Derivation of the Hawking Radiation for the four dimensional Schwarzschild black hole in anti de Sitter space.

by

Tahsin Nahian Bin Quddus
16211001

A thesis submitted to the Department of Mathematics and Natural Sciences
in partial fulfillment of the requirements for the degree of
B.Sc. in Physics

Department of Mathematics and Natural Sciences
BRAC University
December 2019

Declaration

It is hereby declared that

1. The thesis submitted is my/our own original work while completing degree at Brac University.
2. The thesis does not contain material previously published or written by a third party, except where this is appropriately cited through full and accurate referencing.
3. The thesis does not contain material which has been accepted, or submitted, for any other degree or diploma at a university or other institution.
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Student's Full Name & Signature:

Tahsin Nahian Bin Quddus
16211001

Certified by
Dr. Mahbub Majumdar
Supervisor
Chairperson
Department of CSE
BRAC University

Approval

The thesis/project titled “” submitted by

1. Tahsin Nahian Bin Quddus (16211001)

Of Fall, 2020 has been accepted as satisfactory in partial fulfillment of the requirement for the degree of B.Sc. in Physics on December, 2019.

Examining Committee:

Supervisor:
(Dr. Mahbub Majumdar)

Dr. Mahbub Majumdar
Chairperson
Department of Computer Science and Engineering
BRAC University

Program Coordinator:
(Dr. Md. Firoze H. Haque)

Dr. Md. Firoze H. Haque
Associate Professor
Department of Mathematics and Natural Sciences
BRAC University

Head of Department:
(Dr. Abu Mohammad Khan)

Dr. Abu Mohammad Khan
Professor
Department of Computer Science and Engineering
BRAC University

0.1 Acknowledgement

Beginning with my heartfelt gratitude to my thesis advisor, Dr. Mahbub Alam Majumdar for instilling enthusiasm in me and paving the way to finishing this thesis. He has provided me help in navigating me academic goals, and building a solid background in High Energy Physics. Without his skilled assistance and guidance this would not have been possible.

I would also like to thank the chairperson of the Department of Mathematics and Natural Sciences, Dr. A.F.M Yusuf Haider for his relentless advises and lessons that taught me many things throughout my undergraduate life in Physics. My gratitude also goes to my mentor, Dr. Firoze H. Haque for building the backbone of my physics knowledge so far. I thank him for his outstanding lessons in Classical Mechanics, Statistical Mechanics, Classical Electrodynamics and Quantam Mechanics, without which completing this thesis would have been an impossible task.

I would also like to thank the other faculty members who have had and equal share in making this possible. I would like to express gratitude to Murad sir, who has helped me immensely in expanding mathematics knowledge, to Lutfor sir, for mentoring me in applied physics courses.

Next, I would like to thank Ahmed Rakin Kamal, Mishaal Hai, Mehdi Salman Noor, Sayeda Tashnuba Jahan and Shaikot Jahan Shuvo for, as good friends, providing me with the mental support and assistance for this thesis. This undergraduate life would have been difficult without them.

Lastly, but most importantly, I would like to express the highest form of gratitude to my parents for bearing with me through all forms of obstacles and supporting me throughout my life.

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Chapter 1

Introduction

This paper guides us through the derivation of the Hawking radiation of a Schwarzschild AdS black hole, thereby proving that even in AdS space, black holes evaporate and leaves us with to deal with the dilemma of information paradox.

The first few chapters of the thesis is designed to go through the pre-requisites needed before one dives into the actual derivation. In chapter one, I have briefly but hopefully adeptly explained the adS space at a deeper level by introducing the different kinds of possible solutions of the Klein Gordon equations in said space. Not before first giving a very short and introductory few pages on the basics of dS and adS space where I have talked about the metric and the overall geometry of the space.

In the next chapter, I move on to talk about Rindler space, which will prove to be of paramount importance in the derivation of the Hawking radiation in the later chapters. In order to achieve greater clarity, I have derived the Rindler metric from different spaces thereby further fleshing our the nuances and details of this geometry.

Proceeding, I jump into some mathematics required for this thesis. The knowledge of Path Integrals is of great significance in this derivation and for that, we need to know Functionals and their basic calculations. This chapter is dedicated to those two topics where I have tried, where possible, my best to compare and relate the two for the purpose of ease of understanding.

Finally, we get the heart of the thesis where I derive the Hawking radiation. But even here, I do not immediately derive the radiation for the SAdS black hole but for a few other spaces before that to lead up to it and for better comparison when necessary.

This paper has been written with the hope that some day some other undergraduate student writing his or her thesis on a somewhat similar topic can stumble onto this paper and find all the necessary information written in a clean, organised manner that is easy to understand and follow. Hopefully any reader of this will be able to appreciate that effort and forgive any mishaps.

Chapter 2

Solutions to K-G equation in adS Space

2.1 De Sitter and Anti-de Sitter spacetime

De Sitter and Anti-de Sitter space, or as they are better known as, dS and adS spacetimes are solutions to the Einestein field equations. Put forth by William de Sitter, a Dutch mathematician, physicist and astronomer, it remains one of the most widely studied frontiers of modern theoretical physics. The correspondence of adS space with conformal field theory, otherwise known as the adS/cft correspondence, is one of the front runner theories attempting to quantize gravity and shed some light on the physics in the interior of a black hole. For the purpose of this thesis, we shall retain our focus more on the Anti de-Sitter spacetime. In the following section, we will introduce these spacetimes.

Introduction

We will approach our studiy of the de-Sitter and Anti de-Sitter spacetimes by giving the analogy of a sphere. Let, S^d be a d-dimensional sphere with radius L, this sphere, then, is defined as the set of all points $(X^1, X^2, \ldots X^{d+1})$ in E^{d+1}, a $(d+1)$ dimensional Euclidean space. The line element in this space would look like [4]:

$$ds^2 = (dX^1)^2 + (dX^1)^2 + \cdots + (dX^{d+1})^2 = L^2 \tag{2.1}$$

In the same manner, we have a d-dimensional de Sitter spacetime, namely, dS^d. The set of all points $(X^0, X^1, \ldots X^d)$ embedded in a $(d+1)$ dimensional Minkowskian space, $M^{d,1}$. This spacetime will have the following line element [4]:

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + \cdots + (dX^d)^2 = L^2 \tag{2.2}$$

Equations 2.1 and 2.3 look very much alike. We have taken the X^{d+1} term and renamed it as X^0 and by putting a minus sign before it, turned it into a timelike co-ordinate. Therefore, what we have here is essentially that same sphere from 2.1 except it is now a Minkowskian version of itself that resides within a Minkowskian spacetime. This spacetime, then, is known as the de Sitter spacetime. We can also write equation 2.3 as: $(\mathrm{dX}^1)^2 + \cdots + (dX^d)^2 = L^2 + (dX^0)^2$ Equation 2.1 showcases the importance of maintaining the difference in sign between the spatial and the timelike co-ordinates. Equation 2.1 is the definition of a $(d-1)$ dimensional sphere, S^{d-1}.

The time co-ordinate, X^0, in 2.1 runs from $-\infty$ to $+\infty$, here, the radius, $\sqrt{L^2+(X^0)^2}$ of the $(d-1)$ dimensional sphere has maximum values at the infinities while contracting to a minimum of L in the middle, as shown in 2.1. Similarly, the d-dimensional anti de Sitter spacetime, AdS^d, is defined as the set of all points $(X^0, X^1, \ldots X^d)$ embedded in a $(d+1)$ dimensional Minkowskian like spacetime, except one that has two timelike co-ordinates, known as $M^{d-1,2}$. This spacetime, will have the following line element [4]:

$$ds^2 = -(dX^0)^2 + (dX^1)^2 + \cdots + (dX^{d-1})^2 - (dX^d)^2 \tag{2.3}$$

Note that we have added another minus sign before the last term of the above equation as, as was earlier mentioned, it has two timelike co-ordinates. This then satisfies the following equation:

$$-(X^0)^2 + (X^1)^2 + \cdots + (X^{d-1})^2 - (X^d)^2 = -L^2 \tag{2.4}$$

We can now write the line elements of both the spacetiemes just discussed using summation notation as follows:

$$(X^0)^2 - \sum_{i=1}^{d-1}(X^1)^2 + (X^d)^2 = L^2 \quad \text{(anti de Sitter spacetime)} \tag{2.5}$$

$$-(X^0)^2 + \sum_{i=1}^{d-1}(X^1)^2 + (X^d)^2 = L^2 \quad \text{(de Sitter spacetime)} \tag{2.6}$$

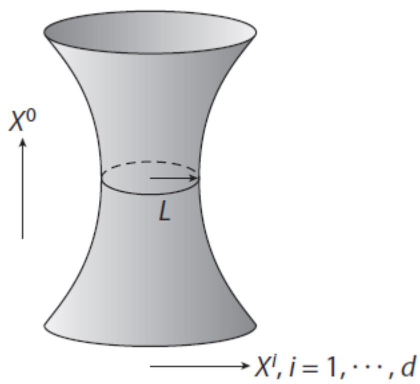

Figure 2.1: d-dimensional de Sitter spacetime ds^d [4]

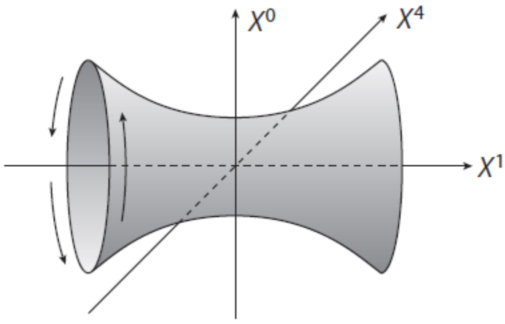

Figure 2.2: d-dimensional anti de Sitter spacetime AdS^d [4]

Poincare patch in adS Spacetime

Let us now consider a 3 dimensional adS spacetime, adS^3, in three dimensions, given that there are two time like slices, our metric, instead of the usual and familiar Minkowski metric, $\eta = (-1, +1, +1, +1)$, will now be written as $\eta = (-1, +1, +1, +1)$, using this, we can rewrite equation 2.40 for 3-dimensional adS spacetime as:

$$(T^2 - X^2) + (W^2 - Y^2) = 1 \tag{2.7}$$

Where we have rescaled the radius L to be unitary. This means the line element takes the form:

$$ds^2 = -dT^2 + dX^2 - dW^2 + dY^2 \tag{2.8}$$

From here on, we make the transformation that:

$$T^2 - X^2 = \frac{t^2 - x^2}{w^2}$$

We can then write:

$$W^2 - Y^2 = 1 + \frac{t^2 - x^2}{w^2} \tag{2.9}$$

We can deduce from these the following equations:

$$T = \frac{t}{w} \tag{2.10}$$

$$X = \frac{x}{w} \tag{2.11}$$

This allows to relabel Y and W as the following:

$$Y = \frac{1}{2}(\frac{x^2 - t^2}{w} + w - \frac{1}{w}) \tag{2.12}$$

$$= \frac{1}{2w}(x^2 - t^2 + w^2 - 1) \tag{2.13}$$

$$W = \frac{1}{2}(\frac{x^2 - t^2}{w} + w + \frac{1}{w}) \tag{2.14}$$

$$= \frac{1}{2w}(x^2 - t^2 + w^2 + 1) \tag{2.15}$$

$$\tag{2.16}$$

We can see now that we no longer have the initial four dimensions, differentiating, 2.10 and 2.11, we get:

$$dT = \frac{dt}{w} \implies (dT)^2 = \frac{dt^2}{w^2}$$
$$dX = \frac{dx}{w} \implies (dX)^2 = \frac{dx^2}{w^2}$$

Furthermore, we differentiate 2.13 and 2.15 to get:

$$dY = \frac{1}{w}(x^2 - t^2 + w^2 - 1)dw \tag{2.17}$$
$$dY^2 = \frac{1}{w^2}(x^2 - t^2 + w^2 - 1)dw^2 \tag{2.18}$$
$$dW = \frac{1}{w}(x^2 - t^2 + w^2 + 1)dw \tag{2.19}$$
$$dW^2 = \frac{1}{w^2}(x^2 - t^2 + w^2 + 1)^2dw^2 \tag{2.20}$$

Making all the necessary substitutions into 2.8, we get a very innocent metric in the form of:

$$ds^2 = -\frac{dt^2}{w^2} + \frac{dx^2}{w^2} + \frac{1}{w^2}[(x^2 - t^2 + w^2 + 1)^2) - (x^2 - t^2 + w^2 - 1)^2)] \tag{2.21}$$
$$= \frac{1}{w^2}(-dt^2 + dx^2 + dw^2) \tag{2.22}$$

The above equation is the Poincare half plane in the Minkowskian space on a higher dimension. We can repeat the above steps while adding more and more dimensions and would get similar results, for example, adS^5 would be described by the following line element:

$$ds^2 = \frac{L^2}{w^2}(-dt^2 + dx^2 + dy^2 + dz^2 + dw^2) \tag{2.23}$$

Where we have generalized the unitary value of L to any value. If we now make the transformation $w = \frac{L^2}{r}$, we get the following equation:

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + dx^2 + dy^2 + dz^2) + \frac{L^2}{r^2}dr^2 \tag{2.24}$$

We can see that the first term of equation 2.24 can be generalized to arbitrary dimensions. Applying our knowledge from general relativity, we can rewrite the equation as the following:

$$ds^2 = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}(\eta_{\mu\nu}dx^\mu dx^\nu) \tag{2.25}$$

This is the representation of the adS space in the Poincare patch [1]. We can now visualize the adS space in a different way than before as is exhibited in 2.3.

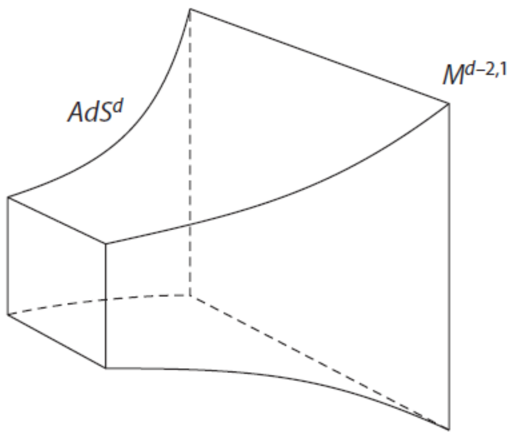

Figure 2.3: A slice of AdS^d at some specific value of w [4]

2.2 The Klein-Gordon equation

One of the most widely well knows equations of motions of the world of high energy physics is the Klein-Gordon(K-G) equation. This was one of the first and arguably one of the most successful attempts at relativistic quantum mechanics. Proposed by Oskar Klein and Walter Gordon in 1926, it was claimed that this equation successfully describes the motion of relativistic electrons. In this section, we will explore some of the solutions obtained from the Klein-Gordon equation in the adS geomtery. We start with a scalar field action:

$$S = -\frac{1}{2}\int d^{d+1}X\sqrt{g}(g^{AB}\delta_A\phi\delta_B\phi + m^2\phi^2) \tag{2.26}$$

As mentioned above, this is a scalar field and hence the partial derivatives can be replaced with covariant derivatives, D_μ. After that, we can perform an integration by parts and obtain the following equation:

$$S = -\frac{1}{2}\int d^{d+1}X\sqrt{g}\phi(-\Delta + m^2)\phi + \frac{1}{2}\int d^{d+1}X\delta_A(\sqrt{g}g^{AB}\phi\delta_B\phi) \tag{2.27}$$

If ϕ satisfies the equation of motion of 2.3, we get the Klein-Gordon equation:

$$(-\Delta + m^2)\phi = 0 \tag{2.28}$$

Next, we move on to derive the laplacian in an adS Space. In order to do that, we will have to derive the metric of the adS space in the Poincare patch. It is defined as:

$$ds^2 = \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu \tag{2.29}$$

In the above equation, if we rescale and replace $r = Lu$, we then get:

$$ds^2 = L^2(\frac{du^2}{u^2} + u^2\eta_{\mu\nu}dx^\mu dx^\nu) \tag{2.30}$$

A further change of variables of $z = \frac{L^2}{r} = \frac{1}{u}$, brings the metric to the form:

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu) \tag{2.31}$$

If we put this definition of the metric in the laplacian which is defined as [5]:

$$\Delta = \frac{1}{\sqrt{g}}\delta_A(\sqrt{g}g^{AB}\delta_B) \tag{2.32}$$

We get:

$$= \frac{z^{d+1}}{L^{d+1}}\left[\delta_z\left(\frac{L^{d+1}}{z^{d+1}}\frac{z^2}{L^2}\delta_z\right) + \frac{L^{d+1}}{z^{d+1}}\frac{z^2}{L^2}{\delta_x}^2\right] \tag{2.33}$$

And further simplification gives us:

$$\Delta = \frac{z^2}{L^2}\left({\delta_z}^2 - (d - 1z^{-1})\delta_z + {\delta_x}^2\right) \tag{2.34}$$

With these tools at hand, we now explore the various solutions of 2.28 in the next section.

2.3 Solutions

2.3.1 Separation of variables

The first approach we are going to look at is utilizing the separation of variables [5]. We write the fields ϕ as:

$$\phi(z, x) = f(z)\Phi(x) \tag{2.35}$$

If we put this in place of ϕ in 2.1, we get:

$$-\frac{z^2}{L^2}\left(z^{d-1}\delta_z(z^{-d+1}f')\Phi + f\delta^2\Phi\right) + m^2 f\Phi = 0 \tag{2.36}$$

After dividing this equation by $f\Phi$ we can proceed to separate the variables to get:

$$-\frac{z^{d-1}}{f}\delta_z(z^{-d+1}f') + \frac{m^2L^2}{z^2} = \frac{\delta^2\Phi}{\Phi} = -k^2 \tag{2.37}$$

Where k^2 is a constant. We can separate this into two equations:

$$(\delta_x^2 - k^2)\Phi_k = 0 \tag{2.38}$$

$$[-z^{d+1}\delta_z(z^{-d+1}\delta_z) + m^2L^2 + k^2z^2]f_k = 0 \tag{2.39}$$

The solutions now depend on the parameter f_k. One will get modes that will depend on the parameter k, meaning that the full solution is the superposition of all of them:

$$\phi(z, x) = \int d^dk f_k(z)\Phi_k(x) \tag{2.40}$$

We know that in Euclidean space, the solution to the Klein-Gordon equation in d-dimensional spacetime are place waves:

$$\Phi(x) = \frac{e^{ikx}}{(2\pi)^d} \tag{2.41}$$

Putting this in equation 2.40 we can get:

$$\phi(z, x) = \int \frac{d^dk}{(2\pi)^d} f_k(z)e^{ikx} \tag{2.42}$$

Which is our solution. We can see from this that $\phi(z, x)$ is the fourier transform of f_k. We see that $f_k(z)$ is the solution in momentum space if we simply invert the transformation.

2.3.2 Solution for the radial direction

We can take 2.39 and rewrite it as [5]:

$$z^2 f"_k - (d-1)z f'_k - (m^2L^2 + k^2z^2)f_k = 0 \tag{2.43}$$

We can see that it looks very much like the modified Bessel Equation, we now perform some actions on this equation to make it identical to a modified Bessel Equation. First, we do a change of variable such that:

$$f_k = z^{\frac{d}{2}} g_k \tag{2.44}$$

And a further transformation where we take g_k as a function of kz instead of a function of z to finally get:

$$(kz)^2 g"_k + (kz)g'_k - \big(\frac{d^2}{4} + (m^2L^2 + k^2z^2\big)g_k = 0 \tag{2.45}$$

This now looks exactly like a modified Bessel Equation, allowing us to use the solutions that are applied to them:

$$g_k(kz) = a_k K_\nu(kz) + b_k I_\nu(kz) \tag{2.46}$$

Where the ν parameter is defined as:

$$\nu = \sqrt{\frac{d^2}{4} + m^2L^2} \tag{2.47}$$

Now, since $f_k = z^{d/2} g_k$, we get the solution for f_k to be:

$$f_k(z) = a_k(kz)^{d/2} K_\nu(kz) + b_k(kz)^{d/2} I_\nu(kz) \tag{2.48}$$

Now, we know that the modifed Bessel Equation has exponential behavior, giving asymptotic forms:

$$I_\nu(kz) \implies \frac{e^{kz}}{\sqrt{kz}}$$

$$K_\nu(kz) \implies \frac{e^-kz}{\sqrt{kz}}$$

We can now see that for $z \to \infty$, I_ν diverges meaning $b_k = 0$, giving us:

$$f_k(z) = a_k(kz)^{d/2} K_\nu(kz) \tag{2.49}$$

Furthermore, we know that the complete limits of K_n in the modified Bessel Function is:

$$K_n(x) \implies \frac{\Gamma(n)}{2}\Big(\frac{2}{x}\Big)^2 + \frac{\Gamma(-n)}{2}\Big(\frac{x}{2}\Big)^2$$

Using this we can see that near the boundary the solution behaves like:

$$f_k(z) \implies a_k(kz)^{d/2}\Big[\frac{\Gamma(\nu)}{2}\Big(\frac{2}{kz}\Big)^\nu + \frac{\Gamma(-\nu)}{2}\Big(\frac{kz}{2}\Big)^\nu\Big] \tag{2.50}$$

Which can be simplified to:

$$f_k(z) \implies \phi_0(k) z^{\Delta_-} + \phi_1(k) z^{\Delta_+} \tag{2.51}$$

Where the following definitions are used:

$$\phi_0(k) = a_k 2^{\nu-1}\Gamma(\nu)k^{\Delta_-}$$

$$\phi_1(k) = a_k 2^{\nu+1}\Gamma(-\nu)k^{\Delta_+}$$

$$\Delta_\pm = \frac{d}{2} \pm \nu$$

We can also note here that this derivation helps us realize the Breitenlohner-Freedman bound from the positivity of the square-root:

$$(mL)^2 > -\frac{d^2}{4}$$

2.3.3 Complete (free) solution

In order to derive the complete solution [5], we first need to take a look at 2.51. In this equation, $\Delta_+ > 0$ is the normalizable solution as $z \to 0$ and $\Delta_- < 0$ is the non-normalizable solution as $z \to \infty$. Therefore, we can write:

$$\phi_0(k) = \lim_{z\to 0} z^{-\Delta_-} f_k(z) \tag{2.52}$$

Or in position space:

$$\phi_0(x) = \lim_{z\to 0} z^{-\Delta_-} \phi(z,x) \tag{2.53}$$

Moving, on, if we substitute 2.49 into 2.42 we get the full solution in the position space, which looks like:

$$\phi(z,x) = \int \frac{d^dk}{(2\pi)^d} a_k (kz)^{d/2} K_\nu(kz) e^{ikx} \tag{2.54}$$

We can now use boundary conditions to deduce the value of of a_k. Here we introduce a cut off $z = \epsilon$ in order to avoid divergences. Using 2.52, we get:

$$\phi_0(k) = \epsilon^{-\Delta_-} f_k(\epsilon) = \epsilon^{-\Delta_-} a_{k]} (k\epsilon)^{d/2} K_\nu(k\epsilon) \tag{2.55}$$

Which after rearranging gives us:

$$a_k = \frac{\epsilon^{\Delta_- - d/2}}{k^{d/2} K_\nu(k\epsilon)} \phi_0(k) \tag{2.56}$$

Then finally we get f_k by:

$$f_k(z) = \epsilon^{\Delta_-} (\frac{z}{\epsilon})^{d/2} \frac{K_{\nu(kz)}}{K_{\nu(k\epsilon)}} \phi_0(k) \tag{2.57}$$

If we use the equation

$$f_k(\epsilon) = \epsilon^{\Delta_-} \phi_0(k) \tag{2.58}$$

and put this in 2.57, we then get:

$$f_k(z) = (\frac{z}{\epsilon})^{d/2} \frac{K_{\nu(kz)}}{K_{\nu(k\epsilon)}} f_k(\epsilon) \tag{2.59}$$

Then the final solution in position space becomes:

$$\phi(z,x) = \epsilon^{\Delta_-} (\frac{z}{\epsilon})^{d/2 d} x' \frac{d^dk}{(2\pi)^d} \frac{K_{\nu(kz)}}{K_{\nu(k\epsilon)}} \phi_0(x') e^{ik(x-x')} \tag{2.60}$$

Chapter 3

Rindler Space

In this chapter we shall look at the basics of Rindler space and some of it's properties. We will set this chapter up for the later ones where we will see the application of Rindler geometry to derive the Hawking Temperature.
Wolfgang Rindler was an Austrian physicist who first coined the term "Event Horizon."

3.1 Surface Gravity

Before moving onto Rindler Geometry, we need to introduce the term surface gravity. Hypothetically, surface gravity can be thought of as the gravitational acceleration on a text particle situated very close to the surface of an astronomical body. We assume that the test particle does not have any mass. However, that definition of surface gravity is incomplete as it can not be applied to black holes. The entire hypothesis breaks down on account of the fact that black holes do not have a surface to begin with. Due to the nature of black holes, the acceleration that a test particle will experience near the event horizon is infinite. Hence, a renormalized value of the surface gravity is necessary. The value for surface gravity varies from one black hole to other depending on it's properties. In the following subsection, we will derive the surface gravity from the Schwarzschild metric.

Schwarzschild Solution

We start by assuming a particle with a finite and unit mass, situated at r. The work done to accelerate this particle in the upwards direction by a force F through a distance δr is:

$$\delta W = F(r)\delta r = ma(r)\delta r\Big|_{m=1} \tag{3.1}$$

The equation for the 4-acceleration is defined by:

$$a(r) = \frac{M}{r^2\sqrt{1-\frac{2M}{r}}} \tag{3.2}$$

3.2 can also be considered the proper acceleration of the observer, Alice, necessary to keep her stationary in the geometry. Substituting 3.2 to 3.15, we get

$$\delta W = \frac{M}{r^2\sqrt{1-\frac{2M}{r}}} \tag{3.3}$$

Let us now assume that this work done is being converted to a high energy photon, being transmitted to another observer, Bob, sitting at infinity, $\delta\omega$ would be the angular frequency of this photon. Imposing 100 percent efficiency, we get:

$$\delta\omega = \frac{M}{r^2\sqrt{1-\frac{2M}{r}}}\delta r \tag{3.4}$$

Due to gravitational redshift, Bob would intercept this photon having a much lower energy than that it was transmitted with. We now attempt to deduce this gravitational redshift.

Perhaps the most popular solution of the Einestein equation to ever exist, schwarzschild geometry was the first exact solution to the equations to have ever been proposed, and applies to a single spherical non-rotating mass. It was done by Karl Schwarzschild, a German physicist and astronomer, who accomplished this great feat in 1915, while serving in the World War 1. Unfortunately, he did not survive the war but his accomplishment has reverberated throughout history and not a day goes by that a physics student does not resort to his solution in order to further their grasp on black hole studies. We start with the metric,

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{3.5}$$

We then impose the following approximations:

$$dr = d\Omega = 0$$
$$d\tau^2 = -ds^2$$

This gives us:

$$d\tau = \sqrt{1-\frac{2M}{r}}dt \tag{3.6}$$

From this equation, we see that the proper time of Bob and Alice are equal to the time coordinate of the Schwarzschild metric, meaning:

$$d\tau_\infty = dt \tag{3.7}$$

Therefore, we can write

$$d\tau_r = \sqrt{1-\frac{2M}{r}}d\tau_\infty \tag{3.8}$$

Where τ_r is the proper time of Alice at a spatial distance r.

We can now apply the general relation between frequency and time to get the proper frequency of the photon at infinity:

$$\delta\omega = \frac{1}{\delta\tau} \tag{3.9}$$

This gives us the relation between the proper frequency of the photon at a distance d and that of a photon at infinity as:

$$\delta\omega_\infty = \sqrt{1 - \frac{2M}{r}}\delta\omega_r \tag{3.10}$$

We can now rewrite the proper frequency at infinity as:

$$\delta\omega_\infty = \frac{M}{r^2}\delta r \tag{3.11}$$

Using dimensional analysis, we can see that the right hand side of 3.11 gives us units of energy in terms of [force][distance].
Thus, dividing both sides of 3.11 by δr, we get and equation of force:

$$\delta F = \frac{M}{r^2} \tag{3.12}$$

Remembering that we set the mass of the observer as unitary, which let's us consider this force to be acceleration. We now write:

$$\kappa(r) = \frac{M}{r^2} \tag{3.13}$$

This acceleration that we just derived can be thought of as the surface gravity. In Schwarzschild geometry, the event horizon is at $r = 2M$. Therefore,

$$\kappa(r = 2M) = \frac{1}{4M} \tag{3.14}$$

3.2 Rindler Geometry

We shall now look at a few ways of deriving the Rindler Metric and some of it's properties. To put it in layman terms, Rindler metric is the representation of the Minkowski space in hyperbolic co-ordinates. But that could be considered as an over simplified definition.

3.2.1 From Minkowski Space

To begin, let us start with a two dimensional Minkowski space, we know that the line element looks like [8]:

$$ds^2 = -dt^2 + dx^2 \tag{3.15}$$

We now introduce the following substitutions of the co-ordinates:

$$x = \rho \cosh \alpha \tag{3.16}$$

$$t = \rho \sinh \alpha \tag{3.17}$$

Evidently, we have switched to hyperbolic polar co-ordinates. This enable the following two identities:

$$\rho^2 = x^2 - t^2 \tag{3.18}$$

$$\tanh \alpha = \frac{t}{x} \tag{3.19}$$

Expressing the line element in terms of these co-ordinates, we get:

$$ds^2 = d\rho^2 - \rho^2 d\alpha^2 \tag{3.20}$$

3.20 is known as the Rindler Metric and the co-ordinates (p, α) are known as Rindler co-ordinates.

3.2.2 From Schwarzschild Geometry

Another method of deriving the Rindler metric starts with the Schwarzschild geometry [10]. We know:

$$ds^2 = -(1 - \frac{2M}{R})dt^2 + (1 - \frac{2M}{R})^{-1}dr^2 + r^2 d\omega^2 \tag{3.21}$$

Implementing the following transformation:

$$r - 2M = \frac{x^2}{8M} \tag{3.22}$$

This allows us to incorporate the surface gravity term ($\kappa = \frac{1}{4M}$) in the equation:

$$r - 2M = \frac{x^2}{8M} \tag{3.23}$$
$$\therefore r = \frac{x^2}{8M} + 2M \tag{3.24}$$
$$\therefore dr = \frac{2xdx}{8M} \tag{3.25}$$
$$\therefore dr = \kappa x dx \tag{3.26}$$
$$\therefore (dr)^2 = (\kappa x)^2 dx^2 \tag{3.27}$$

Therefore, for $r \approx 2M$ we get:

$$ds^2 \approx -(\kappa x)^2 dt^2 + dx^2 + \frac{1}{4\kappa^2} d\Omega^2 \tag{3.28}$$

Where $\frac{1}{4\kappa^2} d\Omega^2$ is the 2-dimensional Rindler spacetime and $\frac{1}{4\kappa^2} d\Omega^2$ is a 2-sphere of radius $\frac{1}{2\kappa}$. It is very easy to show that this metric is the same as 3.15, just in unusual co-ordinates. In order to that, we introduce the Kruskal-type co-ordinates:

$$U' = -xe^{-\kappa t}$$
$$V' = xe^{\kappa t}$$

And writing the Rindler metric in terms of these co-ordinates gives us:

$$ds^2 = -dU'dV' \tag{3.29}$$

Now, if we make the following change:

$$U' = T - X$$
$$V' = T + X$$

Meaning:

$$dU' = dT - dX$$
$$dV' = dT + dX$$

Then we can use these to write the Rindler metric in a form that is identical to the Minkowski line element:

$$ds^2 = -dT^2 + dX^2 \tag{3.30}$$

To proceed, we rewrite the Schwarzschild metric, and get:

$$ds^2 = -f(r)dt^2 + \frac{}{1}f(r)dr^2 + r^2 d\Omega^2 \tag{3.31}$$

Where we assume the following:

$$r_s = 2M$$
$$f = 1 - \frac{r_s}{r}$$

If we now expand the function $f(r)$, considering the space near the horizon of the black hole, we get:

$$f(r) = f(r_s) + f'(r_s)(r - r_s) + \text{higher order terms} \tag{3.32}$$

Ignoring the higher order terms and noting that when r_s, $f(r_s) = 0$, we simplify the above equation to:

$$f(r) = f'(r_s)(r - r_s) \tag{3.33}$$

If we now have an observer at a distance r, and we want to know his proper distance from the horizon, we can get that integrating the second term in 3.5, to get:

$$\begin{aligned} d\rho &= \frac{dr}{\sqrt{f}} \\ &= \frac{dr}{\sqrt{f'(r_s)(r - r_s)}} \\ \rho &= \frac{2}{\sqrt{f'(r_s)}}\sqrt{r - r_s} \end{aligned}$$

We can then express f in terms of ρ:

$$f(r) = f'(r_s)(r - r_s) \tag{3.34}$$
$$= (\frac{1}{1}f'(r_s))^2 \rho^2 \tag{3.35}$$
$$= \kappa^2 \rho^2 \tag{3.36}$$

Where we have made the definition, $\kappa = (\frac{1}{1}f'(r_s))^2$. Putting this back in the equation 3.5 we have, near the horizon of the black hole:

$$ds^2 = -\kappa^2 \rho^2 + d\rho^2 + {r_s}^2 d{\Omega_2}^2 \tag{3.37}$$
$$= -\rho^2 d\eta^2 + {r_s}^2 d{\Omega_2}^2 \tag{3.38}$$

Here we have defined $\eta = \kappa t = \frac{t}{2r_s}$. If we now equate this equation with 3.30, and ignore the 2 dimensional sphere, it becomes self evident that this space does not cover all four quadrants of the Minkowski space. Since $X^2 - T^2 = \rho^2$, and ρ is a positive quantity, the equation only corresponds to the area given by $X \geq 0$, or region I, as is shown in 3.1.

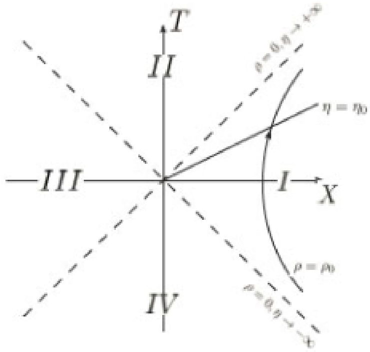

Figure 3.1: Region of Rindler Geometry [8]

We can see from the figure that the line of constant ρ, i.e, for $X^2 - T^2 = constant$ is a hyprebola.

Chapter 4

Functionals and Path Integrals

In this chapter we will introduce and elaborate where necessary, some of the prerequisites for the upcoming chapters. We will briefly talk about functional derivatives before moving onto path integrals, providing with examples where suitable.

4.1 Functionals and Functional Derivatives

4.1.1 Functionals

Functionals are defined by the rule that associates a real or complex number with a function that can have more than one variables. To put it simply, a functional is a function of a function. Where a function takes numbers as input and provides numbers as output, a functional takes functions as input and provides numbers as output. Some examples of functionals would be, a definite integral over a continuous function [9]:

$$F[f] = w(x) \int_{x_1}^{x_2} f(x)dx \tag{4.1}$$

Where $w(x)$ is a fixed weight function and x is called the parameter. If we replace this fixed weight function with a generalized function, or a distribution, we get:

$$F[f] = \int_{x_1}^{x_2} \delta(x - x_0)f(x)dx \tag{4.2}$$

Where we have the w with the δ-function. It is evident from 4.2 that functionals can themselves be functions of the parameters. We can see this if we simply 4.2 further, to get:

$$F[f] = f(x_0) \tag{4.3}$$

Examples of functionals are abundant in physics, one of which would be in the Thomas-Fermi model. This theory provides us with the functional form of the kinetic energy of a non-interacting electron gas, as a function of density. Another example of functionals is the Wheeler–De Witt equation. It is a cornerstone of theoretical physics, especially in quantum gravity.

4.1.2 Functional Derivatives

As the name suggests, functional derivatives is the product of taking a functional and differentiating it with respect to it's variables [9]. Let us take the following functional as an example:

$$F[f] = w(x)\int w(x)f(x)dx \tag{4.4}$$

We can take the weight function inside the integral as it is fixed. The most general relation that can be used to take functional derivatives is:

$$\frac{\delta f(x)}{\delta f(x_0)} = \delta(x - x_0) \tag{4.5}$$
$$\frac{\delta F}{\delta f(x_0)} = \int w(x)\frac{\delta f(x)}{\delta f(x_0)}dx \tag{4.6}$$
$$\frac{\delta F}{\delta f(x_0)} = (x)\delta(x - x_0)dx \tag{4.7}$$
$$\frac{\delta F}{\delta f(x_0)} = w(x_0) \tag{4.8}$$

Where on the right we have used the Dirac delta function, as was introduced in the previous section. We now do an example of computing functional derivatives. Let us take the following functional:

$$T[F] = exp[a\int t(p', p")F(p')F(p")dp'dp"] = expZ[F] \tag{4.9}$$

We know,

$$\frac{\delta T}{\delta F(p)} = (\frac{\delta Z}{\delta F(p)})T \tag{4.10}$$

We now need to deduce $(\frac{\delta Z}{\delta F(p)})$, we start by using product rule of differentiation, to get:

$$\frac{\delta Z}{\delta F(p)} = a\int t(p', p")F(p')(\frac{\delta F(p")}{\delta F(p)}) + (\frac{\delta F(p")}{\delta F(p)})a\int t(p', p")F(p') \tag{4.11}$$

Now, using 4.5, and after conducting further simplification, we get:

$$\frac{\delta Z}{\delta F(p)} = a\int (t(p', p) + t(p, p'))F(p')dp' \tag{4.12}$$

We now make the assumption that the function $t(p', p)$ is symmetric in it's variables, meaning that $t(p', p) = t(p, p')$ and end up with the following equation:

$$\frac{\delta Z}{\delta F(p)} = 2a(p', p)F(p')dp' \tag{4.13}$$

Putting this back in 4.10, we get:

$$\frac{\delta T}{\delta F(p)} = 2a(p', p)F(p')dp'T \tag{4.14}$$

We can now repeat this process on the functional $T[F]$ to get higher order functional derivatives.

4.1.3 Functional Integrals

Functional integration was developed by Percy Daniell in an article of 1919 and Norbert Wiener in a series of studies culminating in his articles of 1921 on Brownian motion. Together they developed a rigorous method for assigning a probability to a particle's random path, a process now known as the Wiener measure.

In functional integration, the domain of an integral is not a region of space anymore, but a space of functions. It is a collection of results that are of much significance in the studies of mathematics and physics. They are most useful in to solve problems involving probability, and one of the most famous and useful functional integrals is the path integral, which we will be talking about in the next section.

In functional integration, a functional $G[A]$ is summed over a continuous range of functions A. Using perturbative methods is usually the go to method to solve most functional integrals as most of them can not be evaluated exactly. The handful ones that can are done so using the Gaussian Integral. The formal definition of a functional integral is [2]:

$$\int D[A]G[A] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G[A]\Pi_n dA_n \tag{4.15}$$

4.2 Path Integral

"The electron does anything it likes. It goes in any direction at any speed, forward and backward in time, however it likes, and then you add up the amplitudes and it gives you the wave-function."

The stated quote was delivered by none other than Richard Feynman during a conversation with Freeman Dyson. He was speaking, of course, about his path integral method. This ingeneous and extremely intuitive method takes account of every single possible trajectory that a particle can take when going from point A to point B and sums over all of them to give us the propagator amplitude. In this section, we derive the path integral method which will come in handy to us later on.

According to Feynman, all of the trajectories that the electron will take to get from one point to another, will contribute exactly a complex factor of $e^{iS\hbar}$, where the S is the action of the trajectory. Our job here is to add up all of those contributions to get the amplitude. Let us start be defining the following terms:

$$\begin{aligned} A &= (t_a, q_a) \quad \text{(a spacetime point)} \\ B &= (t_b, q_b) \quad \text{(another spacetime point)} \end{aligned}$$

And let

$$G = \hat{U}(t_b, t_a) \tag{4.16}$$

be the time-evolution operator that takes the particle at time t_a and evolves it to time t_b [3]. This time-evolution operator is:

$$\hat{U}(t_b, t_a) = e^{-i\hat{H}(t_b - t_a)} \tag{4.17}$$

It is important to note that we will employ natural units from this point on. We can then say:

$$G = \langle q_b| e^{-i\hat{H}(t_b - t_a)} |q_a\rangle \tag{4.18}$$

Where $|q\rangle$ is an eigenstate of position. Now let us divide the total time taken for this trajectory into N infinitesimal steps, as shown in figure 4.1. This process is known as time-slicing. We are able to time-slice as $\hat{U}(t)$ is a unitary operator. This gives us [9]:

$$G = \langle q_b| e^{-i\hat{H}(t_b - t_a)N} |q_a\rangle \tag{4.19}$$

$$= \langle q_b| e^{-i\hat{H}\Delta t} \dots e^{-i\hat{H}\Delta t} \dots e^{-i\hat{H}\Delta t} |q_a\rangle \tag{4.20}$$

Where $\Delta t = \frac{t}{N}$. We now insert an identity in equation 4.20, using the completeness theorem. The completeness theorem states that:

$$\int dq_n |q_n\rangle \langle q_n| = 1 \tag{4.21}$$

This is also sometimes known as *fat unity* [9]. The idea is to insert a fat unity and sandwich them between each mini time-evolution operator. We insert $N - 1$ fat unities to get:

$$G = \langle q_b| e^{-i\hat{H}\Delta t}[\int dq_{N-1} |q_{N-1}\rangle \langle q_{N-1}|]e^{-i\hat{H}\Delta t} \dots \tag{4.22}$$

$$\dots e^{-i\hat{H}\Delta t}[\int dq_{N-2} |q_{N-2}\rangle \langle q_{N-2}|]e^{-i\hat{H}\Delta t} \dots \tag{4.23}$$

$$\dots e^{-i\hat{H}\Delta t}[\int dq_1 |q_1\rangle \langle q_1|]e^{-i\hat{H}\Delta t} |q_a\rangle \tag{4.24}$$

After rearranging this equation a little bit, we get:

$$G = \int dq_1 \dots dq_{N-1} \langle q_b| e^{-i\hat{H}\Delta t} |q_{N-1}\rangle \dots \langle q_1| e^{-i\hat{H}\Delta t} |q_a\rangle \tag{4.25}$$

We now split the Hamiltonian operator into it's two terms, namely:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V\hat{(q)}. \tag{4.26}$$

Where the terms $\hat{p}$ and $\hat{V}$ are the momentum and potential operators respectively. We know that the position states $|q_n\rangle$ are eigenvalues of the potential operator, meaning we can have the operator act through the state and we will be left with numbers. However, we can not say the same for the momentum operators. In order to bypass through this problem, we expand the position states in terms of momentum eigenstates in the usual way:

$$\int \frac{dp}{(2\pi)^{\frac{1}{2}}} |p\rangle |q_n\rangle \langle p| = \int \frac{dp}{(2\pi)} |p\rangle e^{-ipq_n} \tag{4.27}$$

Replacing $|q_n\rangle$ this way and making using the eigenvalues of the momentum and potential operators, we look at one individual mini time evolution operator and get:

$$G_n = \int \frac{dp}{(2\pi)^{\frac{1}{2}}} \langle q_{n+1}| \, |p\rangle \, e^{-i\frac{p^2}{2m}\Delta t} e^{-ipq_n} e^{-iV(q_n)\Delta t} \tag{4.28}$$

$$= \int \frac{dp}{(2\pi)} e^{-ipq_{n+1}} e^{-i\frac{p^2}{2m}\Delta t} e^{-ipq_n} e^{-iV(q_n)\Delta t} \tag{4.29}$$

$$= \int \frac{dp}{(2\pi)} e^{-i\frac{p^2}{2m}\Delta t + ip(q_{n+1}-q_n)} e^{-iV(q_n)\Delta t} \tag{4.30}$$

We can see now that we have successfully gotten rid of all the operators that we began this derivation with [9]. All we have left to do is the integration over p. We use Gaussian integral method to solve this and end up with an exact solution, which is:

$$G_n = (\frac{-im}{2\pi\Delta t})^{\frac{1}{2}} e^{\frac{im}{2}\frac{(q_{n+1}-q_n)}{2}\Delta t e^{-iV(q_n)\Delta t}} \tag{4.31}$$

To simplify, we write the factor $(\frac{-im}{2\pi\Delta t})^{\frac{1}{2}}$ as ζ^{-1}, after putting this back into equation 4.25, we get the propagator amplitude, in the form of:

$$G =_{n=1}^{N-1} \frac{dq_n}{\zeta} e^{\frac{im}{2}\frac{(q_{n+1}-q_n)^2}{\Delta t^2}} \Delta t e^{-iV(q_n)\Delta t} \tag{4.32}$$

Finally, we take the limit $N \to \infty$. This turns makes the jagged shape of the trajectory to a more smooth one. Our summation now turns into an integration and $\frac{(q_{n+1}-q_n)^2}{\Delta t^2}$ into $\dot{q}^2$. Implementing these changes, we can finally write the propagator amplitude in the form that it is usually seen as [3]:

$$G = \int D[q(t)] e^{i\int dt[\frac{m\dot{q}^2}{2} - V(q)]} \tag{4.33}$$

Where we have made the change in notation, making:

$$\int D[q(t)] = \lim_{N\to\infty}^{N-1} {}_{n=1} \frac{dq_n}{\zeta} \tag{4.34}$$

This is an example of a functional integral, where we sum over all the possible paths coded as a massive multiple integral over all the time-sliced co-ordinates q_n. We can take this one step further and see in equation 4.33, the term $\frac{m\dot{q}^2}{2} - V(q)]$ is simply the Lagrangian describing the motion. And we know that the action of any path is given as $S = \int dt L[q(t)]$, we can write:

$$G = \int D[q(t)] e^{i\frac{S}{\hbar}} \tag{4.35}$$

Where we have restored the value of $\hbar$. Equation 4.35 is one that we will come back to in the next chapter.

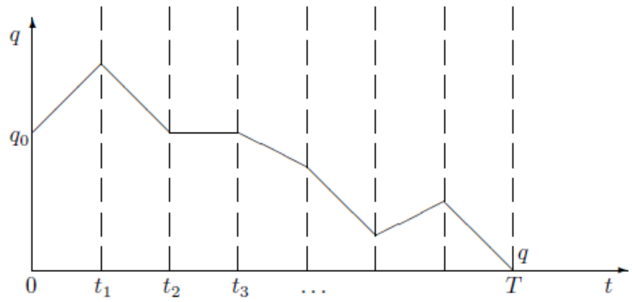

Figure 4.1: Time-slice of a particle's trajectory [3]

Chapter 5

Hawking Radiation

Black holes evaporate.

This single statement has been the cause of many sleepless nights for physicists all over the world for the past fifty years or so. Ever since Stephen Hawking showed in his paper, "Particle Creation by Black Holes", he showed us that black holes radiate particles as their mass decreases to eventually disappear. This of course with it brought the dilemma of the information paradox, a problem that to this day has not been solved. In this chapter, we will derive the Hawking Temperature in a few ways starting at different places.

5.1 In Euclidean Spacetime

The following two derivations will give us the Hawking Temperature in the Euclidean Space time.The first method we will try is going to be an analytical continuation of the Schwarzschild metric to the Euclidean signature, we then associate periodicity to the time parameter, in the same way as we go about describing a field theory in finite temperature. We then use the fact that the event horizon is nothing more than a co-ordinate singularity in the Lorentzian signature to eventually derive the temperature of a black hole in Euclidean signature. This is quite a simple derivation of the Hawking temperature compared to his original one, however, it serves it's purpose quite well.

Analytic

Let us start by performing a Wick rotation, which takes us to the Euclidean space [6], in equation 3.31, this gives us:

$$ds^2 = f(r)d\tau^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega^2 \tag{5.1}$$

Where, we have made the following change:

$$t = -i\tau \tag{5.2}$$

If we now assing periodicity to our time parameter such that:

$$\tau \approx \tau + \hbar\beta \tag{5.3}$$

Where $\beta = \frac{1}{k_b T}$, with k_b being the Boltzmann constant and T is temperature. In this signature, if we follow the same steps we did going from equation 3.31 to 3.37, we get the following expression for the line element in the Euclidean signature, ds_E^2:

$$ds_E^2 = \kappa^2 \rho^2 d\tau^2 + d\rho^2 + {r_s}^2 d{\Omega_2}^2 \tag{5.4}$$

Which is essentially the same equation as 3.37, except for in Euclidean signature. We now introduce a transformation:

$$\theta = \kappa\tau \tag{5.5}$$

This gives us:

$$ds_E^2 = \rho^2 d\theta^2 + d\rho^2 + {r_s}^2 d{\Omega_2}^2 \tag{5.6}$$

Where immediately we can see that the first two terms of the above metric is the 2 dimensional Euclidean flat space written in flat co-ordinates. Now, we know that in standard Euclidean geometry, the θ parameter is periodic in $\theta = \theta + 2\pi$, however, we can see from 5.5 that there is no bound we can set on τ [6]. Luckily, we notice that from the metric in 5.6, that there is a conical singularity at $\rho = 0$ unless the following is true:

$$\theta = \theta + 2\pi \tag{5.7}$$

And since we know θ and τ are related according to equation 5.5, we can say the following:

$$\tau = \tau + \frac{2\pi}{\kappa} \tag{5.8}$$

If we now equate 5.3 with 5.8, we can say:

$$\hbar\beta = \frac{2\pi}{\kappa} \tag{5.9}$$

$$\frac{\kappa}{\hbar} = 2\pi \tag{5.10}$$

$$\frac{1}{T} = \frac{2\pi k_B}{\kappa\hbar} \tag{5.11}$$

$$T = \frac{\kappa\hbar}{2\pi k_B} \tag{5.12}$$

$$T = \frac{\kappa}{2\pi} \tag{5.13}$$

Where in the last line we have resorted to using natural units. We have now successfully derived the expression for Hawking radiation for a black hole. We see from equation 5.13, using our definition of surface gravity from chapter 3, equation 3.14, that the temperature of a black hole is inversely proportional to it's mass. Therefore, as a black hole loses mass due to particle creation near it's horizon, it keeps radiating according to 5.13 until it eventually evaporates. It is important to note here that even though we derived this temperature using the near horizon metric, this is the finite temperature that will be experienced by an observer at infinity as the time parameter we used is the one for said observer.

5.1.1 Path Integral

We start this derivation with the same steps as the previous one. We write the Schwarzschild metric in Euclidean spacetime, after performing the Wick rotation:

$$ds^2 = (1 - \frac{2M}{R})d\tau^2 + (1 - \frac{2M}{R})^{-1}dr^2 + r^2 d\omega^2 \tag{5.14}$$

Using the definition of κ from equation 3.14, we get near horizon,

$$ds_E^2 = (\kappa x)^2 d\tau^2 + dx^2 + {r_s}^2 d{\Omega_2}^2 \tag{5.15}$$

And again, we see that the first two terms of this equation is just the 2 dimensional flat Euclidean spacetime, sometimes also known as Euclidean Rindler spacetime. If we make the periodic identification $\tau = \tau + \frac{2\pi}{\kappa}$, we can remove the singularity that arises when $r = 2M$ in equation 5.16 (or at $x = 0$ in equation 5.17), since it is just a co-ordinate singularity. Another way of putting that is that we need to solve the Euclidean functional integral over fields, $\Phi(\vec{x}, \tau)$, that are periodic in τ over the period $\frac{2\pi}{\kappa}$. We can now write:

$$Z = \int [D\phi] e^{-S_E[\Phi]} \tag{5.16}$$

Which we can recognise immediately as a functional integral, more specifically, the Euclidean functional integral, where the term $-S_E$ is simply the Euclidean action in Hamiltonian form, given by:

$$S_E = \int (-ip\dot{q} + H) \tag{5.17}$$

We now take the Gaussian integral of the above equation over fields Φ which are periodic over the period $\hbar\beta$ in Euclidean signature [7]. This gives us:

$$Z = tre^{\beta H} \tag{5.18}$$

Where we have use the fact that the exponential of the Hamiltonian can be written as a diagonal matrix, which, due to the basic properties of the exponential have the trace equal to it's determinant. Equation 5.18 is the partition function of a quantum mechanical system with Hamiltonian H. We now take notice of the two periodic and can immediate come to the relation:

$$\hbar\beta = \frac{2\pi}{\kappa} \tag{5.19}$$

This equation is identical to equation 5.9, which means we can now come to the conclusion:

$$T = \frac{\kappa}{2\pi} \tag{5.20}$$

And that, again, is the Hawking Temperature of a Black Hole for an observer at infinity.

5.1.2 Schwarzschild Anti de Sitter Black Hole

In this section we will take our calculations of deriving the Hawking temperature into the anti de-Sitter space. We first work out the meric of the four dimensional Schwarzschild anti de Sitter black hole, namely the $SadS_4$, then move onto define surface gravity in this space and work out the event horizon, finally we make Rindler approximation to derive the Hawking Temperature of $SadS_4$.

The $SadS_4$ metric

We start by writing equation 3.5 in exponential form:

$$ds^2 = e^{2\nu}dt^2 - e^{2\lambda}dr^2 - r^2 d{\Omega_2}^2 \tag{5.21}$$

The above is a general form of a static spherically symmetric metric. We can write the above equation with no loss of generality as the exponential function will never yield negative results for real inputs. Analogous to what we had in equation 3.5, in equation 5.21 we have the following:

$$\nu = \nu(r)$$
$$\lambda = \lambda(r)$$

We will choose the co-ordinates such that $(x^0, x^1, x^2, x^3) = (t, r, \theta, \psi)$. It is evident from 5.21 that the metric, $g_{\mu\nu}$ is of the following form [11]:

$$\mathbf{g}_{\mu\nu} = \begin{pmatrix} e^{2\nu} & 0 & 0 & 0 \\ 0 & -e^{2\nu} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta \end{pmatrix}$$

Meaning that $diag(g_{00}, g_{11}, g_{22}, g_{33}) = (e^{2\nu}, -e^{2\nu}, -r^2, -r^2 sin^2\theta)$ And using the relation between co-variant and contravariant tensors, we can write:

$$\mathbf{g}^{-1}_{\mu\nu} = \mathbf{g}^{\mu\nu} = \begin{pmatrix} e^{-2\nu} & 0 & 0 & 0 \\ 0 & -e^{-2\nu} & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ 0 & 0 & 0 & -r^{-2} sin^{-2}\theta \end{pmatrix}$$

i.e, $diag(g^{00}, g^{11}, g^{22}, g^{33}) = (e^{-2\nu}, -e^{-2\nu}, -r^{-2}, -r^{-2}sin^{-2}\theta)$. To proceed, we recall from general relativity that the affine connections, or Christoffel symbols, can be written in terms of the metric tensor as the following:

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda}(\delta_\nu g_{\rho\lambda} + \delta_\rho g_{\nu\lambda} - \delta_\lambda g_{\nu\rho}) \tag{5.22}$$
$$= \frac{1}{2} g^{\mu\lambda}(\frac{\delta g_{\rho\lambda}}{\delta x^\nu} + \frac{\delta g_{\nu\lambda}}{\delta x^\rho} - \frac{\delta g_{\nu\rho}}{\delta x^\lambda}) \tag{5.23}$$

If we now work out the chirstoffel symbols for the above metric, we end up with nine non-zero terms that are independent algebraically:

$$\Gamma^0_{01} = \nu'$$
$$\Gamma^2_{12} = \Gamma^3_{13} \qquad = \frac{1}{r}$$
$$\Gamma^1_{11} = \lambda'$$
$$\Gamma^2_{33} = -\sin\theta\cos\theta$$
$$\Gamma^1_{22} = -re^{-2\lambda}$$
$$\Gamma^3_{23} = \cot g\theta$$
$$\Gamma^1_{33} = -re^{-2\lambda}\sin^2\theta$$
$$\Gamma^1_{00} = \nu' e^{2(\nu-\lambda)}$$

Where $\nu' = \frac{d\nu}{dr}$ and $\lambda' = \frac{d\lambda}{dr}$. Next we work out the non vanishing Ricci Tensors from the above mentioned Christoffel symbols. We know the formula for Ricci tensor looks like:

$$R_{\mu\nu} = R^{\rho}_{\sigma\rho\nu} = \Gamma^{\rho}_{\sigma\nu,\rho} - \Gamma^{\rho}_{\sigma\rho,\nu} + \Gamma^{\lambda}_{\sigma\nu}\Gamma^{\rho}_{\lambda\sigma} - \Gamma^{\lambda}_{\sigma\rho}\Gamma^{\rho}_{\lambda\nu} \tag{5.24}$$

Using the symmetry components of Ricci tensor, we obtain the following non-zero components:

$$R_{00} = (\nu'' - \nu'\lambda' + \nu'^2 + \frac{2\nu'}{r})e^{2(\nu-\lambda)} \tag{5.25}$$

$$R_{11} = -\nu'' + \frac{2\lambda'}{r} - +\nu'^2 + \lambda'\nu' \tag{5.26}$$

$$R_{22} = (r\lambda' - r\nu' - 1)e^{-2\lambda} + 1 \tag{5.27}$$

$$R_{33} = \sin^2\theta R_{22} \tag{5.28}$$

The vacuum Einstein equations equipped with the cosmological constant looks like

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = 0 \tag{5.29}$$

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \tag{5.30}$$

Where in the second equation we have used contraction of indices. To proceed, we recall from the metric that $g_{00} = e^{2\nu}$. Using this and 5.25 we can write 5.30 as:

$$(\nu'' - \lambda'\nu' + \nu'^2 + \frac{2\nu'}{r})e^{2(\nu-\lambda)} = -\Lambda e^{2\nu} \tag{5.31}$$

$$\nu'' - \lambda'\nu' + \nu'^2 + \frac{2\nu'}{r}) = -\Lambda e^{2\lambda} \tag{5.32}$$

Similarly if now use 5.26 in place of 5.25, we come up with the following equation:

$$-\nu'' + \frac{2\lambda'}{r} - +\nu'^2 + \lambda'\nu' = \Lambda e^{2\nu} \tag{5.33}$$

We can now add 5.32 and 5.33 to get the following:

$$\nu' + \lambda' = 0 \tag{5.34}$$

Which tells us that $\lambda(r) = -\nu(r) + constant$. However, in the asymptotically flat Schwarzschild geometry it is a requirement that λ,$\nu \to 0$ as $r \to 0$. Which is only possible if the constant in the above equation is zero [11]. Therefore from this we can write

$$\lambda(r) = -\nu(r) \tag{5.35}$$

We now put 5.27 in 5.30 to get:

$$(r\lambda' - r\nu' - 1)e^{-2\lambda} + 1 = \Lambda g_{\mu\nu} \tag{5.36}$$

$$(r\lambda' - r\nu' - 1)e^{-2\lambda} + 1 = \Lambda r^2 \tag{5.37}$$

If we then use the fact that $\lambda(r) = -\nu(r)$ in the above equation we end up with the following equation:

$$(1 + 2r\nu')e^{2\nu} - 1 = \Lambda r^2 \tag{5.38}$$

We can see that the right hand side of 5.38 can be written as $\frac{d}{dr}(re^{2\nu})$, using this we can now get the following equation:

$$\frac{d}{dr}(re^{2\nu}) = 1 - \Lambda r^2 \tag{5.39}$$

$$(re^{2\nu}) = r - \frac{\Lambda r^3}{3} + constant \tag{5.40}$$

$$e^{2\nu} = 1 + \frac{constant}{r} - \frac{\Lambda r^2}{3} \tag{5.41}$$

$$g_{00} = 1 + \frac{constant}{r} - \frac{\Lambda r^2}{3} \tag{5.42}$$

Where the singularity lies in $r = 0$. We know that for small r, the Newtonian approximation tell us that the constant is $2M$, where M is the gravitational mass, putting this in the above equation, we get the following:

$$g_{00} = 1 + \frac{2M}{r} - \frac{\Lambda r^2}{3} \tag{5.43}$$

We now recall that the cosmological constant is defined by

$$\Lambda = \frac{1}{2}R \tag{5.44}$$

Where R is the scalar curvature of AdS_4, given by $R = -\frac{6}{a^2}$. Putting R back in the eqution we get:

$$\Lambda = -\frac{3}{a^2}R \tag{5.45}$$

We now put this definition of the cosmological constant in our calculation of the metric and get the following:

$$g_{00} = 1 - \frac{2M}{r} + \frac{r^2}{a^2} \tag{5.46}$$

And this gives us the full metric of the Schwarzschild AdS_4 as:

$$ds^2_{SAdS_4} = (1 - \frac{2M}{r} + \frac{r^2}{a^2})dt^2 - \frac{dr^2}{1 - \frac{2M}{r} + \frac{r^2}{a^2}} - r^2 d\Omega_2^2 \tag{5.47}$$

Hawking Radiation of SAds Black Hole

We start with the time-radial part of the $SAdS_4$ metric near the horizon and define a parameter ρ such that $|\rho| << r_h$, $[\rho] = L^1$ and get:

$$r = r_h + \frac{\alpha\rho^2}{r_h} \tag{5.48}$$

Where we determine what α is later, and keep terms leading up to $O(\rho^2)$. If we now take the metric 5.47 and can rewrite it as the following using the time-radial [11]:

$$ds^2_{SAdS_4}|_{t/r} = \frac{2\alpha}{r_h\kappa}((\kappa\rho)^2 dt^2 - d\rho^2) \tag{5.49}$$

Where κ is the surface gravity of this black hole. We can then make the identification that $a_R = \kappa$. We now choose α such that $\alpha = \frac{r_h \kappa}{2}$ to get:

$$ds^2_{SAdS_4}|_{t/r} = ds^2_R \tag{5.50}$$

Where ds^2_R is the Rindler Metric we saw in 3.20, with Rindler acceleration $a_R = \kappa$. Now the time-radial part of the co-ordinate becomes:

$$r = r_h + \frac{\kappa}{2}\rho^2 \tag{5.51}$$

By comparing equations we can see that the Hawking Temperature at the horizon of a $SAdS_4$ black hole is:

$$T_R = \frac{a_R}{2\pi} \tag{5.52}$$

However, we know that for a Schwarzschild Black Hole, the surface gravity is:

$$\kappa = \frac{M}{r_h^2} + \frac{r_h}{a^2} \tag{5.53}$$

Putting this back we get[11]:

$$T_{Hawk}|_{SAdS_4} = \frac{\kappa}{2\pi} = \frac{1}{2\pi}(\frac{M}{r_h^2} + \frac{r_h}{a^2}) \tag{5.54}$$

Which, finally, is the Hawking Temperature for a Schwarzschild Anti de Sitter black hole.

Chapter 6

Conclusion

This paper is more of a review and analysis of work that has already been done on the premise of black holes and Hawking Radiation. I have tried to build a self sufficient work which ultimately derives the Hawking radiation for a Schwarzschild black hole in four dimensional AdS space. Taking help from various textbooks, articles and in desperate situations even youtube videos, I have tried my best to make this a paper worth putting some time to read.

However, undoubtedly, there are areas where the explanation, the analysis or even the math might not sit right or fit in well. I ask for the reader's forgiveness in advance for that. Being an undergraduate student, tackling a topic like information paradox was not an easy feat for me. The field of black holes is one of extreme debate and virulent interest in the world of physics right now, there is much more work left to be done and more mysteries to uncover than lack thereof. In the future, hopefully we finally unmask the entity known as black holes in all it's glory.

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