alpha'\gamma^{-1}F. \tag{2.20}$$

Let us now calculate the product of determinants. Since M is antisymmetric, we get

$$\begin{aligned}\sqrt{\det(\mathbb{1}+M)} &= \Big[\sqrt{\det(\mathbb{1}+M)}\sqrt{\det(\mathbb{1}+M)^T}\Big]^{\frac{1}{2}} \\ &= \det(\mathbb{1}-M^2)^{\frac{1}{4}} \\ &= \exp\Big(\frac{1}{4}\operatorname{Tr}\log(\mathbb{1}-M^2)\Big) \\ &= \exp\Big(-\frac{1}{4}\operatorname{Tr}(M^2+\frac{1}{2}M^4\dots)\Big) \\ &= 1-\frac{1}{4}\operatorname{Tr}M^2-\dots\end{aligned} \tag{2.21}$$

$$\text{and } \sqrt{\det(1+2\pi\alpha'\gamma^{ab}F_{ab})} = 1-\frac{(2\pi\alpha')^2}{4}F_{ab}F^{ab}. \tag{2.22}$$

Now consider

$$\begin{aligned}\sqrt{\det(-\gamma_{ab})} &= \exp\left[\frac{1}{2}\operatorname{Tr}\log\Big(1+\Big(\frac{\partial X}{\partial\xi}\Big)^2\Big)\right], \\ &= \sqrt{\det(-\eta_{ab})}\sqrt{\det\left(1+\frac{\partial X^\alpha}{\partial\xi^a}\frac{\partial X^\beta}{\partial\xi^b}\eta^{ab}\delta_{\alpha\beta}\right)}, \\ &= 1+\frac{1}{2}(2\pi\alpha')^2\partial_a\phi^\alpha\partial_a\phi^\alpha+\dots.\end{aligned} \tag{2.23}$$

Combining 2.23 and 2.22, we see that the action in 2.20 is the same as the action in 2.19 up to an additive constant. When there are no D-brane excitations, we have a purely Maxwell theory on the brane.

2.2.3 Nonabelian Generalization

Now consider what happens when we have multiple Dp-branes. Let us take the case of 2 Dp-branes for simplicity. Each has a gauge field living on it, leading to a $U(1)\times U(1)$ theory. There are strings that start on brane 1 and end on brane 2 ([12] strings), strings that start one brane 2 and end on brane 1 ([21] strings) and strings that start and end on the same brane ([11] and [22]) string sectors. The Dp-brane excitations are given by the matrix,

$$\phi_\alpha = \begin{pmatrix}(\phi_\alpha)^1_1 & (\phi_\alpha)^1_2 \\ (\phi_\alpha)^2_1 & (\phi_\alpha)^2_2\end{pmatrix}, \tag{2.24}$$

where $(\phi_\alpha)^i_j$ represents a string in the α direction that starts on the i^{th} Dp brane, *i.e.* has its $\sigma=0$ endpoint there and ends on the j^{th} Dp brane, *i.e.* has its $\sigma = \pi$ endpoint there. The matrix components transform as $(\phi_\alpha)^i_j \to \exp[i(\theta_i - \theta_j)](\phi_\alpha)^i_j$.

When we have N coincident D-branes, the DBI action generalizes into a non-Abelian $U(N)$ gauge theory. Consider firstly the 2 case of two Dp branes. There are strings in the $[11], [22], [12]$ and $[21]$ sectors. These correspond to the four massless gauge bosons that are the generators of the $U(2)$ gauge field. These cannot be the generators of the $U(1) \times U(1)$ gauge field because the latter are massive, with string mass, as we shall soon see. Since the D-branes are one and the same, we have the freedom to shuffle their indices around using unitary transformations $|\Psi; ij\rangle \to |\Psi'; kl\rangle = U_{ik}U^*_{jl}|\Psi; ij\rangle = U\Psi U^\dagger$, which is a transformation in the adjoint representation of $U(N)$.

The bosonic part of the $U(N)$ Yang-Mills theory is,

$$S^{\text{bosonic}}_{\text{YM}} \sim \int d^{p+1}\xi \,\text{Tr}\, F_{\mu\nu}F^{\mu\nu}. \tag{2.25}$$

Here, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ as μ, ν run over all directions in spacetime. Now we will dimensionally reduce the 10 dimensional superstring theory to $p+1$ dimensions. We start off with a space-filling Dp brane and the directions orthogonal to the brane become bosonic fields.

The action in 2.25 may be expanded as ($a, b = 0, \ldots, p$ and $\alpha, \beta = p+1, \ldots, D-1$),

$$S^{\text{bosonic}}_{\text{YM}} \sim \int d^{p+1}\xi \,\text{Tr}(F_{ab}F^{ab} + 2F_{\alpha a}F^{\alpha a} + F_{\alpha\beta}F^{\alpha\beta}).$$

The derivatives $\partial_\alpha A_a$ are 0, $\because$ the Dirichlet conditions take out the zero modes orthogonal to the brane. Also, $A_\alpha = \phi_\alpha = \frac{1}{2\pi\alpha'}X_\alpha$. For the components of $F_{\mu\nu}$, we then have,

$$\begin{aligned} F_{a\alpha} &= \frac{1}{2\pi\alpha'}\partial_a X_\alpha - \partial_\alpha A_a + \frac{i}{2\pi\alpha'}[A_a, X_\alpha] \equiv \frac{1}{2\pi\alpha'}D_a X_\alpha, \\ \text{and } F_{\alpha\beta} &= \frac{i}{(2\pi\alpha')^2}[X_\alpha, X_\beta]. \end{aligned} \tag{2.26}$$

Therefore, the action 2.25 becomes,

$$S^{\text{bosonic}}_{\text{YM}} \sim \int d^{p+1}\xi \,\text{Tr}[F^2_{ab} + \left(\frac{1}{2\pi\alpha'}\right)^2 (D_a X_\alpha)^2 - \underbrace{\left(\frac{1}{2\pi\alpha'}\right)^4 [X_\alpha, X_\beta]^2]}_{\text{Higgs Term}} \tag{2.27}$$

The coefficients are determined by disk diagrams by summing over topologies.

The low energy bosonic Dp brane action is given as

$$S = -(2\pi\alpha')^2 T_p \int d^{p+1}\xi \mathrm{Tr}\Big(\underbrace{\frac{1}{4}F_{ab}F^{ab}}_{\text{Gauge Field Term}} + \underbrace{\frac{1}{2}\mathcal{D}_a\phi^\alpha\mathcal{D}^a\phi^\alpha}_{\text{Kinetic Term}} - \underbrace{\frac{1}{4}[\phi^\alpha,\phi^\beta]^2}_{\text{Higgs Potential Term}} \Big). \tag{2.28}$$

We have $\frac{1}{g_{YM}^2} = \alpha\prime^2 T_p = \frac{1}{l_s^{p-3}g_s}$, where g_{YM} is the Yang-Mills coupling. Now consider the commutator term, the Higgs potential. We may write it as: $-[\phi_\alpha, \phi_\beta]^2$. The minimum value is when ϕ_α commutes with ϕ_β. This only happens if the two are simultaneously diagonalizable, *i.e.*

$$\phi^\alpha = \begin{pmatrix} \phi_1^\alpha & & \\ & \ddots & \\ & & \phi_N^\alpha \end{pmatrix}. \tag{2.29}$$

The diagonal components give us the positions of the N Dp-branes. Now when the D-branes are separated, we get the string mass from the Higgs mechanism just as we get the mass of the W-boson. Consider the case of two separated D-branes. The Higgs VEV is

$$\phi = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}. \tag{2.30}$$

The gauge field matrix is

$$A_a = \begin{pmatrix} A_a^{11} & W_a \\ W_a^\dagger & A_a^{22} \end{pmatrix}. \tag{2.31}$$

From the term $\frac{1}{2}\,\mathrm{Tr}[A_a,\phi]^2 = -W_a^\dagger(\phi_2-\phi_1)^2 W_a$. This gives a W boson mass of $M_W^2 = \frac{1}{(2\pi\alpha')^2}|X_2-X_1|^2 = T^2|X_2-X_1|^2$ for string tension T. We can interpret this to mean that the mass of a string is the tension times the length.

We may write the full general non-Abelian Dirac-Born-Infeld action as:

$$S_{DBI} = -T_p \int d^{p+1}\xi \exp(-\Phi)\,\mathrm{Tr}\,\sqrt{-\det(G_{ab}+B_{ab}+2\pi\alpha' F_{ab})}. \tag{2.32}$$

The excitations of the closed string include the symmetric metric $G_{\mu\nu}(X)$, the antisymmetric Kalb Ramond field $B_{\mu\nu}(X)$ and the scalar dilaton field $\Phi(X)$.The string moves in a background of these fields. The Kalb Ramond field is a two-form field. In 2.32 the field strength $H = dB$ is analogous to the two-form field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$. Thus, $H_{\mu\nu\omega} = \partial_\omega B_{\mu\nu} + \partial_\nu B_{\mu\omega} + \partial_\nu B_{\omega\mu}$.The gauge transformation is given by $\delta B_{\mu\nu}(X) = \partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu$, which keeps the field strength invariant. The spin-2 field $G_{ab} = G_{\mu\nu}\frac{\partial X^\mu}{\partial\xi^a}\frac{\partial X^\nu}{\partial\xi^b}$ where $a = 0,\ldots p$ are the directions on the D-brane is the pullback of the spacetime metric onto the Dp-brane. The factor of $\exp(-\Phi)$ arises because this is a tree-level action.

The combination $B_{ab} + 2\pi\alpha' F_{ab}$ is due to how a string moves in a background of these fields. On a string worldsheet $\mathcal{M}$ the string couples to these two fields by, $\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} + \int_{\partial\mathcal{M}} d\tau A_a \dot{X}^a$. This is invariant under the gauge transformations of $F_{\mu\nu}$ and $B_{\mu\nu}$.

The derivation of 2.28 from 2.32 is a non-trivial calculation. This is worked out in [2], using the invariance of the theory under T-duality.

2.2.4 D-branes in superstring theories

Type IIA superstring theories contain Dp branes with p even and Type IIB theories contain Dp branes with p odd. D branes in superstring theories couple to $p+1$ form fields known as Ramond Ramond (RR) fields. In type IIA theory, D0-branes couple to A_μ, D2-branes to $A_{\mu\nu}$, D4-branes to $A_{\mu\nu\rho\sigma\lambda}$ and so on and so forth. In type IIB theory, D(-1)-branes couple to the RR fields A, D1-branes to A_μ, D3-branes to $A_{\mu\nu\rho\sigma}$.

In addition to the usual $U(N)$ symmetries, our new Yang Mills action is invariant under supersymmetric transformations. Type II string theories have 32 supersymmetry generators (supercharges). D-branes are invariant under half of these - BPS states. The action now contains 16 component real spinors which transform in the adjoint representation of $U(N)$.

The string tension is given by $T_p = \frac{(2\pi\sqrt{\alpha'})^{1-p}}{2\pi\alpha'}$. In particular, for $D0$ branes, we have $T_0 = \frac{1}{\sqrt{\alpha'}}$. The low energy action for a Dp brane is 10 dimensional, $\mathcal{N}=1$ SUSY Yang Mills action reduced to $p+1$ dimensions. This is done by assuming that all the fields are independent of the coordinates $p+1\dots 9$. The dimensionally reduced action for $D0$ branes is

$$S^{D0} = \int dt \mathrm{Tr}\Big(-\frac{1}{4g_s c^2}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\Gamma^\mu D_\mu \psi\Big). \tag{2.33}$$

Here, $c = \frac{1}{2\pi\alpha'}$ and g_s is the string coupling. The prefactor on the fermion term has been absorbed into the definition of ψ. Here, ψ is a Majorana-Weyl spinor

$$\psi = \begin{pmatrix}\theta \\ 0\end{pmatrix}.$$

And the 32×32 gamma matrices are given by

$$\Gamma^0 = \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}, \Gamma^j = \begin{pmatrix}0 & \gamma^j \\ \gamma^j & 0\end{pmatrix}.$$

Where γ^j are the 16×16 gamma matrices. This action is invariant under the supersymmetry transformations generated by 16 supercharges.

Now consider the directions $m, n = 0\dots p$ on the brane and $i, j = p+1\dots D-1$

transverse to the brane. On the brane, the derivatives with respect to directions normal to the brane are 0. Therefore, the corresponding field strength components are $F_{ij} = ic^2[X^i, X^j], F_{0j} = c(\partial_0 X_j + i[A_0, X_j]) = cD_0 X_j$ and $D_0\theta = \partial_0\theta + i[A_0, \theta]$.

$$S^{D0} = T_0 \int dt \mathrm{Tr}\Big(\frac{1}{2g_s}(D_0 X^i)^2 - i\theta^T D_0\theta + \frac{c^2}{4g_s}([X_i, X_j])^2 + c\theta^T\gamma^j[X_j, \theta]\Big). \tag{2.34}$$

This is a supersymmetric $N \times N$ quantum mechanics, in which X^i and θ are in the adjoint of $U(N)$ - and these are Hermitian matrices. Each component of θ is a 16 component real spinor.

3 M-theory

The following five different ten dimensional superstring theories are related through S-dualities and T-dualities,

i Type II A

ii Type II B

iii $E_8 \times E_8$ heterotic

iv $SO(32)$ heterotic and

v Type I

S-duality (or strong weak duality) in string theory relates a theory with coupling g_s with a theory with coupling $\frac{1}{g_s}$.

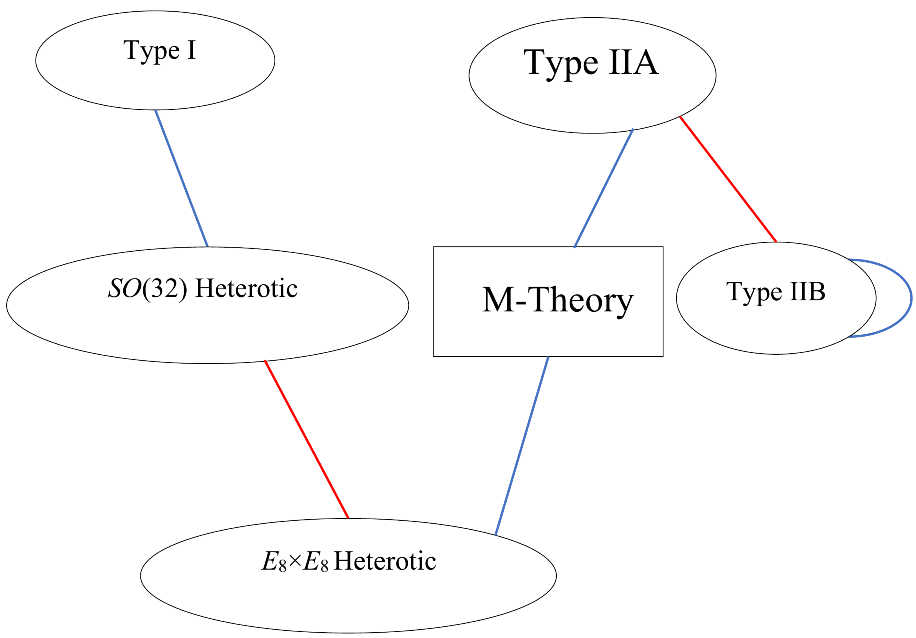

Figure 5: Relationships between the various superstring theories: blue lines indicate S-duality and red lines indicate T-duality

At strong coupling, the Type IIA and $E_8 \times E_8$ heterotic string theories exhibit the 11^{th} dimension and approach a theory called M-theory. Thus, M theory unifies all versions of superstring theory and led to the superstring revolution in the mid-90s.

First, we examine the low energy effective action of M-theory, which is 11 Dimensional supergravity.

3.1 Eleven-Dimensional Supergravity

Eleven dimensional supergravity contains the following fields: a metric G_{MN} with $M, N = 0, \ldots, 9, 11$ or an elevenbein e_M^A (the M directions are in the curved space and A in the tangent space), a three form A_{MNP} and a Majorana gravitino $\psi_{M\alpha}$, where M is a vector index and α a spinor index. This gravitino is a superpartner to the graviton and has spin $\frac{3}{2}$. Here, we calculate the number of on-shell degrees of freedom for each field.

The metric G_{MN} transforms in the symmetric traceless tensor representation of $SO(D-2)$ in D dimensions. Therefore, the number of degrees of freedom is $\frac{(D-2)^2+(D-2)}{2} - 1$. We subtract 1 because of the traceless nature of the metric. Plugging in $D = 11$ dimensions, we have 44 degrees of freedom. The 3-form field A_{MNP} transforms in the antisymmetric representation of the $SO(D-2)$ and thus it has $\frac{1}{3!}(D-2)(D-3)(D-4)$ degrees of freedom (dofs) or 84 dofs in 11 dimensions. Thus the bosonic part of the action has $84 + 44 = 128$ dofs.

Since we are working in D dimensions, the Clifford algebra has $2^{\lfloor \frac{D}{2} \rfloor}$ spinor components. For even D, this equals $2^{\frac{D}{2}}$ and for odd D, $2^{\frac{D-1}{2}}$. Since the components are complex, there are $2.2^{\lfloor \frac{D}{2} \rfloor}$ degrees of freedom for each vector index. Imposing the Majorana condition $\psi = \psi^c = C\Gamma^0\psi$ where C is the charge conjugation matrix gives us $2^{\lfloor \frac{D}{2} \rfloor}$ constraints and we have $2^{\lfloor \frac{D}{2} \rfloor}$ degrees of freedom per vector index left. Now consider the massless Rarita Schwinger action for spin-$\frac{3}{2}$ particles:

$$S_{RS} = \int d^D x \bar{\Psi}_M \Gamma^{MNP} \partial_N \Psi_P. \tag{3.1}$$

The massless equations of motion are:

$$\Gamma^{MNP} \partial_N \Psi_P = 0. \tag{3.2}$$

Using the gamma-matrix identities $\Gamma_M \Gamma^{MNP} = (D-2)\Gamma^{NP}$ and $\Gamma^{MNP} = \Gamma^M \Gamma^{NP} - 2\eta^{M[N}\Gamma^{P]}$, 3.2 becomes,

$$\Gamma^M (\partial_M \Psi_N - \partial_N \Psi_M) = 0. \tag{3.3}$$

Taking the derivative ∂_P on both sides, we have,

$$\not{\partial}(\partial_P \Psi_N - \partial_N \Psi_P) = 0 \tag{3.4}$$

This gives us $2^{\lfloor \frac{D}{2} \rfloor}$ components. Now we fix the gauge $\Gamma^i \Psi_i = 0$ (Coulomb gauge). The $N = 0$ and $N = i$ components are given by,

$$\begin{aligned} \Gamma^i \partial_i \Psi_0 - \partial_0 \Gamma^i \Psi_i = 0, \\ \text{and } \Gamma \cdot \partial \Psi_i - \partial_i \Gamma \cdot . \Psi = 0 \end{aligned} \tag{3.5}$$

When we set the Coulomb gauge condition on the first of 3.5, we get $\nabla^2\Psi_0 = 0 \implies \Psi_0 = 0$. For the spatial components of Ψ_i, we get $\Gamma \cdot \partial\Psi_i - \partial_i\Gamma^0\cancelto{0}{\Psi_0} - \partial_i\cancelto{0}{\Gamma^j\Psi_j} = 0$ or $\Gamma \cdot \partial\Psi_i = 0$. Contracting this equation with Γ^i, we end up with $2\eta^{i\mu}\partial_\mu\Psi_i - \Gamma^\mu\partial_\mu\cancelto{0}{\partial^i\Psi_i} = 0 \implies \partial^i\Psi_i = 0$. Hence we end up with three independent constraints:

$$\begin{aligned}
&\Gamma^i\Psi_i(\underline{\mathrm{x}}, 0) = 0 \\
&\Psi_0(\underline{\mathrm{x}}, 0) = 0 \\
&\partial^i\Psi_i(\underline{\mathrm{x}}, 0) = 0
\end{aligned} \tag{3.6}$$

Because of these three initial conditions, we have $(D-3)2^{\lfloor \frac{D}{2} \rfloor}$ total components. This leads to $\frac{1}{2}(D-3)2^{\lfloor \frac{D}{2} \rfloor}$ on-shell degrees of freedom. In 11 dimensions, this works out to be 128 degrees of freedom. Thus the full action had $128+128 = 256$ degrees of freedom.

$$S = \frac{1}{2}\int \underbrace{d^{11}x}_{\text{11-form}} \sqrt{G}\Big(\underbrace{R}_{\text{curvature scalar}} + |dA_3|^2\Big) + \int \underbrace{A_3}_{\text{3-form}} \wedge \underbrace{dA_3}_{\text{4-form}} \wedge \underbrace{dA_3}_{\text{4-form}} \tag{3.7}$$

The fermionic terms are obtained by supersymmetry considerations.

3.2 Dimensional Reduction to IIA Supergravity in 10 Dimensions

We can dimensionally reduce the 11D supergravity action to a 10D IIA action by compactifying the x^{11} directon on a circle and assuming that nothing depends on x^{11}. We also take only the zero modes in the Fourier expansion of the fields. This latter fact distinguishes dimensional reduction from *compactification*, in which all the Fourier modes are accounted for.

First we see what happens to the fermionic part. We use Greek indices for 10-D Type IIA Supergravity. The Majorana gravitino in 11 dimensions $\Psi_M \equiv \begin{pmatrix}\psi^1_M & \psi^2_M\end{pmatrix}^{\mathrm{T}}$ gives rise to a pair of Majorna-Weyl gravitinos ψ^a_μ and a pair of Majorana- Weyl dilatinos in 10 dimensions, $\psi^a \equiv \psi^a_{11}, a = 1, 2$. To calculate the degrees of freedom of each Majorana-Weyl gravitino, we carry out a calculation analogous to the calculation for Majorana gravitino in 11 dimensions. In this case, everything is identical except for the fact that for each gravitino, the Majorana-Weyl condition has 8 dimensional spinors and the vector indices can take on 10 possible values. Therefore, each Majorana-Weyl gravitino has 56 degrees of freedom. Furthermore, each Majorana-Weyl dilatino has 8 degrees of freedom and we end up with the same $56\times 2+8\times 2 = 128$ degrees of freedom.

Now consider the bosonic degrees of freedom. We still have the three form field $A_{\mu\nu\rho}$ in ten dimensions. This gives us $\frac{1}{3!}(8)(7)(6) = 56$ dofs and $A_{\mu\nu 11} \to B_{\mu\nu}$

(Kalb Ramond), which has $\frac{1}{2}(8)(7) = 28$ dofs. The 11-dimensional metric G_{MN} becomes a 10-dimensional metric $G_{\mu\nu}$, which has $\frac{1}{2}(10-2)(10-1)-1 = 35$ dofs, $G_{11\ 11} \equiv \exp(\frac{-4\phi}{3})$ (1 dof) and $G_{\mu 11} \equiv -\exp(\frac{4\phi}{3})A_\mu$ (8 dofs). In total, we have 128 bosonic degrees of freedom and combined with the 128 fermionic dofs, we get the 256 dofs of 11 dimensional supergravity.

The 11D metric in terms of the 10D metric $G_{\mu\nu}$ is given by (Note that $G_{\mu\nu}$ is NOT the same as G_{MN} with $M = \mu$ and $N = \nu$),

$$G_{MN} = \exp\left(\frac{-2\phi}{3}\right)\left(\begin{array}{c|c} g_{\mu\nu} + \exp(2\phi)A_\mu A_\nu & \exp(2\phi)A_\mu \\ \hline \exp(2\phi)A_\nu & \exp(2\phi) \end{array}\right). \tag{3.8}$$

The line element is given by

$$\begin{aligned} ds^2 = G_{MN}dx^M dx^N =& G_{\mu\nu}dx^\mu dx^\nu + \exp\left(\frac{2\phi}{3}\right)(dx^{11})^2 \\ &+ 2\exp\left(\frac{2\phi}{3}\right)A_\mu dx^\mu dx^{11} + \exp\left(\frac{2\phi}{3}\right)(A_\mu dx^\mu)^2, \\ =& G_{\mu\nu}dx^\mu dx^\nu + \exp\left(\frac{4\phi}{3}\right)\Big(dx^{11} + A_\mu dx^\mu\Big)^2. \end{aligned} \tag{3.9}$$

Hence, x^{11} is compactified on a circle of radius $\exp(2\phi/3)$. Integrating it out yields a factor of 2π. Using the expression for the determinant of a block matrix, you have $\det G_{MN} = \exp(\frac{4\phi}{3})\det G_{\mu\nu}$. Hence we have:

$$\int d^{11}x\sqrt{\det G_{MN}}\cdots = 2\pi\int d^{10}x\exp\left(\frac{2\phi}{3}\right)\sqrt{\det G_{\mu\nu}}\ldots \tag{3.10}$$

The bosonic part of 10D Type IIA SUGRA is then (we are not concerned with the exact numerical factors),

$$\begin{aligned} &\int d^{10}x\sqrt{\det G_{\mu\nu}}\Big[\exp\left(\frac{2\phi}{3}\right)(R + |\partial_\mu\phi\partial^\mu\phi| + |dA_3|^2) + \exp\big(2\phi\big)|dA|^2 \\ &+ \exp\left(\frac{-2\phi}{3}\right)|dB|^2\Big] + \int B\wedge dA_3\wedge dA_3. \end{aligned} \tag{3.11}$$

Now a word about the exponents.The square root of the determinant gives us a factor of $\exp\left(\frac{2\phi}{3}\right)$. From the Ricci scalar, we have terms of the form $G^{1111}\partial_\mu G_{\nu 11}\partial_\rho G_{\sigma 11}\ \exp\left(-\frac{4\phi}{3}\right)\exp\left(\frac{4\phi}{3}\right)\exp\left(\frac{4\phi}{3}\right)\partial_\mu A_\nu\partial_\rho A_\sigma + \partial\phi$ terms. This gives us a term of the form $\exp\left(\frac{4\phi}{3}\right)\partial A\partial A$ which combines with the factor from the determinant to give you $\exp(2\phi)$ in front of $|dA|^2$. Also, since $dA_3 = \partial_{[M}A_{NPQ]}$, we get terms like $G^{1111}\partial A_{\mu\nu 11}\partial A_{\rho\sigma 11}\ \exp\Big(-\frac{4\phi}{3}\Big)|dB|^2$. Combine this with the factor from the determinant and we get $\exp(-\frac{2\phi}{3})$.

However, the action for bosonic part of Type IIA gravity is typically written as,

$$\int d^{10}x\sqrt{g}\Big[\exp\big(-2\phi\big)(R+|\partial_\mu\phi\partial^\mu\phi|+|dB|^2)+|dA_3|^2+|dA|^2\Big]+\int B\wedge dA_3\wedge dA_3. \tag{3.12}$$

In order to bring the action to this form, we must perform a Weyl rescaling $G_{\mu\nu} = \exp\Big(-\frac{2\phi}{3}\Big)g_{\mu\nu}$. This gives us a factor of $\exp(-\frac{10\phi}{3})$ because the metric is 10-D and we are taking a square root. The Ricci scalar scales as follows: $R[G_{10}] = \exp\big(\frac{2\phi}{3}\big)R[g]+ \sim |\partial_\mu\phi\partial^\mu\phi|$. This results in an exponential to the power of $\frac{2\phi}{3} - \frac{10\phi}{3} + \frac{2\phi}{3} = -2\phi$. Thus, we have the correct exponential in front of the Ricci scalar. The terms of the form $|dA_p|^2$ yield factors of $\exp\Big((p+1)\frac{2\phi}{3}\Big)$.These exactly cancel out the prefactors of $|dA_3|^2$ and $|dA|^2$ in (3.11).

3.3 String coupling and radius

Consider the metric (3.9). In terms of the rescaled 10-D metric $g_{\mu\nu}$ and the string coupling $g_s = \exp\phi$, we may write this as,

$$ds^2 = g_s^{-\frac{2}{3}}g_{\mu\nu}dx^\mu dx^\nu + g_s^{\frac{4}{3}}(dx^{11} + A_\mu dx^\mu)^2. \tag{3.13}$$

From this metric, we see that the relationship between the 11D Planck length ℓ_p and the 10D string length ℓ_s is,

$$\ell_p = g_s^{1/3}\ell_s, \text{ where } \ell_s = \sqrt{\alpha'}. \tag{3.14}$$

Newton's constant in 11 and 10 dimensions is given as, [4]

$$G_{11} = 16\pi^7\ell_p^9 \text{ and } G_{10} = 8\pi^6\ell_p^8 g_s^{-5/3}. \tag{3.15}$$

When we compactify in the 11 direction with radius R_{11}, [8],

$$G_{11} = 2\pi R_{11}G_{10}. \tag{3.16}$$

Combining the last 3 equations, the 11-D radius is given by

$$R_{11} = g_s^{2/3}\ell_p = g_s\ell_s. \tag{3.17}$$

The nonperturbative excitations of of Type IIA superstring theory are D0 branes with mass (in the string frame) is $\frac{1}{\ell_s g_s}$. This can be interpreted from the viewpoint of M-theory compactified on a circle as the first Kaluza Klein excitation of the massless supergravity multiplet. There are 256 degrees of freedom that correspond to the $2^{\frac{16}{2}}$ dimensional Clifford algebra of the Majorana Weyl fermions θ of the D0 brane. This 256 dimensional multiplet is sometimes referred to as the *supergraviton*.

In 11 dimensions, the mass of the supergraviton is zero,

$$M_{11}^2 = -p_M p^M = 0, M = 0, 1, \ldots, 9, 11. \tag{3.18}$$

This leads to $M_{11}^2 = -p_M p^M = 0 = -p_\mu p^\mu - p_{11}^2 = M_{10}^2 - p_{11}^2$, where M_{10} is the 10 dimensional mass and $\mu = 0, 1, \ldots, 9$. Thus, we have

$$M_{10}^2 = p_{11}^2 \tag{3.19}$$

Because we have quantized in the 11 direction, the corresponding momentum is $p_{11} = \frac{N}{R_{11}}, N \in \mathbb{Z}$ and the masses of the tower of Kaluza Klein excitations is

$$M_N = \frac{N}{R_{11}} \tag{3.20}$$

If we set $N = 1$ and $R_{11} = g_S \ell_S$, we get the D0-brane mass. We can also see this from the fact that $D0$ branes saturate the BPS bound, i.e. $M = |Z|$, where $|Z|$ is the central charge of the $\mathcal{N} = 2$ supersymmetry algebra. For the $D0$ brane, the central charge $Z = \tau_0 = \frac{1}{\ell_s g_s}$ or the brane tension. Since these type IIA states are BPS states, they come in short supermultiplets of $2^8 = 256$ states.

Witten showed that a system of N D0 branes has a threshold bound state of mass $\frac{N}{\ell_s g_s}$. There is no binding energy left since the D0 branes saturate the BPS state, i.e. they are as light as they are allowed to be if they carry N units of $U(1)$ charge that couple to the one form field A in the IIA action.

Consider what happens as $g_s \to \infty$. The states become very light and we get infinitely many light states, which is the Kaluza Klein spectrum of 11D supergravity. Furthermore, in this case, the compactified dimension of radius $R_{11} = g_s \ell_s$ becomes large and we get back the full 11 dimensional supergravity theory. Thus, *all* the Kaluza Klein states of 11 D supergravity on $\mathbb{M}^{10} \times S^1$ is contained in the type IIA supergravity. Clearly, 11 dimensional supergravity is the strong coupling, low energy limit of Type IIA superstring theory. This is the S duality between 11D SUGRA and IIA superstring theory.

11D supergravity, however, is not a consistent quantum theory. Beyond two loops, the scattering amplitude of two gravitons diverges. Thus, it must be the low energy limit of another theory, called M-theory, which generalizes SUGRA beyond the UV cutoff. Furthermore, M theory with its 11^{th} dimension compactified on a circle of radius R_{11} must be Type IIA superstring theory with coupling $g_s = \frac{R_{11}}{\sqrt{\alpha'}}$. Because it describes type IIA superstrings, M theory must contain D0, D2, D4. D6, D8 branes and the fundamental string, the F1 brane from IIA string theory. The correspondence between objects in M theory and type IIA superstring theory are given in 1.

M-Theory	Type IIA
KK Photon ($g_{\mu 11}$)	RR gauge field A_μ
supergraviton with $p_{11} = \frac{1}{R_{11}}$	D0 brane
wrapped membrane	IIA string
unwrapped membrane	IIA D2 - brane
wrapped 5-brane	IIA D4-brane
unwrapped 5-brane	IIA NS 5-brane (charged under $B_{\mu\nu}$)

Table 1: Correspondence between M-theory and Type IIA superstring theory

How do we account for D6 branes? Magnetically, the D0 brane is the dual of the D6 brane. Since the D0 brane couples electrically to the RR field A_μ, the D6 brane must couple magnetically to the same gauge field. Correspondence with the D8 brane is still an open question.

The three form gauge field A_3 couples electrically to the M2 brane and magnetically to the $\mathrm{M}(11-2-4) = \mathrm{M5}$ brane. The tensions in the M2 and M5 branes are (we will show this later using M(atrix) theory),

$$T_{M2} = 2\pi(2\pi\ell_p)^{-3} \text{ and } T_{M5} = 2\pi(2\pi\ell_p)^{-6} \tag{3.21}$$

Since tension is energy per unit area and compactifying the M2 brane in the 11 direction gives us the F1 string, the tension in the F1 string is

$$T_{F1} = 2\pi R_{11} T_{M2} = \frac{1}{2\pi\ell_s^2}, \text{ where } R_{11} = \ell_s g_s \text{ and } \ell_p = g_s^{1/3}\ell_s. \tag{3.22}$$

4 BFSS Conjecture

Let us recap what we have discussed so far,

I) If we take the large coupling limit of Type IIA superstring theory the compactified dimension becomes large and the 11th dimension appears. The Kaluza Klein states of 11D supergravity correspond to threshold bound states of n D0 branes.

II) The collection of N D0 branes can be described by a $U(N)$ super Yang Mills theory dimensionally reduced to $0 + 1$ dimensions, as in 2.34.

In the BFSS conjecture, the 9 *space* dimensions ($X^i, i = 1, \dots 9$) on which the D0 branes live are the transverse directions in the light cone frame, also known as the infinite momentum frame (IMF). We will call the momentum in the 9 transverse directions $p_\perp$. Thus, the total momentum is given by $(p^0, p_\perp, p^{11})$. The statement of the BFSS conjecture is,

"*M-theory, in the infinite momentum frame is exactly described by the large N limit of $U(N)$ matrix quantum mechanics.*"

4.1 The Infinite Momentum Frame

The infinite momentum frame was introduced by Weinberg [11] to simplify calculations in perturbative quantum field theory. If we have a collection of particles labeled a, the individual momenta can be written as,

$$p_a = \eta_a P + p_\perp^a \tag{4.1}$$

where P is the total momentum of the system. Now $P \equiv \sum p_a = P \sum \eta_a + \sum p_\perp^a \implies \sum \eta_a = 1, \sum p_\perp^a = 0$. Furthermore, (no sum over a), $p_\perp^a . p_a = \eta_a p_\perp^a . P + p_\perp^a . p_\perp^a \implies p_\perp^a . (p_a - p_\perp^a) = \eta_a p_\perp^a . P$. The LHS in the last equation is 0 since subtracting $p_\perp^a$ from p_a leaves only the longitudinal component of the momentum. Therefore, $p_\perp^a . P = 0$.

Now consider a large boost in the P direction, i.e. $P \to \infty$. The observer is moving in the $-P$ direction. Weinberg [11] showed, in the perturbative formalism, Feynman diagrams with internal $\eta > 0$, contribute finitely. If an internal $\eta < 0$, the diagram does not contribute anything because it is suppressed by a factor of $\frac{1}{P^2}$. Such diagrams correspond to physical situations in which multiple particles are created from a vacuum or destroyed into a vacuum. Thus, we will consider $\eta > 0$ only.

The total energy of a particle is then given by

$$E_a = \sqrt{p_a^2 + m_a^2} = \sqrt{(p_{\perp,a} + \eta_a P)^2 + m_a^2} = \eta_a P + \frac{(p_\perp^a)^2 + m_a^2}{2\eta_a P} + \mathcal{O}(P^{-2}). \tag{4.2}$$

This is basically the non-relativistic $d-2$ dimensional energy $\frac{(p_\perp^a)^2}{2\eta_a P}+$ constant terms, with mass $\eta_a P$.

We now move to the IMF frame in which we boost in the 11, longitudinal direction The other spatial (transverse) components of the momentum are $p_i, i = 1 \dots 9$. We will use $p_\perp$ to refer to these transverse components. Now we compactify the x_{11} direction on a circle of radius R. Thus, we have $p_{11}^a = \frac{n_a}{R}$ with $n_a > 0$. The mass shell relation is then

$$\begin{aligned} E &= \sum_a \sqrt{p_a^2 + m_a^2} = \sum_a \sqrt{p_{\perp,a}^2 + p_{11,a}^2 + m_a^2} = \sum_a p_{11,a}\sqrt{1 + \frac{p_{\perp,a}^2 + m_a^2}{p_{11,a}^2}}, \\ &\approx \sum_a p_{11,a}\Big(1 + \frac{p_{\perp,a}^2 + m_a^2}{2p_{11,a}^2}\Big). \end{aligned} \tag{4.3}$$

For massless particles, the mass shell relation is then

$$E - p_{11(\mathrm{tot})} = \sum_a \frac{(p_{\perp,a})^2}{2p_{11,a}}. \tag{4.4}$$

This is reflective of the non-relativistic structure. The Galilean transformation takes the form

$$p_\perp \to p_\perp + p_{11}v_\perp. \tag{4.5}$$

The Infinite Momentum Frame therefore confers the following advantages:

1. Because of the Galilean energy momentum relationship, we can näively use concepts such as wavefunctions, bound states and mass conservation.

2. $p_{11,a}$ is not negative because of the large boost in the 11 direction means that modes with negative p_{11} are very high enegy and will be integrated out in a low energy effective action. This is good news since we don't have to deal with anti D0 branes ($N < 0$) and perturbative states ($N = 0$).

The 32 real supersymmetry generators split up into two groups of 16 : Q_α, q_A with $\alpha, A = 1, \dots 16$. Each of them transforms as a spinor of $SO(9)$.

$$\{Q_\alpha, Q_\beta\} = \delta_{\alpha\beta}, \{q_A, q_B\} = \delta_{AB}p_{11}, \{Q_\alpha, q_A\} = \gamma^i_{A\alpha}p_i. \tag{4.6}$$

For the purposes of this thesis, we will also be using another framework for M -theory: Discrete Light Cone Quantization (DLCQ) [12]. In summary (details to follow) we switch to light cone coordinates and set $x^+ = \tau$ (timelike direction), x^- = transverse direction, and x^i the longitudinal 9 dimensions. Then we compactify x^- on a circle of radius R. Under DLCQ, we are allowed to use the $U(N)$ SYM action 2.33 for finite N.

4.2 Matrix Model Hamiltonian

The action for $ND0$ branes is given by 2.34. We set $c = \frac{1}{2\pi\alpha'}, T_0 = \frac{1}{\sqrt{\alpha'}} = \frac{1}{\ell_s}$. X^i and θ transform in the adjoint representation of $U(N)$. We make a gauge choice $A_0 = 0$ and thus the kinetic term becomes $\int dt \frac{M}{2}(\frac{dX^i}{dt})^2$, where $M = \frac{T_0}{g_s}$ is the D0 brane mass. $\because$ we Weyl-rescaled the metric, we Weyl rescale the fields as $X^i = g_s^{1/3} Y^i$ and $t = g_s^{1/3}\ell_p\tilde{\tau} = g_s^{2/3}\ell_s\tilde{\tau} = \frac{g_s\ell_s}{g_s^{1/3}}\tilde{\tau} = \frac{R_{11}}{g_s^{1/3}}\tilde{\tau}$. But we need the Hamiltonian to have dimensions of energy or inverse length. Therefore, we have

$$t = g_s^{2/3}\tau = \frac{T_0 R_{11}}{g_s^{1/3}}\tau. \tag{4.7}$$

We denote $\dot{Y} \equiv \frac{\partial Y}{\partial \tau}$. The action is then

$$S = T_0^2 \int d\tau \,\mathrm{Tr}\Big(\frac{1}{2R_{11}T_0^2}(\dot{Y}^i)^2 - i\frac{1}{T_0}\theta^T\dot{\theta} + \frac{c^2 R_{11}}{4}([Y^i, Y^j])^2 + cR_{11}\theta^T\gamma^j[Y_j,\theta]\Big). \tag{4.8}$$

The momentum conjugate to Y^i and θ are given by $\Pi = \frac{\partial \mathcal{L}}{\partial \dot{Y}^i} = \frac{\dot{Y}^i}{R_{11}}$ and $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = -iT_0\theta^T$. The Hamiltonian is, therefore,

$$\mathcal{H} = R_{11}\mathrm{tr}\Big(\frac{1}{2}\Pi_i^2 - \frac{c^2T_0^2}{4}([Y^i,Y^j])^2 - cT_0^2\theta^T\gamma^j[Y_j,\theta]\Big) \equiv R_{11}\tilde{\mathcal{H}} \tag{4.9}$$

Consider the Higgs potential term $V(Y) = -\frac{1}{4}R_{11}c^2T_0^2\,\mathrm{Tr}([Y^i,Y^j])^2$ $= \frac{1}{4}R_{11}c^2T_0^2\,\mathrm{Tr}(i[Y^i,Y^j])(i[Y^i,Y^j])$. Since Y^i is Hermitian, so is $i[Y^i,Y^j]$ and $(i[Y^i,Y^j])^2$. Thus the, trace is non-negative and so is $V(Y)$.

As the compactified dimension becomes large ($R_{11} \to \infty$), the finite energy eigenvalues of the Hamiltonian correspond to vanishing $\tilde{\mathcal{H}}$ energy. In other words, we are looking for states for which

$$\tilde{\mathcal{H}}\,|\psi\rangle = \frac{\epsilon}{N}\,|\psi\rangle \iff \mathcal{H}\,|\psi\rangle = \frac{R_{11}\epsilon}{N}\,|\psi\rangle \tag{4.10}$$

where ϵ is a finite number. The minima for $V(Y)$ is for $[Y^i,Y^j] = 0$, for which the $Y^i = \mathrm{diag}(y_1^i, y_2^i, \dots y_N^i)$ - y_a^i is the i^{th} coordinate of the a^{th} D0 brane.

If the D0 branes are far apart, then the matrices Y^i commute, to a good approximation. In this case, being far from each other costs a lot of energy. As the branes get closer, non commutativity becomes more important. We will justify these statements now.

Consider a configuration in which Y^i is an $N \times N$ matrix in block-diagonal form, which n blocks of size $N_1, N_2 \dots N_n$ with $\sum_a N_a = N$. Each block corresponds to a cluster of N_aD0 branes and the distance between the clusters labeled a and b is,

$$r_{ab} = \Big|\frac{\mathrm{tr}Y_a}{N_a} - \frac{\mathrm{tr}Y_b}{N_b}\Big| = \Big[\sum_{i=1}^{9}\Big(\frac{\mathrm{tr}Y_a^i}{N_a} - \frac{\mathrm{tr}Y_b^i}{N_b}\Big)^2\Big]^{1/2}. \tag{4.11}$$

Now consider two clusters of D0 branes with $N_a = N_b = 1, N = 2$. Let $Y^i = \begin{pmatrix} \alpha^i & \beta^i \\ \beta^{*i} & \delta^i \end{pmatrix} \implies r_{1,2} = [\sum_{i=1}^{9} (\alpha^i - \delta^i)^2]^{1/2}$. This is large if, say, for any $i = i_0, |\alpha_{i_0} - \delta_{i_0}|$ is large, $|\alpha_{i_0} - \delta_{i_0}| \geq \frac{1}{3} r_{1,2}$. Then, we have,

$$-\frac{1}{2}\mathrm{tr}([Y^{i_0}, Y_j])^2 = 4\,\mathrm{Im}(\beta_i \beta_j^*)^2 + \left|\beta_j(\alpha_{i_0} - \delta_{i_0}) - \beta_{i_0}(\alpha_j - \delta_j)\right|^2. \tag{4.12}$$

This is, in general, of the order of

$$|\beta_j|^2(\alpha_{i_0} - \delta_{i_0})^2 \geq \frac{1}{9}|\beta_j|^2 r_{1,2}^2. \tag{4.13}$$

This idea generalizes to larger block diagonal matrices. $\mathrm{tr}([Y^i, Y^j])^2$ is at least of the order of $r_{ab}^2 \times$ off-diagonal elements. Therefore, either the off-diagonal elements or the distances between clusters must be small in order to limit the size of the Higgs potential term.

4.3 The spectrum of $\mathcal{H}$

A D0 brane carries longitudinal momentum $\frac{1}{R_{11}}$. A *supergraviton* of longitudinal momentum $\frac{N}{R_{11}}$ corresponds to bound states at threshold of $ND0$ branes.The 256 dimensional representation of the supersymmetry algebra corresponds exactly to the $44 + 84 + 128 = 256$ Kaluza Klein modes that we have seen before.

For $N > 1$, we separate our coordinates into center of mass and relative coordinates:

$$\begin{aligned} Y^i = Y^i_{rel} + Y^i_{cm}\mathbf{1}, \quad Y^i_{cm} = \frac{1}{N}\mathrm{tr}Y^i, \\ \text{and } \Pi^i = \Pi^i_{rel} + \frac{1}{N}P^i_{cm}\mathbf{1}, \quad P^i_{cm} = \mathrm{tr}\Pi^i. \end{aligned} \tag{4.14}$$

with $\mathrm{tr}Y^i_{rel} = \mathrm{tr}\Pi_{rel} = 0$. We plug these expressions into the Hamiltonian 4.9 to obtain

$$\mathcal{H} = \mathcal{H}_{cm} + \mathcal{H}_{rel}. \tag{4.15}$$

Where $\mathcal{H}_{cm} = \frac{R_{11}}{2N}(P_i^{cm})^2 = \frac{1}{2p_{11}}(P_i^{cm})^2$. The relative part of the Hamiltonian looks just like the original Hamiltonian except that the matrices are $SU(N)$ traceless. Witten showed [14] that duality between type IIA string theory and M theory requires the relative Hamiltonian Schrodinger equation $\mathcal{H}_{rel}\left|\psi\right\rangle = E_{rel}\left|\psi\right\rangle$ must have zero energy normalizable threshold bound states. Thus, the total energy is the total center-of-mass energy:

$$E = E_{cm} = \frac{R_{11}}{2N}(p_\perp^{cm})^2 = \frac{1}{2p_{11}}(p_\perp^{cm})^2, \tag{4.16}$$

which is a full supergravity multiplet of 256 states. For any N, the spectrum contains single supergraviton states of momentum $p_{11} = \frac{N}{R_{11}}$.

Now consider what happens when we have block diagonal matrices Y^i.

$$Y^i = \begin{pmatrix} Y_1^i & 0 & 0 & \dots \\ 0 & Y_2^i & 0 & \dots \\ 0 & 0 & Y_3^i & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.17}$$

Each Y_a^i is an $N_a \times N_a$ matrix, where $\sum_{a=1}^{n} N_a = N$. The Schrödinger equation decouples into n different uncoupled Schrodinger equations, with N degrees of freedom. Interactions come about via the off-diagonal terms.

In the next chapter, we will look at some reasons why this conjecture may be correct. In particular, we will see that the effective potential of two supergravitons scattering at transverse velocity v and impact parameter b, at one loop, is $V(r) = -\frac{15v^4}{16r^7}$, a result confirmed by 11-dimensional supergravity [4]. Here. $r = \sqrt{b^2 + (v\tau)^2}$. Furthermore, we will see that, at two loops, no renormalization of the v^4 term of the effective potential occur. Finally, we will use M(atrix) theory to calculate the M2 brane tension and compare this with the result from M theory.

5 M(atrix) Theory

In this chapter, we will verify three correspondences between M theory and M(atrix) theory, using the ten-dimensional Super Yang Mills theory, dimensionally reduced to $(0+1)$ dimensions: the effective (velocity dependent) scattering of two D0 branes (supergravitons) at one and two loops and the membrane tension. We will verify the coefficient of the $\frac{v^4}{r^7}$ term at one loop. We will then show that,at two loops, the coefficient of $\frac{v^4}{r^{10}}$ disappears. Lastly, we will show that the membrane tension, T_{M2}, is given by 3.21.

5.1 Background Field Method

The background field method [15] and [16] is an efficient way to calculate the effective action while retaining explicit gauge invariance.In quantum field theory, the most important quantity is the S-matrix. This can be obtained from the Green's functions using the LSZ reduction formula. The generating functional for a quantum field theory is defined as

$$Z[J] = \int \mathcal{D}Q \exp i(S[Q] + J.Q), \tag{5.1}$$

where $J.Q \equiv \int d^d x J(x) Q(x)$. By taking successive functional derivatives of Z with respect to J, one obtains Green's functions with higher numbers of endpoints. Now consider the generating functional $W = -i \log Z$. This can be shown to generate Feynman graphs that are connected, i.e. graphs in which all vertices and propagators are linked. Now, it simplifies matters a great deal if one expresses the connected graphs in terms of 1PI subgraphs plus connected pieces. A 1PI (one-particle irreducible) diagram is a diagram that stays connected even when an internal line is cut. It can be shown that the generator of 1PI subgraphs, $\Gamma[\bar{Q}]$ is the Legendre transformation of $W[J]$, i.e.

$$\Gamma[\bar{Q}] = W[J] - J.\bar{Q}, \text{ with } \bar{Q} := \frac{\delta W[J]}{\delta J(x)}, \tag{5.2}$$

Successive functional derivatives of $\Gamma[\bar{Q}]$ with respect to $\bar{Q}$ generate 1PI diagrams.

Let us now define a quantity analogous to 5.1 for disconnected graphs:

$$\tilde{Z}[J, \phi] = \int \mathcal{D}Q \exp i(S[Q + \phi] + J.Q) \tag{5.3}$$

We call ϕ the *background field* . By analogy with the generator for connected graphs, we define

$$\tilde{W}[J, \phi] = -i \log \tilde{Z}[J, \phi] \tag{5.4}$$

and the background field effective action as,

$$\tilde{\Gamma}[\tilde{Q}, \phi] = \tilde{W}[J, \phi] - J.\tilde{Q}, \text{ where } \tilde{Q} = \frac{\delta \tilde{W}}{\delta J}. \tag{5.5}$$

To see the relationship between the quantities with tildes and those without, make the substitution $Q \to Q - \phi$ in 5.3. Performing the integration over Q, we have

$$\tilde{Z}[J, \phi] = Z[J] \exp(-iJ.\phi). \tag{5.6}$$

Taking the natural logarithm on both sides gives us,

$$\tilde{W}[J, \phi] = W[J] - J.\phi. \tag{5.7}$$

Taking the functional derivative of both sides of 5.7 with respect to J and invoking the definitions of $\tilde{Q}$ and $\bar{Q}$ yield

$$\tilde{Q} = \bar{Q} - \phi. \tag{5.8}$$

Using the above relation between $\bar{Q}$ and $\tilde{Q}$ and the definitions of $\tilde{\Gamma}$, 5.5 and 5.2, the relationship between Γ and $\tilde{\Gamma}$ is,

$$\tilde{\Gamma}[\tilde{Q}, \phi] = W[J] - J.\phi - J.\bar{Q} + J.\phi = W[J] - J.\bar{Q} = \Gamma[\tilde{Q} + \phi]. \tag{5.9}$$

If one sets $\tilde{Q} = 0$,

$$\tilde{\Gamma}[0, \phi] = \Gamma[\phi]. \tag{5.10}$$

In other words, the background field effective action is a regular effective action with the presence of the background field as a source, ϕ. Here we see an advantage of the background field method that massively simplifies computations: The n-point 1PI Green's function is the n^{th} functional derivative of $\tilde{\Gamma}[\tilde{Q}, \phi]$ with respect to $\tilde{Q}$. Since we have made $\tilde{Q}$ constant, the Green's functions are all 0, the Feynman graphs that this generates are all vacuum diagrams, i.e. they are all without external lines.

How do we calculate $\tilde{\Gamma}[0, \phi]$? There are two ways to do this:

1. We treat the background field ϕ exactly, then read off the Feynman rules from the shifted action $S[Q + \phi]$. Then we sum up all the 1PI graphs using the Feynman rules. We follow this first approach in our thesis.

2. We treat the background field perturbatively. The ϕ fields show up as external lines. We use the shifted action $S[Q + \phi]$ to generate Feynman rules from the quadratic parts of $S[Q + \phi]$ and interactions from higher powers. Vertices corresponding to powers of Q have internal lines between them. The ϕ fields generate external lines.

5.1.1 Background Field Method for Gauge Fields

Consider the gauge theory with gauge-fixing G,

$$Z[J] = \int \mathcal{D}Q \det \left[\frac{\delta G^a}{\delta \omega^b} \right] \exp i[S[Q] - \underbrace{\frac{1}{2\alpha} G.G}_{\text{gauge fixing term}} + J.Q]. \tag{5.11}$$

Here, $S = -\frac{1}{4} \int d^d x (F^a_{\mu\nu})^2$ and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} Q^b_\mu Q^c_\nu$ where the f^{abc} are the structure constants of the generators of the gauge group. Furthermore, the term $J.Q := \int d^d x J^a_\mu(x) Q^a_\mu(x)$ and the gauge fixing term $G.G = \int d^d x G^a(x) G^a(x)$. $\frac{\delta G^a}{\delta \omega^b}$ is the functional derivative of the gauge with respect to a gauge transformation: $\delta Q^a_\mu = -f^{abc} \omega^b Q^c_\mu + \frac{1}{g} \partial_\mu \omega^a$.

We define a quantity $\tilde{Z}[J, A]$ by analogy with 5.3 for gauge fields:

$$\tilde{Z}[J, A] = \int \mathcal{D}Q \det \left[\frac{\delta \tilde{G}^a}{\delta \omega^b} \right] \exp i[S[Q + A] - \underbrace{\frac{1}{2\alpha} \tilde{G}.\tilde{G}}_{\text{gauge fixing term}} + J.Q]. \tag{5.12}$$

Here, $\frac{\delta \tilde{G}^a}{\delta \omega^b}$ is the functional derivative of the gauge with respect to a gauge transformation: $\delta Q^a_\mu = -f^{abc} \omega^b (Q^c_\mu + A^c_\mu) + \frac{1}{g} \partial_\mu \omega^a$. Just as with the scalar fields, making the shift of variables $Q^a_\mu \to Q^a_\mu - A^a_\mu$ in the functional integral 5.12 and following through with the procedure for calculate the background field effective action,

$$\tilde{\Gamma}[0, A] = \Gamma[A]. \tag{5.13}$$

If we calculate $\tilde{\Gamma}[0, A]$ using the gauge-fixing $\tilde{G}^a = \tilde{G}^a(Q, A)$, you get the conventional effective action $\Gamma[\bar{Q}]$ with $G^a = \tilde{G}^a(Q - A, A)|_{\bar{Q}=A}$. For the following background gauge choice, the background field effective action $\tilde{\Gamma}[0, A]$ is a gauge invariant functional of A.

$$\tilde{G}^a = \partial_\mu Q^a_\mu + g f^{abc} A^b_\mu . Q^c_\mu . \tag{5.14}$$

To see this, we only need to prove that the background field generating functional 5.12 is invariant under the following transformations,

$$\delta A^a_\mu = -f^{abc} \omega^b A^c_\mu + \frac{1}{g} \partial_\mu \omega^a \text{ and } \delta J^a_\mu = -f^{abc} \omega^b J^c_\mu . \tag{5.15}$$

Then make the change of variables $Q^a_\mu = Q^a_\mu - f^{abc} \omega^b Q^c_\mu$ to get,

$$\delta(Q^a_\mu + A^a_\mu) = -f^{abc} \omega^b (Q^c_\mu + A^c_\mu) + \frac{1}{g} \partial_\mu \omega^a . \tag{5.16}$$

Now, 5.16 is a gauge transformation, we expect $S[Q + A]$ to be invariant under this transformation. By definition, the gauge fixing term is invariant under a gauge transformation. Furthermore, it can be shown that the determinant of the derivative is invariant under these transformations. This leads to the fact that $\tilde{\Gamma}[0, A]$ is invariant under gauge transformations.

5.2 SYM action in (0+1) dimesnions

We will be using the normalization for the SYM action followed in [10] and choose units in which $2\pi\alpha' = 1$. The $\mathcal{N} = 1$ ten-dimensional supersymmetric gauge theory becomes, after dimensional reduction to $(0+1)$ dimensions and gauge fixing,

$$\mathcal{S} = \frac{1}{g}\int dt \mathrm{Tr}\Big(\frac{1}{2g}F_{\mu\nu}F^{\mu\nu} - i\bar{\psi}\not{D}\psi + \underbrace{\frac{1}{g}(\bar{D}^{\mu}A_{\mu})^2}_{\text{gauge-fixing term}}\Big) + \mathcal{S}_{ghost}. \tag{5.17}$$

We want to calculate the effective potential due to low-velocity scattering of 2 supergravitons. If the impact parameter is b and the distance between two clusters of $D0$ branes with $N_1 = N_2 = 1$ is $r \equiv r_{1,2} = \sqrt{b^2 + (v\tau)^2}$. At a first approximation, the supergravitons do not interact. The interactions only manifest when we include the Heisenberg fluctuations in an expansion around a classical background field B^i:

$$X^i = B^i + \sqrt{g}Y^i, i = 1, \dots, 9. \tag{5.18}$$

To describe the motion of two D0-branes in straight lines, we choose the classical background:

$$B^1 = i\frac{v\tau}{2}\sigma^3 \text{ and } B^2 = i\frac{b}{2}\sigma^3 \text{ and } B^0, B^k = 0 \text{ for } k = 3, \dots, 9. \tag{5.19}$$

Here σ^j is the j^{th} Pauli matrix, where $j = 1, 2, 3$. Along with $\mathbb{1}_2$, they form the set of $U(2)$ generators. In a basis of these generators, we may write the fields as:

$$X = \frac{i}{2}(X_0\mathbb{1} + X_a\sigma^a). \tag{5.20}$$

with analogous expressions for the gauge field A and the fermionic field ψ. In this notation, we have:

$$B_3^1 = v\tau, B_3^2 = b. \tag{5.21}$$

The 0 components describe the center of mass motion and we ignore them. The Lagrangian then becomes a sum of four terms:

$$\mathcal{L} = \mathcal{L}_Y + \mathcal{L}_A + \mathcal{L}_{GH} + \mathcal{L}_{\text{fermi}}. \tag{5.22}$$

The gauge field bosonic Lagrangian $\mathcal{L}_A$ is:

$$\begin{aligned}\mathcal{L}_A =& i\Big(\frac{1}{2}A_1(\partial_\tau^2 - r^2)A_1 + \frac{1}{2}A_2(\partial_\tau^2 - r^2)A_2 + \frac{1}{2}A_3\partial_\tau^2 A_3 + 2\epsilon^{ab3}\partial_\tau B_3^i A_a Y_b^i \\ &+ \sqrt{g}\epsilon^{abc}\partial_\tau Y_a^i A_b Y_c^i - \sqrt{g}\epsilon^{a3x}\epsilon^{bcx}B_3^i A_a A_b Y_c^i - \frac{g}{2}\epsilon^{abx}\epsilon^{cdx}A_a Y_b^i A_c Y_d^i\Big).\end{aligned} \tag{5.23}$$

The Lagrangian $\mathcal{L}_Y$ for fluctuations is:

$$\begin{aligned}\mathcal{L}_Y =& i\Big(\frac{1}{2}Y_1^i(\partial_\tau^2 - r^2)Y_1^i + \frac{1}{2}Y_2^i(\partial_\tau^2 - r^2)Y_2^i + \frac{1}{2}Y_3^i(\partial_\tau^2 - r^2)Y_3^i \\ &- \sqrt{g}\epsilon^{a3d}\epsilon^{cbd}B_3^i Y_a^j Y_b^i Y_c^j - \frac{g}{4}\epsilon^{abe}\epsilon^{cde}Y_a^i Y_b^j Y_c^i Y_d^j\Big).\end{aligned} \tag{5.24}$$

For the fermionic terms, we define

$$\psi_+ = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \text{ and } \psi_- = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2). \tag{5.25}$$

And decompose the gamma matrices in 5.17 as:

$$\Gamma^0 = \sigma^3 \otimes \mathbb{1}_{16} \text{ and } \Gamma^i = i\sigma^i \otimes \gamma^i. \tag{5.26}$$

The fermionic term is then given by:

$$\begin{aligned} \mathcal{L}_{\text{fermi}} =& i\Big(\psi_+^T(\partial_\tau - v\tau\gamma_1 - b\gamma_2)\psi_- + \sqrt{\frac{g}{2}}(Y_1^i - iY_2^i)\psi_+^T\gamma^i\psi_3 \\ &+ \frac{1}{2}\psi_3^T\partial_\tau\psi_3 + \sqrt{\frac{g}{2}}(Y_1^i + iY_2^i)\psi_3^T\gamma^i\psi_- - i\sqrt{\frac{g}{2}}(A_1 - iA_2)\psi_+^T\psi_3 \\ &+ i\sqrt{\frac{g}{2}}(A_1 + iA_2)\psi_-^T\psi_3 - \sqrt{g}Y_3^i\psi_+^T\gamma^i\psi_- + i\sqrt{g}A_3\psi_+^T\psi_-\Big). \end{aligned} \tag{5.27}$$

5.3 Ghost Action

The background condition is:

$$G^a t^a = \partial^\mu A_\mu^a t^a + [\, B^{\mu r} t^r, A_\mu^s t^s],$$

where, in our case $t^a = \sigma^a$ are the basis matrices for our fields. The gauge condition is therefore given by:

$$G^a = \partial^\nu A_\nu^a t^a + B^{\nu r} A_\nu^s \epsilon^{rsa} t^a.$$

From [5], we have the expression for the ghost term in the Lagrangian:

$$\mathcal{L}_{GH} = \overline{c}^a \frac{\delta G^a}{\delta A_\mu^b} D_\mu^{bc} c^c$$

for Grassman variables c^a. The gauge covariant derivative is:

$$D_\mu^{bc} c^c = (\delta^{bc}\partial_\mu + \epsilon^{bsc} A_\mu^s) c^c = \partial_\mu c^b + \epsilon^{bsc} A_\mu^s c^c.$$

And also

$$\frac{\delta G^a}{\delta A_\mu^b} = \delta^{ab}\partial^\mu + B^{\mu r}\epsilon^{rba}.$$

Putting it all together, we obtain,

$$\begin{aligned} \mathcal{L}_{GH} &= \overline{c}^a(\delta^{ab}\partial^\mu + B^{\mu r}\epsilon^{rba})(\partial_\mu c^b + \epsilon^{bsc} A_\mu^s c^c), \\ \mathcal{L}_{GH} &= \overline{c}^a \Box c^a + \epsilon^{asc}\overline{c}^a \partial^\mu (A_\mu^s c^c) + B^{\mu r}\epsilon^{rca}\overline{c}^a \partial_\mu c^c + B^{\mu r}\epsilon^{bsc}\epsilon^{rba} A_\mu^s \overline{c}^a c^c. \end{aligned} \tag{5.28}$$

This is a dimensionally reduced Yang-Mills theory, with all the space derivaties disappearing. Also, $B^0 = 0$.

$$\mathcal{L}_{GH} = \overline{c}^a \partial^t \partial_t c^a + \epsilon^{asc}\overline{c}^a \partial^t (A^s c^c) + \cancelto{0}{B^{0r}\epsilon^{rca}\overline{c}^a} \partial_t c^c + \cancelto{0}{B^{0r}\epsilon^{bsc}\epsilon^{rba}} A^s + B^{ir}\epsilon^{bsc}\epsilon^{rba} A_i^s \overline{c}^a c^c.$$

We make a Wick rotation $t \to -i\tau \implies \partial_t \to i\partial_\tau$. Also, $A^c \to -iA^c$. Then, upto a total derivative, in Euclidean space,

$$\mathcal{L}_{GH} = -\bar{c}^a \partial_\tau^2 c^a + \epsilon^{abc}(\partial_\tau \bar{c}^a) c^b A^c - B^{ir} \epsilon^{cbx} \epsilon^{arx} A_i^c \bar{c}^a c^b$$

We can expand the last term above about the background field,

$$\begin{aligned}
&\epsilon^{arx}\epsilon^{cbx} B^{ir}(B_i^c + \sqrt{g} Y_i^c)\bar{c}^a c^b \\
&= (\delta^{ac}\delta^{rb} - \delta^{ab}\delta^{rc}) B^{ir}(B_i^c + \sqrt{g} Y_i^c)\bar{c}^a c^b, \\
&= B^{ia}B^{ib}\bar{c}^a c^b + \sqrt{g} B^{ib} Y^{ia} \bar{c}^a c^b - B^{ic}B^{ic}\bar{c}^a c^a - \sqrt{g} B^{ic} Y^{ic} \bar{c}^b c^b, \\
&= (B^{13}B^{13} + B^{23}B^{23})(\bar{c}^a c^a - \bar{c}^3 c^3) + \sqrt{g}(Y^{ia}B^{ib}\bar{c}^a c^b - Y^{ic}B^{ic}\bar{c}^b c^b), \\
&= -r^2(\bar{c}_1 c_1 + \bar{c}_2 c_2) + \sqrt{g}(B^{13}Y^{1a}\bar{c}^a c^3 + B^{23}Y^{2a}\bar{c}^a c^3 - B^{13}Y^{13}\bar{c}^a c^a \\
&- B^{23}Y^{23}\bar{c}^a c^a), \\
&= -r^2(\bar{c}_1 c_1 + \bar{c}_2 c_2) + \sqrt{g}\epsilon^{a3x}\epsilon^{cbx} B_3^i \bar{c}^a c^b Y_c^i.
\end{aligned} \tag{5.29}$$

Combining everything,the ghost action is

$$\begin{aligned}
S_{GH} = i\int d\tau \bar{c}_1(-\partial_\tau^2 + r^2)c_1 + \bar{c}_2(-\partial_\tau^2 + r^2)c_2 - \bar{c}_3\partial_\tau^2 c_3 + \epsilon^{abc}(\partial_\tau \bar{c}^a)c^b A^c - \\
\sqrt{g}\epsilon^{a3x}\epsilon^{cbx} B_3^i \bar{c}^a c^b Y_c^i.
\end{aligned} \tag{5.30}$$

There is a difference in normalization of the Yang-Mills action between [5] and this thesis. Since we have a factor of $\sim \frac{1}{g}$ as opposed to $\frac{1}{g^2}$, we must rescale our gauge fields $A^c \to \sqrt{g}A^c$. Therefore, the final form of our ghost action is:

$$\begin{aligned}
\mathcal{S}_{ghost} = i\int d\tau \bar{c}_1(-\partial_\tau^2 + r^2)c_1 + \bar{c}_2(-\partial_\tau^2 + r^2)c_2 - \bar{c}_3\partial_\tau^2 c_3 + \sqrt{g}\epsilon^{abc}(\partial_\tau \bar{c}^a)c^b A^c - \\
\sqrt{g}\epsilon^{a3x}\epsilon^{cbx} B_3^i \bar{c}^a c^b Y_c^i.
\end{aligned} \tag{5.31}$$

5.4 Bosonic and Ghost Particles

The free terms in the bosonic Lagrangian 5.23, 5.24 are:

$$\begin{aligned}
&\frac{1}{2}Y_1^i(\partial_\tau^2 - r^2)Y_1^i + \frac{1}{2}Y_2^i(\partial_\tau^2 - r^2)Y_2^i + \frac{1}{2}Y_3^i\partial_\tau^2 Y_3^i + \frac{1}{2}A_1(\partial_\tau^2 - r^2)A_1 \\
&+ \frac{1}{2}A_2(\partial_\tau^2 - r^2)A_2 + \frac{1}{2}A_3\partial_\tau^2 A_3 + 2\epsilon^{ab3}\partial_\tau B_3^i A_a Y_i^b.
\end{aligned} \tag{5.32}$$

The last term in 5.32 may be written as $2v(A_1 Y_1^2 - A_2 Y_1^1) = \frac{1}{2}(2v)A_1 Y_1^2 + \frac{1}{2}(2v)A_1 Y_1^2 - \frac{1}{2}(2v)A_2 Y_1^1 - \frac{1}{2}(2v)A_2 Y_1^1$. Let $k = 3, 4, \ldots, 9$. Therefore 5.32 can be expressed in matrix form as follows:

$$
\begin{aligned}
&\frac{1}{2}Y_1^k(\partial_\tau^2 - r^2)Y_1^k + \frac{1}{2}Y_2^i(\partial_\tau^2 - r^2)Y_2^i + \frac{1}{2}Y_3^i\partial_\tau^2 Y_3^i + \frac{1}{2}A_3\partial_\tau^2 A_3 \\
&-\frac{1}{2}\begin{pmatrix} Y_1^1 & A_2 \end{pmatrix}\begin{pmatrix} -\partial_\tau^2 + r^2 & 2v \\ 2v & -\partial_\tau^2 + r^2 \end{pmatrix}\begin{pmatrix} Y_1^1 \\ A_2 \end{pmatrix} \\
&-\frac{1}{2}\begin{pmatrix} Y_1^2 & A_1 \end{pmatrix}\begin{pmatrix} -\partial_\tau^2 + r^2 & -2v \\ -2v & -\partial_\tau^2 + r^2 \end{pmatrix}\begin{pmatrix} Y_1^2 \\ A_1 \end{pmatrix}.
\end{aligned}
\tag{5.33}
$$

From the first two terms, we get $7 + 9 = 16$ bosons of $m^2 = r^2$. The third term gives us 9 massless bosons and the fourth term 1. If we diagonalize the mass squared matrices in the last two terms, we get 2 bosons with $m^2 = r^2 - 2v$ and two bosons with $m^2 = r^2 + 2v$.

Lastly, the ghost action 5.31 gives us 2 complex bosons with $m^2 = r^2$ and one complex massless boson, as is apparent from the action.

5.5 Bosonic Feynman Rules

In general, bosonic propagators take the form

$$(-\partial_\tau^2 + \mu^2 + (v\tau)^2)^{-1} = \Delta_{\mathcal{B}}(\tau, \tau'|\mu^2 + (v\tau)^2), \tag{5.34}$$

such that

$$(-\partial_\tau^2 + \mu^2 + (v\tau)^2)\Delta_{\mathcal{B}}(\tau, \tau'|\mu^2 + (v\tau)^2) = \delta(\tau - \tau'), \tag{5.35}$$

and $\mu^2 = b^2$ or $b^2 \pm 2v$, depending on the mass of the bosonic field. This is just the propagator for the harmonic oscillator and therefore,

$$
\begin{aligned}
\Delta_{\mathcal{B}}(\tau, \tau'|\mu^2 + (v\tau)^2) = &\int_0^\infty ds \exp(-\mu^2 s)\Big(\frac{v}{2\pi \sinh 2sv}\Big)^{1/2} \\
&\times \exp\Big(-\frac{v}{2}\frac{(\tau^2 + \tau'^2)\cosh 2sv - 2\tau\tau'}{\sinh 2sv}\Big).
\end{aligned}
\tag{5.36}
$$

Expanding up to the leading order in v, we have,

$$\Delta_{\mathcal{B}}(\tau, \tau'|b^2) = \frac{1}{2b}\exp(-b|\tau - \tau'|). \tag{5.37}$$

This is the propagator for a particle of mass $m^2 = b^2$.For $b^2 = 0$,

$$\Delta_{\mathcal{B}}(\tau, \tau'|0) = -(\tau' - \tau)\theta(\tau' - \tau). \tag{5.38}$$

We can find the propagators from the action with diagonalized mass matrices. From 5.32, we can read off the propagators for the bosonic fields by inverting the mass matrices,

$$\langle Y_a^i(\tau)Y_b^j(\tau')\rangle = \delta_{ab}\delta^{ij}\Delta_{\mathcal{B}}(\tau, \tau'|r^2) \text{ for } a, b = 1, 2 \text{ and } i, j = 2\dots 9. \tag{5.39}$$

For the massless field $Y_3^i, i = 1, \ldots, 9$

$$\langle Y_3^i(\tau)Y_3^j(\tau')\rangle = \delta^{ij}\Delta_{\mathcal{B}}(\tau,\tau'|0) \text{ and } \langle A_3(\tau)A_3(\tau')\rangle = \Delta_{\mathcal{B}}(\tau,\tau'|0). \tag{5.40}$$

Furthermore, for $a, b = 1, 2$,

$$\langle Y_a^1(\tau)Y_b^1(\tau')\rangle = \langle A_a(\tau)A_b(\tau')\rangle = \frac{1}{2}\delta_{ab}\Big(\Delta_{\mathcal{B}}(\tau,\tau'|r^2+2v) + \Delta_{\mathcal{B}}(\tau,\tau'|r^2-2v)\Big). \tag{5.41}$$

And for the mixed fields,

$$\langle A_1(\tau)Y_1^2(\tau')\rangle = -\langle A_2(\tau)Y_1^1(\tau')\rangle = \frac{1}{2}\Big(\Delta_{\mathcal{B}}(\tau,\tau'|r^2+2v) - \Delta_{\mathcal{B}}(\tau,\tau'|r^2-2v)\Big). \tag{5.42}$$

The quartic vertices do not contribute anything when they are massless, because the integral

$$\int \frac{d^d p}{p^2}$$

disappears under dimensional regularization. To see why this is the case, we note that dimensional regularization of poorly defined integrals requires the following three properties, by definition [7]:

1. Translation symmetry: $\int d^d p F(p+q) = \int d^d p F(p)$.
2. Dilatation: $\int d^d p F(\lambda p) = |\lambda|^{-d} \int d^d p F(p)$.
3. Factorization: $\int d^d p d^{d'} q f(p) g(q) = \int d^d p f(p) \int d^{d'} q g(q)$.

By property (2) above, we have, for $|\lambda| \neq 1, \int \frac{d^d p}{(\lambda p)^{2n}} = |\lambda|^{-d} \int \frac{d^d p}{p^{2n}} \implies \int \frac{d^d p}{p^{2n}} = 0$, for $2n \neq d$. This can be interpreted as UV and IR divergences canceling:

$$\int \frac{d^d p}{2n} = \frac{2\pi^{d/2}}{\Gamma(d/2)}\Big(\int_1^\infty p^{d-1-2n} dp + \int_0^1 p^{d-1-2n} dp\Big).$$

5.6 Fermionic Feynman Rules

From 5.27, the fermionic propagator $\Delta_{\mathcal{F}}$ is given by $(-\partial_\tau + m_{\mathcal{F}})\Delta_{\mathcal{F}}(\tau,\tau'|m_{\mathcal{F}}) = \delta(\tau-\tau')$ with the mass matrix $m_{\mathcal{F}} = v\tau\gamma_1 + b\gamma_2$. One way to find the fermionic massses is to express $\Delta_{\mathcal{F}}$ in terms of a bosonic propagator. We claim that $\Delta_{\mathcal{F}}(\tau,\tau'|m_{\mathcal{F}}) = (\partial_\tau + m_{\mathcal{F}})\Delta_{\mathcal{B}}(\tau,\tau'|r^2 - v\gamma_1)$. We can verify this if we can show that $(-\partial_\tau + m_{\mathcal{F}})\Delta_{\mathcal{F}}(\tau,\tau'|m_{\mathcal{F}}) = \delta(\tau-\tau') = (\partial_\tau^2 - r^2 + v\gamma_1)\Delta_{\mathcal{B}}(\tau,\tau'|r^2 - v\gamma_1)$. We will proceed with the proof using gamma-matrix technology:

$$(\partial_\tau - v\tau\gamma_1 - b\gamma_2)(\partial_\tau + v\tau\gamma_1 + b\gamma_2)\Delta_{\mathcal{B}} = \partial_\tau^2\Delta_{\mathcal{B}} + \partial_\tau(m_{\mathcal{F}}\Delta_{\mathcal{B}}) - m_{\mathcal{F}}\partial_\tau\Delta_B - m_{\mathcal{F}}^2\Delta_{\mathcal{B}}.$$

We can write $m_{\mathcal{F}}^2 = (v\tau)^2\gamma_1^2 + b^2\gamma_2^2 + bv\tau\{\gamma_1,\gamma_2\} = r^2 1, \because$ gamma matrices square to the identity and anticommute with each other. Also, $\partial_\tau(m_{\mathcal{F}}\Delta_{\mathcal{B}}) = m_{\mathcal{F}}\partial_\tau\Delta_{\mathcal{B}} + (\partial_\tau m_{\mathcal{F}})\Delta_{\mathcal{B}} = m_{\mathcal{F}}\partial_\tau\Delta_{\mathcal{B}} + v\gamma_1\Delta_{\mathcal{B}}$. Thus, we have

$$(\partial_\tau - v\tau\gamma_1 - b\gamma_2)(\partial_\tau + v\tau\gamma_1 + b\gamma_2) = \partial_\tau^2 - r^2 + v\gamma_1.$$

By diagonalizing the mass squared matrix $M^2 = r^2 - v\gamma_1$, we get 8 fermions with mass squared $r^2 - v$ and 8 fermions with mass squared $r^2 + v$. From the fermionic terms in the action,

$$\begin{aligned}\langle\psi_+(\tau)\psi_-(\tau')\rangle &= (\partial_\tau + v\tau\gamma_1 + b\gamma_2)\Delta_{\mathcal{B}}(\tau,\tau'|r^2 - v\gamma_1)\\ \langle\psi_3(\tau)\psi_3(\tau')\rangle &= \partial_\tau\Delta_{\mathcal{B}}(\tau,\tau'|0)\end{aligned} \tag{5.43}$$

5.7 Dimensional Analysis

The loop expansion of the Lagrangian is given as:

$$g\mathcal{L} = \sum_{m=0}^{\infty} g^m \mathcal{L}_m = c_{00}v^2 + \sum_{m,n=1}^{\infty} c_{mn} g^m \frac{v^{2n+2}}{r^{3m+4n}} \tag{5.44}$$

Therefore

$$\begin{aligned}\mathcal{L}_0 &= c_{00}v^2,\\ \mathcal{L}_1 &= c_{11}\frac{v^4}{r^7} + c_{12}\frac{v^6}{r^{11}} + c_{13}\frac{v^8}{r^{15}} + \ldots,\\ \mathcal{L}_2 &= c_{21}\frac{v^4}{r^{10}} + c_{22}\frac{v^6}{r^{14}} + c_{23}\frac{v^8}{r^{18}} + \ldots,\\ \mathcal{L}_3 &= c_{31}\frac{v^4}{r^{13}} + c_{32}\frac{v^6}{r^{17}} + c_{33}\frac{v^8}{r^{21}} + \ldots.\end{aligned} \tag{5.45}$$

We can verify these powers of v and r by dimensional analysis. The action 5.17 has a length dimension of 0 and thus the Lagrangian has a length dimension of -1, i.e. $[\mathcal{L}] = -1$. But we know that $[D_\tau] = -1 \implies [D_\tau]^2 = -2$. Therefore, we have $2[X^i] - 2 - [g] = -1$ and from the commutator term, $4[X^i] - [g] = -1$, which leads to $[X^i] = -1$ and $[g] = -3$. Now, $\because m^2, r^2$ and $(v\tau)^2$ have the same units, we must have $[r] = -1$ and $[v] = -2$. As a result $[g^m \frac{v^{2n+2}}{r^{3m+4n}}] = -4 = [g\mathcal{L}]$ as expected.

5.8 One-loop effective potential

The calculation in this section is based on [3].

At first, we integrate out the massive fields, starting with the fermionic ones. For fermionic fields η and $\bar{\eta}$, and an opertator $\mathcal{O}$ we have $\int \mathcal{D}\eta\mathcal{D}\bar{\eta}\exp\left[i\int d^n x\bar{\eta}\mathcal{O}\eta\right] = \det(\mathcal{O})$, and noting that $\bar{\psi}_+ = \psi_-$, the integration gives us a factor of $\det(\partial_\tau - v\tau\gamma_1 - b\gamma_2)$.

$$\begin{aligned}\det(\partial_\tau - v\tau\gamma_1 - b\gamma_2) &= \sqrt{\det(\partial_\tau - v\tau\gamma_1 - b\gamma_2)\det(\partial_\tau + v\tau\gamma_1 + b\gamma_2)}.\\ &= \sqrt{\det(\partial_\tau^2 - r^2 + v\gamma_1)} = \det{}^4(\partial_\tau^2 - r^2 + v)\det{}^4(\partial_\tau^2 + r^2 - v).\end{aligned}$$

Since we are not concerned about any numerical prefactors, we use $\int \mathcal{D}\phi\exp(\phi\mathcal{O}\phi) \propto \sqrt{\frac{1}{\det(\mathcal{O})}}$ for a bosonic field ϕ to get the determinants $\det^{-1}(\partial_\tau^2 - r^2 + 2v)\det^{-1}(\partial_\tau^2 +$

$r^2 - 2v$) from the matrices in 5.32.

Next, we deal with the bosons of mass squared $m^2 = r^2$. Integrating out Y_1^k andY_2^k from 5.32, we get a determinant of $\det^{-8}(\partial_\tau^2 - r^2)$ $\because$, we have $7+9=16$ bosons of mass $m^2 = r^2$. Noting that the ghost fields are Grassman variables, integrating them out gives us a factor of $\det^2(\partial_\tau^2 - r^2)$. Therefore, integrating out the massive fields gives the following powers of the determinant:

$$\begin{aligned}&\det{}^4(-\partial_\tau^2 + r^2 - v)\det{}^4(-\partial_\tau^2 - r^2 + v)\det{}^{-1}(-\partial_\tau^2 + r^2 - 2v)\\&\det{}^{-1}(-\partial_\tau^2 - r^2 + 2v)\det{}^{-6}(-\partial_\tau^2 + r^2).\end{aligned} \tag{5.46}$$

Let D_{tot} be the total determinant in 5.46. Then the Euclidean effective action at one loop is

$$S_{\text{eff}} = S_0 - \log D_{\text{tot}}. \tag{5.47}$$

Thus, the one-loop effective potential is

$$-\log D_{\text{tot}} = \int d\tau V_{\text{eff}}(r(\tau)) \equiv \int d\tau V_{\text{eff}}(\sqrt{b^2 + (v\tau)^2}). \tag{5.48}$$

In order to calculate V_{eff}, we must first calculate the determinants of the different masses. For this, we consider the harmonic oscillator Hamiltonian of unit mass and angular frequency ω:

$$H_\omega = \frac{1}{2}(P^2 + \omega^2 Q^2), [P, Q] = i. \tag{5.49}$$

We invoke this Hamiltonian because if we set $Q = \tau, v = \omega$ and move into momentum space, our determinant operators are of the form $-\partial_\tau^2 + v^2\tau^2 + \lambda = 2H_\omega + \lambda$. The time evolution amplitude for a time interval $2s$ is:

$$\begin{aligned}\langle q'|\exp(-2sH_\omega)|q\rangle \equiv U(\omega, 2s, q', q) =& \Big(\frac{\omega}{2\pi\sinh 2s\omega}\Big)^{1/2}\\&\times\exp\Big(-\frac{\omega}{2}\frac{(q^2+q'^2)\cosh 2s\omega - 2qq'}{\sinh 2s\omega}\Big).\end{aligned} \tag{5.50}$$

Now, $\log(\det(2H_\omega + \lambda)) = \operatorname{Tr}\log(2H_\omega + \lambda)$ We will use the operator identity

$$\operatorname{Tr}\log A = \lim_{\epsilon\to 0^+}\int_\epsilon^\infty \frac{ds}{s}\operatorname{Tr}\exp(-sA). \tag{5.51}$$

Setting $A = 2H_\omega + \lambda, 5.51$ becomes, up to a divergent term,

$$-\operatorname{Tr}\int_0^\infty \frac{ds}{s}\exp(-2sH_\omega - s\lambda) = \int_0^\infty \frac{ds}{s}\exp(-s\lambda)\int_{-\infty}^\infty dq U(\omega, 2s, q, q). \tag{5.52}$$

In the last line we have used the continuous version of the fact that the trace of an operator is the sum of eigenvalues of that operator then applied the first part of 5.50, with $q = q'$ and then summed over q. The second integral in 5.52 is then Gaussian in q^2:

$$\int_{-\infty}^{\infty} dq U(\omega, 2s, q, q) = \int_{-\infty}^{\infty} dq \frac{\omega^{1/2}}{2\pi \sinh 2s\omega} \exp(-\omega q^2 \tanh s\omega) = \frac{1}{2\sinh s\omega}.$$

Hence, we have,

$$-\log\det(2H_\omega + \lambda) \approx \int_0^\infty \frac{ds}{s} \frac{\exp(-s\lambda)}{2\sinh s\omega}. \tag{5.53}$$

Owing to the disappearing powers in the determinants, any (potentially divergent) additive constants to 5.53 will cancel out. Taking the logarithm of 5.46, we get the exact relation

$$-\log D_{\text{tot}} = \int_0^\infty \frac{ds}{s} \frac{\exp(-sb^2)}{2\sinh sv} (-6 - 2\cosh 2sv + 8\cosh sv). \tag{5.54}$$

We will expand 5.54 in powers of $\frac{v^n}{b^{2n}}$. For this low energy scattering process, only small s contribute, since b is large.

$$-\log D_{\text{tot}} = \int_0^\infty ds \exp(-sb^2)\Big(-\frac{v^3 s^2}{2} - \mathcal{O}(s^6)\Big) = -\frac{v^3}{b^6} + \mathcal{O}\left(\frac{v^7}{b^{14}}\right).$$

The $\frac{v^4}{r^7}$ term in V_{eff} is:

$$a\int_{-\infty}^{\infty} d\tau \frac{v^4}{r^7} = a\int_{-\infty}^{\infty} d\tau \frac{v^4}{(b^2 + (v\tau)^2)^{7/2}} = a\Big(\frac{16v^3}{15b^6}\Big).$$

Equating the last two expressions above, we get $a = -\frac{15}{16}$ and obtain the well-known result that

$$V_{\text{eff}}(r) = -\frac{15v^4}{16r^7} + \mathcal{O}\Big(\frac{v^6}{r^{11}}\Big). \tag{5.55}$$

This is exactly the result from M theory, using probe and source gravitons, calculated in [4].

5.9 Effective Potential At Two Loops

In the background field formalism, the diagrams that contribute at two loops are given by Figure 6. The wavy lines represent propagators of the gauge fields A and fluctuations Y. The dotted lines represent the ghost propagators and the solid lines represent fermionic propagators.

By a SUSY non-renormalization theorem, [17] we expect the total contribution (bosonic + fermionic) up to the $\frac{v^4}{r^{10}}$ coefficient to cancel out. The contributions from the 17 individual vertices in 5.24, 5.23, 5.27 and 5.31 are listed in [10]. We then use the resuts from sections 5.5 and 5.6 to determine the propagators. We will outline how to evaluate them individual diagrams in this section.

For a diagram with a quartic vertex λ_4, we have two propagators Δ_1 and Δ_2, the explicit expression for the graph is:

$$\int d\tau \lambda_4 \Delta_1(\tau, \tau|m_1)\Delta_2(\tau, \tau|m_2). \tag{5.56}$$

For diagrams with two cubic vertices, $\lambda_3^{(1)}$ and $\lambda_3^{(2)}$,the diagram is:

$$\int d\tau d\tau' \lambda_3^{(1)} \lambda_3^{(2)} \Delta_1(\tau, \tau'|m_1)\Delta_2(\tau, \tau'|m_2)\Delta_3(\tau, \tau'|m_3). \tag{5.57}$$

The bosonic and fermionic contributions sum to zero up to the v^4 term, confiriming another correspondence between M theory and M(atrix) theory [10].

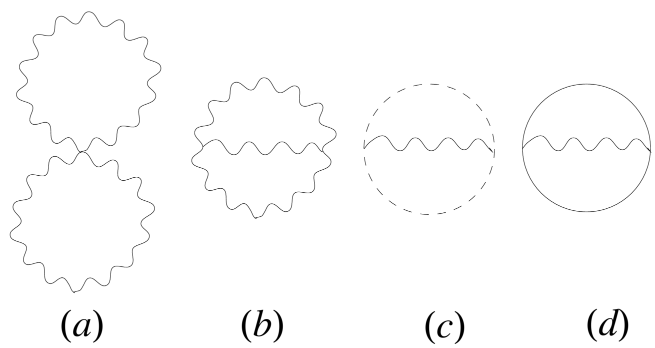

Figure 6: Diagrams with two loops that contribute to the effective action. Wavy lines indicate bosonic and gauge propagators. Broken lines indicate ghost propagators and solid lines fermionic propagators.

Consider for example the of quartic term in 5.24:

$$\begin{aligned} & -\frac{g}{4}\epsilon^{abe}\epsilon^{cde}\int d\tau Y_a^i(\tau)Y_b^j(\tau)Y_c^i(\tau)Y_d^j(\tau) \\ & = -\frac{g}{4}(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc})\int d\tau Y_a^i(\tau)Y_b^j(\tau)Y_c^i(\tau)Y_d^j(\tau) \\ & = -\frac{g}{4}\int d\tau \Big(Y_a^i(\tau)Y_b^j(\tau)Y_a^i(\tau)Y_b^j(\tau) - Y_a^i(\tau)Y_b^j(\tau)Y_b^i(\tau)Y_a^j(\tau)\Big). \end{aligned} \tag{5.58}$$

Here, we sum over i, j, a and b. Since there are two propagator loops, we use 5.56 to evaluate the diagram. The corresponding scattering amplitude is given by

$$\begin{aligned} -\frac{g}{8\times 4}\int d\tau \langle Y_a^i(\tau)Y_b^j(\tau)\rangle\langle Y_a^i(\tau)Y_b^j(\tau)\rangle + \langle Y_a^i(\tau)Y_a^i(\tau)\rangle\langle Y_b^j(\tau)Y_b^j(\tau)\rangle \\ - \langle Y_a^i(\tau)Y_b^j(\tau)\rangle\langle Y_b^i(\tau)Y_a^j(\tau)\rangle - \langle Y_a^i(\tau)Y_b^i(\tau)\rangle\langle Y_b^j(\tau)Y_a^j(\tau)\rangle \end{aligned} \tag{5.59}$$

The prefactor of $\frac{1}{8}$ is due to the symmetry factor of the diagram. To evaluate 5.59, we use the bosonic propagators in section 5.5. From [10], we know that this diagram has an expansion of,

$$-\frac{9}{r^2} - \frac{3v^2}{8r^6} - \frac{4239v^4}{640r^{10}} + \dots. \tag{5.60}$$

5.10 (Super)membrane tension

The calculations in this section are based on [3]. Before proceeding with the membrane tension calculations, we will describe some mathematical machinery.

Consider a pair of unitary operators U and V, such that

$$U^N = 1, V^N = 1, UV = \exp\left(\frac{2\pi i}{N}\right)VU. \tag{5.61}$$

Set, for matrices p and q, $U = \exp(ip)$ and $V = \exp(iq)$ with $[q, p] = \frac{2\pi i}{N}$. This is consistent with 5.61, since, by the Baker-Campbell-Hausdorff formula,

$$\begin{aligned} UV &= \exp(ip)\exp(iq) = \exp(i(p+q) + \frac{1}{2}[ip, iq] + \text{nested commutators}) \\ &= \exp\left(i(p+q) + \frac{\pi i}{N}\right). \end{aligned} \tag{5.62}$$

Similarly,

$$VU = \exp(iq)\exp(ip) = \exp\left(i(q+p) - \frac{\pi i}{N}\right).$$

This leads to the last part of 5.61. The eigenvalues of p and q are $0, \frac{2\pi}{N}, 2\frac{2\pi}{N}, \dots, (N-1)\frac{2\pi}{N}$, $\because U^N = V^N = \mathbb{1}_N$. The eigenvalues of U^nV^m are then $1, \exp(i\frac{2\pi}{N}(n+m)), \exp(i2\frac{2\pi}{N}(n+m)), \dots, \exp(i(N-1)\frac{2\pi}{N}(n+m))$. Thus, $\operatorname{Tr} U^nV^m = N\delta_{n,0 \mod N}\delta_{m,0 \mod N}$. Using this fact, we may expand any $N \times N$ matrix Z as follows:

$$Z = \sum_{n,m=-N/2-1}^{N/2} z_{nm}U^nV^m \text{ with } z_{nm} = \frac{1}{N}\operatorname{Tr} U^{-n}ZV^{-m}. \tag{5.63}$$

We can verify the value of z_{nm} as follows:

$$U^{-k}ZV^{-l} = \sum_{n,m=-N/2-1}^{N/2} z_{nm}U^{n-k}V^{m-l}.$$

Take the trace on both sides and we have

$$\operatorname{Tr}(U^{-k}ZV^{-l}) = \sum_{n,m=-N/2-1}^{N/2} z_{nm}N\delta_{n-k,0 \mod N}\delta_{m-l,0 \mod N}.$$

Therefore, $z_{kl} = \frac{1}{N}\,\mathrm{Tr}(U^{-k}ZV^{-l})$, as required. We have the expressions for U and V and can write 5.63 as

$$Z = \sum_{-N/2-1}^{N/2} z_{nm} \exp(inp) \exp(imq). \tag{5.64}$$

As $N \to \infty$, $[q,p] = \frac{2\pi i}{N} \to 0$ and thus they commute. The eigenvalues of p and q then fill up all the values in $[0,2\pi] \times [0,2\pi]$, with $0 \sim 2\pi$. We can Fourier expand $z(p,q)$, periodic in both p and q:

$$z(p,q) = \sum_{m,n=-\infty}^{\infty} z_{nm} \exp(inp) \exp(imq), \tag{5.65}$$

where

$$z_{nm} = \int_0^{2\pi}\int_0^{2\pi} \frac{dp}{2\pi}\frac{dq}{2\pi} z(p,q) \exp(-inp) \exp(-imq), \tag{5.66}$$

are the standard Fourier coefficients. From the matrix expression for z_{mn}, 5.63, we have $\mathrm{Tr}\,Z = Nz_{00}$ and in the $N \to \infty$ limit,

$$\mathrm{Tr}\,Z \to N \int_0^{2\pi}\int_0^{2\pi} \frac{dp}{2\pi}\frac{dq}{2\pi} z(p,q). \tag{5.67}$$

We will show that the commutator of two $N \times N$ matrices in the $N \to \infty$ limit becomes a Poisson bracket of the corresponding functions, i.e.

$$\{A(p,q), B(p,q)\} = \partial_q A(p,q)\partial_p B(p,q) - \partial_p A(p,q)\partial_q B(p,q). \tag{5.68}$$

Now, using the expressions for U and V, $[U^n, V^m] = 2i\sin(\frac{nm\pi}{N})\exp(inp+imq)$. We have $\therefore$, upon expanding the sine function,

$$\frac{N}{2\pi i}[U^nV^k, U^mV^l] = (nl-km)\exp[i(n+m)p + i(k+l)q] + \mathcal{O}(\frac{1}{N}). \tag{5.69}$$

Let $N \to \infty$ and set $u(p,q) = \exp(ip)$ and $v(p,q) = \exp(iq)$ and plug this into the definition of the Poisson commutator 5.68, we have

$$\frac{N}{2\pi i}[U^nV^k, U^mV^l] = \{u^nv^k, u^mv^l\}. \tag{5.70}$$

We will exploit the bilinearity of both the commutator and the Poisson bracket. Multiply both sides by z_{nk}, w_{ml} and sum over n, k, m and l to get, in the limit as $N \to \infty$:

$$\frac{N}{2\pi i}[Z,W] \to \{z(p,q), w(p,q)\}, \tag{5.71}$$

where we have used 5.63 and 5.65 for Z and $z(p,q)$ respectively and analogous expressions for W and $w(p,q)$.
M2 branes or supermembranes exist in 11 dimensions. They are described by bosnic coordinates $y^\mu(p,q,\tau)$, where $p = \sigma_1, q = \sigma_2$ and y^μ describes how the

membrane is embedded in 11 dimensional target space. In the Hamiltonian formalism, any dependence on τ disappears. Using 5.71, we may write the Hamiltonian for matrix theory 4.9 as:

$$\begin{aligned} H_m =& \frac{1}{2p_{11}} \int \frac{dp}{2\pi}\frac{dq}{2\pi}\Pi_i^2(p,q) + \frac{(2\pi T_2^m)^2}{4p_{11}} \int dpdq\Big(\{y^i(p,q), y^j(p,q)\}\Big)^2 \\ & + \text{ fermionic terms }. \end{aligned} \tag{5.72}$$

The $\Pi_i, i = 1, 2, \ldots 9$, are the transverse momenta conjugate to the y^i. We have used the fact that $p_{11} = \frac{N}{R}$. T_2^m is the membrane tension. If the transverse momenta all disappear, i.e. $\Pi_i = 0$, the membrane mass $\mathcal{M}$ is given by $\mathcal{M}^2 = 2p_{11}H$. If there are no fermionic excitations,

$$\mathcal{M}^2 = \frac{(2\pi T_2^m)^2}{2} \int dpdq\Big(\{y^i(p,q), y^j(p,q)\}\Big)^2. \tag{5.73}$$

We will now check the normalization of this term. The area $\mathcal{A}$ of the membrane is

$$\begin{aligned} \mathcal{A}^2 &= (2\pi)^2 \int dpdq \sum_{i<j} \Big(\{y^i(p,q), y^j(p,q)\}\Big)^2, \\ &= \frac{1}{2}(2\pi)^2 \int dpdq\Big(\{y^i(p,q), y^j(p,q)\}\Big)^2. \end{aligned} \tag{5.74}$$

Now plug in $y^8(p,q) = \frac{p}{2\pi}L_8$ and $y^9(p,q) = \frac{q}{2\pi}L_9, p, q \in [0, 2\pi]$ with $0 \sim 2\pi$. We then have $\mathcal{A} = L_8 L_9$ and $\mathcal{M}^2 = (T_2^m \mathcal{A})^2$. This is just the expected relation for branes: energy = tension $\times$ volume. $\therefore$, we have the correct normalization constant.

Now we rewrite the bosonic part of the Lagrangian, 4.8 in the $N \to \infty$ limit, using 5.71.

$$L_{\text{matrix}}^{\text{bos}} \to \frac{N}{2R} \int \frac{dp}{2\pi}\frac{dq}{2\pi}(\dot{y}^i(p,q))^2 - \frac{R}{4N}c^2 T_0^2 \int dpdq(\{y^i, y^j\})^2. \tag{5.75}$$

Equating the last term of 5.75 with the last term in 5.72, we have

$$\frac{R}{4N}c^2T_0^2 = \frac{(2\pi T_2^m)^2}{4p_{11}} \implies 2\pi T_2^m = cT_0 = \frac{1}{2\pi\alpha'}T_0 = \frac{1}{2\pi\alpha'}(4\pi^2\alpha')T_2,$$
$$\implies T_2^m = T_2.$$

We see that membrane tension equals D2 brane tension. Therefore,

$$T_2^m = \frac{2\pi}{g_s(2\pi)^3\alpha^{3/2}} = \frac{2\pi}{g_s(2\pi\ell_s)^3}.$$

We use the fact that $\ell_p = g_s^{1/3}\ell_s$ to get 3.21:

$$T_2^m = \frac{2\pi}{(2\pi\ell_p)^3}. \tag{5.76}$$

as promised.

6 A M(atrix) Big Bang Model

In this chapter, we use non perturbative string theory, where the degrees of freedom are the D branes, to investigate the structure of the big bang singularity. In particular, we will use the M(atrix) version of M-theory to probe the singularity.

6.1 Singularities

General relativity predicts spacetime singularities. Intuitively speaking, a singularity is a "place of infinite curvature" (we are excluding curvature singularities, which can be dealt with by changing the coordinate system). A singularity is a point at which a geodesic ends: GR cannot predict the behaviour of a particle at the singularity.

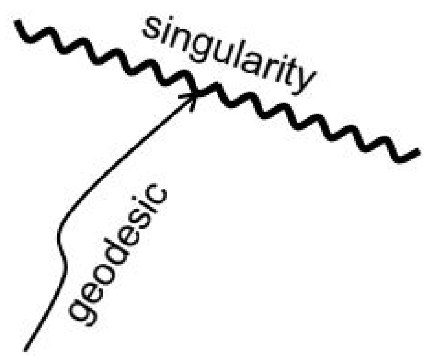

Figure 7: A singularity is the point at which the worldline (geodesic) of a particle ends, according to GR

Consider the FLRW metric with scale factor $a(t)$:

$$ds^2 = -dt^2 + a^2(t)dx^i dx_i \text{ where } i = 1, 2, 3. \tag{6.1}$$

As $a \to 0$ at $t = 0$, the curvature blows up and this cannot be done away with by a change of coordinates. This singularity is known as the big bang singularity. General relativity cannot tell us what happens at singularities. Therefore, we must find complete, quantum theory, for which general relativity is a low energy effective theory.

6.2 Renormalizing General Relativity

Consider the Einstein-Hilbert action

$$S_{\text{EH}} = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g} R. \tag{6.2}$$

The measure has a length dimension of D and the curvature scalar a length dimension of -2. $\therefore$, for the action to have a length dimension of 0, Newton's constant in D dimensions must have a length dimension of $D-2$, which means that canceling out the infinities requires an infinite number of counterterms. Thus, for $D > 2$, G_D has a negative mass dimension. $\because G_D \propto \kappa^2$, which is the coupling constant, the Einstein Hilbert action probably cannot be renormalized, from power counting arguments. This is not true for $D = 3$, but by explicit calculation the $D = 4$ (pure gravity) theory is nonrenormalizable.

So long as we are using a theory as the low-energy effective action of a full quantum theory, non-renormalizability is not a problem. However, the infinite number of counterterms cannot be ignored in high-energy interactions. Therefore, we must find an underlying theory for which general relativity is a low energy effective theory.

An effective low-energy theory is obtained from a a complete quantum theory by integrating out the heavy degrees of freedom.

$$S_{\text{eff}}(\phi_{\text{light}}) = \log\left[\int \mathcal{D}\phi_{\text{heavy}} \exp(S(\phi_{\text{heavy}}, \phi_{\text{light}}))\right]. \tag{6.3}$$

In order to find a complete theory applicable to interactions at the Planck scale, we need to integrate out fewer low-energy degrees of freedom.

6.3 Light-like linear dilaton

We use as our starting point, a type IIA string propagating in a light like linear dilaton background. This defines a conformally invariant theory with string coupling g_s given by

$$g_s = \exp(-\phi) = \exp(-QX^+) \text{ where } Q = \text{ a constant }. \tag{6.4}$$

At early times, i.e. as $X^+ \to -\infty$, this theory is very strongly coupled, but the coupling becomes weaker at later times (as $X^+ \to +\infty$).

This background preserves 16 of the 32 supersymmetries. To see why, we note that only the dilatino feels the presence of the linear dilaton background in the supersymmetry variation. The dilatino variation is $\delta\lambda = \Gamma^+\partial_+\phi\epsilon = 0$ which gives us 16 solutions to the supersymmetry parameter ϵ:

$$\Gamma^+\epsilon = 0. \tag{6.5}$$

where Γ^+ is a Dirac 32×32 matrix, since we are dealing with a superstring theory here in 10 dimensions.While we can use perturbative string theory as $X^+ \to \infty$, the supersymmetry that remains will aid us in non perturbative calculations as $X^+ \to -\infty$, as we shall soon see. In all of this, we are assuming that $Q > 0$ (a big bang scenario). $Q < 0$ would correspond to a big crunch scenario.

The linear dilaton background metric is

$$ds^2 = -2dX^+dX^- + \sum_{i=1}^{8} dX^i dX_i. \tag{6.6}$$

If the radius of compactification becomes large,then using 3.13, we can lift this solution to one of M-theory:

$$ds_{11}^2 = \exp\Big(-\frac{2}{3}\phi\Big)ds_{10}^2 + \exp\Big(\frac{4}{3}\phi\Big)dy^2. \tag{6.7}$$

Here, ds_{10}^2 is the metric in 6.6. We now switch to tetrad formalism in order to calculate the curvature components. Define the following orthonormal basis of one-forms

$$\begin{aligned} e^i &= \exp\Big(\frac{QX^+}{3}\Big)dX^i, e^+ = \exp\Big(\frac{QX^+}{3}\Big)dX^+ \\ \text{and } e^- &= \exp\Big(\frac{QX^+}{3}\Big)dX^-, e^y = \exp\Big(\frac{-2QX^+}{3}\Big)dX^y. \end{aligned} \tag{6.8}$$

We plug in the above basis vectors into the metric 6.7:

$$ds_{11}^2 = -2e^+e^- + (e^i)^2 + (e^y)^2. \tag{6.9}$$

The spin connection components are then:

$$\begin{aligned} \omega_{i+} &= \frac{Q}{3}\exp\big(-\frac{QX^+}{3}\big)e^i, \\ \omega_{y+} &= \frac{-2Q}{3}\exp\big(-\frac{QX^+}{3}\big)e^y, \\ \omega_{-+} &= -\frac{Q}{3}\exp\big(-\frac{QX^+}{3}\big)e^+. \end{aligned} \tag{6.10}$$

We can use the Cartan structure equations to find the curvature two-forms

$$R^a_{\ b} = d\omega^a_{\ b} + \omega^a_{\ c}\wedge\omega^c_{\ b} = \frac{1}{2}R^a_{\ bcd}e^c\wedge e^d. \tag{6.11}$$

The non zero components of the curvature two-form are then given by:

$$\begin{aligned} R_{+i} &= \frac{Q^2}{9}\exp(2QX^+/3) = e^+\wedge e^i, \\ R_{y+} &= -\frac{8Q^2}{9}\exp(-4QX^+/3) = e^+\wedge e^y. \end{aligned} \tag{6.12}$$

The nonzero components of the Riemann curvature tensor are then,

$$\begin{aligned} &R_{+i+i}\frac{Q^2}{9}\exp(2QX^+/3), \\ \text{and } &R_{+y+y} - \frac{8Q^2}{9}\exp(-4QX^+/3). \end{aligned} \tag{6.13}$$

The Ricci tensor components, given by $R_{ab} = R^{c}{}_{acb}$ all disappear. The singularity at $X^+ \to \infty$ is at infinite geodesic distance and hence we will not consider it. The only non zero Christoffel symbol is $\Gamma^+_{++} = \frac{2Q}{3}$. Given an affine parameter $\lambda = \exp(\frac{2Q}{3}X^+)$, the geodesic, up to an affine transformation, is given by,

$$\frac{d^2X^+}{d\lambda^2} + \lambda\Big(\frac{dX^+}{d\lambda}\Big)^2 = 0. \tag{6.14}$$

The left hand side is a total derivative of $\lambda\frac{dX^+}{d\lambda}$ and therefore, integrating both sides gives us:

$$\lambda\frac{dX^+}{d\lambda} = \text{some constant}. \tag{6.15}$$

At the singularity $X^+ \to -\infty$, the affine parameter $\lambda \to 0$. When $X^+ \to +\infty, \lambda \to \infty$. This corresponds to the radius of the 11^{th} dimension curling up to zero size. Now, in the lifted M theory metric, $\lambda dX^+ = \frac{3}{2Q}d\lambda$ and by taking a derivative with respect to X^+. In terms of the affine parameter and the constant $Q > 0$, we may write the lifted M-theory metric 6.7 as:

$$ds^2 = -\frac{3}{Q}d\lambda dX^+ + \sum_{i=1}^{i=8}(dX^i)^2 + \frac{1}{\lambda^2}dY^2. \tag{6.16}$$

The nonzero Riemann tensor components in these affine coordinates are:

$$\begin{aligned} R_{\lambda i\lambda i} &= \frac{1}{4\lambda}, \\ \text{and } R_{\lambda y\lambda y} &= -\frac{2}{\lambda^4}. \end{aligned} \tag{6.17}$$

Even upon changing coordinates, the singularity remains. Thus, the singularity at $\lambda = 0$ is a curvature singularity and not a coordinate one.

6.4 Einstein and String Frame

We get the low energy effective action for a superstring coupled to a graviton and dilaton background, by setting the one-loop beta functions for the background string metric $G_{\mu\nu}$, the dilaton Φ and the Kalb-Ramond 2-form $B_{\mu\nu}$ equal to 0 [13]. The first term in the bosonic part of the action for type IIA theory is given as,

$$S = \frac{1}{2\kappa_0^2}\int d^{10}X\sqrt{-G}\exp(-2\Phi)\left(\mathcal{R} - \frac{1}{2}|H_3|^2 + 4\partial_\mu\Phi\partial^\mu\Phi\right). \tag{6.18}$$

Here $\mathcal{R}$ is the curvature scalar and $H_3 = dB_2$. The first term in 6.18 is almost the Einstein-Hilbert action with except for the factor of $\exp(-2\Phi)$ which comes from the fact that this is a tree level action. To get the Einstein Hilbert action, we need to get rid of this factor. First, we break the dilaton into a constant part and a part that varies: $\Phi = \tilde{\Phi} + \Phi_0$, with Φ_0 constant. Define a new metric in D dimensions by,

$$\tilde{G}_{\mu\nu}(X) = \exp(-4\tilde{\Phi}/(D-2))G_{\mu\nu}(X), \tag{6.19}$$

where D is the number of dimensions. In $D = 10$ dimensions, the square root of the metric determinant becomes

$$\sqrt{-G} = \sqrt{-\exp(5\tilde{\Phi})\tilde{G}} = \exp\left(\frac{5\tilde{\Phi}}{2}\right)\sqrt{-\tilde{G}}. \tag{6.20}$$

For a given Weyl transformation of the metric, $\tilde{G}_{\mu\nu} = \exp(2\omega)G_{\mu\nu}$, the Ricci scalar transforms as,

$$\tilde{\mathcal{R}} = \exp(-2\omega)(\mathcal{R} - 2(D-1)\nabla^2\omega - (D-2)(D-1)\partial_\mu\omega\partial^\mu\omega). \tag{6.21}$$

We are only interested in the first term of the transformation with $2\omega = -\frac{4\tilde{\Phi}}{D-2}$. Therefore, we have $\tilde{\mathcal{R}} = \exp\left(\frac{-4\tilde{\Phi}}{10-2}\right)(\mathcal{R} + \dots) = \exp\left(-\frac{\tilde{\Phi}}{2}\right)(\mathcal{R} + \dots)$ in ten dimensions. The $\dots$ represents terms in the dilaton, its derivatives and the field strength of the Kalb Ramond field. The first term of 6.18 then becomes,

$$S_{GR} = \frac{1}{2\kappa_0^2\exp(2\Phi_0)}\int d^{10}X\sqrt{-\tilde{G}}(\tilde{\mathcal{R}} + \dots). \tag{6.22}$$

This is exactly the Einstein Hilbert action in $D = 10$ dimensions with $\kappa_0^2\exp(2\Phi_0) = 8\pi G_N^{(10)}$. Therefore, we may call the metric $\tilde{G}_{\mu\nu}$ the Einstein frame metric. It is the metric that gravity sees, as opposed to $G_{\mu\nu}$, which is the string frame metric. In IIA superstring theory with linear dilaton background, the relationship 6.19 is given as

$$ds_E^2 = \exp\left(\frac{QX^+}{2}\right)ds_{10}^2. \tag{6.23}$$

We introduce the affine parameter u given by,

$$u = \exp(QX^+/2), \tag{6.24}$$

and set the coordinate $v = X^-$. The Einstein metric 6.23 is given by,

$$ds_E^2 = -\frac{4}{Q}dudv + u\sum_i(dX^i)^2. \tag{6.25}$$

Define an orthonormal basis by,

$$e^i = u^{1/2}dX^i,\ e^u = \frac{2}{Q}du,\ e^v = dv. \tag{6.26}$$

And just as with the superstring metric, we have a non-vanishing spin connection

$$\omega_u^i = \frac{Q}{4u}e^i, \tag{6.27}$$

and the curvature two-form is

$$R^i_u = \frac{Q^2}{16u^2} e^i \wedge e^u. \tag{6.28}$$

The nonzero component of the Riemann tensor in a coordinate basis is then

$$R_{iuiu} = \frac{1}{4u}, \tag{6.29}$$

and the Ricci tensor is,

$$R_{uu} = \frac{2}{u^2}. \tag{6.30}$$

For $\phi = -2 \log u$, the energy momentum tensor then becomes,

$$T_{uu} = \frac{1}{2}(\partial_u \phi)^2 = \frac{2}{u^2}. \tag{6.31}$$

This is also true in the Einstein picture as well. The singularity in the Riemann tensor $R_{\lambda y \lambda y}$ shows up in the energy momentum tensor in the dilaton and by Einstein's equations, to the Ricci tensor 6.30.

6.5 Perturbative String Theory

Now consider the light-like linear dilaton solution in perturbative string theory. The energy-momentum tensor on the worldsheet is given in (2.5.1) in [1] with $\alpha' = 1$ is

$$T(z) = -\partial X_i \partial X^i + 2\partial X^+ \partial X^- - Q\partial^2 X^+, \tag{6.32}$$

where the central charge $c = D$ and $Q > 0$ a free parameter. From the state-operator correspondence, we may construct the vertex operators corresponding to 6.32are given by

$$V = \exp(ip_\mu X^\mu) P_N(\partial X^\mu, \bar{\partial} X^\mu), \tag{6.33}$$

where P_N is a polynomial of scaling dimension N.The CFT Hamiltonian equals the Virasoro generator L_0 where

$$L_0 = \frac{1}{4} p_i^2 - \frac{1}{2} p^+(p^- + iQ) + N. \tag{6.34}$$

The zero mode part of vertex operators for the emission of string modes has the general form

$$V = g_s \Psi, \tag{6.35}$$

where Ψ is the wavefunction and $g_s = \exp(-QX^+)$. The wavefunction is of the form $\exp(ip.X)$The, the momentum conjugate to X^+ is $p^- + iQ$. Therefore,

$$\Psi(X^k, X^+, X^-) = \exp(ip^k X^k - ip^+ X^- - i(p^- + iQ)X^+). \tag{6.36}$$

Hence, $E^- = p^- + iQ$. Since $L_0 = 1$ for physical states,the mass shell condition in light cone coordinates becomes

$$m_{eff}^2 \equiv 2p^+E^- - p^kp^k = 4(N-1). \tag{6.37}$$

Any free scalar field ϕ in a linear dilation background of mass m will have a Lagrangian that is proportional to

$$\mathcal{L} = \frac{1}{2}\exp(2QX^+)(\underbrace{2\partial_+\phi\partial_-\phi - \partial_k\phi\partial_k\phi}_{\text{Kinetic Term}} - \underbrace{m^2\phi^2}_{\text{Potential Term}}). \tag{6.38}$$

The Euler-Lagrange equations yield

$$(2\partial_-\partial_+ - \partial_k\partial_k + 2Q\partial_- + m^2)\phi = 0, \tag{6.39}$$

and the solutions can be expanded in the basis

$$\phi(X^+, X^-, X^k) = \exp(-QX^+ - ip^+X^- - iE^-X^+ + ip_kX_k), \tag{6.40}$$

with the mass shell condition,

$$-2p^+E^- + p_kp_k + m^2 = 0. \tag{6.41}$$

Now we attempt to calculate string scattering amplitudes in a light-like linear dilaton background. We make a gauge choice $X^+ = p^+\tau$ on the worldsheet and over light cone diagrams to get the amplitude. For a genus (number of handles)g contribution to n-string scattering, the number of vertex operators is $2g-2+n$. Every joining/splitting amplitude is multiplied by $\exp(Qp^+\tau_i)$.We are going to perform an integration over the average of all the positions τ_* and the relative insertion points τ_i.The string scattering amplitude integrand is,

$$\prod_{i=1}^{2g-2+n} \exp(-Qp^+\tau_i) \equiv \exp[-(2g-2+n)Qp^+\tau_*]. \tag{6.42}$$

Integrating over the τ_i gives us the amplitude in flat space $A^{g,n}_{\text{flat}}$. The full amplitude in the light like linear background is then,

$$A^{g,n} = A^{g,n}_{\text{flat}} \int_{-\infty}^{+\infty} d\tau_* \exp[-(2g-2+n)Qp^+\tau_*]. \tag{6.43}$$

Even before summing over the genus, we see that $A^{g,n}$ diverges. Thus, we introduce a cutoff τ_c. But even when $\tau_* > \tau_c$, for τ_c negative, the effective coupling $g_s \sim \exp(-Qp^+\tau_c)$ becomes large. Clearly, we need a non perturbative description.

6.6 Matrix String Description

Before providing a full derivation, we quote the result: Matrix theory with the light-like linear dilaton background is described by the flat-space matrix string theory with the string coupling $g_s = \exp(-QX^+)$. We dimensionally reduce a $(9+1)$ dimensional super Yang Mills (SYM) theory to a $(1+1)$ dimensional SYM. The 8 matrix fields X^i represent the transverse bosonic coordinates and the 8 matrix -valued spinor coordinates Θ^a the fermionic part of the action.

$$S = \frac{1}{2\pi\ell_S^2}\int \mathrm{Tr}\left(\frac{1}{2}(D_\mu X^i)^2 + \theta^T \not{D}\theta + g_s^2\ell_s^4\pi^2 F_{\mu\nu}^2 - \frac{1}{4\pi^2 g_s^2\ell_s^4}[X^i, X^j]^2 + \frac{1}{2\pi g_s\ell_S^2}\theta^T\gamma_i[X^i,\theta]\right). \tag{6.44}$$

The worldsheet metric $\eta_{\mu\nu}$ =diag$(-1,+1)$ and the spatial coordinate $\sigma \sim \sigma + 2\pi\ell_S$. The Yang-Mills coupling constant g_{YM} is defined as,

$$g_{YM} \equiv \frac{1}{g_s\ell_s}, \tag{6.45}$$

where ℓ_s is the string length. The fields X^i, θ^a and $\theta^{\dot{a}}$ are $N\times N$ Hermitian matrices that transform in the $\mathbf{8}_v, \mathbf{8}_s$ and $\mathbf{8}_c$ representations of $SO(8)$ respectively.

There are two equivalent ways we may look at matrix theory. The first is

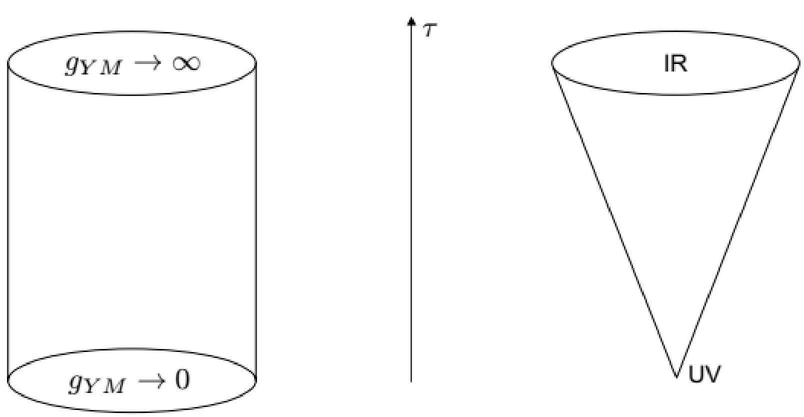

Figure 8: We can either think of Matrix theory as a theory on a cylinder going from the UV to IR or as a theory with constant coupling on a Milne orbifold

to think of it as a $(1+1)$ dimensional Yang-Mills theory on a cyclinder, with coupling $g_{YM} = \frac{1}{\ell_s}\exp(Q\tau)$ with a compactified coordinate. As $\tau \to \infty$, this corresponds to a renormalization group flow from the UV to the IR phase.

A second way to view this is to transfer the $\exp(Q\tau)$ dependence to the worldsheet metric. If the metric is rescaled by a function $f(\tau)^2$, the coupling g_s is reslaled by $f(\tau)^{-1}$.We then have a Super Yang Mills theory with a fixed coupling but a worldsheet with metric given by

$$ds^2 = \exp(2Q\tau)(-d\tau^2 + d\sigma^2) \tag{6.46}$$

This describes an FLRW cosmology, with a scale factor of $\exp(2Q\tau)$. This is known as a Milne orbifold. If we define the light-cone world sheet coordinates

$$\xi^{\pm} = \frac{1}{\sqrt{2}Q}\exp[Q(\tau \pm \sigma)], \tag{6.47}$$

the metric 6.46 becomes a flat light cone metric

$$ds^2 = -2d\xi^+\xi^-. \tag{6.48}$$

This is an *orbifold* because of the identifications

$$\xi^{\pm} \sim \exp(\pm 2\pi Q\ell_s)\xi^{\pm}. \tag{6.49}$$

6.7 The Emergence of Spacetime

As we have already mentioned,in the Matrix theory picture, the passage of time corresponds to renormalization group flow. Near the big bang, the Yang-Mills coupling 6.45 is very weak and hence the potential terms turn off. We need to replace our spacetime theory by a theory of non-commuting matrices.The off-diagonal modes which are, as before, identified with strings stretched between different D-branes, provide extra light degrees of freedom, other than the degrees of freedom in general relativity. This theory is of non commuting matrices is the matrix big bang model referred to in the beginning of this chapter.

However, as time grows large, the Yang Mills coupling becomes larger, and the matrices in the potential term commute, since, in the low energy theory, off diagonal modes that represent the distance between well-separated D-branes or clusters of supergravitons become heavy and therefore can be integrated out. While these do give rise to a static, effective, harmonic oscillator potential, supersymmetry comes to our rescue. The bosonic contribution to the zero-point SHO energy can be shown to cancel the fermionic contributions. Thus, we are left with a purely velocity dependent potential between two supergravitons that is essential to describe M(atrix) theory in flat space time. In fact, we have already seen this chapter 5.

6.8 Derivation of Matrix Theory in the DLCQ

Discrete Light Cone Quantization (DLCQ) involves taking the null direction X^- and compactifying it in a circle of radius R and therefore giving it a conjugate momentum of $P^+ = \frac{2\pi N}{R}$, then quantizing the theory. Usually, this is accomplished by making taking a large boost limit of a space-like identification of radius R_s to get a space-like identification on a radius of $R >> R_s$, *i.e.* we take the high boost spacelike compactification

$$\begin{pmatrix} x \\ t \end{pmatrix} \sim \begin{pmatrix} x \\ t \end{pmatrix} + \begin{pmatrix} -\sqrt{R_s^2 + \frac{R^2}{2}} \\ \frac{R}{\sqrt{2}} \end{pmatrix}. \tag{6.50}$$

This leads to

$$X^- \sim X^- + R. \tag{6.51}$$

However, this breaks the symmetry of the linear dilaton background

$$\phi = -QX^+. \tag{6.52}$$

We single out a direction in space, say X^1 and make the identification

$$(X^+, X^-, X^1) \sim (X^+, X^-, X^1) + (0, R, \epsilon R). \tag{6.53}$$

We will take the $\epsilon \to 0$. Now consider the Lorentz transformations

$$\begin{aligned} X^+ &= \epsilon x^+, \\ X^- &= \frac{x^+}{2\epsilon} + \frac{x^-}{\epsilon} + \frac{x^1}{\epsilon}, \\ \text{and } X^1 &= x^+ + x^1. \end{aligned} \tag{6.54}$$

Of course, this leaves the linear dilaton background metric 6.6 invariant.

$$\begin{aligned} ds^2 &= -2dx^+dx^- + \sum_{i=1}^{8}(dx^i)^2 \\ \phi &= -\epsilon Q x^+. \end{aligned} \tag{6.55}$$

We still have the identification

$$x^1 \sim x^1 + \epsilon R. \tag{6.56}$$

This leads to a momentum in the x^1 direction $p^1 = \frac{2\pi N}{\epsilon R}$. When we do a T-duality, x^1 is now compactified on a circle of $\frac{1}{\epsilon R}$ and we have converted a Type IIA background to a Type IIB background. Define,

$$r \equiv \frac{\epsilon R}{2\pi \ell_s}. \tag{6.57}$$

This leads to the identification

$$x^1 \sim x^1 + \frac{2\pi \ell_s}{r}. \tag{6.58}$$

Now we perform an S-duality $g_s \rightarrow \frac{1}{g_s}$ and get

$$\begin{aligned} ds^2 &= r\exp(\epsilon Q x^+)\big\{-2dx^+dx^- + \sum_{i=1}^{8}(dx^i)^2\big\}, \\ \text{and } \phi &= \epsilon Q x^+ + \log r. \end{aligned} \tag{6.59}$$

This gives us a theory of N D1 branes in a background in which the coupling becomes weak close to the big bang and strong at later times. This behaviour is exactly the opposite that of the original coupling g_s.The bosonic action for the ground state of the D1 brane at low energies is given by the DBI action 2.19, coupled to the dilaton background at tree level.In the following action,μ and ν are directions in spacetime and $\{\alpha, \beta\} = \{\sigma, \tau\}$ are coordinates on the brane. Also $\ell_s = \sqrt{\alpha'}$. The DBI action for $ND1$ branes is then,

$$S_{\mathrm{D1}} = -\frac{1}{2\pi\ell_s^2}\int d\tau d\sigma \exp(-\phi)\sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} + 2\pi\ell_s^2 F_{\alpha\beta}F^{\alpha\beta})}. \tag{6.60}$$

We make the gauge choice 6.8,

$$\begin{aligned} x^1 &= \frac{1}{r}\sigma, \\ \text{and } x^+ &= \frac{1}{r}\frac{\tau}{\sqrt{2}}. \end{aligned} \tag{6.61}$$

We define a new coordinate y by

$$x^- = \frac{1}{r}\frac{\tau}{\sqrt{2}} + \sqrt{2}y. \tag{6.62}$$

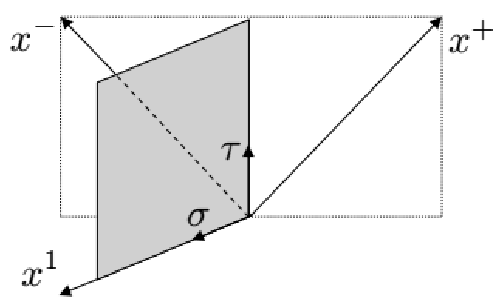

Figure 9: D1 brane is a Type IIB string background

Using the gauge choices in 6.61, and going through the same analysis as in section 2.2.2, we get the action,

$$S_{\rm D1} = \frac{1}{2\pi\ell_s^2}\int d\tau d\sigma\bigg(-\frac{1}{r^2} + \frac{1}{2}[(\partial_\tau y)^2 + (\partial_\tau x^i)^2 - (\partial_\sigma y)^2 - (\partial_\sigma x^i)^2] \\ + 2\pi^2\ell_s^4 \exp\bigg(-\frac{\sqrt{2}\epsilon Q\tau}{r}\bigg) F_{\tau\sigma}F^{\tau\sigma} + \dots \bigg). \tag{6.63}$$

After a rescaling of fields, this is the same as the bosonic terms in 6.44 for $N = 1$. As before, we can derive the commutator term using T-duality [2].

6.9 Validity of the Matrix Direction

The modes of a scalar field 6.40 after the Lorentz transformations 6.54 are given by

$$\phi(x^+, x^-, x^k) = \exp(-\epsilon Q x^+)\exp\Big[-i(\epsilon E^- + \frac{p^+}{2\epsilon} - k_1)x^+ - i\frac{p^+}{\epsilon}x^- + \\ i(k_1 - \frac{p^+}{\epsilon})x^1 + i\sum_{j=2}^{8} k_j x^j\Big]. \tag{6.64}$$

Given the identification 6.56, we can write the momentum conjugate to x^1 as $k_1 - \frac{p^+}{\epsilon} = \frac{2\pi n}{\epsilon R}$. We look for quantum fluctuations with $n = 0$ which leads to

$$p^+ = \epsilon k_1. \tag{6.65}$$

We now invoke the mass shell condition 6.41 with $m^2 > 0$. This gives us

$$2p^+E^- - \sum_{j=1}^{8} k_j k^j \geq 2\epsilon k_1 E^- - |k_1|^2 \geq 0. \tag{6.66}$$

The latter condition leads to,

$$|k_1| \leq 2\epsilon |E^-|. \tag{6.67}$$

This tells us that energy and momentum in the new Lorentz boosted coordinate system is $\sim \epsilon E^-$. Given our gauge choices 6.61, the world sheet energy and momentum are of order

$$E_{\rm typicalworldsheet} \sim \frac{\epsilon E^-}{r} \sim \frac{E^-\ell_s}{r}. \tag{6.68}$$

Effectively, the time-dependent string length is given by,

$$\ell_s^{\rm eff} = \frac{\ell_s \exp(-\epsilon Q x^+/2)}{\sqrt{r}}. \tag{6.69}$$

Open string oscillatory modes decouple when the energy scale of the string is much greater than the worldsheet energy scale, *i.e.*,

$$\epsilon E^- \ell_s^{\rm eff} << 1. \tag{6.70}$$

This is true when $\epsilon \to 0$.

The corresponding effective Newton constant is

$$G_N^{\text{eff}} \sim g_s^2 \ell_s^8. \tag{6.71}$$

The closed strings decouple when

$$(\epsilon E^-)^8 G_N^{\text{eff}} << 1. \tag{6.72}$$

This is also true as $\epsilon \to 0$. Therefore, matrix theory is valid for small ϵ.

6.10 Effective Potential

At one loop of the supersymmetric M(atrix) theory, the following potential is generated between two supergravitons a distance b apart,

$$\int \sqrt{g} V_{eff}(b) \sim \int d\sigma d\tau \sqrt{\frac{b}{g_s}} \exp\Big(-\frac{Cb}{g_s}\Big). \tag{6.73}$$

This potential is generated because of twisted boundary conditions on the Milne orbifold. These twisted boundary conditions lead to a mismatch between bosonic and fermionic frequencies in the Matrix model and we end up with a velocity independent potential. However, at late times, this potential decays rapidly and we are free to use the D0 brane scattering effective potentials that we calculated in the previous chapter, which is good news.

In this thesis, we will not be using the Wilsonian effective action since we are dealing with a time dependent background. At early times, matrix theory 6.44 is a theory of non-commuting matrices. This leads to a non-abelian gauge theory. But notice what happens when we integrate out massive degrees of freedom at early times. The mass scale the W bosons is,

$$m_W^2 \sim \exp(2Q\tau) b^2. \tag{6.74}$$

Where b is the $SO(8)$invariant distance between the eigenvalues of X^i. At early times, $i.e. \tau \to -\infty$, this mass becomes very small and hence is not integrated out. If we introduce a cutoff Λ for our energy scale, the effective action is non-abelian for times for which $m_W > \Lambda$. In other words, the action is non-abelian for times after $\tau_{\text{nonabelian}}$ such that

$$\tau_{\text{nonabelian}} > \frac{1}{Q} \ln\Big(\frac{\Lambda}{b}\Big). \tag{6.75}$$

There is a second time scale about which we transition from non perturbative to perturbative string theory. This occurs roughly when $\frac{g_{\text{YM}}}{b} \sim 1$. Now $\because \frac{g_s^2}{b^2} \sim \frac{1}{m_{\text{W}}^2}$ by 6.74, $\frac{1}{\ell_s^2 g_{\text{YM}}^2 b^2} \sim \frac{1}{m_{\text{W}}^2}, \frac{1}{\ell_s^2 b^4} \sim \frac{1}{b^2 \exp(2Q\tau_{\text{string}})}$. This leads to

$$\tau_{\text{string}} \sim \frac{1}{Q} \ln(\ell_S b). \tag{6.76}$$

Beyond this time scale, we get a perturbative string theory, corresponding to DLCQ type IIA string theory in a linear dilaton background.

6.11 One-Loop Effective Potential

To calculate the 1PI effective action, in a loop expansion, we rescale the parameters in 6.44 as follows:

$$X^i \to \ell_s^2 X^i, \text{ and } \psi \to \ell_s^2 \psi, \ A_\mu \to A_\mu. \tag{6.77}$$

The rescaled action then becomes

$$\begin{aligned} S = \frac{\ell_S^2}{2\pi} \int \mathrm{Tr} \Big(& \frac{1}{2}(D_\mu X^i)^2 + \bar{\psi} \not{D} \psi + \exp(-2Q\tau)\pi^2 F_{\mu\nu}^2 \\ & - \frac{1}{4\pi^2} \exp(2Q\tau)[X^i, X^j]^2 + \frac{1}{2\pi} \exp(Q\tau) \bar{\psi} \Gamma_i [X^i, \psi] \Big). \end{aligned} \tag{6.78}$$

On a cylinder, the massive off-diagonal fields with W-boson mass 6.74 are given by,

$$S = \frac{\ell_S^2}{2\pi} \int d\tau d\sigma \big(\dot{X}^2 - X'^2 - b^2 \exp(2Q\tau) X^2 \big). \tag{6.79}$$

When we switch to the conical Milne orbifold and light cone coordinates $\xi^\pm$, with identification 6.49 the Klein Gordon operator H becomes,

$$H = 2 \frac{\partial^2}{\partial \xi^+ \partial \xi^-} + b^2. \tag{6.80}$$

Path integrating out this boson gives us the determinant

$$\det{}^{-1/2}(H). \tag{6.81}$$

The effective action generates all the 1PI diagrams. From standard QFT, the one loop effective potential $V_{\text{1-loop}}$ is given by

$$-i \int V_{\text{1-loop}} = \log \det{}^{-1/2}(H) = -\frac{1}{2} \mathrm{Tr} \log(H). \tag{6.82}$$

Call the propagator of mass b, $G(\xi, \xi', b^2)$.The heat kernel can then be expressed as,

$$\begin{aligned} \exp(tH)(\xi, \xi') &= \oint \frac{dz}{2\pi i} \frac{\exp(tz)}{z - H}, \\ &= \oint \frac{dz}{2\pi i} \frac{\exp(tz)}{z - (p^2 + b^2)}, \\ &= -\oint \frac{dz}{2\pi i} \frac{\exp(tz)}{p^2 + (b^2 - z)}. \end{aligned} \tag{6.83}$$

The the expression multiplied to $\exp(tz)$ is the propagator with mass $b^2 - z$. Therefore, the heat kernel becomes:

$$\exp(tH)(\xi, \xi') = -\oint \frac{dz}{2\pi i} \exp(tz) G(\xi, \xi', b^2 - z). \tag{6.84}$$

Using the identity 5.51 and the fact that the trace of an operator is the sum of its eigenvalues,we have,

$$-\frac{1}{2} \operatorname{Tr} \log(H) = \frac{1}{2} \int d^2\xi \int \frac{dt}{t} \exp(-it(H - i\epsilon))(\xi, \xi'). \tag{6.85}$$

The $i\epsilon$ factor is inserted to ensure convergence. ϵ is then taken to go to 0. Now we will find the propagator for off-diagonal modes of SYM theory on the Milne orbifold with a boost identification. To invert the kinetic operator 6.80, we use the method of images, noting that the action of a boost depends on spin s. The images are under orbifold identification.

$$\begin{aligned} G(\xi, \xi', b^2) = &\sum_n \int \frac{dp^+ dp^-}{(2\pi)^2} \frac{\exp\Big(- ip^-(\xi^+ - \exp(2\pi Q \ell_s n)\xi^{+'})\Big)}{-2p^+p^- + b^2} \\ &\times \exp\Big(- ip^+(\xi^- - \exp(-2\pi Q \ell_S n)\xi^{-'}) + 2\pi Q \ell_S n s\Big). \end{aligned} \tag{6.86}$$

According to 6.84 this leads to a heat kernel of:

$$\begin{aligned} \exp(-itH_s) = &- \oint \frac{dz}{2\pi i} \exp(-itz) \\ &\times \sum_n \int \frac{dp^+ dp^-}{(2\pi)^2} \exp\Big(- ip^- \xi^+(1 - \exp(2\pi Q \ell_s n))\Big) \\ &\times \exp\Big(- ip^+ \xi^-(1 - \exp(-2\pi Q \ell_S n)) + 2\pi Q \ell_S n s\Big) \\ &\times \frac{1}{-2p^+p^- + b^2 - z}. \end{aligned} \tag{6.87}$$

The contour integral in 6.87 has a pole at $z = -b^2 + 2p^+p^-$. This leads to

$$\begin{aligned} \exp(-itH_s) = &\sum_n \int \frac{dp^+ dp^-}{(2\pi)^2} \exp\Big(- ip^- \xi^+(1 - \exp(2\pi Q \ell_s n))\Big) \\ &\times \exp\Big(- ip^+ \xi^-(1 - \exp(-2\pi Q \ell_S n)) + 2\pi Q \ell_S n s\Big) \\ &\times \exp\Big(- it(b^2 - 2p^-p^+)\Big). \end{aligned} \tag{6.88}$$

Performing the integrals over $p^\pm$ and simplifying,

$$\begin{aligned} \exp(-itH_s) = \sum_n \frac{1}{(2\pi)2t} \exp\Big(&- itb^2 + 2i\frac{\xi^-\xi^+}{t} \sinh^2(\pi Q \ell_S n) \\ &+ 2\pi Q \ell_s n s\Big). \end{aligned} \tag{6.89}$$

Therefore, the heat kernel 6.89 leads to,

$$\int V_{\text{eff}}(b) = i\int d^2\xi \int \frac{dt}{2t} \sum_{\text{helicities}} \exp\Big(it(H_s - i\epsilon)\Big)(\xi,\xi). \tag{6.90}$$

Each supersymmetry multiplet effectively contributes one $s=1$, four $s=\frac{1}{2}$, six $s=0$, four $s=-\frac{1}{2}$ and one $s=1$ states. $\because$ ghosts cancel out two of the scalars. Therefore, summing up the only term with helicities gives us

$$\begin{aligned}\sum_{\text{helicities}} (-)^{2s}\exp(2\pi Q\ell_s ns) &= (\exp(\pi Q\ell_s n/2) - \exp(-\pi Q\ell_s n/2))^4,\\ &= 16\sinh^4(\pi Q\ell_s n/2).\end{aligned} \tag{6.91}$$

The potential in equation 6.90 then becomes,

$$\begin{aligned}&\int d^2\xi \sum_{n=-\infty}^{+\infty} \left(\frac{2i}{\pi}\right)\sinh^4(\pi Q\ell_s n/2)\\ &\times \int_0^\infty \frac{dt}{t^2}\exp\Big(-itb^2 + \frac{i}{t}2\sinh^2(\pi Q\ell_s n)\xi^+\xi^-\Big).\end{aligned} \tag{6.92}$$

We analytically continue the Schwinger parameter $t = it'$,

$$\begin{aligned}&-\int d^2\xi \sum_{n=-\infty}^{+\infty} \frac{2}{\pi}\sinh^4(\pi Q\ell_s n/2)\\ &\times \int_0^\infty \frac{dt'}{t'^2}\exp\Big(-it'b^2 - \frac{i}{t'}2\sinh^2(\pi Q\ell_s n)\xi^+\xi^-\Big),\\ =&-\int d^2\xi \sum_{n=-\infty}^{+\infty} \frac{2}{\pi}\frac{b\sinh^4(\pi Q\ell_S n/2)}{[2\sinh^2(\pi Q\ell_S n)\xi^+\xi^-]^{1/2}}\\ &\times K_1\Big(\sqrt{8b^2\sinh^2(\pi Q\ell_S n)\xi^+\xi^-}\Big),\end{aligned} \tag{6.93}$$

where K_1 is a modified Bessel function with the following asymptotic behaviour,

$$\begin{aligned}K_1(z) &\approx \frac{1}{\sqrt{z}}\exp(-z) &\qquad (z >> 1),\\ K_1(z) &\approx \frac{1}{z} &\qquad (z << 1).\end{aligned} \tag{6.94}$$

Plugging in 6.94 for $b^2\xi^+\xi^- >> 1$, we see the effective potential for late times is,

$$\begin{aligned}\int V_{\text{eff}} \approx= -\int d^2\xi \frac{2^{3/4}b^{1/2}\sinh^4(\pi Q\ell_s/2)}{\pi(\xi^+\xi^-)^{3/4}\sinh^{3/2}|\pi Q\ell_s|},\\ \times\exp\Big(-\sqrt{8b^2\sinh^2(\pi Q\ell_s)\xi^+\xi^-}\Big).\end{aligned} \tag{6.95}$$

The contributions to the sum in 6.93 are dominated by $n = \pm 1$.At late times, the circle on which σ becomes large and full supersymmetry is restored.

Plugging in 6.47 yields,

$$\xi^+\xi^- = \frac{1}{2Q^2}\exp(2Q\tau) = \frac{1}{2Q^2 g_s^2}. \tag{6.96}$$

Furthermore, the measure in terms of the cylindrical coordinates τ, σ becomes,

$$d\xi^+ d\xi^- = \frac{1}{g_s^2} d\tau d\sigma. \tag{6.97}$$

Combining 6.93, 6.96 and 6.97, the effective potential in the light-like linear dilaton background becomes, schematically,

$$\int \sqrt{g} V_{\text{1-loop}}(b) \sim -\int d\tau d\sigma \left(\frac{b}{g_s}\right)^{1/2} \exp\left(-\frac{Cb}{g_s}\right), \tag{6.98}$$

for a positive constant $C > 0$.

In order to compute the **early time potential**, we set $b^2\xi^+\xi^- << 1$ and use the approximation 6.94 to get,

$$\begin{aligned}
\int V_{\text{eff}} &\approx -\int d^2\xi \sum_n \frac{2}{\pi} \frac{b\sinh^4(\pi Q\ell_s n/2)}{[2\sinh^2(\pi Q\ell_S n)\xi^+\xi^-]^{1/2}\sqrt{8b^2\sinh^2(\pi Q\ell_s n)\xi^+\xi^-}}, \\
&= -\int d^2\xi \frac{1}{8\pi\xi^+\xi^-}\sum_n \tanh^2(\pi Q\ell_s n/2), \\
&\approx \int d^2\xi \frac{1}{8\pi^2 Q\ell_s\xi^+\xi^-}\log(2b^2\xi^+\xi^-).
\end{aligned} \tag{6.99}$$

In the last line, we have summed only over the values of n for which the argument of the modified Bessel function $\sqrt{8b^2\sinh^2(\pi Q\ell_S n)\xi^+\xi^-} < 1$, since the summation decays rapidly otherwise.

Let us recap what we have done so far.

The off-diagonal modes in Matrix Theory that correspond to stretched strings between D1 branes are massive with masses poroportional to the distances between the D1 branes. Integrating out these off-diagonal modes gave us an effective potential for the diagonal modes. The Green's function for the propagator may be calculated using the method of images under orbifold identification. This one-loop potential vanishes at late times, which is exactly the behaviour that we expected.

7 Conclusion and Further Developments

In this thesis, we started with bosonic string theory and applied ideas about D-branes to superstring theories. We then established a correspondence between M-theory and type IIA string theory, in order to demonstrate that M theory was the lynchpin that connects all five superstring theories.

We then demonstrated the correspondence between M theory and the hypothesized BFSS Matrix model by calculating the velocity dependent potential of scattering potential between two D0 branes or supergravitons at one and two loops using background field formalism. Additionally we managed to derive the correct membrane tension, which provides additional support for the BFSS conjecture.

Subsequently, we looked at finite N version of Matrix theory, in which one of the null directions was compactified, in a type IIA linear dilaton background. Using the Dirac-Born-Infeld action, we derived the Matrix theory action, a $U(N)$ non-abelian Super Yang Mills theory, compactified on the Milne circle. From the effective dynamics of this theory, we derived a one-loop potential at distance b, given schematically by 6.98. This expression corresponds nicely with our superscattering potentials in one and two loops, since, as time goes to infinity, we expect the static potential to go to zero.

At this point, the following avenues of research are open to us:

1. We can extend the computation to higher loops on the Milne orbifold just as we did in flat spacetime. This calculation, although tedious, will help us see if higher loop effects are suppressed due to supersymmetry,

2. Is it possible to extend our computations to the large N limit? Can we carry out the DLCQ computation at finite N and then set $N \to \infty$ at the very end?

3. We can consider the light like linear dilaton in type IIB string theory using IIB Matrix theory. There will be additional complications, since the IIB string theory is S-Dual unto itself.

List of Figures

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