A Two Loop Test of Matrix Big Bang Model

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A thesis submitted to the Department of Mathematics and Natural Sciences in partial fulfilment of the requirements for the degree of B.S. in Physics

Department of Mathematics and Natural Science
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Declaration

It is hereby declared that

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2. The thesis does not contain material previously published or written by a third party, except where this is appropriately cited through full and accurate referencing.
3. The thesis does not contain material which has been accepted, or submitted, for any other degree or diploma at a university or other institution.
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Abstract

In this thesis we expand the effective action for the Matrix Big Bang presented in [9, 10] and indicate the interaction terms. Then we compute the two loop effective potential of this Matrix Big Bang model. We show that this two loop effective potentials are attractive near the big bang and turns off very rapidly in late time. The form of the potential indicates that they are coming from a multi-D-brane contribution establishing a connection between late time physics and cosmological singularity.

Keywords: String Theory, D-brane, Two Loop, Matrix Theory, M-Theory

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Contents

List of Figures

Chapter 1

Introduction and Motivation

1.1 The Impact of Unification

Throughout the history of science the idea of unification has played a vital role in studying diverse phenomenon. The goal was to describe this diverse events with a handful of simple ideas using some fundamental building blocks. James Clerk Maxwell's unification of Electric & magnetic forces later paved the way for Special Relativity.

Another notable successful theory is Quantum Mechanics , whose marriage with Special Relativity gave rise to Quantum Field Theory. Till to this day it is one of the most successful frame work physicists have. Later strong and weak nuclear forces were taken into account and the resulting theory came to be known as the Standard Model , which is also known as $SU(3) \times SU(2) \times U(1)$ theory. However , it does not incorporate gravity in the. General Theory of Relativity is successful and well tested theory. So the attempt to include gravity was taken. But this presented some challenges. One of them is , when one tries to combine gravity and quantum field theory, the resulting theory is a non-renormalizable theory. Moreover, general theory of relativity predicts spacetime and cosmological singularities. In a consistent theory of quantum gravity these singularities should disappear.

To get a remedy of this problem one needs to find a framework that will solve the problem of divergences. Several ideas have proposed and one of them is to study quantum theory of one dimensional objects called strings.

1.2 A First Look at Strings

The ideas that were put forward to solve the problem mentioned in the previous sections contained extra dimensions ,supersymmetry and grand unification. String Theory contains all these features. Also, these ideas are consistent with different tests of Standard Model. Moreover , if we study the quantum theory of one dimensional objects , we can see the emergence of gravity. This is because every string theory must contain massless spin 2 particles called graviton. Furthermore, in the low energy limit the interactions reproduces General Theory of Relativity.

Unlike the standard Model , string Theory contains no free parameters which corresponds to no adjustable coupling constants. String theory also predicts extra dimension. String theory requires the definite number of spacetime dimensions. For weakly coupled string theory the number is 10. For strongly coupled a picture changes. In addition , it reflects upon some of the challenging problems of quantum gravity. One of the interesting examples would be resolving the physics of black holes and singularities in a early universe. One can say this is like a Pandora's Box but only good things came out of it.

On a more profound level, the string hypothesis gives new and extremely amazing techniques to comprehend parts of quantum measure speculations. Of these, the most surprising is the AdS/CFT correspondence. These thoughts have been applied in territories going from atomic material science to condense matter science and have given new understandings into strongly coupled phenomena.

1.3 Superstrings and M-Theory

Early work in string theory shows that bosonic string is unique. However , to describe nature one needs fermions as well . So , one is equipped with a choice on how they can add fermions in the worldsheet. This gives rises to five different types of perturbative superstring theory. The critical dimension of a superstring theory is $D = 10$. The five types of superstring theories are known as **Type I , Type IIA, Type IIB, Heterotic** $SO(32)$ and **Heterotic** $E_8 \times E_8$. These five type of seemingly diferent theories are related to one another via a web of dualities. The following is a diagram to see how they are connected to each other.

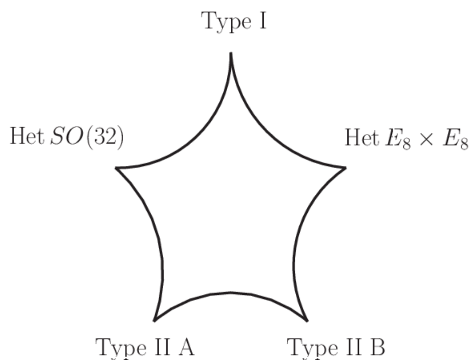

Figure 1.1: Web of dualities connecting superstring theories.

From the above diagram one can infer that all the possible superstring theory could be different phases of a single theory. This theory is called ***"M-Theory"***. This theory would contain two and five dimensional objects called **"Branes"** and the low energy effective theory of M-Theory would be described by 11 dimensional Supergravity. However , the full formulation of this theory is yet unknown. At present , there exists formulation which is based on **AdS/CFT Correspondence** and **"Matrix Theory"**. The Matrix theory of M-theory in the low energy agrees with the results from 11 dimensional supergravity. Furthermore , one can get a non-commutative spacetime geometry and nonperturbative realization of the holographic principle from it.

Chapter 2

A Brief Introduction to Bosonic String Theory

2.1 Action principles for Relativistic Point Particles

The principle of least action is a variational principle that, when applied to the action of a mechanical system, can be used to obtain the equations of motion for that system. This is a well known principle in all arenas of physics. So now we want to write down action functional for our one dimensional objects and find their equation of motions. A good starting point is to see how a relativistic point particle behaves in spacetime with the help of action principles.

Before we dive into the equations we want to fix our conventions.

- We take $\hbar = c = 1$
- D dimensional flat metric has a signature $\eta_{\mu\nu} = (-, +, +, +, ..., +)$
- The spacetime co-ordinates are represented by X^μ

Now, let's suppose our relativistic point particle sweeps out a worldline in spacetime and that line is parameterized by the proper time τ. So in order to get the equations of motion we need to extremize the action along the path the particle has taken. So we write down the action as

$$S = -C \int d\tau \sqrt{-\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} \tag{2.1}$$

where C is a proportionality constant which has a dimension of mass m and it is easily seen from a dimensional analysis .

In the right hand side of (2.1) , we have a length term $ds = d\tau\sqrt{-\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}$ which have a dimension of $[L] = \frac{1}{[M]}$. But the actions S is dimensionless so we have $[C] = [M]$ and safely replace $C = m$. Also , we fix a frame with co ordinates $X^\mu = (\tau, x)$ So our action is :

$$S = -m \int d\tau \sqrt{1 - \dot{x} \cdot \dot{x}} \tag{2.2}$$

and here $\dot{x}$ represents the three velocity of the test particle with respect to the chosen co ordinates.

To check whether this action gives us the known results from special relativity we can can calculate the conjugate momentum p . We identify the Lagrangian as $L = -m\sqrt{1 - \dot{x}.\dot{x}}$

Now , we have :

$$p = \frac{\partial L}{\partial \dot{x}} = \frac{m\dot{x}}{\sqrt{1 - \dot{x} \cdot \dot{x}}} \tag{2.3}$$

which matches with special relativity. Also one can calculate the Hamiltonian to end up with the famous mass energy relation $E^2 = m^2 + p^2$. Now quantization of this action is very simple and we can do it using canonical formalism. Now we apply the ideas from the point particle case in the arena of strings.

2.2 Actions For Strings

Now we consider a one dimensional object called string . For now we will focus on closed strings. Just like the particle sweeps out a wordline in Minkowski space , a string will sweep out a surface. We call this surface *world-sheet* of the string. The world-sheet can be thought of as a curved 2 dimensional surface and we identify two parameters for this world sheet. One being τ and one being σ . Where $\sigma \to \sigma + 2\pi$. So , we can think of a map from the world-sheet to Minkowski spacetime.

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau). \tag{2.4}$$

We call the X^μ to be coordinates of the *Target Spacetime*. Now , since the closed string gives rise to worldsheet , we can make a good guess that the action for the string will be proportional to the area of the world-sheet.

First we introduce a world-sheet metric $\gamma_{\alpha\beta}$. Then we can find a transformation of the target spacetime to world-sheet by the following:

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu}. \tag{2.5}$$

So, the action is,

$$S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\det\, \gamma}. \tag{2.6}$$

Here α' is known as the *Regge slope* and this action is known as *Nambu-Goto Action*. To find the equations of motion we vary the action with respect to the metric on the world-sheet.

$$\begin{aligned}
\delta S &= -\frac{1}{2\pi\alpha'}\int d^2\sigma\, \delta\sqrt{-\gamma} \\
&= -\frac{1}{2\pi\alpha'}\int d^2\sigma\, \frac{1}{2}\sqrt{-\gamma}\gamma^{\alpha\beta}\delta\gamma_{\alpha\beta} \\
&= -\frac{1}{2\pi\alpha'}\int d^2\sigma\, \frac{1}{2}\sqrt{-\gamma}\gamma^{\alpha\beta}\delta(\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}) \\
&= -\frac{1}{2\pi\alpha'}\int d^2\sigma\, \sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu \eta_{\mu\nu}\, \partial_\beta(\delta X^\nu) \\
&= \frac{1}{2\pi\alpha'}\int d^2\sigma\, \partial_\beta(\sqrt{\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu)\eta_{\mu\nu}\delta X^\nu.
\end{aligned}$$

In the last line we have discarded the boundary terms.

To get the equation of motion we do the following

$$\frac{\delta S}{\delta X^\nu} = 0. \tag{2.7}$$

So we get

$$\partial_\beta(\sqrt{-\gamma}\gamma^{\alpha\beta}\partial_\alpha X^\mu) = 0. \tag{2.8}$$

This is our equation of motion. In terms of X^μ they are really messy. However , we can still talk about the symmetries of action (2.6). But Before doing that we will look at the *Polyakov action*.

The Polyakov action eliminates the square root in Nambu-Goto action in the expense of introducing another field $g^{\alpha\beta}$ which is a metric on the tangent manifold. The action is given by

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{2.9}$$

The advantage of Polyakov action is that it is easier to work with actions that don't have square root in it in the path integral framework. The action in (2.9) gives the equation of motion when varied with respect to X^ν

$$\partial_\beta(\sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu) = 0. \tag{2.10}$$

Which has a similar structure to equation (2.8) but here $g^{\alpha\beta}$ is an independent variable and it will have its equation of motion as well. Varying the action with respect to the metric we have,

$$\delta S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \delta g^{\alpha\beta}(\sqrt{-g}\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2}\sqrt{-g}g_{\alpha\beta}g^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X^\nu)\eta_{\mu\nu} = 0. \tag{2.11}$$

Now, we choose conformal gauge which is $g_{\alpha\beta} = e^{2\phi}\,\eta_{\alpha\beta}$. Also, after making this gauge choice , we can do a Weyl transformation and make $\phi = 0$ to get $g_{\alpha\beta} = \eta_{\alpha\beta}$. The reason we could do this ,is coming from the gauge symmetry of the Polyakov action. We define the stress-energy tensor as the following

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}} \tag{2.12}$$

Finally we have using (2.11) and (2.12)

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\eta^{\rho\sigma}\partial_\rho X^\mu \partial_\sigma X_\mu = 0 \tag{2.13}$$

Equation (2.13) gives rises to some constraints . We notice that $T_{\alpha\beta} = T_{\beta\alpha}$. More explicitly the constrains are

$$\begin{aligned} T_{01} &= \dot{X}.X' = 0 \\ T_{11} = T_{00} &= \frac{1}{2}(\dot{X}^2 + X'^2) = 0 \end{aligned} \tag{2.14}$$

Now use this conformal gauge in equation (2.10) and we get

$$\partial_\alpha \partial^\alpha X^\mu = 0 \tag{2.15}$$

Which is simply the wave equation.

2.3 Mode Expansion for Closed String

Firstly to solve (2.15) we introduce worldsheet lightcone co-ordinates as $\sigma^\pm = \tau \pm \sigma$ and the derivatives in the lightcone co-ordinates take the form as $\partial_\pm = \partial_\tau \pm \partial_\sigma$. Using these equation (2.15) takes the form

$$\partial_+ \partial_- X^\mu = 0. \tag{2.16}$$

The solution of (2.16) can be written in terms of left-moving$X^\mu_L(\sigma^+)$ and right-moving part $X^\mu_R(\sigma^-)$. So , we write the most general solution as

$$X(\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-). \tag{2.17}$$

We also remember this solution is still subject to constraints (2.13) and the periodicity condition in this choice of coordinates becomes $X(\sigma^{\pm}, \tau) = X(\sigma^{\pm} + 2\pi, \tau)$.

Since the solution has a translational invariance in $\sigma^{\pm}$ direction we can write the most general solution in terms of the Fourier modes.

$$X^{\mu}_{L}(\sigma^{+}) = \frac{1}{2}x^{\mu} + \frac{1}{2}p^{\mu}\sigma^{+} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^{\mu}_{n}e^{-in\sigma^{+}}. \tag{2.18}$$

$$X^{\mu}_{R}(\sigma^{-}) = \frac{1}{2}x^{\mu} + \frac{1}{2}p^{\mu}\sigma^{-} + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^{\mu}_{n}e^{-in\sigma^{-}}. \tag{2.19}$$

These solutions plays critical role in the quantum theory of the strings. We will look at open strings in the next section and will analyze different boundary conditions for them.

2.4 Open Strings and Emergence of D-Branes

In this section, we will look at the dynamics of open strings. Just like before we denote target spacetime by $X^{\mu}(\sigma, \tau)$. The only difference is there is no periodicity condition on σ. One can think of parameterizing the length of the string by σ where

$$\sigma \ \epsilon \ [0, \pi]. \tag{2.20}$$

Now, We want to use Polyakov action to describe our open strings. To do that, we require locality. By locality we mean no point in the string can know whether it's a part of an open string or closed string. It will be the boundary conditions which will tell us how the end points are behaving.

With all this in our mind , we write down the Polyakov action in the conformal gauge

$$S \ = \ -\frac{1}{4\pi\alpha'}\int d^{2}\sigma \, \partial_{\alpha}X^{\mu}\partial^{\alpha}X_{\mu}.$$

Next we take the variation of this actions with to get equations of motion

$$\begin{aligned} \delta S \ &= \ -\frac{1}{2\pi\alpha'}\int_{\tau i}^{\tau_f} d\tau \int_{0}^{\pi} d\sigma \, \partial_{\alpha}X^{\mu} \, \partial^{\alpha}\delta X_{\mu} \\ &= \frac{1}{2\pi\alpha'}\int d^{2}\sigma \, \partial_{\alpha}\partial^{\alpha}X^{\mu} \, \delta X_{\mu} + \frac{1}{2\pi\alpha'}\left[\int_{0}^{\pi} d\sigma \, \dot{X}^{\mu}\delta X_{\mu}\right]_{\tau_i}^{\tau_f} - \frac{1}{2\pi\alpha'}\left[\int_{\tau_f}^{\tau_f} d\tau X'^{\mu}\delta X_{\mu}\right]_{0}^{\pi}. \end{aligned} \tag{2.21}$$

The first term in (2.21) is the term from where we get the equations of motion. The second and third terms are the boundary terms. In general we require $\delta X = 0$ on the boundary points. However we have something more here. The third term appearing on (2.21) also needs to vanish to get the equations of motion. So we write the following

$$\begin{aligned} X'^{\mu}\delta X_{\mu} &= 0 \\ \implies \partial_{\sigma} X^{\mu}\delta X_{\mu} &= 0. \end{aligned} \tag{2.22}$$

So we can say either $\delta X_{\mu} = 0$ or $\partial_{\sigma} X_{\mu} = 0$ or they are both zero at $\sigma = 0, \pi$.

The condition $\partial_{\sigma} X^{\mu} = 0$ is known as the **Neumann boundary conditions**. As in this condition there is no restriction on δX_{μ}, the open end of the string moves freely. With the help of the constraints in (2.14) one can show that open ends of the strings moves at the speed of light.

We now look the other type of boundary condition we can impose on the endpoints of the open string. The boundary condition is $\delta X^{\mu} = 0$. This is known as the **Dirichlet boundary conditions**. This implies that the endpoint of the string is at a constant position $X^{\mu} = h^{\mu}$. To understand the physical significance of this let's consider Neumann boundary conditions for some coordinates and Dirichlet boundary conditions for the other coordinates.

$$\begin{aligned} &\partial_{\sigma} X^{a} = 0 \qquad \text{for } a = 0, ..., p \\ &X^{I} = h^{I} \qquad \text{for } I = p+1, ..., D-1. \end{aligned}$$

We recall that we are working with D dimensional spacetime. The conditions mentioned above fixes the endpoints of the string to lie in a $(p{+}1)$ dimensional hypersurface in spacetime. Furthermore , the $SO(1, D-1)$ Lorentz group gets broken into

$$SO(1, D-1) \to SO(1,p) \times SO(D-p-1). \tag{2.23}$$

This hypersurface is known as *Dirichlet Brane* or *D-brane*. If we want to specify its dimension we will call it as *Dp-brane*. With this terminology we can identify a D0-brane as particles, D1-brane as a string itself and so on.

Though we started with strings only , because of these boundary conditions we have seen the emergence of hypersurfaces that we call D-branes. It turns out that these are dynamical objects in their own right. One quick way to check this is that when we apply Dirichlet boundary conditions in some coordinates , the momentum associated with that coordinate is not zero. Mathematically we write $\partial_{\sigma} X^{\nu} \neq 0$. So we conclude that string theory not only contains one dimensional objects but also higher dimensional branes. Also one can ask what if there is only Neumann boundary conditions and no Dirichlet boundary conditions? The answer is that the open ends of the string is free to move throughout spacetime. What this implies is that spacetime is space-filling D-branes , which is an amazing fact.

What we have discussed so far can be put inside a nice picture and we give it below.

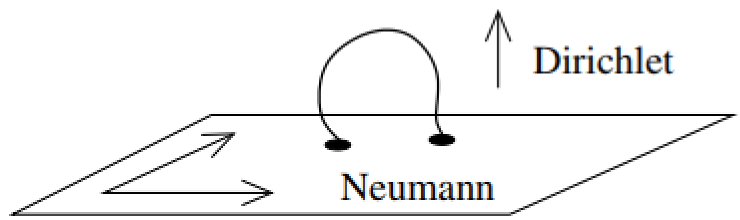

Figure 2.1: Dirichlet boundary and Neumann Boundary Conditions. [23]

2.4.1 An Aside : D(-1)-brane

So far , what we have discussed is imposing Dirichlet boundary condition on the spatial coordinates. What if we impose this condition on X^0? It can obviously be done and physically it means that the string's endpoints are localized in a fixed point in time. Though it seems a bit out of place, lucky for us this object indeed has a physical significance. They are called **D-instantons**. They relate to to tunneling phenomenons in the quantum theory.

There is a ton of reasons for which D-branes are objects of interest. We have only scratched the surface here. In chapter 4 we will have much more to say about D-branes and study its dynamics. But In the next chapter we will focus on the quantaization of open and closed strings. The framework and tools developed there will be important for us in the later chapters.

Chapter 3

Quantum Theory of Strings

3.1 Canonical Quantaization of Open & Closed Strings

So far we have only looked at the classical solutions for a string. The open string solution with Neumann boundary condition is given by

$$X^\mu(\tau,\sigma) = x^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu \cos(n\sigma)\, e^{-i\,n\,\tau}. \tag{3.1}$$

We can take the σ and τ derivative of the above equation. Denoting X' as σ derivative and $\dot{X}$ as τ derivative and doing a little algebra one can easily show that

$$\dot{X}^\mu \pm X'^\mu = \sqrt{2\alpha'}\sum_{n\epsilon Z}\alpha_n^\mu\, e^{-i\,n\,\sigma^\pm}. \tag{3.2}$$

The result (3.2) will be useful later. Now to quantize we upgrade these X^μ to field operators. Then we impose the equal time canonical commutation relations

$$[X^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \qquad [X^\mu(\tau,\sigma),X^\nu(\tau,\sigma')] = 0 \qquad [\Pi^\mu(\tau,\sigma),\Pi^\nu(\tau,\sigma')] = 0$$

here Π^μ denotes the conjugate momentum.

The conjugate momentum Π^μ are defined as the following

$$\Pi^\mu \;=\; \frac{\delta S}{\delta \dot{X}^\mu}.$$

We recall our action

$$S = -\frac{1}{2\pi\alpha'}\int d^2\sigma(-\dot{X}^2 + X'^2).$$

So our momenta are

$$\Pi^\mu = \frac{\dot{X}^\mu}{2\pi\alpha'} \tag{3.3}$$

so the nonzero commutation relation becomes

$$[X^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')] = 2\pi i\alpha'\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{3.4}$$

then taking the σ derivative of the above equation gives

$$[X'^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')] = 2\pi i\alpha'\eta^{\mu\nu}\frac{d}{d\sigma}\delta(\sigma-\sigma'). \tag{3.5}$$

Now we want to know the the following

$$[\alpha_n^\mu, \alpha_m^\nu] =?$$

this motivates us to examine the following commutator relationship

$$[X'^\mu(\tau,\sigma) + \dot{X}^\mu(\tau,\sigma), X'^\nu(\tau,\sigma') + \dot{X}^\nu(\tau,\sigma')]. \tag{3.6}$$

Now using the following

$$[A+B, C+D] \ = \ [A,C] + [A,D] + [B,C] + [B,D]$$

and the canonical commutation relation we arrive from (3.6) to

$$[\dot{X}^\mu(\tau,\sigma), X'^\nu(\tau,\sigma')] + [X'^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')]. \tag{3.7}$$

Now we do the following ,

$$\begin{aligned}
[\dot{X}^\mu(\tau,\sigma), X'^\nu(\tau,\sigma')] + [X'^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')] &= 2\pi i\alpha'\eta^{\mu\nu}\frac{d}{d\sigma}\delta(\sigma-\sigma') - [\dot{X}^\nu(\tau,\sigma'), X'^\mu(\tau,\sigma)] \\
&= 2\pi i\alpha'\eta^{\mu\nu}\frac{d}{d\sigma}\delta(\sigma-\sigma') - 2\pi i\alpha'\eta^{\nu\mu}\frac{d}{d\sigma'}\delta(\sigma'-\sigma) \\
&= 2\pi i\alpha'\eta^{\mu\nu}\frac{d}{d\sigma}\delta(\sigma-\sigma') + 2\pi i\alpha'\eta^{\nu\mu}\frac{d}{d(-\sigma')}\delta(\sigma'-\sigma) \\
&= 2\pi i\alpha'\eta^{\mu\nu}\frac{d}{d\sigma}\delta(\sigma-\sigma') + 2\pi i\alpha'\eta^{\nu\mu}\frac{d}{d\sigma}\delta(\sigma-\sigma') \\
&= 4\pi i\alpha'\eta^{\mu\nu}\frac{d}{d\sigma}\delta(\sigma-\sigma')
\end{aligned}$$

so we finally have

$$[\dot{X}^\mu(\tau,\sigma), X'^\nu(\tau,\sigma')] + [X'^\mu(\tau,\sigma), \dot{X}^\nu(\tau,\sigma')] = 4\pi i\alpha'\eta^{\mu\nu}\frac{d}{d\sigma}\delta(\sigma-\sigma') \tag{3.8}$$

equation (3.8) tells us

$$[X'^{\mu}(\tau,\sigma) + \dot{X}^{\mu}(\tau,\sigma), X'^{\nu}(\tau,\sigma') + \dot{X}^{\nu}(\tau,\sigma')] = 4\pi i\alpha'\eta^{\mu\nu}\frac{d}{d\sigma}\delta(\sigma-\sigma') \tag{3.9}$$

but, the left hand side can be expand with the help of (3.2) and we will get

$$2\alpha' \sum_{n,m\,\epsilon\,Z} e^{-i(m+n)\tau}\, e^{-i\sigma n}\, e^{i\sigma' m}[\alpha_n^{\mu}, \alpha_m^{\nu}] \;=\; 4\pi i\alpha'\eta^{\mu\nu}\frac{d}{d\sigma}\delta(\sigma-\sigma') \tag{3.10}$$

now we integrate both side by

$$\int_0^{2\pi} d\sigma \frac{1}{2\pi} \int_0^{2\pi} d\sigma' \frac{1}{2\pi}$$

and arrive finally at

$$[\alpha_n^{\mu}, \alpha_m^{\nu}] \;=\; n\eta^{\mu\nu}\delta_{m+n,0}. \tag{3.11}$$

This is our commutation relation for the oscillator modes which were present in the mode expansion.

Now we can do more with equation (3.11) . First we notice that the delta function forces us to take $m = -n$. Also the reality condition of X^{μ} gives us $\alpha_n^{\mu} = (\alpha_{-n}^{\mu})^{\dagger}$ and defining the following

$$a_n^{\mu} \;=\; \frac{\alpha_n^{\mu}}{\sqrt{n}} \qquad a_n^{\mu\,\dagger} \;=\; \frac{\alpha_{-n}^{\mu}}{\sqrt{n}} \qquad n > 0$$

and then we get

$$[a_n^{\mu}, a_{-n}^{\nu\,\dagger}] \;=\eta^{\mu\nu} \tag{3.12}$$

which is exactly the commutator of *creation* and *annihilation* operator for an Harmonic Oscillator. Also just like the harmonic oscillator the following can also be shown to be true

$$[x^{\mu}, p^{\nu}] \;= i\,\eta^{\mu\nu}.$$

In a similar fashion , one can show that the for the closed string the following commutation relations are true as well

$$[x^{\rho}, p_{\sigma}] = i\,\delta_{\sigma}^{\rho}$$
$$[\alpha_n^{\mu}, \alpha_m^{\nu}] = [\tilde{\alpha}_n^{\mu}, \tilde{\alpha}_m^{\nu}] = n\eta^{\mu\nu}\delta_{m+n,0}.$$

We have all the tools necessary to build out fock space. For the closed strings we can identify

$$\alpha_n^{\mu}\,|0\rangle = \tilde{\alpha}_n^{\mu}\,|0\rangle = 0 \qquad n \;>\; 0. \tag{3.13}$$

The vacuum for the string is not the same vaccum of the spacetime. The ground state of the string is $|0\rangle$ tensored with the wavefunction $\Psi(x)$.

Now we can build up the Fock space using the creation operators acting on the vacuum. Mathematically, getting this done is doing the following

$$\prod_{n=1}^{\infty}\prod_{i,j} \alpha_n^{\mu_i}\tilde{\alpha}_n^{\nu_j}\,|0;p\rangle\,. \tag{3.14}$$

Now one gets in trouble when one considers the X^0 fields . They come with a minus sign in the metric. As a result we can get negative norm states. This is not desirable . We can see this is true not only for the closed string but for the open string as well .

To see this we can take equation (3.11) and put $\mu\,,\nu\;=\;0$ and we have for $n=-1$

$$\langle p'\,;\,0|\,\alpha_1^0\alpha_{-1}^0\,|0\,;\,p\rangle\;=\;-\delta^D(p-p')$$

These negative norm-states are referred as *ghosts*.

So , we tried to quantize the string by the covariant approach but now we are getting the negative norm states. This is only the beginning of the troubles. If one studies the constraints of the equation of motion , then after a careful analysis one can end up with a mass formula which looks like the following for a closed string

$$M^2\;=\;\frac{4}{\alpha'}\left(-a+\sum_{n=1}^{\infty}\alpha_{-n}.\alpha_n\right)\;=\;\frac{4}{\alpha'}\left(-a+\sum_{n=1}^{\infty}\tilde{\alpha}_{-n}.\tilde{\alpha}_n\right) \tag{3.15}$$

where a is some normal ordering constant.

3.2 Lightcone Quantization Scheme

In the previous section we noticed that the negative norm states appearing when we tried to quantize in the covariant approach. Now we take another approach which is known as lightcone quantization. First we introduce our lightcone coordinates as

$$\sigma^{\pm}=\tau\pm\sigma$$

then our flat worldsheet metric takes the form

$$ds^2=-d\sigma^+d\sigma^-.$$

Then we introduce our spacetime lightcone coordinates

$$X^{\pm}=\sqrt{\frac{1}{2}}(X^0\pm X^{D-1}). \tag{3.16}$$

However notice that this choice of coordinates chooses one spatial direction and the time direction , all the coordinates in spacetime is not treated on equal footing. As a result any calculation involving the $X^{\pm}$ will not be Lorentz invariant.

The choice of (3.16) takes out spacetime metric to the following form

$$ds^2 \;=\; -2dX^+dX^- + dX^i dX_i \tag{3.17}$$

where $i \;=\; 1, ..., D-2$

So any scalar product $\eta_{\mu\nu}A^\mu B^\nu$ is written as

$$A.B \;=\; -A^- B_+ - A^+ B_- + A^i B_i$$

and the indices are lowered and raised as the following

$$A_\pm = -A^\mp \text{ and } A_i = A^i.$$

Now we fix our gauge as

$$X^+ \;=\; x^+ + \alpha' p^+ \tau \tag{3.18}$$

this is the *lightcone gauge*.

Solving for $\mathbf{X}^-$

Our wave equation in the light coordinates after the gauge fixing look like

$$\partial_+ \partial_- X^- \;=\; 0$$

which can be easily solved by taking

$$X^- \;=\; X_L^-(\sigma^+) = X_R^-(\sigma^-) \tag{3.19}$$

and from the constraints we have are

$$(\partial_+ X)^2 \;=\; (\partial_- X)^2 \;=\; 0.$$

Then from our first constraint we have

$$2\partial_+ X^- \partial_+ X^+ \;=\; \partial_+ X^i \partial_+ X_i \tag{3.20}$$

now using equation (3.18) in the baove equation we arrive at

$$\partial_+ X^- \;=\; \frac{1}{\alpha' p^+} \partial_+ X^i \partial_+ X_i \tag{3.21}$$

also one can get the following by using the second constraint

$$\partial_- X^- \;=\; \frac{1}{\alpha' p^+} \partial_- X^i \partial_- X_i. \tag{3.22}$$

So from the above two equations we can easily determine the solution for X^- upto an integration constant. Then we can write down the mode expansion in the following manner

$$X_L^-(\sigma^+) \ = \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^- \, e^{-in\sigma^+} \tag{3.23}$$

and

$$X_L^-(\sigma^-) \ = \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha_n^- \, e^{-in\sigma^-} \tag{3.24}$$

and the using the constraints in (3.21) and (3.22) we can have the oscillator modes α_n^- written as

$$\alpha_n^- \ = \sqrt{\frac{1}{2\alpha'}}\frac{1}{p^+} \sum_{m=-\infty}^{+\infty} \sum_{i=1}^{D-2} \alpha_{n-m}^i \alpha_m^i. \tag{3.25}$$

We can look into a special case of (3.25) by taking $n = 0$ and this corresponds to $\alpha_0'^- = \sqrt{\frac{\alpha'}{2}} p^-$ which yields ,

$$\frac{\alpha' p^-}{2} \ = \ \frac{1}{2p^+} \sum_{i=1}^{D-2} (\frac{1}{2}\alpha' \, p^i p^i + \sum_{n\neq 0} \alpha_{-n}^i \alpha_n^i). \tag{3.26}$$

Also using (3.21) we get the following

$$\frac{\alpha' p^-}{2} \ = \ \frac{1}{2p^+} \sum_{i=1}^{D-2} (\frac{1}{2}\alpha' \, p^i p^i + \sum_{n\neq 0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i). \tag{3.27}$$

Using (3.26) and (3.27) we see that ,

$$\sum_{n\neq 0} \alpha_{-n}^i \alpha_n^i = \sum_{n\neq 0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i. \tag{3.28}$$

This equation tells us that the right moving modes of the closed string and the left moving modes are equal to each other. This is known as *level matching*.

Now write the mass as

$$M^2 \ = \ 2p^+ p^- - p^i p_i \tag{3.29}$$

and using (3.27) or (3.26) we can see that

$$M^2 \ = \ \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \alpha_{-n}^i \alpha_n^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i. \tag{3.30}$$

Since i runs from $1, ..., D-2$ which are *transverse* to our lightcone coordinates , we will call these oscillators as transverse oscillators. However , there is something to notice here. Since the formula above is still not quantized there is no ordering ambiguity of the α s. That's why we could do write $n > 0$ in the lower limit of the sum.

Now it is time to promote the α and $\tilde{\alpha}$ to operators and impose the commutation relations as follows

$$[x^i, p^j] \,=\, i\delta^{ij} \qquad [x^-, p^+] \,=\, -i$$
$$[\alpha^i_n, \alpha^j_m] \,=\, [\tilde{\alpha}^i_n, \tilde{\alpha}^j_m] \,=\, n\,\delta^{ij}\delta_{m+n}.$$

Now remembering that α s are now operators our mass formula in (3.29) suffers from an *ordering* ambiguity since the oscillator modes don't commute. Now, we break up the sum that is apprearing in the mass formula as

$$\begin{aligned} M^2 \,&=\, \frac{4}{\alpha'}\frac{1}{2}\sum_{n=\neq 0} \alpha^i_{-n}\alpha^i_n \\ &=\, \frac{4}{\alpha'}\left(\frac{1}{2}\sum_{n>0} \alpha^i_{-n}\alpha^i_n + \frac{1}{2}\sum_{n<0} \alpha^i_{-n}\alpha^i_n\right). \end{aligned}$$

Now we can use the commutation relation imposed on the α s to show that

$$M^2 \,=\, \sum_{n>0} \alpha^i_n\alpha^i_{-n} + n(D-2)\sum_{n>0} n. \tag{3.31}$$

Now we need to find a way to evaluate $\sum\limits_{n>0} n$.

3.2.1 A Regularised Sum

Even though the sum $\sum\limits_{n>0} n$ is divergent , there is a way to get something meaningful out of it. The idea is to introduce a regulator and carry out the calculation. So to get that we do the following

$$\sum_{n>0} n \,=\, \lim_{\epsilon\to 0}\sum_{n>0} n\, e^{-\epsilon\, n}. \tag{3.32}$$

Now we calculate the right hand side of equation (3.32) and while we do this we don't write the limit explicitly.

Now we do the following :

$$\begin{aligned}
\sum_{n>0} n\, e^{\epsilon\, n} &= -\sum_{n>0} \frac{d}{d\epsilon}\, e^{-\epsilon\, n} \\
&= \frac{d}{dx}\frac{1}{1-e^{-\epsilon}} \\
&= \frac{e^{\epsilon}}{(1-e^{\epsilon})^2} \\
&= \frac{(1+\epsilon+\epsilon^2+\frac{\epsilon^2}{2})+\mathcal{O}(\epsilon^3)}{\epsilon^2(1+\frac{\epsilon^2}{2}+\frac{\epsilon^2}{6})^2+\mathcal{O}(\epsilon^3)} \\
&= \frac{(1+\epsilon+\frac{\epsilon^2}{2})}{\epsilon^2(1+\epsilon+\frac{7\epsilon^2}{12})} \\
&= \frac{1}{\epsilon^2}(1+\epsilon+\frac{\epsilon^2}{2})(1+\epsilon+\frac{7\epsilon^2}{12})^{-1} \\
&= \frac{1}{\epsilon^2}(1+\epsilon+\frac{\epsilon^2}{2})(1-\{\epsilon+\frac{7\epsilon^2}{12}\}+\{\epsilon+\frac{7\epsilon^2}{12}\}^2) \\
&= \frac{1}{\epsilon^2}(1+\epsilon+\frac{\epsilon^2}{2})(1-\epsilon+\frac{5\epsilon^2}{12}) \\
&= \frac{1}{\epsilon^2}(1-\frac{\epsilon^2}{12}) \\
&= \frac{1}{\epsilon^2}-\frac{1}{12}.
\end{aligned}$$

So we finally write

$$\sum_{n>0} n \;=\; \lim_{\epsilon\to 0} \frac{1}{\epsilon^2}-\frac{1}{12}. \tag{3.33}$$

We note that the $\frac{1}{\epsilon^2}$ term is divergent and should be *renormalised* away. After this we conclude ,

$$\sum_{n>0} n \;=-\frac{1}{12}. \tag{3.34}$$

this looks a bit odd at first but one should remember that this is a *regularised* sum , not the *exact* sum. One can also come to the same result using other regulators. One of the popular methods is to use the *zeta function regularisation.* But we don't want to go there now.

The ordering we used for the mass formula is known as *normal ordering* . Using that prescription and the regularised sum result from (3.34) our mass formula looks like the following

$$M^2 \;=\; \frac{4}{\alpha'}(N-\frac{D-2}{24}) \tag{3.35}$$

where we call N as the number operator and it is denoted by

$$N = \sum_{n>0} \alpha^i_{-n}\alpha^i_n. \tag{3.36}$$

Now we notice something in equation (3.33). The number operator consists of operators of the *transverse* oscillators only. We will have to say more about these when we discuss open strings and D-branes in the next chapter. For now we want to take a look at the closed string spectrum.

3.3 Closed String Spectrum

Now with all the tools we have developed so far , we can look at the spectrum of the closed string. Let's consider the ground state first where $N = 0$ the using (3.35) we have

$$M^2 = -\frac{1}{\alpha'}\frac{D-2}{6} \tag{3.37}$$

which gives us a negative mass squared term where we assume $D > 2$. Now this is problematic. Exicitations with negative masses are called **tacyons**. They signal to an unstable vacuum. So clearly we don't want them in our theory. The fastest particle we can have in any physical theory are photons which travel at the speed of light and hence are massless. Since the mass term depends on both the N and D we need to fix them appropriately. Now we take the first excited state where $N = 1$ and we require them to be massless . So setting $M = 0$ we get

$$\frac{D-2}{24} = 1$$

which implies $D = 26$.

This $D = 26$ is called the critical dimension of *bosonic string theory.* The reason it is called critical because it restores the $SO(1, D-1)$ Lorentz Symmetry. Also, massless particles are interesting to study as they mediate long range forces. The states from the first excited state transform in the representation of $SO(24)$. They decompose into three irreducible representations. One of them is traceless and symmetric and the filed associated with this excitation is denoted by $G_{\mu\nu}(X)$. This points to a massless spin-2 particle which is known as graviton.

Another exictation is of the from traceless and anti-symmetric and the associated field $B_{\mu\nu}(X)$ and known as "Kalb-Ramond field". It can be thought of as a generalised Maxwell field. The third exciatation is coming from a scalar and its called the "dilaton" and written as $\Phi(X)$. The string coupling is given as the expectation value of the dilaton.

This is a brief summary of the closed string spectrum. In the next section we look at the open string spectrum.

3.4 Open String Spectrum

We have already seen the commutation relation between the oscillator modes of an open string in section (3.1). Now we note that due to Neumann boundary conditions we have

$$\tilde{\alpha}_n^a = \alpha_n^a$$

where $a = 0, ..., p$ and the Dirichlet boundary conditions $X^I = c^I$ where $I = p+1, ..., D-1$ gives us

$$x^I = c^I \qquad p^I = 0 \qquad \tilde{\alpha}_n^I = -\alpha_n^I.$$

Now to see the spectrum we will work in the lightcone gauge but now we choose the spacetime lightcone coordinates to lie on the brane. So we take

$$X^{\pm} = \sqrt{\frac{1}{2}}(X^0 \pm X^p). \tag{3.38}$$

Again we write down the mass formula as

$$M^2 = \frac{1}{\alpha'}(\sum_{i=1}^{p-1}\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \sum_{i=p+1}^{D-1}\sum_{n>0}\alpha_{-n}^i\alpha_n^i - a) \tag{3.39}$$

where a is the normal ordering constant. However now one can see that the mass is written in terms of the oscillator modes perpendicular and parallel to the brane. The first sum is parallel to the branes and this modes are called *longitudinal* modes. The second term is perpendicular to the brane and they are called *transverse* modes. It turns out that the critical dimension for open strings are also $D = 26$. So one can safely say that open and closed strings are part of the same theory.

Just like the closed string , the first excited states of the open string are massless. Now the oscillator longitudinal to the brane α_{-1}^a gives rise to states that transform under $SO(1,p)$ Lorentz group. These are spin 1 particle that live on the brane. Hence the associated fields are gauge fields denoted as A_a with $a = 0, ..., p$. The transverse oscillator gives rises to scalar fields ϕ^I with $I = p+1, ..., D-1$. These fluctuations can be thought of as D-brane fluctuation in the transverse direction. We will discuss more about D-branes and their properties in the next chapter.

Chapter 4

Physics of D-Branes

We have seen that string theory is not only a theory of strings but it contains extended objects like D-branes. D-branes are topological defects and have the defining property that open strings can end on them. The study of D-branes is a huge subject and an excellent reading on this subject can be found in [20]. In this chapter I will try to introduce the concepts which will be relevant in out study of the M-atrix model.

4.1 T-Duality

One way of motivating the necessity of D-branes in string theory is based on T-duality. T-duality means *target space duality*. Under T-duality transformations , closed bosonic strings transforms into the closed bosonic string in T-dual geometry. The picture is different for open strings. For them the only Lorentz invariant boundary conditions in the Neumann boundary conditions. The Dirichlet boundary condition breaks the Lorentz invariance. The reason why we still consider them is that they give to physical objects like Dp-branes. Here , Dp-brane means that this hypersurface has p spatial dimensions. If time is also taken into account it has $p+1$ dimensions. In this way of speaking a D0-brane corresponds to a point particle , a D1-brane is called a D-string and so on.

The importance of Dp-branes comes from the fact that they provide a remarkable way of introducing non-abelian gauge symmetries in string theory, Non-abelian gauge fields naturally appear confined to the world volume of multiple coincident Dp-branes. The concept of d-branes are useful in realising string dualities. T-duality is introduced here, because it can be understood in perturbative string theory. Most other string dualities are nonperturbative.

4.2 T-Duality for Closed Strings

Now, lets consider the simplest case. A closed string living on a 26 dimensional spacetime with 25 spatial directions. Now if we compactify the 25 th dimensions on a circle with a identification

$$X^{25}(\sigma+2\pi,\tau) \;=\; X^{25}(\sigma,\tau)+2\pi wR \tag{4.1}$$

where we call w the **winding number**. Also, we require the string wave function to be single valued under this identification. So we get the following

$$\psi(x^{25}) \sim \psi(x^{25}+2\pi R).$$

Now , the wavefunction will carry a factor of $e^{ip.x}$. So we get

$$e^{ip^{25}x^{25}} \;\sim\; e^{ip^{25}(x^{25}+2\pi R)}$$

and this relation gives us

$$p^{25} \;=\; \frac{n}{R} \tag{4.2}$$

where $n \,\epsilon\, \mathbf{Z}$. These n are also called Kaluza–Klein modes and R denotes the radius of the compactified dimension.

Now we introduce new coordinates as follows

$$z \;=\; e^{(\tau-i\sigma)} \qquad\quad \bar{z} \;=\; e^{(\tau+i\sigma)}$$

and in these coordinates our wave equation becomes

$$\partial_z\partial_{\bar{z}}X^{\mu} \;=\; 0 \tag{4.3}$$

and the solution to the above equation can be written as follows

$$X^{\mu}(z,\,\bar{z}) \;=\; X^{\mu}(z)+X^{\mu}(\bar{z}). \tag{4.4}$$

So the solution splits into holomorphic and anti-holomorphic parts.

The mode expansion can be written as follows

$$X^{25}(z,\bar{z}) \;=\; x^{\mu}-i\sqrt{\frac{\alpha'}{2}}(\alpha_0^{\mu}+\tilde{\alpha}_0^{\mu})\tau+\sqrt{\frac{\alpha'}{2}}(\alpha_0^{\mu}-\tilde{\alpha}_0^{\mu})\sigma+i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}(\alpha_m^{\mu}z^{-m}+\tilde{\alpha}_m^{\mu}\bar{z}^{-m}) \tag{4.5}$$

and we see that the canonical momentum is

$$p^{\mu} \;=\; \frac{1}{\sqrt{2\alpha'}}(\alpha_0^{\mu}+\tilde{\alpha}_0^{\mu}) \tag{4.6}$$

also for a closed string we have the identification

$$X(\sigma+2\pi,\tau) \;=\; X(\sigma,\tau)$$

which gives us

$$\alpha_0^{\mu} \;=\; \tilde{\alpha}_0^{\mu}. \tag{4.7}$$

However this is not the case where one of the dimensions is compactified. To see this we take into account both equation (4.1) and the closed string identification on the 25th direction. So we have

$$X^{25}(\sigma,\tau) \;=\; X^{25}(\sigma+2\pi,\tau)+2\pi Rw. \tag{4.8}$$

Before we impose this condition to our solution we note that z and $\bar{z}$ does not change. As a result the oscillation modes stay the same and cancel each other out . Keeping this in mind we have

$$\begin{gathered} x^{25}-i\sqrt{\frac{\alpha'}{2}}(\alpha_0^{25}+\tilde{\alpha}_0^{25})\tau+\sqrt{\frac{\alpha'}{2}}(\alpha_0^{25}-\tilde{\alpha}_0^{25}) = x^{25}-i\sqrt{\frac{\alpha'}{2}}(\alpha_0^{25}+\tilde{\alpha}_0^{25})\tau+\sqrt{\frac{\alpha'}{2}}(\alpha_0^{25}-\tilde{\alpha}_0^{25})+2\pi wR \\ \sqrt{\frac{\alpha'}{2}}(\alpha_0^{25}-\tilde{\alpha}_0^{25})+Rw=0. \end{gathered} \tag{4.9}$$

So we have

$$\tilde{\alpha}_0^{25}-\alpha_0^{25} \;=\; \sqrt{\frac{2}{\alpha'}}.wR \tag{4.10}$$

Using (4.6) and (4.10) we can solve for the zero modes and after a bit of manipulation we end up having the following

$$\alpha_0^{25} \;=\; \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R}+\frac{wR}{\alpha'}\right) \tag{4.11}$$

and

$$\tilde{\alpha}_0^{25} \;=\; \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R}-\frac{wR}{\alpha'}\right). \tag{4.12}$$

Notice that the zero mode α_0^{25} is invariant under the exchange of n with w provided one also makes the change $R \to \frac{\alpha'}{R}$. But we get

$$\tilde{\alpha}_0^{25} \;=\; -\,\tilde{\alpha}_0^{25}. \tag{4.13}$$

This is known as **T-duality** of the closed string. Let's denote the **T-dual Radius** as

$$R' \;=\; \frac{\alpha'}{R}. \tag{4.14}$$

Now we note that as $R \to 0$ the KK modes become infinitely heavy and only $n = 0$ contributes to the spectrum. But it does not effect the winding modes. Also when we consider $R \to \infty$, $w \to 0$. To see what this effects mean we consider the mass spectrum for the closed string

$$\begin{aligned} M^2 &= -p^\mu p_\mu \\ &= \frac{2}{\alpha'}(\alpha_0^{25})^2 + \frac{4}{\alpha'}(N-1) \\ &= \frac{2}{\alpha'}(\tilde{\alpha}_0^{25})^2 + \frac{4}{\alpha'}(\bar{N}-1). \end{aligned} \tag{4.15}$$

So under T-dual transformations the mass spectrum is invariant. Now adding both the left moving and right moving modes we get

$$M^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{wR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N + \bar{N} - 2) \tag{4.16}$$

and subtracting gives

$$\bar{N} = N + wn. \tag{4.17}$$

The key feature is that there are terms apart from the oscillator terms and the level matching condition is does not hold anymore as we have the presence of winding modes and the momentum modes. Also in the mass formula we can see that apart from the usual Kaluza-Klein tower of momentum states , the winding modes give rise to tower of winding states. This is indeed a stringy phenomenon.

Now one can go further by using

$$\alpha_0^{25} \to \alpha_0^{25} \qquad \tilde{\alpha}_0^{25} \to -\tilde{\alpha}_0^{25}$$

to introduce the **T-dual fields** as

$$\hat{X}^{25}(z) = X^{25}(z) - X^{25}(\bar{z}) \tag{4.18}$$

and this leaves the stress-energy tensor , all correlation functions invariant. This T-dulaity for closed string can have a interpretation as a parity operation on the right moving degrees of freedom only.

4.3 T-Duality for Open Strings

For open strings, to get the equations of motion from varying the action one has to choose boundary conditions. In our previous discussion we have seen two types of boundary conditions namely Dirichlet and Neumann boundary conditions. However the only boundary conditions which respect the spacetime poincare invariance is the Neumann boundary conditions. Now let us start with Neumann Boundary condition both endpoints for the string. The mode expansion gives us

$$X^{\mu}(z,\bar{z}) \;=\; x^{\mu} - i\alpha' p^{\mu}\ln(z\bar{z}) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}(\alpha^{\mu}_{n}z^{-n} + \tilde{\alpha}^{\mu}_{n}\bar{z}^{-n}). \tag{4.19}$$

Like before we compactify along the 25 th dimension and since we are dealing with open string we say no winding mode appears.We also recall that $\alpha = \tilde{\alpha}$ for open strings.

Now we want to see the T-dual picture of open string theory. First we write the T-dual field for this open string and it is given by

$$\begin{aligned}\hat{X}^{25}(z,\bar{z}) \;&=\; X^{25}(z) - X^{25}(\bar{z})\\ &= x^{25} - i\alpha' p^{25}\ln\left(\frac{z}{\bar{z}}\right) + i\alpha'\sum_{n\neq 0}\frac{\alpha^{25}_{0}}{n}(z^{-n} - \bar{z}^{-n})\\ &= x^{25} - i\alpha' p^{25}\ln\left(\frac{z}{\bar{z}}\right) + i\alpha'\sum_{n\neq 0}\frac{\alpha^{25}_{0}}{n}e^{-n\tau}\sin(n\sigma).\end{aligned}$$

The wave function is single valed and we get

$$p^{25} \;=\; \frac{n}{R}$$

for open string as well. So Finally out T-dual field becomes

$$\hat{X}^{25}(z,\bar{z}) \;=\; x^{25} - i\alpha'\frac{n}{R}\ln(\frac{z}{\bar{z}}) + i\alpha'\sum_{n\neq 0}\frac{\alpha^{25}_{0}}{n}e^{-n\tau}\sin(n\sigma). \tag{4.20}$$

Now as before we study the limit $R \to 0$ and as a result only $n = 0$ momentum mode contributes. So we have

$$\hat{X}^{25}(z,\bar{z}) \;=\; x^{25} - i\alpha'\sum_{n\neq 0}\frac{\alpha^{25}_{0}}{n}e^{-n\tau}\sin(n\sigma) \tag{4.21}$$

and when we evaluate the boundary terms , that is at $\sigma = 0$ and $\sigma = \pi$ we see that X^{25} is constant . Which is precisely our Dirichlet boundary conditions! So even though we started with a Neumann boundary condition we were able to recover the Dirichlet condition in the dual picture and that indicates D-branes are indeed an essential part of string theory.

Also we note that

$$\hat{X}^{25}(\pi) - \hat{X}^{25}(0) \;=\; 2\pi nR'. \tag{4.22}$$

One can also interpret it as the string in the dual theory wind as the following diagram

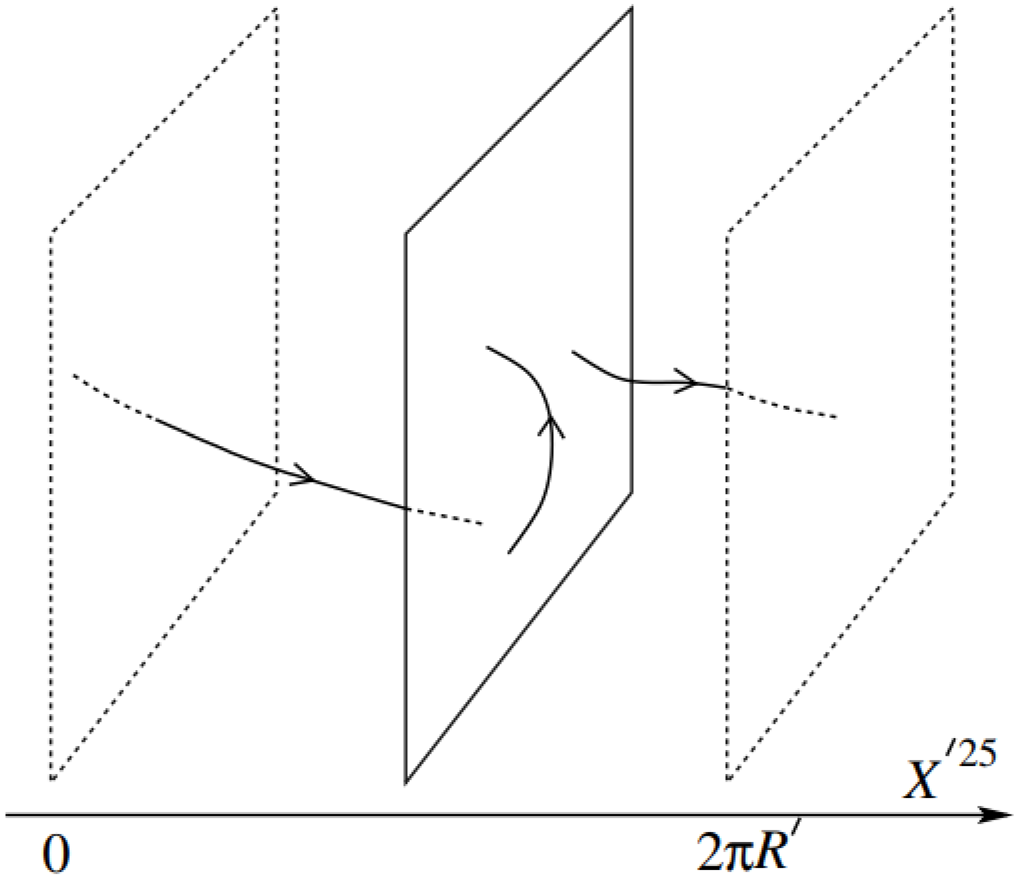

Figure 4.1: Open strings in the dual theory. Dashed planes are periodically identified. Figure taken from Johnson. [16]

what this means is that , in the dual picture we have a identification as

$$\hat{X}^{25} \to \hat{X}^{25} + 2\pi n R'. \tag{4.23}$$

This identification means that both ends of the string lie in a 24+1 dimensional hyperplane. This T-dual open strings is a logical consequence of T-duality of closed string sector contained in open strings.

Now we can take $R' \to \infty$ and see that Dirichlet boundary condition is allowed whether or not $\hat{X}^{25}$ is compactified or not. T-dualising more than one dimensions , say k dimensions leads to a **D p-branes** where $p = 25 - k$. A $D\,25$ brane is a $25 + 1$ dimensional hypersurface and we just get our ordinary open strings back.

4.4 Chan-Paton Factors

From our earlier discussions we have seen that when we have Dirichlet conditions, string endpoints are fixed on D-branes. So , with this in mind one can add non-dynamical degrees of freedom to open string endpoints. Then one can go on and label this endpoint as **Chan-Paton Factors**. Since they are non-dynamical their Hamiltonian is zero.

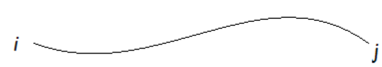

Figure 4.2: Chan-Paton Factor labelled as i and j.

After adding these Chan-Paton factors , apart from the usual momentum label , the Fock space carres additional labels that represent this factors. So one writes the full wavefunction as following

$$|k, a\rangle \ = \ \sum_{i,j=1}^{N} |k, ij\rangle \ \lambda^a_{ij}. \tag{4.24}$$

Here , the λ denotes the $N \times N$ matrices. They serve as a basis to decompose the above wavefunction. All open strings vertex operators carry such factors. For example , we can consider the tree level diagram for four open oriented strings in the following figure

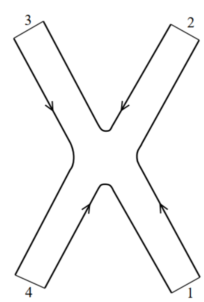

Figure 4.3: Four point scattering of open strings. [20]

Also this diagram is conformally related with the following disc diagram

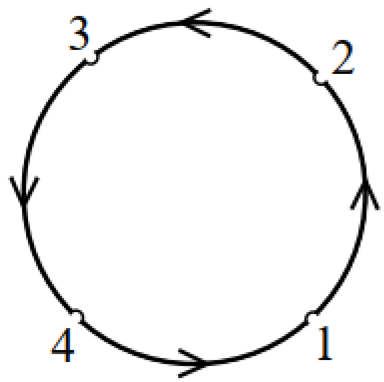

Figure 4.4: Conformal disc diagram of four point scattering. [20]

Since the Chan-Paton degrees of freedom are non-dynamical the right end of string denoted by number 1 must be in the same state as left end of string denoted by number 2 and so on. After summing over all possible states involved we get a trace of the Chan-Paton factors and it is given by

$$\lambda^1_{ij}\lambda^2_{jk}\lambda^3_{kl}\lambda^4_{li} = \mathrm{Tr}(\lambda^1\lambda^2\lambda^3\lambda^4). \tag{4.25}$$

It turns out that all open string amplitudes carry a trace like this and it is easy to see that they are invariant under a global world-sheet $U(N)$ transformation :

$$\lambda^i \to U\lambda^i U^{-1}. \tag{4.26}$$

Closed-string vertex operators are the product of left- and right-handed ones, which are functions of z and $\bar{z}$, respectively, and thus take the form of the product of 2 independent open-string vertex operators. The massless vertex operators one can have the form

$$V^{a\mu} = \lambda^a_{ij}\partial_t X^\mu e^{ik\cdot X}$$

transforms in the adjoint under $U(N)$ symmetry. This implies that *the global symmetry on the worldsheet is now promoted to a gauge symmetry in spacetime* asone can do different $U(N)$ rotations in spacetime at every separate points.

4.5 Wilson Lines and Breaking of U(N) gauge symmetry

From our earlier discussions, we saw that the presence of Chan-Paton factors at each end of the string leads us to a $U(N)$ gauge symmetry. This means we can include a background gauge field configuration which corresponds to a Wilson line defined as the following

$$W = \exp(iq \oint dx^{25} A_{25}), \tag{4.27}$$

here x^{25} is the compact direction.

This is a gauge invariant quantity and is useful for constructing various gauge invariant objects in field theory as well. Now one can write

$$A_{25} = -\frac{1}{2\pi R}\mathrm{diag}(\theta_1, \theta_2, ..., \theta_n) \tag{4.28}$$

which breaks the $U(N)$ to $U(1)^N$. To see this we look at the following example.

We begin by writing '

$$A_{25}(x^{25}) = -\frac{\theta}{2\pi R} = -i\Lambda^{-1}\frac{\partial \Lambda}{\partial x^{25}} \tag{4.29}$$

where

$$\Lambda = \exp(-\frac{i\theta x^{25}}{2\pi})$$

which is just a pure gauge so the field strength vanishes. But there exists non trivial physics here when we go around a circle. After going around a circle we pick up a phase

$$W = \exp(-iq\theta).$$

Now consider a particle under $U(1)$ whose action is given by

$$S = \int d\tau \, (\frac{1}{2}\dot{X}^\mu \dot{X}_\mu - iqA_\mu \dot{X}^\mu). \tag{4.30}$$

Now we can easily obtain the canonical momentum from this action and we have

$$\Pi^\mu = i\dot{X}^\mu$$

except for the compactified dimension, where we get

$$\Pi^{25} = i\dot{X}^{25} - iqA_{25} = \frac{n}{R} \tag{4.31}$$

which in turn gives the following

$$p^{25} = \frac{n}{R} + \frac{\theta}{2\pi R}. \tag{4.32}$$

Now one can gauge away A_{25} using λ^{-1} but this corresponds to the fact that when we move around the circle, the particle will pick up a phase $\exp(iq\theta)$. So eventually , we end up having (4.32) again.

Now we can generalise for the case where we have a $U(N)$ gauge case and this general case gives us

$$p^{25} = \frac{n}{R} + \frac{\theta_i - \theta_j}{2\pi R}. \tag{4.33}$$

So pictorially we can see there will be N hyperplanes now .

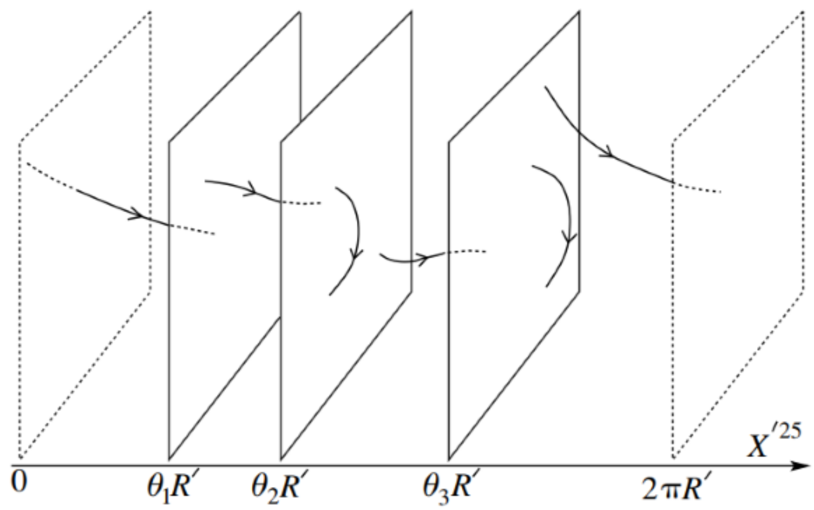

Figure 4.5: D-branes in different positions. [16]

Also we can see that

$$\hat{X}^{25}(\pi) - \hat{X}^{25}(0) = (2\pi n + \theta_i - \theta_j)R' \tag{4.34}$$

which means that the winding numbers in the dual theory can be **fractional**. This signals that now we do not just have a single D-brane but N D-branes at positions $\theta_i R'$. Moreover we see that

$$\begin{aligned}\theta_i R' &= \frac{\theta_i \alpha'}{R} \\ &= 2\pi\alpha' \frac{\theta_i}{2\pi R} \\ &= 2\pi\alpha'(A_{25})_{ii}.\end{aligned}$$

So, the position of the D-branes are eigenvalues of A_{25} times $2\pi\alpha'$. This relation will play a key role later when we talk about the D-brane coordinates.

We have N separated branes which corresponds to the gauge group $U(1)^N$. If one lets all the branes stick together, U(N) symmetry is restored. The way to see it is to calculate the mass spectra as before. So we get,

$$\begin{aligned} M^{25} &= (p^{25})^2 + \frac{1}{\alpha'}(N-1) \\ &= \left(\frac{(2\pi n + \theta_i - \theta_j)R'}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}(N-1). \end{aligned} \tag{4.35}$$

Now considering the $N = 1$ where $n = 0$ we have the mass as

$$M = \frac{(\theta_i - \theta_j)R'}{2\pi\alpha'}. \tag{4.36}$$

What this means is that the mass is proportional to the distance covered by a stretched string between two D-branes. As $\theta_i \to \theta_j$ new massless vectors appear which corresponds to heavy vector bosons becoming massless and thus increasing the gauge group to $U(2)$direct product with an $U(1)$ factor. Hence, when all the D-branes coincide, the full $U(N)$ theory emerges.

4.6 D-branes as Dynamical Objects

Open strings end on D-branes , that was the solgan we used before. From there it might occur to someone that D-branes are rigid object that fill spacetime. Interestingly , this is not true. Rather D-branes are dynamical objects. To see it in a shorter way , we need to noitce that for $\theta_i \neq \theta_j$ the massless vector states only come from strings with endpoints on the same D-brane. These strings have vertex operators $V^\mu = \partial_t X^\mu$ and $V^{25} = \partial_t X^{25} = \partial_n \hat{X}^{25}$. We could write this because under T-duality , the derivative in the tangential direction becomes a derivative in the normal direction to the dual field. More details can be found in [16].

The vertex operator $V^\mu = \partial_t X^\mu$ where $\mu = 0, 1, 2, ...24$ describes the tangential fluctuations. These fluctuations are on the D-branes. Here we ar actually taking about a D string. These give rise to the gauge fields. On the other hand, V^{25} describes the fluctuations normal to the D-branes as it involves the normal derivative to the brane. These give rise to the Scalar fields. These things indicate that D-branes are indeed dynamical.

The mechanism is is similar to starting with closed strings in the Minkowski spacetime and finding the massless modes describing the fluctuations of spacetime itself. Since D-branes are dynamical objects it would be interesting to study their dynamics . To do that we will need to set up a framework where we can comment it's coordinates. We will also want to know what type of action we can write for it and what will be the effective action that is induced on it.

Also we will want to know the low energy processes which describe what is going on the D-p brane world volume etc etc. In the next section we will talk about the D-brane actions. Later we will also see that D-brane coordinates are non-commutating matrices. This is a very interesting result and this gives rise to non-commutative geometry.

4.7 Non commutative matrices as D-brane corodinates & D-Brane Actions

To determine the action for the D-branes one can start working in the same way as to construct the action for the closed strings. But now , one has to keep in mind the boundary terms as well. The ansatz for a D-p brane the action is the following

$$S = -T_p \int d^{p+1}\xi \, e^{-\phi} \sqrt{-\det(g_{mn} + b_{mn} + 2\pi\alpha' F_{mn})} \tag{4.37}$$

where

$$g_{mn} = \frac{\partial X^\mu}{\partial \xi^m} \frac{\partial X^\nu}{\partial \xi^n} G_{\mu\nu} \tag{4.38}$$

is the pullback from the spacetime fields $G_{\mu\nu}$ to the brane. Same can be said for b_{mn} which is the pullback from $B_{\mu\nu}$ spacetime field. The string coupling constant g_s is defined as the expectation value of the dilation field and in this case we have

$$g_s = e^{\phi}. \tag{4.39}$$

The term T_p represents the tension of a D-p brane. For constant dilaton , trivial metric and antisymmetric background gauge fields the action is just the Born-Infled action which was computed in [2].

So we can write the action as

$$S = -\frac{T_p}{g_s} \int d^{p+1}\xi \sqrt{-det(\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}. \tag{4.40}$$

To get the action in the form that is convenient for us to work with one can expand the determinant around the metric. So mathematically it becomes a problem of expanding the following

$$\sqrt{-\det(\mathbf{1} + A)} \tag{4.41}$$

where $\mathbf{1}$ is the $N \times N$ identity matrix and A is any general matrix. Later we will require A to be antisymmetric which will correspond to $2\pi\alpha' F$.

Now to start the derivation we first use an identity

$$\ln(\det(B)) = \mathrm{Tr}(\ln(B)) \tag{4.42}$$

and solving for $\det(B)$ gives us

$$\det(\mathrm{B}) = \exp(\mathrm{Tr}(\ln B)) \tag{4.43}$$

Now we expand the exponential and in our case B is identified with $\mathbf{1}+A$. So we just need to do the following

$$\begin{aligned}\exp(\mathrm{Tr}(\ln(\mathbf{1}+A))) &= \mathrm{Tr}\left(\mathbf{1}+\ln(\mathbf{1}+F)+\frac{1}{2}(\ln(\mathbf{1}+F))^2+......\right)\\ &= \left(\mathrm{Tr}(\mathbf{1})-\mathrm{Tr}(F)+\mathrm{Tr}\left(\frac{F^2}{2}\right)-\left(\frac{\mathrm{Tr}(F)^2}{2}\right)+...\right)\end{aligned}$$

and we can see the non trivial terms are coming from the trace of F. But in our case F is antisymmetric and hence its trace vanishes. Also notice that in deriving the above relation we took $2\pi\alpha' = 1$. Now putting these factors and using the fact that $\mathrm{Tr}(F)=0$, we have the leading order action as

$$S \sim -\frac{T_p(2\pi\alpha')^2}{4g_s}\int d^{p+1}\xi\,\mathrm{Tr}(F_{\mu\nu}^2) \tag{4.44}$$

and expanding the indices gives us

$$S \sim -\frac{T_p(2\pi\alpha')^2}{4g_s}\int d^{p+1}\xi\,\mathrm{Tr}(F_{mn}^2+2F_{mj}^2+F_{ij}^2) \tag{4.45}$$

here the metric has a signature of $(-,+,+,...,+)$. On the D-brane there is no dependence on the zero-modes as the D-boundary condition has removed the zero-modes, that is α_0^μ, in the direction which is normal to the brane. What this means is that all derivatives in the i th direction disappear. Also, note that in the above term we have not written $\mathrm{Tr}\,(\mathbf{1})$ explicitly as it is a constant and hence does not effect the equations of motion. So we have the following equations :

$$F_{mn} = \partial_m A_n - \partial_n A_m + i[A_m, A_n] \tag{4.46}$$

$$F_{mj} = \frac{1}{2\pi\alpha'}\left(\partial_m X_j + i[A_m, X_j]\right) \tag{4.47}$$

$$F_{ij} = \frac{1}{(2\pi\alpha')^2} i[X_i, X_j]. \tag{4.48}$$

Note that we have the *Covariant derivative* as

$$D_m X_j = \partial_m X_j + i[A_m, X_j] \tag{4.49}$$

and putting all this back into our action in (4.45) we get

$$S_{YM} \sim -\frac{T_p(2\pi\alpha')^2}{4g_s}\int d^{p+1}\xi\,\mathrm{Tr}(F_{mn}^2)+\frac{T_p}{g_s}\int d^{p+1}\xi\,\mathrm{Tr}\left(-\frac{1}{2}(D_mX^i)^2+\frac{1}{4}\frac{([X_i,X_j])^2}{(2\pi\alpha')^2}\right). \tag{4.50}$$

The first term on the action (4.50) is just the $p+1$ dimensional Yang-Mills action on the brane. It can also be thought of as the non-abelian generalisation of the Born-Infled action of a single brane. The extra terms that we have govern the dynamics of the D-brane. While in the bosonic string theory we see no fermionic terms, here we will have to include fermionic terms when we discuss D-branes in superstring theory. The third term represents the kinetic term for the X^i and the potential term is $-\mathrm{Tr}([X^i, X^j])^2$. However in , superstring theory , a vaccum with unbroken supersymmetry must have vanishing potential, meaning

$$[X^i, X^j] \;=\; 0$$

as a result the X^i matrices are simultaneously diagonalisable . If we denote the eigenvalues of the matrices as b_i then we write

$$X^i \;=\; \mathrm{diag}(b_1^i, b_2^i, b_3^i, ..., b_n^i)$$

and these eigenvales b_k^i where $i = (p+1, ..., 25)$ represents the coordinates of the k^{th} brane. Now one is faced with this **noncommutative matrices as coordinates** . This seems a bit odd at first , but following E.Witten [25] will make it clear that this happens very naturally.

We have already seen that N separated D-branes correspond to a gauge group of $U(1)^N$ with the massless vector being denoted by $\alpha_{-1}^{\mu} \otimes |ij\rangle$. When we have N coincident D-branes we recover the full $U(N)$ gauge theory. For separated D-branes the breaking of gauge symmetry is related with the mass term associated with the off diagonal $(A_\mu)_{ij}$ components which are given by the product of the string tension and the distance between i and j.

Now when N D-brane lie together , that theory will be given by a $U(N)$ gauge theory. However the key point is to note that the world volume is same for all$N$ D-brane and it is not dependent on whether they coincide or not . So, even in the case of separated D-branes , the effective action should still be a $U(N)$ theory.

Now the gauge fields with A_m where $m = 0, 1, ..p$ are the gauge fields that live on the brane and the rest will give transverse fluctuations and they are labelled as

$$A^i \;=\; \frac{1}{2\pi\alpha'} X^i. \tag{4.51}$$

This indicates that these X^i are $U(N)$ matrices as well . So one has these positions now as noncommutating matrices. For more details one can refer to [6].

In the next section we will take a brief view at D-branes in Superstring theory. It will be important to us as it will help us in building the action functional we need to study *D0-branes* which will be center point of study of the M-atrix model of M-theory.

4.8 Superstring Theory and D-branes

Supersymmetry is a symmetry between bosons and fermions. Up untill now we have only discussed bosonic string theory. Now we will take a very brief review about the essentials we will need from superstring theory.

The mass shell condition we get from bosonic string theory is the following

$$M^2 \;=\; -p^\mu p_\mu.$$

Now to get spacetime fermions we will need the Dirac equation

$$ip_\mu \Gamma^\mu + M \;=\; 0 \tag{4.52}$$

and this generally leads to all known consistent string theories.

The gamma matrices satisfy the following algebra :

$$\{\Gamma^\mu, \Gamma^\nu\} \;=\; 2\eta^{\mu\nu}. \tag{4.53}$$

Now we can write down the world sheet action which includes fermions as well in the following manner

$$S \;=\; \frac{1}{4\pi}\int d^2z \left(\frac{2}{\alpha'}\partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial}\psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu\right) \tag{4.54}$$

where we have

$$\psi^\mu(z)\psi^\nu(0) \;\sim\; \frac{\eta^{\mu\nu}}{z} \tag{4.55}$$

the holomorphic part and

$$\tilde{\psi}^\mu(\bar{z})\tilde{\psi}^\nu \;\sim \frac{\eta^{\mu\nu}}{\bar{z}} \tag{4.56}$$

the anti-holomorphic part. These two equations equations (4.54) and (4.55) represent the operator products.

One can dive deeper and consult any standard superstring theory book to know more [21].

Now we go on and discuss D-branes in superstring theory. We want to follow the footsteps of [6].

One can introduce the concept of D-branes into supersymmetric string theory the same way it was introduced in bosonic string theory . In type**IIA** or type**IIB** D-branes are identified where type**I** superstrings can end. They couple naturally to the $p+1$ form Ramond-Ramond gauge field. Ramond–Ramond fields are differential form fields in the 10-dimensional space-time of type II supergravity theories, which are the classical limits of type II string theory. It is notable at this point that D-branes are BPS states and they conserve half of the super-symmetries. Since D-branes are BPS states, they carry RR charges which are also known as conserved abelian charges.

For type**IIB** string theory one can have the gauge fields of the form A, $A_{\mu\nu}$, $A_{\mu\nu\rho}$ etc. So in this case these fields couple to D(-1) branes ,D1-branes , D3-branes etc. However in type**IIA** theory , the gauge fields take the form of A_μ, $A_{\mu\nu\rho}$ etc. These fields couple to D0-branes , D2-branes etc. Here D0 branes are particles , D1 branes are called D strings and so on.

In our previous discussions we have worked out the action for the D branes in the landscape of bosonic string theory. All the things also hold in a supersymmetric theory as well but the only extra thing we need now is the addition of fermions in our action. So the Yang -Mills action will be replaced by a super Yang-Mills action and it will carry a 16 real component spinor ψ which will transform in the adjoin of $U(N)$. Moreover , there will be terms which will describe couplings to RR gauge fields. Also , by calculating the disk diagram one can compute the string tension T_p and the charge μ_p.

Next we will write down the effective action for a D0 brane which will play vital role in our study of M-atrix theory.

4.9 D0-Brane Effective Action

W have already derived the Dp brane action before. Now when we consider D0 brane one just needs to put $p = 0$ in that action and add the corresponding fermionic terms. Since the superstring theory lives in p dimension as we discussed before , our ten dimensional super Yang-Mills theory is reduced to $p + 1$ dimensions. Now we set $p = 0$ and we get

$$S_{D0} \;=\; T_0 \int dt \, \mathrm{Tr} \left(- \frac{(2\pi\alpha')^2}{4g_s} F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\Gamma^\mu\Psi \right) \tag{4.57}$$

and this action can also be found in literature [6].

The general representation of the Clifford algebra is 32 dimensional but Ψ is a Majorana-Weyl spinor with the following property

$$\Psi = \begin{pmatrix} \theta \\ 0 \end{pmatrix} . \tag{4.58}$$

Here , θ is a real , 16 component spinor and one can choose a representation for the gamma matrices. Also, note that this is a basis dependent structure. We choose the representation given in [6]

$$\Gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.59}$$

$$\Gamma^j = \begin{pmatrix} 0 & \gamma^j \\ \gamma^j & 0 \end{pmatrix} \tag{4.60}$$

where γ^i , $i = 1, 2, ..., 9$. These are 16×16 gamma matrices of $SO(9)$ [6]. Now we can work it out and one ends up with the following [6]

$$F_{ij} = i(2\pi\alpha')^2 \left[X^i, X^j\right] \tag{4.61}$$

and one also gets

$$\begin{aligned} F_{0i} &= \frac{1}{2\pi\alpha'}\partial_0 X^j + i\frac{1}{2\pi\alpha'}[A_0, X^j] \\ &= \frac{1}{2\pi\alpha'} D_0 X^j \end{aligned} \tag{4.62}$$

and finally the action looks like the one that is presented in [6]

$$S_{D0} = T_0 \int dt \, \mathrm{Tr} \left(\frac{1}{g_s}(D_0 X^i)^2 - i\theta^T D_0 \theta + \frac{1}{4g_s(2\pi\alpha')^2}[X^i, X^j]^2 + \frac{1}{2\pi\alpha'}\theta^T \gamma^j [X_j, \theta] \right). \tag{4.63}$$

One can see that this is a supersymmetric quantum mechanics[6] for X^i and θ in the adjoint of $U(N)$. The potential term $\theta^T \gamma^j [X_j, \theta]$ can be viewed as the Yukawa interaction Also the term $[X^i, X^j]^2$ is like the Higgs potential term.

With this , we conclude this chapter. In the next chapter we will briefly look at M-Theory and its Matrix Model that was proposed in [3]

Chapter 5

M-Theory and The M-atrix Model

5.1 Appearance of Eleventh Dimension

Type IIA supergravity gives us long wavelenght effective theory of type IIA superstring theory. Also, one can get IIA supergravity from dimensional reduction of 11 dimensional supergravity. Now, next natural question to ask is there any theory whose long wavelength limit is 11 dimensional supergravity? It turns out that such theory might exist and has been called "M-Theory" by Witten in [25]. So, this M-theory could be studied as a short wavelength coupling limit of 11 dimensional supergravity.

To see this, we have to go through some steps. We follow the procedure by [6]. Firstly , we will study the theories of supergravity in 10 and 11 dimensions. Then we will study type IIA string theory and its strong coupling limits. Then,we will introduce the Infinite momentum frame(IMF) and finally we will see how studying M-theory in the infinite momentum frame will lead us to M-atrix model.

5.2 Supergravity in 11 dimensions & type IIA supergravity

The 11 dimensional supergravity contains the metric G_{MN}, a 3 form potential A_3 which has components of the form A_{MNP} and Majorana gravitino Ψ_M. Now with all these objects one can go on and form the action of this theory. The Bosonic part which is presented in [6] is

$$S = \frac{1}{2}\int d^{11}\sqrt{G}(R + |dA_3|^2) + \int A_3 \wedge dA_3 \wedge dA_3 \tag{5.1}$$

where we have used the notations from differential forms.

One can go on and get the fermionic terms by using supersymmetry. Now, we will reduce this theory to ten dimensions. To do that we can do what we have done in our earlier discussions on compactifications. That is, we can take the direction x^{11} to be periodic and in turn we will assume nothing will depend on x^{11}. Now note that after this reduction , the Majaorana garvitino

$$\Psi_M \sim \begin{pmatrix} \psi^1_M \\ \psi^2_M \end{pmatrix} \tag{5.2}$$

gives rises to Majorana-Weyl gravitinos of opposite chirality in 10 dimensions. The 3 form A_3 gives birth to one 3 form and a 2 form denoted respectively as $A_{\mu\nu\rho}$ and $B_{\mu\nu}$. Now one can finally look at what the 11D metric gives rise to when its reduced to 10 dimensions. The 11D metric decomposes into a tensor $G_{\mu\nu}$ which is nothing but the 10D metric. A Scalar G_{11} and a vector A_μ and the vector is related as the following which was shown in [6]

$$A_\mu \;=\; -e^{-2\gamma} G_{\mu 11}.$$

Note that we have used M,N and P to decompose the 11th dimensional part and used μ, ν etc to deonote the 10 dimensional parts. One can easily check that under this dimensional reduction, the physical degrees of freedom are conserved. Now one thing is important that is the interpretation of $e^{2\gamma}$. To interpret this we will first write down the line element as follows :

$$\begin{aligned} ds^2 \;&=\; G_{MN}\, dx^M\, dx^N \\ &= G_{\mu\nu}\, dx^\mu\, dx^\nu + e^{2\gamma}(dx^{11} - A_\mu dx^\mu)^2 \end{aligned} \tag{5.3}$$

where $\mu, \nu \;=\; 0, 1, ...9$. Since we took x^{11} to be periodic , that is a circle , we can identify it with 0 and 2π. Now let's consider the line element on a circle of radius R. We will have

$$ds^2 \;=\; R^2\, d\theta^2 \tag{5.4}$$

and since x^{11} acts as the angular parameter in our compactification scheme, we can interpret e^γ as the radius of the compact dimension. Also we can see that the invariant world volume relation goes as follows

$$\int d^{11}x\, \sqrt{G_{11}} \;=\; 2\pi \int d^{10}x e^\gamma \sqrt{G_{10}}. \tag{5.5}$$

Now from action (5.1) we get in 10 dimensions [6]

$$\int d^{10}x \sqrt{G_{10}} \left(e^\gamma (R + |\nabla\gamma|^2 + |dA_3|^2) + e^{3\gamma}|dA|^2 + e^{-\gamma}|dB|^2 \right) + \int B \wedge dA_3 \wedge dA_3. \tag{5.6}$$

The explanation of the powers of the exponential comes from different contributions which is given in details in [6]. We won't go there now.

Also, the usual type of action in type IIA supergravity is given by the following action

$$\int d^{10}x \sqrt{g} \left(e^{-2\phi}(R + |\nabla\gamma|^2 + |dB|^2) + |dA|^2 + |dA_3|^2 \right) + \int B \wedge dA_3 \wedge dA_3. \tag{5.7}$$

One can get (5.6) from (5.7). To do that one needs to use the following identity

$$G_{\mu\nu} = e^{-\gamma} g_{\mu\nu} \tag{5.8}$$

Also after doing this identification one can reach to a conclusion as [6]

$$e^{\gamma} = e^{\frac{2\phi}{3}} \tag{5.9}$$

5.3 String Coupling

In the last section we wrote the radius of the compact dimension as $R = e^{\gamma}$. We also had an identification of this radius with the exponential of the dilaton. Now, we remind ourselves that, the coupling of string is given by the following

$$g_s = e^{\phi}$$

and in the previous section we used $\alpha' = 1$. So restoring α' in our equations we have the following relationship between the radius of the compact dimension and the string coupling.

$$R_{11} = \sqrt{\alpha'} g_s^{\frac{2}{3}}. \tag{5.10}$$

One thing should be noted carefully that the radius in the above equation gives us the radius of the 11th dimension where the distance is measured with the 11D metric G_{MN}. However one can measure the distances with Weyl scale metric $g_{\mu\nu}$ and in that case one has

$$ds_{11}^2 = e^{\gamma} \, ds_G^2$$

and hence one gets

$$R_{11}^{(g)} = \sqrt{\alpha'} g_s \tag{5.11}$$

and after all this one introduces the string length scale by $l_s = \sqrt{\alpha'}$ and ends up having

$$R_{11}^{(g)} = l_s g_s. \tag{5.12}$$

Now since we have compactified the 11 th direction we will have the usual KK modes and they are of the form

$$p^{11} = \frac{n}{R_{11}^{(g)}} = \frac{n}{l_s g_s}. \tag{5.13}$$

But the eleven dimensional states are massless, these K-K momenta contributes in the ten dimensional effective mass. So, we get ,

$$M = \frac{n}{l_s g_s}$$

Next we look at the strong coupling limit of the type IIA string theory in the next section.

5.4 Strong Coupling Limit of Type IIA String Theory & M-Theory

The low energy effective action of type IIA string theory is governed by the action in (5.7). The ϕ is the dilaton field and the expectation value of e^{ϕ} gives us the string coupling constant g_s. Let's write the action again here for our convenience.

$$\int d^{10}x\sqrt{g}\left(e^{-2\phi}(R+|\nabla\gamma|^2+|dB|^2)+|dA|^2+|dA_3|^2\right)+\int B\wedge dA_3\wedge dA_3.$$

now we can notice that the metric , dilaton and the the two form potential have a factor of $e^{-2\phi}$ that is g_s^{-2} with them. On the other hand the one form and the three form potential have no ϕ dependence. We know that $D\,p+1$ branes naturally couples to a p form field. So, there will D0 branes which will couple to the one form fields. We also notice the fact that D-branes are BPS states. Which means they preserve half of the supersymmetries. Also, they saturate the bound on the masses. This means the following

$$M = |Z| \tag{5.14}$$

where |Z| is the central charge of the $\mathcal{N} = 2$ supersymmetry algebra. Since this is the central charge , it will commute with all the generators of the algebra. In ten dimensions the algebra schematically reads as follows

$$\{Q,Q'\} \sim Z.$$

If the algebra is derived from eleven dimensions, then the central charge is just the momentum component of the eleventh direction. It is denoted as P. Since the central charge commutes with every generators, it must be made of abelian charge. From the supersymmetric algebra one can find that as shown in [6] that

$$Z = \frac{1}{\sqrt{\alpha'}g_s}. \tag{5.15}$$

And since the central charge is just the mass in ten dimensional case we can write

$$M = \frac{1}{g_s l_s} \tag{5.16}$$

where we used $\sqrt{\alpha'} = l_s$. Now, we can see that type IIA superstring theory gives us the same physics that we obtained by KK compactification of eleven dimensional supergravity.

One can also take the limit $g_s \to \infty$ and see that the massive now become very light. Also , as it was shown in [22] that when one has n D0 branes the mass is exactly given by

$$M = \frac{n}{l_s g_s} \tag{5.17}$$

and recall from our previous discussion that the denominator in this case is nothing but the radius of the compactified dimension. So, when one considers strong coupling limit , the radius become very large and we have an uncompactifed eleven dimensional theory.

Now, the eleven dimensional supergravity can not be the end of story. It is only considered as the long wavelength effective theory of some other consistent quantum theory. Now this is the point where M-theory had its genesis.

5.5 M-Theory : A brief overview

Eleven dimensional supergravity is the long wavelength effective theory **M-Theory**. So, we expect M-theory to describe the strong coupling limit of type IIA superstrings not only at low energy. Now one can give a definition of M-theory as follows :

Definition: M-theory is a eleven dimensional theory with its eleventh direction compactified on a circle of radius R , should be identical to type IIA superstring theory with string coupling $g_s = \frac{R}{\sqrt{\alpha'}}$,where R is the radius of the elventh dimension measured with the string frame metric.

Now lets just take a tour on what we know so far about M-theory. We wont dive into technical details here. Since we expect M-theory to describe type IIA superstring theory, it should contain all the objects that are already present in the type IIA theory. For example, M -theory should also incorporate D0, D2, D4 branes etc. We expect M-theory to contain extended objects as well since higher dmensional D-p branes are dual to their lower dimensional one. The theory will work out if there are extended objects as well as point like particles. Although the full formation of the theory is still not known, the modern approach consists of the M-atrix model and the AdS/CFT correspondence.

In the next section we will look at the M-atrix model approach to M-theory.

5.6 The Infinite Momentum Frame

Before we start discussing the M-atrix model of M-theory , we need a vital tool. This tool is knows as the **Infinite Momentum Frame** ,in short IMF . It is also knows as a light-cone frame. It was first introduced by Weinberg in [24] to unravel theory of perturbation. For perturbation theory in IMF the vacuum is trivial. Vertices of Feynman diagrams , from where the particles are generated from the vacuum are vanished. As mentioned in [6] , one can think of field theory in IMF as non-covariant perturbation theory but now the energy denominators are replaced by covariant denominators. Also , it is the only frame where Hamiltonian formulation of string theory was possible. For more comprehensive review one can take a look at [24].

We will now review some features of IMF that will be useful for us . Let us begin by choosing a particular direction say x^{11}. This means that we are choosing the eleventh direction. Now we call this direction as the *longitudinal* direction. There are directions x^i where $i = 1, ...9$ are called the *transverse* directions. This mechanism is similar from our discussion of lightcone coordinates from previous chapters. Now lets say we have a collection of particles in this frame. We denote their momentum by $(p^j_\perp\,,\, p^j_{11})$. Here j denotes the j^{th} particle and p_{10} is the momentum in the eleventh direction and and $p_\perp$ denotes the momentum in the transverse directions. Now we can boost the system in the x^{11} direction until the longitudinal momentum becomes much larger than all other relevant scales in the system.

After this we want to compactify the x^{11} direction with a radius R. According to [3] this will serve as infrared cutoff. So the momentum in direction in quantized as $p^{11} = \frac{n}{R}$. Now we write the energy formula

$$\begin{aligned}
E &= \sqrt{p_\perp^2 + (p^{11})^2 + m^2} \\
&= \sqrt{p_\perp^2 + \frac{n^2}{R^2}} \\
&= \frac{n}{R}\left(1 + \frac{R^2}{2n^2}(p_\perp^2 + m^2)\right) \\
&= \frac{n}{R} + \frac{R}{2n}(p_\perp^2).
\end{aligned} \tag{5.18}$$

Here we have expanded up to the lowest non-trivial order. Also, since we are taking non relativistic limit, we have dropped the mass term on right hand side. Now we boost the momentum significantly and at the same time expand the radius. So, at the end of calculation we will take $n \to \infty$ and $R \to \infty$. As a result $\frac{n}{R}$ remains constant. It is evident that there is no mass term as $\frac{n}{R} >> m^2$. So, our energy formula becomes

$$E = p_{11} + \frac{p_\perp^2}{2p_{11}} \tag{5.19}$$

and we can see that apart from the constant p_{11} the energy equation has a non-relativistic structure. In this structure p_{11} plays the role of an effective mass. One can call it the eleven dimensional mass.

Since we have a non-relativistic structure, we will have the Galilean symmetry. But since we also have fermions in our theory, we end up having a Supergalilean group. They obey the anticommutation relation as follows from [3]

$$
\begin{aligned}
[Q_\alpha, q_A] &= \gamma^i_{A\alpha} P_i \\
\{q_A, q_B\} &= \delta_{AB} P_{11} \\
\{Q_\alpha, Q_\beta\} &= \delta_{\alpha\beta} H
\end{aligned}
\tag{5.20}
$$

With this we conclude our discussion of IMF. In the next section we will briefly look at M-theory in IMF and the BFSS conjecture which was proposed in [3].

5.7 M-Theory in IMF : The Matrix Model

We now put our illustration of M-theory in infinite momentum frame. This idea came from [3] and this is the famous BFSS conjecture. It states that in IMF , M theory is a theory whose only degrees of freedom are D0-branes which has a minimal quantum of $p_{11} = \frac{1}{R}$. This system of interest is described by the effective action action of N $D0$-branes which is just an $N \times N$ quantum mechanics in the limit of $N \to \infty$.

Now its time time to justify why this seems to be the case. Again for full details one have to consult [3].

From earlier discussion we saw that the radius of the compact dimension is given in terms of the string coupling constant and the string length scale. That is $R = l_s g_s$. Also, this R takes this form when we measure the distance with the string metric g. As shown from earlier discussion and from [6] that the RR photon from type IIA string theory becomes a KK photon from compactifying x^{11} in a circle. The RR charge corresponds to p_{11} . Also , no perturbative string contains RR charge and has a vanishing p_{11}. For a single D0 brane we had $p_{11} = \frac{1}{R}$. Since we saw that in eleven dimensions it is massless , in ten dimension it is BPS saturated. Also, we saw that there are also KK states with $p_{11} = \frac{n}{R}$ where $n \neq 0$. For $n < 0$ we have [6] anti D0 branes and for $n > 1$ we get bound states of D0 branes. So when one takes this to IMF , one only gets the $n > 0$ contribution. As a result , one can come to a decision that M-theory in IMF only contains D0 branes and their bound states. Also we note that in this process the perturbative string which was coming from $n = 0$ and anti D0 branes coming from $n < 0$ is getting integrated out. So *this D0 brane dynamics in the IMF has the memory that before going to IMF there were more to M-theory and type IIA superstrings are just D0 branes.*

One can compare this situation with field theory in IMF where the vacuum is trivial but things like amplitude, cross sections etc incorporates all the subtleties of a quantum field theory. From [6] one can see that M-theory also contains objects like 2-branes and 5-branes.

It is shown in [6] that all of these M-theory membranes can effectively be described by the quantum mechanics of D0 branes.

In the next section we will discuss matrix model Hamiltonian and its spectrum.

5.8 Matrix model Hamiltonian, its Spectrum and Interactions

In this section we want to look at the matrix model Hamiltonian and its eigenvalue spectrum. We want to do it so as to gain ideas about how things actually work in this frame work. We start with an action that was introduced before for D0 branes. But now lets take the convention of [6] and continue.

$$S \,=\, T_0 \int dt\, Tr\left(\frac{1}{2g_s}(D_0X^i)^2 - i\theta^T D_0\theta + \frac{c^2}{4g_s}[X^i, X^j]^2 + c\theta^T\gamma^j[X_j, \theta]\right). \tag{5.21}$$

Here $c = \frac{1}{2\pi\alpha'}$ and $T_0 = \frac{1}{\sqrt{\alpha'}}$. The index i runs over all the transverse direction. Also , X^i, θ are in the adjoint representation of the gauge group $U(N)$. The covarinat derivative contains a gauge field and following [6] we can take it to be 0. Lets rescale our fields as $X^i = g_s^{\frac{1}{3}}Y^i$. This actually means a Weyl rescaling of the metric. As a result , now one measure distance with eleven dimensional supergravity metric G rather than the ten dimensional string metric g.

One can also go on and rescale time and get the eleven dimensional Planck length. In [3] the choice is made like

$$\tilde{\tau} \,=\, \frac{g_s^{\frac{1}{3}}}{R}t \tag{5.22}$$

and this makes $\tilde{\tau}$ dimensionless. So the corresponding Hamiltonian will also be dimensionless. But, it is convenient for us when the Hamiltonian spectrum comes with the dimension of energy. So following [6] we define

$$\tau \,=\, \frac{g_s^{\frac{1}{3}}}{T_0R}. \tag{5.23}$$

After rescaling, the action looks like

$$S \,=\, T_0^2 \int d\tau\ \mathrm{Tr}\left(\frac{1}{2RT_0^2}(\dot{Y^i}^2) - i\frac{1}{T_0}\theta^T\dot{\theta} + \frac{c^2R}{4}[Y^i, Y^j]^2 + cR\theta^T\gamma^j[Y_j, \theta]\right). \tag{5.24}$$

Now we can calculate the Hamiltonian and it turns out as,

$$H \;=\; R\;\mathrm{Tr}\left(\frac{1}{2}\Pi_i^2 - \frac{c^2T_0^2}{4}[Y^i,Y^j]^2 - T_0^2\,c\,\theta^T\,\gamma^j[Y_j,\theta]\right) \tag{5.25}$$

where $\Pi^i = \frac{\delta S}{\delta \dot{X}^i}$ and we took $\pi = \frac{\delta S}{\delta \dot{\theta}}$ and treated it as a constant and dropped it out of the Hamiltonian. We denote

$$\hat{H} \;=\; \mathrm{Tr}\left(\frac{1}{2}\Pi_i^2 - \frac{c^2T_0^2}{4}[Y^i,Y^j]^2 - T_0^2\,c\,\theta^T\,\gamma^j[Y_j,\theta]\right) \tag{5.26}$$

so we have ,

$$\hat{H} \;=\; \frac{H}{R} \tag{5.27}$$

Now lets take some state $|\Psi\rangle$ and we will act it with $\hat{H}$. the result in [6] gives us the following :

$$\hat{H}\,|\Psi\rangle \;=\; \frac{\epsilon}{n} \tag{5.28}$$

which eventually gives

$$H\,|\Psi\rangle \;=\; \frac{R}{n}\,|\Psi\rangle\,. \tag{5.29}$$

But we recall that for n D0 branes we have

$$p_{11} \;=\; \frac{n}{R}.$$

So we can see that the energy takes the following form

$$E \;=\; \frac{\epsilon}{p_{11}} \tag{5.30}$$

and here one can easily check that this ϵ corresponds to $\frac{p_\perp^2}{2}$. This is the dispersion relation for eleven dimension in IMF.

Now since we have seen the spectrum we can now discuss various interaction terms that arises in the Hamiltonian. Firstly , the term in the Hamiltonian that has $\theta^T\gamma^j[Y_j,\theta]$ is like a Yukawa potential term that appears in quantum field theories. This describes the interaction of the boson fermion vertex. Then we can see a potential of the form $[Y^i,Y^j]^2$. This is a familiar Higgs type potential. It is analogous to $\mathcal{N} \,=\, 2$ supersymmetric Yang-Mills theory in 4 dimensions. However, in field theories, when supersymmetry is present, this term vanishes.

Now, notice that we are dealing with D0 branes. From our chapter on D branes we found that the positions of D branes was given by non-commutating matrices. Now here , in this case , we can see that we have quantum mechanics and the expectation values of scalar Y^i do not give us distinct vacua but are collective coordinates of corresponding quantum wavefunctions. Now as argued in [6] this potential can have flat directions along which the different Y matrices commute and can be diagonalised simultaneously. These, diagonal components give the coordinates of the D0 branes. When the branes are far from each other and non- commutativity costs much energy. Also, when the branes approach each other , that means when they are really close, non-commutativity takes places and spacetime becomes intrinsically non-commutative. One can think of this scenario as strings stretching between two branes evolving in time. So from all this one can infer that in *late time physics , our usual notion of commutative spacetime emerges.* This point will be vital for our next chapters and it will be shown to be true in both one and two loop corrections of the Matrix cosmological we will be doing.

Before we conclude this section , it is useful to note that the matrix model incorporates supergravitons and their Fock space [3]. Also, there have been one and two loop test of low energy supergraviton scattering. They are presented in greater details in [4].

Chapter 6

Big Bang Models in String Theory : A Matrix Big Bang

In this chapter I will try to motivate why we need a string theoretic approach to cosmology. First we will discuss about the singularities that arises in general relativity. Then we will see how string theory smear out these singularities. Then we will introduce The Matrix Big Bang model that was introduced in [10].

6.1 Singularities in General Theory of Relativity

From general theory of relativity one can see that particles in spacetime moves in geodesics. However in some cases geodesics do not go on forever. We define singularity of a metric by the divergence of the Ricchi curvature. If geodesics terminate for finite values of affine parameter, the spacetime is geodesically incomplete. For singular spacetime we usually have geodesic incompleteness. Two famous examples can be given here. One of them being the black hole singularity. If we consider the Schwarzschild line element

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{6.1}$$

and then one can see that at $r = 2GM$ and at $r = 0$ the component of the metric diverges. However if one computes the Kretschmann scalar, it is easily seen that at $r = 0$ there is a singularity. This is the essential singularity. Another example is the FLRW line element (for the flat universe) as :

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2). \tag{6.2}$$

Here from the metric , if one considers the radiation dominated era, the scale factor $a(t)$ goes to zero as $t = 0$. It can be thought of as the big bang and we can also see that the metric diverges. It is a cosmological singularity.

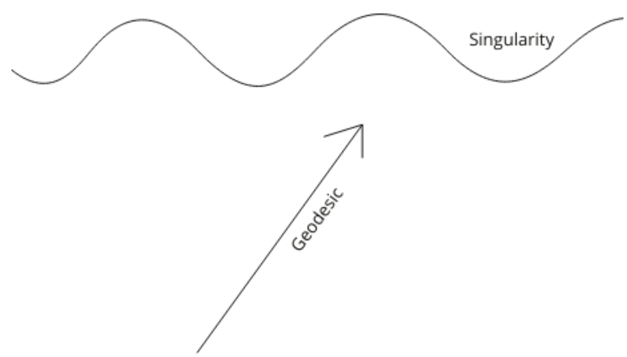

Figure 6.1: Geodesic ending on Singularity.

These spacetime singularities can be thought of as a point of infinite curvature. This is because the metric component diverges and the Riemann tensor is fundamentally dependent on the metric components. As a result general relativity breaks down at those points. Since these are very small length scale things, one also needs to take into account the quantum effects coming from this very small length scale. So, to understand the physics of singularity one needs to find a quantum description of gravity. But as we discussed at the beginning that quantizing general relativity seems to be troublesome since it is not renormalizable.

Keeping all this things in mind, one will want to turn to a quantum description of gravity. Since string theory is a promising candidate for a consistent quantum theory of gravity, it would be interesting to see what insights string theory brings to the table. In the next section we will briefly take a look at this.

6.2 Singularities in String Theory

It is clear by now that one has to go beyond general relativity to describe the spacetime singularities. We want to explore what string theory has to say about spacetime singularities. In particular we will focus on orbifold singularities. An orbifold can be thought of as a manifold with discrete identification. Mathematically, if M is a manifold and there is discrete group G which acts on M then the orbifold is defined as $\Gamma \equiv M/G$. This means, it is the quotient space. Lets look at an example here first. We can take the two dimensional plane which can also be identified with the complex plane C. Now if one considers the group Z_3 and it's action on the manifold, then the orbifold consists of one third of the complex plane and the emerging space is a cone and the head of this cone corresponds to a singularity.

We saw that at singularities general theory of relativity breaks down. However, string theory situation is completely different. As mentioned in [8] string perturbation theory is completely trouble free in orbifold spacetimes. The reason is that it contains *twisted sectors* of closed strings. These string sectors contain light degrees of freedom that was not taken into general relativity. As a result of carrying these light degrees of freedom , perturbative framework of string theory makes sensible result to physical questions. In some way it can be thought of as *resolving the singularity*. Now , one can go on ask which objects will be useful for resolving this kind of singularities? Well, it turns out that D-branes are useful probes to study this singularities. From our previous discussions we saw that when D-branes are close together, spacetime becomes non-commutative. They also carry additional light degrees of freedom that is essential to get a non-singular description.

Also, it is a well known result that D-branes can wrap cycles in extra-dimensions of string theory. As a result one can get charged particles in lower dimensional string theory. The mass of such particles are given by the product of the D-brane tension and volume of the wrapped cycle. If the D-branes wrap very small cycle , only then they can become light otherwise they are very heavy [8].

These wrapped branes are also known as fractional branes. This is illustrated in the figure below:

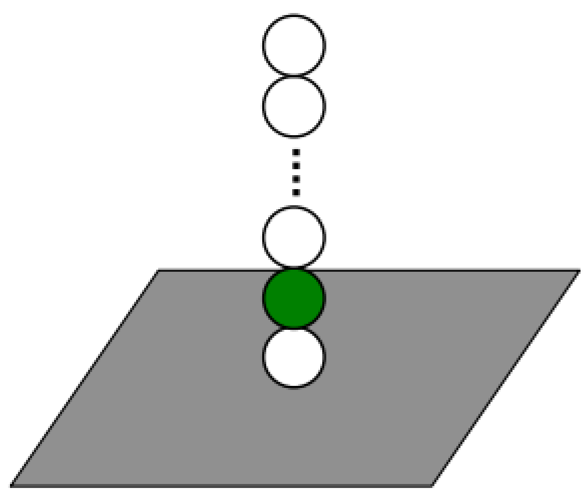

Figure 6.2: Fractional D-branes. [8]

These fractional branes are stuck at orbifold singularities and this is the property which makes them useful probes to study singularities. So we can say that the static orbifold singularities are resolved in perturbative string theory provided we take into account the light degrees of freedom provided by the fractional branes.

Now, we will take a look at cosmological singularities and discuss in a brief manner how they work out in string theory. The concept of big bang comes with a number of questions. For example, one of them would be, does time really begin ? Is it possible to know what was before big bang? If there was something indeed , how one gets past the singularity? Some work has been done in the landscape of pre big-bang scenario in [14].

So, to answer such questions we will turn to string theory now. Orbifold singularities are well understood in string theory. But, all those singularities were static singularities. To study singularities like the big bang singularity we will now make our orbifolds time dependent. Our approach will be to compute the S-matrix elements in the far past and in the far future. The idea is that, in the backgrounds where string theory is well understood, the fundamental observables will always have to be related with asymptotic region of spacetime. However, there are some subtle issues with working in orbifolds that is discussed in [8].

For our discussion of the big bang singularity , we will want to work on Milne orbifold which looks like the following :

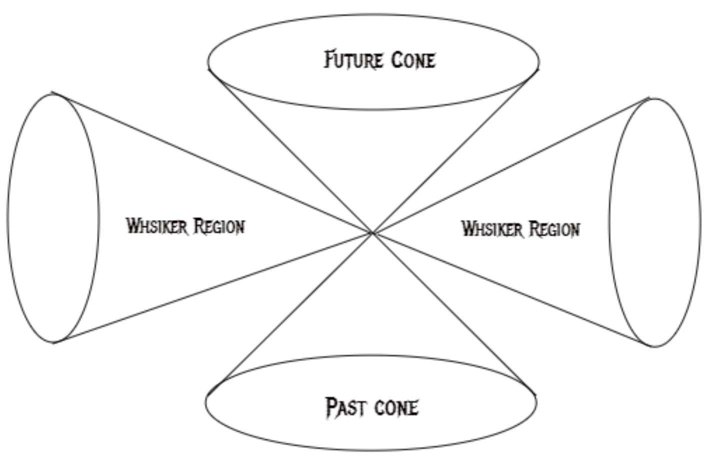

Figure 6.3: The Milne Orbifold.

As one can see there is a future cone and a past cone. It also comes with two whisker regions where one may get closed timelike curves. The tip where four cones meet will represent our **Big Bang Singularity**. Around these singularities string perturbation theory turns out to be a poor description of what is really going on. Taking into account of winding modes which become light near these regions might capture what is happening near this singularity. Another way of resolving this is to find a non-pertubative description of string theory. In the next section we will start our journey by construction a framework that follows from BFSS [3] matrix theory. It is called the *Matrix Big Bang* model proposed in [10].

6.3 Matrix Big Bang Model

In this section we will briefly review the Matrix Big Bang model that was proposed in [10]. It was shown that the light-like linear dilaton background can be studied as it gave a simple time dependent solution to type IIA critical superstring theory in ten dimensions. So to begin we will choose the dilation to be linear and have a lightlike direction. As following [10] we will choose it as

$$\phi = -QX^{+} \tag{6.3}$$

so the string coupling is given by

$$g_s = e^{-QX^{+}}. \tag{6.4}$$

Now to construct a string theory solution one also needs to define a metric. The flat ten dimensional spacetime metric will look like

$$ds_{10}^2 = -2dX^{+}dX^{-} + dX^{i}dX_{i}. \tag{6.5}$$

These are in fact the time dependent solution of a type IIA theory[8].

The Q factor appearing in the dilation can be thought of as a constant having no special physical significance. It can be scaled by using a boost which leaves the metric invariant. One thing is to note that the light like linear dilation contributes nothing to the conformal anomaly which breaks the conformal symmetry between the classical and the quantum theory. Now, if we look at things from the string frame, the only time dependence is seen in the string coupling constant. Now, lets go to the Einstein frame which is particularly useful for comparing different cosmological scenarios to the well known results of general relativity. In this frame the dilaton is canonically normalised as a scalar field which couples minimally with gravity and the Planck length gives the natural length scale. So we write the metric as

$$ds_e^2 = e^{\frac{QX^{+}}{2}} ds_{10}^2 \tag{6.6}$$

as was chosen in [10]. We can see that in the far past that is $X^{+} \rightarrow -\infty$, $ds^2 = 0$is zero. This is a *big bang type singularity* for this model. We will take the future cone/future quadrant of the Milne orbifold as our world sheet which is described by the metric

$$ds^2 = e^{2\eta}(-d\eta^2 + dx^2) \tag{6.7}$$

where we have chosen the conventions of [10]. We also have $x \sim x + 2\pi$. Again the milne circle shrinks to zero when $\eta \to -\infty$. In the dual Matrix description, the time flow from early to late universe can be thought of as an renormalization group flow from the UV sector to the IR sector. So, in late time we will get the conventional spacetime picture. We discussed before that perturbative string theory will not work because of huge gravitational counter reaction from the singularity. This Milne Orbifold is our world sheet in matrix string theory. Since they gave us a non perturbative description, it should capture what is going on around the singularity. Also we note that the metric in (6.7) is actually similar to the cosmological metric in the conformal time while the exponential term is the scale factor.

Now as $X^+ \to -\infty$ the string coupling becomes very strong. So this ten dimensional metric gets lifted into the 11 dimensional M-theory and we write the metric as

$$ds^2 = e^{\frac{2QX^+}{3}} ds_{10}^2 + e^{\frac{-4QX^+}{3}} dY^2 \tag{6.8}$$

where Y is the extra dimension.

Now to calculate the Riemann tensor components , following [10] define the following one form basis

$$\begin{aligned} e^i &= e^{QX^+/3} dX^i \\ e^y &= e^{-2QX^+/3} dY \\ e^+ &= e^{QX^+/3} dX^+ \\ e^- &= e^{QX^+/3} dX^-. \end{aligned}$$

Then one calculates the non vanishing Riemann tensor components [10] as

$$\begin{aligned} R_{+i+i} &= \frac{Q^2}{9} \exp\left(\frac{2QX^+}{3}\right) \\ R_{+y+y} &= -\frac{8Q^2}{9} \exp\left(-\frac{4QX^+}{3}\right). \end{aligned} \tag{6.9}$$

Also one can show that [10] the Ricci tensor vanishes and what we get is a purely gravitational M-theory. Now the Reimann tensor diverges along the lightlike direction in the far past and the far future. So we have two singularities here . To know which one will represent the big bang singularity , we have to check that for which singularity the affine parameter is geodesically incomplete . In short , the "big bang singularity" we are looking for must occur from a finite distance from present. To see this we consider the geodesic equation

$$\frac{d^2X^+}{d\lambda^2} + \frac{2Q}{3}\left(\frac{dX^+}{d\lambda}\right)^2 = 0 \tag{6.10}$$

where the connection Γ^+_{++} is computed as $\frac{2Q}{3}$. Then it is integrated and we get [10]

$$\lambda = e^{\frac{2}{3}QX^+} \tag{6.11}$$

and as we take $X^+ \to -\infty$ we see that $\lambda \to 0$.

To see that this is indeed the curvature singularity we can write the metric in terms of λ and then get the Riemann tensor [10]

$$\begin{aligned} R_{\lambda i \lambda i} &= \frac{1}{4\lambda} \\ R_{\lambda y \lambda y} &= -\frac{2}{4\lambda} \end{aligned} \tag{6.12}$$

So at $\lambda = 0$ we indeed have a curvature singularity. So one can conclude that the big big type singularity does occur in the light like linear dilaton background.

Also we recall that , we will be interested in computing the S-matrix elements. In principle one gets a really simple formula for string amplitudes in terms of the flat space amplitudes in the case of a light like linear dilaton. More details of how one can get this can be found in [10]. We just present the result here

$$A^{g,n} = A^{g,n}_{flat} \int_{-\infty}^{\infty} d\tau_* \, e^{(-2g-2+n)Qp^+\tau_*} \tag{6.13}$$

here , τ_* is the average insertion points , n denotes the number of strings we want to scatter and g denotes the genus. One effect of the linear dilation is that now the coupling is that now it dependent on the worldsheet coordinate τ, which is the lightcone cordinate on the worldsheet. For more details one needs to see [10].

Next thing we want to do is to generalize out matrix string description for the time dependent background. This is a very long derivation. So , we will just state the result here. Full details can be found in [10, 8].

To generalize matrix string description in the time dependent background and derive the action from the matrix big bang model we will take the same route which was taken into [10]. The recipe for this looks like the following :

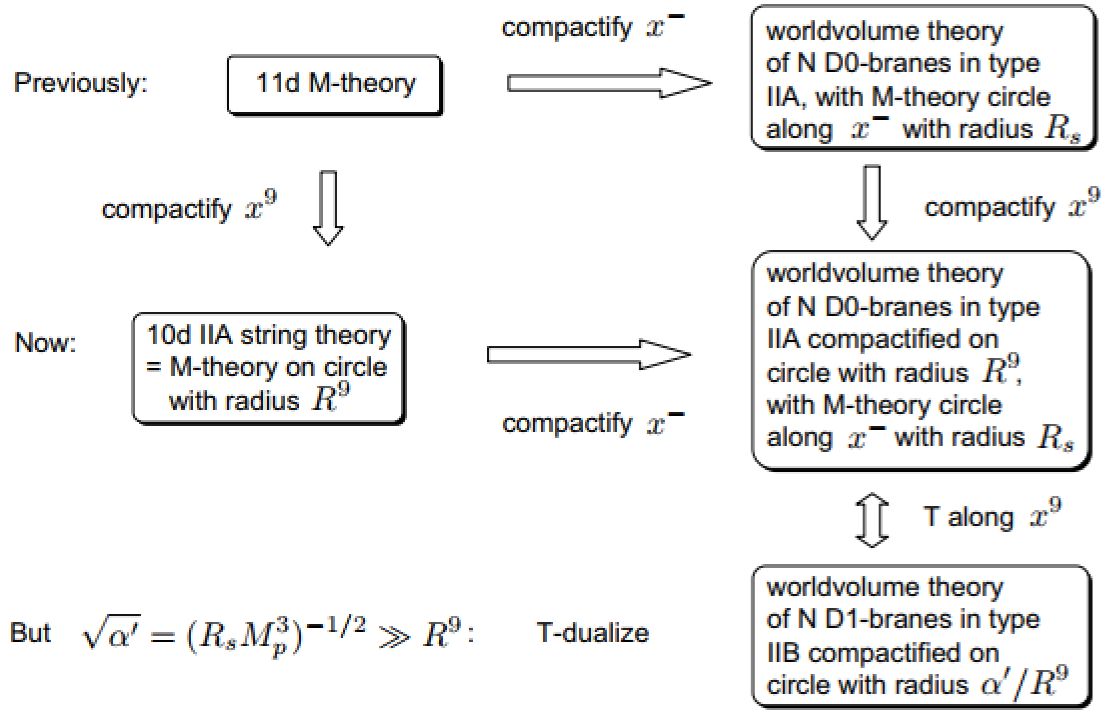

Figure 6.4: Recipe to get Matrix Big Bang Model Action. [8]

The action for the matrix big bang is [10, 9, 8]

$$S = \int d\tau \, d\sigma \, \mathrm{Tr}\left(\left(D_\mu X^i\right)^2 + \theta^T \not{D}\theta + g_s^2 F_{\mu\nu}^2 - \frac{1}{g_s^2}\left[X^i, X^j\right]^2 + \frac{1}{g_s}\theta^T \gamma_i \left[X^i, \theta\right]\right) \tag{6.14}$$

Here this action comes from the discrete light cone quantaization of type IIA string theory in a sector with N units of lightcone momentum is given by the low-energy limit of the world volume theory of N D1-branes in type IIB string theory. This is the dimensional reduction of $(9+1)$dimensional super-Yang Mills theory to $(1+1)$ dimensions with $\mathcal{N} = 8$. To get uncompactified type IIA string theory, one has to take a large N limit [8].

So now as we have our action in hand , the next thing we want to do is to look for the quantum dynamics of this matrix big bang model. This is studied in details in [9] where the authors found a one loop effective potential which near the big bang turned out to be attractive and in late time dies away. The result is as it should be for the dynamical emergence of spacetime. In the next chapter I will review on how one can get this one loop potential and after that I will explain my findings of the two loops effective potential for this model.

Chapter 7

One Loop Effective Potential For The Matrix Big Bang Model

In this chapter , I will briefly review the concept of effective potential in quantum field theories. Some of the tools we will see here will be useful to study the effective dynamics of our model. Then I will go on and provide a full technical detail of how one can get the one loop effective potential for the matrix big bang model that was first computed in [9].

7.1 Effective Potential in Field Theories

In quantum field theories , one cat start with the generating functional to compute things like two point functions , four point functions etc. Let us write our generating functional as

$$Z[J] \;=\; \int \mathcal{D}\phi \;\; exp\left[-\frac{1}{\hbar}(S[\phi] - J \cdot \phi)\right]. \tag{7.1}$$

Here the action $S[\phi]$ is for a scalar field with interactions and J is our source term. However, this generating functional will give us the connected and the disconnected Feynman graphs. We will only be interested in the connected one particle irreducible graphs. So to get only connected and one particle irreducible graphs we need to define something that is known as the effective action.

The effective action $\Gamma[\phi]$ is defined as the following

$$\Gamma[\varphi] = W[J] - \int d^d x' \varphi\left(x'\right) J\left(x'\right) \tag{7.2}$$

where $W[J]$ is defined as

$$Z[J] = e^{iW[J]} \tag{7.3}$$

and it gives the connected graphs only. Now the φ is the classical part of ϕ. So in principle we are using the saddle point method and we write

$$\phi(x) \;=\; \varphi(x) + \hbar^{\frac{1}{2}}\tilde{\phi}(x) \tag{7.4}$$

and the term $\tilde{\phi}(x)$ denotes the quantum correction and we restored to measure the strength of the quantum correction. Now we expand the terms in the exponential of the generating functional around the classical background and get

$$S[\phi] - J \cdot \phi \ = \ S\left[\varphi\right] - J \cdot \varphi + \hbar \frac{1}{2} \tilde{\phi} \cdot S''\left[\phi_c\right] \cdot \tilde{\phi} + \cdots \tag{7.5}$$

where we have written

$$\tilde{\phi} \cdot S''\left[\phi_c\right] \cdot \tilde{\phi} \ = \ \iint d^d x d^d y\, \tilde{\phi}(x) S''_{xy}\left[\varphi\right] \tilde{\phi}(y) \tag{7.6}$$

where

$$S''_{xy}[\phi] \ = \ \frac{\delta^2 S[\phi]}{\delta\phi(x)\delta\phi(y)}. \tag{7.7}$$

With all this we can put the expansion into the generating functional to obtain

$$Z[J] = \left(\det\left[S''\left[\varphi\right]\right]\right)^{-1/2} \exp\left(\frac{1}{\hbar}\left(J \cdot \varphi - S\left[\varphi\right]\right)\right). \tag{7.8}$$

The determinant is coming from evaluating the Gaussian integral in the generating functional. After this one can compute the $W[J]$ and after that putting all this in equation (7.2) one can get

$$\Gamma[\varphi] = S[\varphi] + \frac{\hbar}{2}\,\mathrm{Tr}\left[\log\left(S''[\varphi]\right)\right]. \tag{7.9}$$

Now that we have got an effective action , we can go on and define the *Effective Potential*. But lets try to understand what this effective action means. This is indeed important as this gives us the 1PI diagrams. But at the same time it is important for the discussion renormalisation and renormalisation group etc. In summary , we can state that *effective action carries the things that one can see in a theory.*

Now that we have a basic idea what effective action means we define our effective potential as the following way

$$\Gamma[\varphi] \ = \ \int d^d x\, \mathcal{V}(\varphi) \tag{7.10}$$

this $\mathcal{V}(\varphi)$ is our effective potential . If one draws analogy with classical mechanics , we can think of this effective potential as the thing that fixes the quantum field theory ground state. And perturbation around the vaccum , gives us the loop amplitudes. Here , for one loop case , the contribution is coming from the determinant part of the effective action . So for one loop case we can finally write the expression of the effective potential as

$$\int d^d x\, \mathcal{V}_{1-Loop}(\varphi) \ = \ \frac{1}{2}\,\mathrm{Tr}\left[\log\left(S''[\varphi]\right)\right] \tag{7.11}$$

and we have selected to work in natural units again.

The next sections of this chapter will describe the quantum dynamics of the matrix big bang model. Then we will take a look at the effective world sheet action that describes this model and finally will compute the one and two loop effective potentials.

7.2 Fermions and Wavefunctions on Milne Orbifold

Let us first briefly recall what we have been doing so far. We are studying a type IIA spacetime theory with a light like linear dilation background $\phi = -QX^+$. Here we took the big bang to occur when $X^+ \to -\infty$. So , near the big bang the theory is strongly coupled at at late times we can have a perturbative description. Then one can compactify $X^- \sim X^- + R$ and get a non-perturbative definition of string theory in this linear dilaton background which is given by the two dimensional supersymmetric Yang-Mills theory on the Milne Orbifold. As in [9] one can see that the Yang Mills theory is weakly coupled in early times and strongly coupled in late times.

We can obtain the Milne Orbifold from the usual Minkowski spacetime as was done in [9]

$$ds^2 = -2d\xi^+ d\xi^- \tag{7.12}$$

with the boost identification

$$\xi^\pm \sim \xi^\pm e^{\pm 2\pi Q \ell_s}. \tag{7.13}$$

Then we can go on and define new coordinates as σ and τ where

$$\sigma \sim \sigma + 2\pi l_s \tag{7.14}$$

and

$$\xi^\pm = \frac{1}{\sqrt{2}Q} e^{Q(\tau \pm \sigma)}. \tag{7.15}$$

So the metric in terms of the new coordinates will be

$$ds^2 = e^{2Q\tau}\left(-d\tau^2 + d\sigma^2\right) \tag{7.16}$$

and we can see that only at $\tau \to -\infty$ the metric vanishes and we have our big bang singularity. So from this metric we can see that the presence of singularity can be seen by the late time physics . It is the case as we have a large circle [9] that varies in time and have vanishing local curvatures. This is what we will call the *Milne Circle* and because of this circle we have supersymmetry breaking.

Now we ill turn out attention to the Fermions that are on the Milne Orbifold. In [9] it was shown that this are indeed the Fermions on the Milne space. Also , from [9] we can see that because of this the Lorentz invariance is spontaneously broken and one can also determine the choice of spin structure on world volume circle. To study the Fermions , we will need to study the type IIB background which is related to type IIA string with a linear light like dilaton via U-duality. Now in [9] the authors considered a single D-string wrapped on x^1. Where

$$x^1 \sim x^1 + \frac{2\pi l_s}{r} \tag{7.17}$$

and r is defined as

$$r \equiv \frac{\epsilon R}{2\pi l_s}. \tag{7.18}$$

Also we have the metric as [9]

$$ds^2 = r e^{\epsilon Q x^+} \left\{ -2dx^+ dx^- + \sum_{i=1}^{8} \left(dx^i \right)^2 \right\} \tag{7.19}$$

where we get [9]

$$\phi = \epsilon Q x^+ + \log r. \tag{7.20}$$

Then after fixing the world volume coordinates with spacetime coordinates as was done in [9] , we can write the action for the Fermions in the DBI form as

$$S_F = \frac{1}{2\pi \ell_s^2} \int d\tau \, d\sigma \, e^{-\phi} \sqrt{\det(-g)} \bar{\psi} \not{D} \psi. \tag{7.21}$$

This was done using making the gauge fields to be zero. Then one can expand similarly as we did while deriving actions for D-branes in early chapters and after doing this one ends up with [9] kinetic terms as below

$$\frac{1}{r^{3/2}} \, e^{-\frac{\epsilon Q \tau}{2\sqrt{2r}}} \bar{\psi} \left(\gamma^0 \partial_\tau + \gamma^1 \partial_\sigma \right) \psi. \tag{7.22}$$

In our case we will want the non-abelian version of this and the answer from [9] is

$$\frac{1}{r^{3/2}} \operatorname{Tr} \left\{ e^{-\frac{\epsilon Q \tau}{2\sqrt{2r}}} \bar{\psi} \left(\gamma^0 D_\tau + \gamma^1 D_\sigma \right) \psi + e^{\frac{\epsilon Q \tau}{2\sqrt{2r}}} \bar{\psi} \gamma^i \left[X^i, \psi \right] \right\}. \tag{7.23}$$

Here the trace is over the gauge index and we also have *Yukawa Type Interaction.*

The γ^i are gamma matrices for $Spin(8)$. The scalar X^i are in the adjoint representation of gauge group $U(N)$. Also rescaling the ψ gives [9]

$$\mathrm{Tr}\left\{\bar{\psi}\left(\gamma^0 D_\tau + \gamma^1 D_\sigma\right)\psi + e^{\frac{\epsilon Q\tau}{\sqrt{2}r}}\bar{\psi}\gamma^i\left[X^i,\psi\right]\right\}. \tag{7.24}$$

This is the canonical form of the fermionic kinetic terms. One thing to notice is that the fermions are periodic in the σ direction.

The next natural thing to consider is the wavefunctions on the Milne orbifold. This orbifold s identified by the boos identification (7.13). So we want wave functions ϕ_s which have a conserved norm on the Milne orbifold. This boost invariant wavefunction will satisfy [5]

$$\phi_s\left(e^{2\pi Q l_s}\xi^+, e^{-2\pi Q l_s}\xi^-\right) = e^{2\pi Q l_s s}\phi_s\left(\xi^+,\xi^-\right) \tag{7.25}$$

and the wave functions have the solution as [5]

$$\phi_{j,s}\left(\xi^+,\xi^-\right) = \int_{-\infty}^{\infty} dv \exp\left[i\left(p^+\xi^- e^{-v} + p^-\xi^+ e^{v}\right) + ivj - vs\right]. \tag{7.26}$$

Here one can see that the momentum in the σ direction is $j + is$ and for any non zero spin , the solutions are not periodic. With this we conclude this section. In the next section we will look at the effective action for the matrix big bang [10] and the loop expansion for this case .

7.3 Effective Action and Loop Expansion

The Matrix String Theory [12] compactified on Milne orbifold is governed by the action that was presented in [10] is

$$S = \frac{1}{2\pi l_s^2}\int d\sigma\, d\tau\, \mathrm{Tr}\left(\frac{1}{2}\left(D_\mu X^i\right)^2 + \bar{\psi}\not{D}\psi + g_s^2 l_s^4\pi^2 F_{\mu\nu}^2 - \frac{1}{4\pi^2 g_s^2 l_s^4}\left[X^i,X^j\right]^2 + \frac{1}{2\pi g_s l_s^2}\bar{\psi}\gamma_i\left[X^i,\psi\right]\right). \tag{7.27}$$

Here we have $g_s = exp(-Q\tau)$. We will be using this simpler relation rather than the exact relation [10]. As there is explicit time dependence , the world sheet energy is not conserved. Now we can define the yang mills coupling constant by the following identification

$$g_{YM} \equiv \frac{1}{g_s l_s}. \tag{7.28}$$

In early times that is $\tau \to -\infty$ we have a weakly coupled Yang-Mills theory. So we can do perturbation of this theory to describe the dominant quantum effects. For that we need to have a choice of **vacuum**. From [9] we note that the vacuum solution is still described by chossing a constant matrix configuration on the Cartan of the gauge group. This means we will integrate out the off diagonal heavy modes first. Then we will ask whether we still get an effective potential for the massless diagonal modes. However , it should be kept in mind that , if the theory has enough supersymmetry present , then the contributions from the bosons gets cancelled by the contribution from the fermions. But in the big bang model we are discussing , supersymmetry is broken on the worldsheet of the matrix string. So there might be a potential.

There exists two natural choice fro the vacuum. One is the *adiabatic vacuum* and another one is *conformal vacuum*. We will choose **adiabatic vacuum** for our computation. This has advantage that [10] , it behaves better in the high energy regime.

Having all the essential tools in our hand , we now begin to start to calculate the one loop effective potential. Firstly we will use the effective action from [9]

$$S = \frac{l_s^2}{2\pi} \int \mathrm{Tr} \left(\frac{1}{2} \left(D_\mu X^i \right)^2 + \bar{\psi} \not{D} \psi + e^{-2Q\tau} \pi^2 F_{\mu\nu}^2 - \frac{1}{4\pi^2} e^{2Q\tau} \left[X^i, X^j \right]^2 + \frac{1}{2\pi} e^{Q\tau} \bar{\psi} \gamma_i \left[X^i, \psi \right] \right) \tag{7.29}$$

where the one has the fields rescaled as follows [9] :

$$\begin{aligned} X^i &\to l_s^2 X^i \\ \psi &\to l_s^2 \psi \\ A_\mu &\to A_\mu . \end{aligned}$$

We can see that one of the dimensionful parameter Q and another one is ℓ_s^2. From [10] , we know that Q has no invariant physical meaning. Also following [9] we identify $\hbar$ with ℓ_s^2. So this ℓ_s^2 will control our loop expansion just as $\hbar$ controlled our quantum corrections in earlier discussions. It should be noted that we are using 1PI effective action. The reasons can be more transparent from [9].

The off-diagonal scalar fields have the action in the form [9]

$$S \;=\; \frac{l_s^2}{2\pi} \int d\tau \, d\sigma \left(\dot{X}^2 - X'^2 - b^2 e^{2Q\tau} X^2 \right). \tag{7.30}$$

Where b is the $SO(8)$ trace invariant distance between two eigenvalues of X^i. From the action one can identify the mass term as $b^2 e^{2Q\tau}$. So , the mass is time dependent. However, from the identification $\sigma \sim \sigma + 2\pi l_s$ we can see that the momentum in the σ direction is quantized as $p_\sigma \sim \frac{1}{l_s}$. for non-zero modes , we will have a conventional mass term [9] and no problem in finding out the effective potential. So we will consider the zeroth mode and we will have

$$S \sim \frac{l_s^2}{2\pi} \int d\tau \left(\dot{X}^2 - b^2 e^{2Q\tau} X^2 \right). \tag{7.31}$$

From this we get the equation of motion

$$\left(\partial_\tau^2 + b^2 e^{2Q\tau} \right) X = 0. \tag{7.32}$$

Now in principle one can solve this and find the corresponding propagator. but lets switch to $\xi^\pm$ coordinates now . After going to this coordinate system the explicit time dependent in the mass term goes away. Also , now we have to keep in mind that the background is now time dependent. So we now have the wave equation as [9].

$$\left(2 \frac{\partial^2}{\partial \xi^+ \partial \xi^-} + b^2 \right) X = 0. \tag{7.33}$$

This equation has invariance under the orbifold identification. The non trivial effects will come of the σ momentum quantized condition which is dependent on the spin of the particle. The fermions also satisfy this wave equation [9] and the only difference is the momentum in the σ direction is given by as we mentioned before $j \pm is$.

We can find the propagator from the above equation . The only difference we have from the usual Minkowski space propagator is that the propagator for a spin s particle is obtained by summing under the images [5]. Also the ghosts cancel the contribution coming from the two scalars [9]. So our propagator looks like the following [9].

$$G_s \left(\xi, \xi'; b^2 \right) = \sum_n \int \frac{dp^+ dp^-}{(2\pi)^2} \frac{\exp \left(-ip^- \left(\xi^+ - e^{2\pi Q \ell_s n} \xi^{+'} \right) - ip^+ \left(\xi^- - e^{-2\pi Q \ell_s n} \xi^{-'} \right) + 2\pi Q \ell_s n s \right)}{-2p^+ p^- + b^2 - i\epsilon} \tag{7.34}$$

and here the sum over n runs from $-\infty$ to ∞. The $i\epsilon$ prescription was added for later convenience.

From our discussion in the first section of this chapter we have seen that to get the one loop effective potential we need to compute the following quantity

$$-\frac{1}{2} \operatorname{Tr} \log(H) \tag{7.35}$$

where $H = \left(-2 \frac{\partial^2}{\partial \xi^+ \partial \xi^-} + b^2 \right)$. This amounts to compute the determinat of the differential operator. For that we will follow the method of **Heat Kernel**.

The heat kernel is the kernel that satisfies the heat equation with certain boundary conditions . Lets denote the heat kernel by K and it satisfies the following

$$\begin{gathered} i\partial_s K\left(t',t;s\right)=\Delta K\left(t',t;s\right) \\ K\left(t',t;0\right)=\delta\left(t-t'\right) \end{gathered} \tag{7.36}$$

where we have

$$K\left(t',t;s\right)\equiv < t'\left|e^{-is\Delta}\right|t > . \tag{7.37}$$

To compute the determinant , we use the identity

$$\ln(\det\Delta)=-\int_0^\infty\frac{ds}{s}Tr\left(e^{-is\Delta}\right) \tag{7.38}$$

and if the trace is written in terms of the integration one can easily see that the propagator will be there inside the expression for the determinant. We will follow the conventions of [9] and one thing more to notice is that the integral can also be done in therms of contour integrals. Following [9] we have the heat kernel as

$$e^{-itH}\left(\xi,\xi'\right)\;=\;-\oint\frac{dz}{2\pi i}e^{-tiz}\,G\left(\xi,\xi';z\right). \tag{7.39}$$

Here this z is to be compared with the $i\epsilon$ in the propagator and s in the previous formula. Then using **Cauchy's theorem** we can do the contour integral and the result is

$$\begin{aligned} e^{-itH_s}(\xi,\xi)=&\sum_n\int\frac{dp^+dp^-}{(2\pi)^2}\exp\Big(-it\left[b^2-2p^+p^-\right] \\ &+\left[-ip^-\xi^+\left(1-e^{2\pi Q\ell_s n}\right)-ip^+\xi^-\left(1-e^{-2\pi Q\ell_s n}\right)+2\pi Q\ell_s ns\right]\Big). \end{aligned} \tag{7.40}$$

Then all we need to do is to evaluate dp^+ and dp^- integrals. Notice that we wrote $l_s\;=\;\ell_s$.

To do the integral I have chosen to write

$$\begin{aligned} A\;&=\;e^{2\pi\ell_s Qn} \\ B\;&=\;e^{-2\pi\ell_s Qn} \end{aligned}$$

and we will also ignore the following terms,

$$e^{-itb^2}\qquad\quad 2\pi Q\ell_s ns \tag{7.41}$$

for now since this wont be relevant in evaluating the integral .

We will put this back at the end of the calculation. Now the the integral becomes

$$
\begin{aligned}
I &= \int \frac{dp^+dp^-}{(2\pi)^2} \exp(-2ip^+p^-t - ip^-\xi^+A - iBp^+\xi^-) \\
&= \int \frac{dp^+dp^-}{(2\pi)^2} \exp\left[-ip^+(2p^-t + \xi^-B)\right] \exp(-ip^-\xi^+A) \\
&= \int \frac{dp^-}{(2\pi)^2} \frac{\exp[-ip^-\xi^+A]}{i(\xi^-B + 2p^-t)} \\
&= \int \frac{du}{(2\pi)^2\, 2t} \frac{e^{\left(\frac{-i\xi^+Au}{2t}\right)}}{u} \exp\left(\frac{-i\xi^-\xi^+AB}{2t}\right) \\
&= \exp\left(\frac{-i\xi^-\xi^+AB}{2t}\right) \frac{1}{(2\pi)^2 2t} \int du \frac{e^{\left(\frac{-i\xi^+Au}{2t}\right)}}{u} \\
&= \frac{1}{(2\pi)2t} \exp(\frac{-i\xi^+\xi^-AB}{2t})
\end{aligned}
$$

now we will put back the terms in (7.41) and the summation over n hence we finally obtain the result which was exactly presented in [9]

$$
\begin{aligned}
e^{-itH_s}(\xi, \xi) &= \sum_n \frac{1}{(2\pi)2t} \exp\left(-itb^2 - i\frac{\xi^-\xi^+}{2t}\left(1 - e^{2\pi Q\ell_s n}\right)\left(1 - e^{-2\pi Q\ell_s n}\right) + 2\pi Q\ell_s ns\right) \\
&= \sum_n \frac{1}{(2\pi)2t} \exp\left(-itb^2 + 2i\frac{\xi^-\xi^+}{t} \sinh^2(\pi Q\ell_s n) + 2\pi Q\ell_s ns\right).
\end{aligned}
\tag{7.42}
$$

The result for the heat kernel we found above has nice features. For $n = 0$ this kernel describes a particle going over a closed loop in time t . This in turns gives rise to Coleman-Weinberg potential [9]. Also one can see that there is a spacetime dependence as $\xi^{\pm}$. In [9] this is interpreted to be the thing that regularizes the ultraviolet divergence coming from small t contribution. This means a particle which is propagating the distance squared $\xi^+\xi^-$ in small time t is exponentially large in τ. More details can be found regarding this in [9].

Now using this kernel we can calculate the one loop effective potential as [9]

$$
\int V_{eff}(b) = i \int d^2\xi \int \frac{dt}{2t} \sum_{\text{helicities}} e^{-it(H_s)}(\xi, \xi). \tag{7.43}
$$

We mentioned before that that the ghosts cancel the contributions coming from the two scalars. So, we have [9]

$$
\sum_{\text{helicities}} (-1)^{2s} e^{2\pi Q\ell_s s} = \left(e^{\pi Q\ell_s n/2} - e^{-\pi Q\ell_s n/2}\right)^4 = 16\sinh^4(\pi Q\ell_s n/2). \tag{7.44}
$$

Now we plug all that we have found in equation (7.43) and we have [9]:

$$
\begin{aligned}
\int V_{eff}(b) &= \int d^2\xi \sum_{n=-\infty}^{\infty} \left(\frac{2i}{\pi}\right) \sinh^4(\pi Q\ell_s n/2) \int_0^{\infty} \frac{dt}{t^2} \exp\left(-itb^2 + \frac{i}{t} 2\sinh^2(\pi Q\ell_s n)\,\xi^+\xi^-\right) \\
&= -\int d^2\xi \sum_{n=-\infty}^{\infty} \frac{2}{\pi} \sinh^4(\pi Q\ell_s n/2) \int_0^{\infty} \frac{dt'}{(t')^2} \exp\left(-t'b^2 - \frac{1}{t'} 2\sinh^2(\pi Q\ell_s n)\,\xi^+\xi^-\right) \\
&= -\int d^2\xi \sum_{n=-\infty}^{\infty} \frac{2}{\pi} \frac{b\sinh^4(\pi Q\ell_s n/2)}{\left[2\sinh^2(\pi Q\ell_s n)\,\xi^+\xi^-\right]^{1/2}} K_1(\sqrt{8b^2\sinh^2(\pi Q\ell_s n)\,\xi^+\xi^-})
\end{aligned}
\tag{7.45}
$$

and in the second line we have switched to the new parameter defined as $t' = it$. Here the K_1 denotes a modified Bessel function of the second kind. Now we have the expression for the one loop effective potential. Now we want to look at the early times and the late time behaviour of this potential.

7.4 Asymtotic Behaviour of The One Loop Potential

7.4.1 Late Time Behaviour

In very late time $\xi^{\pm} \to \infty$. As a result the argument of the modified Bessel function is very large and we have the Bessel function asymptotic behaviour

$$K_1(z) \approx \frac{1}{\sqrt{z}} e^{-z}. \tag{7.46}$$

Using this relation we have [9]

$$\int V_{late} \approx -\int d^2\xi \frac{2^{3/4} b^{1/2} \sinh^4(\pi Q\ell_s/2)}{\pi\,(\xi^+\xi^-)^{3/4} \sinh^{3/2}|\pi Q\ell_s|} \exp(-\sqrt{8b^2\sinh^2(\pi Q\ell_s)\,\xi^+\xi^-}) \tag{7.47}$$

and one can note that the sum over n gives dominant contribution at $n = \pm 1$ [9]. So at very late times the potential dies of really quick. As the Milne circle grows large supersymmetry is effectively restored. Also, as the potential dies off, one gets the regular commutative spacetime. So in this matrix big bang model [10] spacetime emerges dynamically. Next we will look at the early time potential.

7.4.2 Early Time Potential

In early times , that is near the big bang we have the Bessel function behaviours as follows

$$K_1(z) \approx \frac{1}{z} \tag{7.48}$$

as a result we have the early time potential to be [9]

$$
\begin{aligned}
\int V_{early} &\approx -\int d^2\xi \sum_n \frac{2}{\pi} \frac{b\sinh^4\left(\pi Q\ell_s n/2\right)}{\left[2\sinh^2\left(\pi Q\ell_s n\right)\xi^+\xi^-\right]^{1/2}} \frac{1}{\sqrt{8b^2\sinh^2\left(\pi Q\ell_s n\right)\xi^+\xi^-}} \\
&\approx -\int d^2\xi \frac{1}{8\pi\xi^+\xi^-} \sum_n \tanh^2\left(\pi Q\ell_s n/2\right) \\
&\approx \int d^2\xi \frac{1}{8\pi^2 Q\ell_s \xi^+\xi^-} \log\left(2b^2\xi^+\xi^-\right).
\end{aligned}
\tag{7.49}
$$

So in very early times this potential is is very large and attractive. As a result the non commutative structure arises.

The authors in [9] stated that , it would be interesting to know whether the higher loop potential behaves in the same way as the one loop potential. In fact, they speculated that higher loops contribution should decay more rapidly. In the next chapter I will present my calculation which indeed validates this speculation.

Chapter 8

Two Loop Effective Potential of the Matrix Big Bang

In this chapter I will present what I have found in my thesis work. We will see for the two loop effective potential, we have an exponentially suppressed behaviour of the effective potential as was suspected in [9]. To compute the loop diagrams I have used the dimensional regularisation and the following Schwinger's trick

$$\int_0^\infty dx\, e^{-\alpha\, x} \;=\; \frac{1}{\alpha}. \tag{8.1}$$

At this point this might seem trivial but it will be very helpful while doing our loop integrals. So first I will present the relevant 1PI loop diagrams. Then we will use Feynman rules to calculate them and then will see our effective potential from these diagrams.

We could also find out the related diagrams using the path integral and saddle point method. But it's mathematically very long. Since the diagrams give the same answer as them its better to proceed with the diagrams. Making life easier. Now, let us start.

8.1 Diagrams at Two Loops

We recall our action

$$S = \frac{1}{2\pi l_s^2}\int d\sigma\, d\tau\, \mathrm{Tr}\left(\frac{1}{2}\left(D_\mu X^i\right)^2 + \bar{\psi}\not{D}\psi + g_s^2 l_s^4 \pi^2 F_{\mu\nu}^2 - \frac{1}{4\pi^2 g_s^2 l_s^4}\left[X^i, X^j\right]^2 \; + \frac{1}{2\pi g_s l_s^2}\bar{\psi}\gamma_i\left[X^i,\psi\right]\right). \tag{8.2}$$

From this action we see there cubic and quartic interaction terms. For any theory we can have two loops diagram as follows.

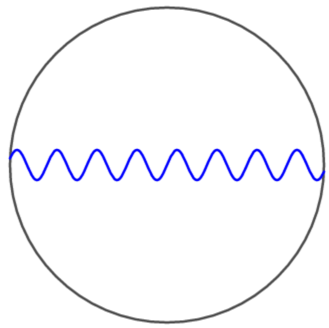

Figure 8.1: Three point interaction in two loop diagram

Here wavy line represents bosons and solid line represents fermions. Also, we can have the following diagram. The diagram in (8.1) has symmetry factor of $\frac{1}{12}$ and the diagram below has a symmetry factor $\frac{1}{8}$. We will not mention them explicitly further.

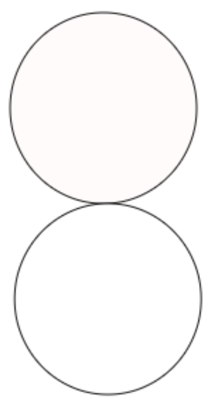

Figure 8.2: Four point interaction in two loop diagram

.

Moreover we can have diagrams like below

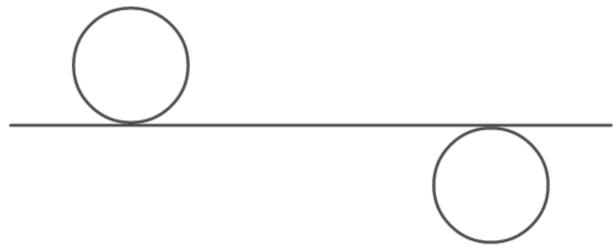

Figure 8.3: Non 1PI two loop diagram.

We can have many more diagrams. However, the first two diagrams are relevant as they are 1PI diagrams. The third diagram we are seeing is not 1PI as cutting the internal line between them generates disconnected graphs.

So, we will only consider the diagram in figure (8.1) which is commonly known as *The Fish Diagram* and the diagram in figure (8.2) which is also known as *Figure of eight*.

8.2 Four Point Interaction

The four point interaction will come from the $[X^i, X^j]^2$ terms. Since we are working with a $U(N)$ gauge theory , we will denote our generators with T^a. Here we have the following commutation relation

$$[T_a, T_b] \;=\; i\, f^{abc} T_c. \tag{8.3}$$

Also we have the following relationship for the " Normalised Anti-commutator"

$$\{T_a, T_b\} \;=\; \frac{1}{N}\delta_{ab}\mathbf{1_N} + \text{Traceless Part.} \tag{8.4}$$

So we get

$$\mathrm{Tr}(T_a T_b) \;=\; \frac{1}{2}\delta_{ab}. \tag{8.5}$$

Now from our action , we see that the quartic potential is $\mathrm{Tr}\left(-\frac{1}{4\pi^2 g_s \ell_s^4}[X^i, X^j]^2\right)$. So , now will evaluate the trace. to do that , we do the following .

We will write the X^i as follows

$$X^i = \frac{i}{2}\left(X_0^i \mathbf{1} + X_a^i T^a\right) \tag{8.6}$$

and we will ignore the 0 index as this is just the center of mass co ordinate. So we have

$$\begin{aligned}[X^i, X^j] &= -\frac{1}{2} X_a^i X_b^j [T^a, T^b] \\ &= -\frac{1}{2} X_a^i X_b^j f^{abc} T^c.\end{aligned} \tag{8.7}$$

So , we get

$$[X^i, X^j]^2 = -\frac{1}{4} X_a^i X_b^i X_d^i X_e^j f^{abc} f^{def} T^c T^f \tag{8.8}$$

and taking the trace we have

$$\mathrm{Tr}\left([X^i, X^j]^2\right) = -\frac{1}{8} f^{abc} f^{dec} X_a^i X_b^i X_d^i X_e^j. \tag{8.9}$$

This above term in equation (8.9) gives rise to four point interactions with vertices denoted by

$$g_4 = \frac{1}{32\pi^2 g_s \ell_s^4} f^{abc} f^{dec}. \tag{8.10}$$

Now lets look at the diagram we are considering.

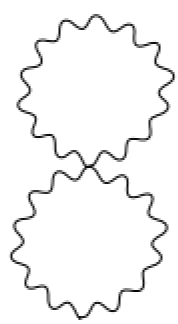

Figure 8.4: Four point scalar interaction. Wavy lines represent bosonic propagators. [4]

We recall that our propagator is as follows

$$G_s\left(\xi,\xi';b^2\right) = \sum_n \int \frac{dp^+dp^-}{(2\pi)^2} \frac{\exp\left(-ip^-\left(\xi^+ - e^{2\pi Q\ell_s n}\xi^{+'}\right) - ip^+\left(\xi^- - e^{-2\pi Q\ell_s n}\xi^{-'}\right) + 2\pi Q\ell_s ns\right)}{-2p^+p^- + b^2}. \tag{8.11}$$

We will ignore the spin s term and will not write the sum over n explicitly for now as they are not necessary to evaluate the loop integral. Also one can notice that there is only one vertex of four point interaction and in this case $\xi' = \xi$. We will put all the things back after doing the integration. Also for this loop integral our propagator will look like the following

$$G_s\left(\xi,\xi;b^2\right) = \sum_n \int \frac{dp^+dp^-}{(2\pi)^2} \frac{\exp\left(-ip^-\left(\xi^+ - e^{2\pi Q\ell_s n}\xi^{+}\right) - ip^+\left(\xi^- - e^{-2\pi Q\ell_s n}\xi^{-}\right) + 2\pi Q\ell_s ns\right)}{-2p^+p^- + b^2}. \tag{8.12}$$

So we now denote the momentum in the upper loop as l and momentum in the lower loop as k. We then proceed to calculate the diagram.

$$\begin{aligned}
&\sum_{n,m} \int d\xi^+ d\xi^-\, dl^+ dl^- dk^+ dk^- \,\frac{1}{(2\pi)^4}\, \frac{1}{-2k^+k^- + b^2}\, \frac{1}{-2l^+l^- + b^2} \\
&\exp\left(-ik^-\xi^+(1-e^{2\pi\ell_s Qn}) - ik^+\xi^-(1-e^{-2\pi\ell_s Qn}\right) \\
&\exp\left(-il^-\xi^+(1-e^{2\pi\ell_s Qm}) - il^+\xi^-(1-e^{-2\pi\ell_s Qm}\right).
\end{aligned} \tag{8.13}$$

Then we denote the followings to minimize the notation clutter.

$$\begin{aligned}
A &= (1-e^{2\pi\ell_s Qn}) \\
A' &= (1-e^{-2\pi\ell_s Qn}) \\
B &= (1-e^{2\pi\ell_s Qm}) \\
B' &= (1-e^{-2\pi\ell_s Qm}).
\end{aligned}$$

Using the above mentioned generalisation, we get

$$\begin{aligned}
&\sum_{n,m} \int d\xi^+ d\xi^-\, dl^+ dl^- dk^+ dk^- \,\frac{1}{(2\pi)^4}\, \frac{1}{-2k^+k^- + b^2}\, \frac{1}{-2l^+l^- + b^2} \\
&\exp\left(-ik^-\xi^+ A - ik^+\xi^- A'\right)\, \exp\left(-il^-\xi^+ B - il^+\xi^- B'\right).
\end{aligned} \tag{8.14}$$

First we do $\xi^\pm$ integration as

$$\begin{aligned}&\int d\xi^+\, d\xi^- \;\exp\left(-ik^-\xi^+A - ik^+\xi^-A'\right)\,\exp\left(-il^-\xi^+B - il^+\xi^-B'\right)\\ =&\int d\xi^+\, d\xi^- \;\exp\left(-i\xi^+(k^-A + l^-B)\right)\exp\left(-i\xi^-(k^+A' + l^+B')\right)\end{aligned} \tag{8.15}$$

thus giving us two delta functions as

$$\delta^+\left(k^-A + l^-B\right) \qquad \delta^-\left(k^+A' + l^+B'\right) \tag{8.16}$$

so using these delta functions we can evaluate $k^\pm$ integration and the delta functions forces us to use the following

$$k^- \;=\; -\frac{B}{A}l^- \qquad k^+ \;=\; -\frac{B'}{A'}l^+$$

Now lets write the following

$$c \;=\; -\frac{B}{A} \qquad c' \;=\; -\frac{B'}{A'} \tag{8.17}$$

and $d^2l \;=\; dl^+\, dl^-$. After a wick rotation our integral becomes

$$\int \frac{d^2l}{(2\pi)^2}\,\frac{1}{-2l^2 + b^2}\,\frac{1}{-2l^2cc' + b^2}. \tag{8.18}$$

Here we note that $l^+l^- \;=\; l^2$.Then we manipulate the denominator as follows

$$\begin{aligned}&(-2l^2 + b^2)\,(-2D^2l^2 + b^2)\\ =&4D^2l^4 - 2b^2l^2 - 2D^2l^2b^2 + b^4\\ =&4D^2l^4 - l^2(2b^2 + 2D^2b^2) + b^4\end{aligned} \tag{8.19}$$

we take

$$\gamma \;=\; -(2b^2 + 2D^2b^2) \tag{8.20}$$

so now we want to put it back and complete the square for the following

$$\begin{aligned}&4D^2l^4 + \gamma l^2 + b^4\\ =&4D^2\left(l^2 + \frac{\gamma}{8D^2}\right)^2 - \left(\frac{\gamma^2}{16D^2} - \frac{b^4}{4D^2}\right).\end{aligned} \tag{8.21}$$

Then we take

$$l'^2 \;=\; \left(l^2 + \frac{\gamma}{8D^2}\right) \tag{8.22}$$

so we have

$$l'\,dl' \;=\; l\,dl$$

and we write

$$M \;=\; -\left(\frac{\gamma^2}{16D^2} - \frac{b^4}{4D^2}\right). \tag{8.23}$$

Finally, we get

$$\int dl'\,\frac{1}{2\pi}\,\frac{l'}{4D^2l'^4 + M} \tag{8.24}$$

where we have integrated over the angular direction after the wick rotation.

So we have

$$V(l) \;=\; \int dl'\,\frac{1}{2\pi}\,\frac{l'}{4D^2l'^4 + M}. \tag{8.25}$$

Now we want to see this potential in the co-ordinate picture. For that, we need to fourier transform the following potential. In the following manner ,

$$\int d^2\xi\, V(\xi^+, \xi^-) \;=\; \int d^2\xi\, d^2k\, e^{-ik\cdot\xi}\, V(k) \tag{8.26}$$

So we need to evaluate the following integral

$$\int dr\, dl'\,\frac{1}{2\pi}\,\frac{l'\,e^{-il'r}}{4D^2l'^4 + M}. \tag{8.27}$$

To evaluate this, we used Wolfarm Mathematica and the details of the code is provided in the Appendix. So we get the result of this integration

$$\int dl'\,\frac{l'\,e^{-il'r}}{4D^2l'^4 + M} \;=\; G^{5,1}_{1,5}\left(\left.\begin{matrix} \frac{1}{2} \\ 0\;\frac{1}{4}\;\frac{1}{2}\;\frac{1}{2}\;\frac{3}{4}\end{matrix}\;\right|\;\frac{M\,r^4}{1024\,D^2}\right) \tag{8.28}$$

The solution comes off as a MeijerG function. This function has an asymptotic behaviour when r is very large , which can be simplified to the following. Here to reduce notational cluster we will write it as just G_4 and by 4 it will be understood that it represents the solution to the four point interaction. So we have the following asymptotic behaviour for large r.

$$G_4 \;\sim\; 4\sqrt{2}\,\pi^{\frac{5}{2}}\,\exp\left\{-\left(\frac{M}{cc'}\right)^{\frac{1}{4}}\frac{r}{2}\right\}\cos\left\{\left(\frac{M}{cc'}\right)^{\frac{1}{4}}\frac{r}{2}\right\} \tag{8.29}$$

Now one needs to find this $\frac{M}{cc'}$. We will do it and find out dominant terms as we recall this terms carries sum over m and n from $-\infty$ to ∞.

We recall that $cc' = \frac{B\,B'}{A\,A'}$. Now putting values of A and B we get

$$\begin{aligned} cc' &= \frac{B\,B'}{C\,C'} \\ &= \sum_{n,m} \frac{(1-e^{2\pi\ell_s Q m})(1-e^{-2\pi\ell_s Q m})}{(1-e^{2\pi\ell_s Q n})(1-e^{-2\pi\ell_s Q n})} \\ &= \sum_{n,m} \frac{1-\cosh(2\pi\ell_s Q m)}{1-\cosh(2\pi\ell_s Q n)} \end{aligned} \tag{8.30}$$

Now we determine M.

We have from (8.23)

$$\begin{aligned} M &= \left(\frac{b^4}{4\,cc'} - \frac{\gamma^2}{16cc'}\right) \\ &= \frac{1}{4cc'}\left(b^4 - \frac{\gamma^2}{4}\right) \\ &= \frac{b^4}{4cc'}\left(1-(1+cc')^2\right) \\ &= \frac{b^4}{4}(cc'-2). \end{aligned}$$

we have

$$\frac{M}{cc'} = \frac{b^4}{4}\left(1-\frac{2}{cc'}\right). \tag{8.31}$$

Thus giving us

$$\begin{aligned} \frac{M}{cc'} &= \frac{b^4}{4}\left(1 - \frac{2(1-\cosh(2\pi\ell_s Q n))}{1-\cosh(2\pi\ell_s Q m)}\right) \\ &= \frac{b^4}{4}\left(1 - \frac{2\sinh^2(\pi\ell_s Q n)}{\sinh^2(\pi\ell_s Q m)}\right). \end{aligned} \tag{8.32}$$

Here the last line following by the identity $\cosh(2x) - 1 = 2\sinh^2(x)$.

Now we recall that while evaluating this integrals we got rid off the sum over the spins s. Now we do that sum and from [9] the result is

$$\sum_{spins} (-1)^{2s}\, e^{2\pi\ell_s Qns} \;=\; 16\sinh^4\left(\frac{\pi Q\ell_s n}{2}\right) \tag{8.33}$$

another term would be $16\sinh^4\left(\frac{\pi Q\ell_s m}{2}\right)$.

So our potential will be

$$\begin{aligned} i\int V_4 \approx i\int\, dr\sum_{n,m} \frac{4\sqrt{2}\pi^{\frac{3}{2}}}{2}\,\sinh^4\left(\frac{\pi Q\ell_s n}{2}\right)\,\sinh^4\left(\frac{\pi Q\ell_s n}{2}\right)\,\exp\left(-\frac{r}{2}\left\{\frac{b^4}{4}\left(1-\frac{2\sinh^2(\pi\ell_s Qn)}{\sinh^2(\pi\ell_s Qm)}\right)\right\}^{\frac{1}{4}}\right) \\ \cos\left(\frac{r}{2}\left\{\frac{b^4}{4}\left(1-\frac{2\sinh^2(\pi\ell_s Qn)}{\sinh^2(\pi\ell_s Qm)}\right)\right\}^{\frac{1}{4}}\right) \end{aligned} \tag{8.34}$$

And the dominant contribution comes from $n=\pm 8$ and $m=\pm 9$. So Finally **late time behaviour of the potential is**

$$\begin{aligned} \int V_4^{late} \approx \int\, dr\, \frac{4\sqrt{2}\pi^{\frac{3}{2}}}{2}\,\sinh^4\left(\frac{8\pi Q\ell_s}{2}\right)\,\sinh^4\left(\frac{9\pi Q\ell_s}{2}\right)\,\exp\left(-\frac{r}{2}\left\{\frac{b^4}{4}\left(1-\frac{2\sinh^2(8\pi\ell_s Q)}{\sinh^2(\pi\ell_s Q9)}\right)\right\}^{\frac{1}{4}}\right) \\ \cos\left(\frac{r}{2}\left\{\frac{b^4}{4}\left(1-\frac{2\sinh^2(8\pi\ell_s Q)}{\sinh^2(9\pi\ell_s Q)}\right)\right\}^{\frac{1}{4}}\right) \end{aligned} \tag{8.35}$$

and in the very late time regime we have $r\to\infty$, we get a vanishing potential. One can check this by plotting this function. For simplicity we have considered the following function

$$f(\mathrm{r}) := \exp\left(-\frac{r}{2}\right)\cos\left(\frac{r}{2}\right). \tag{8.36}$$

The graph included here is plotted for values of r ranging from 0 to 20.

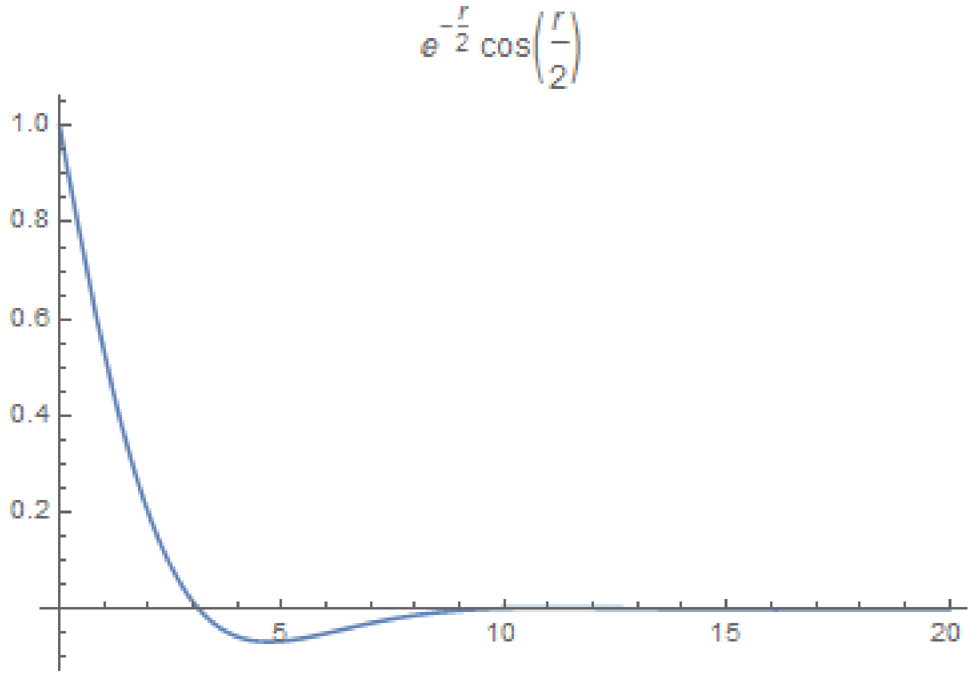

Figure 8.5: Behaviour of the potential function as a function of r.

So , the reason to include this toy model graph to is to show that the potential indeed dies of really quickly. The physical significance of this phenomenon will be discussed later. At the end of the two loop correction we shall have to say more about this.

We will now look at the early time behaviour of the two loop potential that is generated by this four point scalar interaction. So similarly in early times we have an expansion of the G as the following

$$\int V_4^{early} \approx -\int dr \frac{r\left(\pi^{7/2}\sqrt[4]{\frac{M}{D^2}}\right)}{2\left(\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)} \tag{8.37}$$

where we have dropped the constant part appearing in the asymptotic expansion and work with the first order part.

Again we write down the terms explicitly and also add the factors like before. Thus taking the form

$$
\begin{aligned}
\int V_4^{early} \approx & -\int dr \, \frac{1}{32} \frac{\pi^{\frac{7}{2}}}{\left(\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)} \sum_{n,m} \sinh^4\left(\frac{\pi Q \ell_s m}{2}\right) \sinh^4\left(\frac{\pi Q \ell_s n}{2}\right) \\
& \left(\frac{b^4}{4}\left(1 - \frac{2\sinh^2(\pi \ell_s Q n)}{\sinh^2(\pi \ell_s Q m)}\right)\right)^{\frac{1}{4}} r.
\end{aligned} \tag{8.38}
$$

One can clearly see that there is an attractive potential near the big bang.

The way to think is the following. The coordinate r can be thought of as the τ that appears in the boost identification and play the role of *world-sheet time*. When $r \to -\infty$ this means the big bang and this potential becomes positive. Also it has a very large values. At this limit the Milne circle shrinks to zero radius and we have our big bang singularity.

Now one can also find the dominant term as we discussed earlier and we have

$$
\begin{aligned}
\int V_4^{early} \approx & -\int dr \, \frac{1}{32} \frac{\pi^{\frac{7}{2}}}{\left(\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)} \sinh^4\left(\frac{8\pi Q \ell_s}{2}\right) \sinh^4\left(\frac{\pi Q \ell_s 8}{2}\right) \\
& \left(\frac{b^4}{4}\left(1 - \frac{2\sinh^2(8\pi \ell_s Q)}{\sinh^2(9\pi \ell_s Q)}\right)\right)^{\frac{1}{4}} r.
\end{aligned} \tag{8.39}
$$

So in this case we discussed both early and the late time behaviour of the two loop effective potential. In the next Section we will look into what happens for the Yukawa interaction , that is the three point interaction at two loop level.

8.3 Yukawa Interaction

From our action we see that we have the Yukawa interaction of a boson and a fermion in the following way

$$
\mathrm{Tr}\left(\frac{1}{2\pi g_s \ell_s^2}\bar{\psi}\gamma_i\left[X^i, \psi\right].\right) \tag{8.40}
$$

We again make use of the following decomposition as was done before

$$
\begin{aligned}
X^i &= \frac{i}{2}(X_0^i \mathbf{1} + X_a^i T^a) \\
\psi^i &= \frac{i}{2}(\psi_0^i \mathbf{1} + \psi_b^i T^b).
\end{aligned} \tag{8.41}
$$

So we will evaluate the trace in the same manner as was done for the four point interaction in the previous section and this gives us

$$\mathrm{Tr}\left(\frac{1}{2\pi g_s \ell_s^2}\bar{\psi}\gamma_i\left[X^i,\psi\right]\right) = \frac{1}{32\pi\ell_s^2 g_s}\, f^{abc}\bar{\psi}_c^i X_a^i \psi_b^i \tag{8.42}$$

and the three point vertex term can be read of as

$$g_3 = \frac{1}{32\pi\ell_s^2 g_s}\, f^{abc}. \tag{8.43}$$

Now our diagram looks like the following

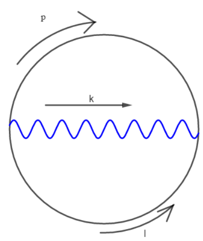

Figure 8.6: Two loop diagram for 3 point interaction.

In this diagram the solid line represents the fermions and the wavy line are for the bosons. Now we write down the loop contribution. Again we don't write the vertex terms explicitly since they wont be relevant in the loop integrals. Also we wont write the helicity terms right now but will add them later as they don't contribute in the loop momentum integral evaluation as well.

$$\begin{aligned} A = \sum_{q,n,m} \int d\xi^+ d\xi^- d\xi'^+ d\xi'^- & \frac{d^2 l\, d^2 k\, d^2 p}{(2\pi)^6} \frac{1}{-2l^+l^- + b^2} \frac{1}{-2p^+p^- + b^2} \frac{1}{-2k^+k^- + b^2} \\ & \exp\left(-il^-(\xi^+ - e^{2\pi\ell_s Qn}\xi'^+) - il^+(\xi^- - e^{2\pi\ell_s Qn}\xi'^-)\right) \\ & \exp\left(-ik^-(\xi^+ - e^{2\pi\ell_s Qm}\xi'^+) - ik^+(\xi^- - e^{2\pi\ell_s Qm}\xi'^-)\right) \\ & \exp\left(-ip^-(\xi^+ - e^{2\pi\ell_s Qq}\xi'^+) - ip^+(\xi^- - e^{2\pi\ell_s Qq}\xi'^-)\right). \end{aligned} \tag{8.44}$$

Firstly we do the $\xi'^{\pm}$ integration as follows ,

$$\int d\xi^{\pm} \exp\left(-i\xi^{\pm}(l^{\mp}\, e^{\pm 2\pi\ell_s Qn} + k^{\mp}\, e^{\pm 2\pi\ell_s Qm} + p^{\mp}\, e^{\pm 2\pi\ell_s Qq})\right) \tag{8.45}$$

which gives

$$\delta^{\pm}\left(p^{\mp}e^{\pm 2\pi\ell_s Qq} + l^{\mp}e^{\pm 2\pi\ell_s Qn} + k^{\mp}e^{\pm 2\pi\ell_s Qm}\right). \tag{8.46}$$

In this notation we wrote the two delta functions in a single line using the $\pm$ notation. From these delta functions we get,

$$p^{\pm} = M^{\mp}(a^{\pm}l^{\pm} + b'^{\pm}k^{\pm}) \tag{8.47}$$

where we wrote

$$\begin{aligned} M^{\mp} &= e^{\mp 2\pi\ell_s Qq} \\ a^{\pm} &= e^{\pm 2\pi\ell_s Qn} \\ b'^{\pm} &= e^{\pm 2\pi\ell_s Qm}. \end{aligned} \tag{8.48}$$

Then we do $p^{\pm}$ integration to get

$$\begin{aligned} A = \int \frac{d^2l\, d^2k}{(2\pi)^4}\, d^2\xi \frac{1}{-2l^+l^- + b^2}\frac{1}{-2k^+k^- + b^2}\frac{1}{-2M^+M^-(a^-l^- + b'^-k^-)(a^+l^+ + b'^+k^+) + b^2} \\ \exp(-il^-\xi^+ - il^+\xi^-)\exp(-ik^-\xi^+ - ik^+\xi^-) \\ \exp(-i\xi^- M^+(l^+a^- + k^+b'^-) + i\xi^+M^-(a^-l^- + k^-b'^-)) \end{aligned} \tag{8.49}$$

then we do the $\xi^{\pm}$ integration as

$$\int d\xi^{\pm}\, \exp\left(i\xi^{\pm}(M^{\mp}(a^{\mp}l^{\mp} + b'^{\mp}k^{\mp}) - l^{\mp} - k^{\mp})\right) \tag{8.50}$$

which gives a delta function

$$\delta^{\pm}\left\{M^{\mp}(l^{\mp}a^{\mp} + k^{\mp}b'^{\mp}) - (k^{\mp} + l^{\mp})\right\} \tag{8.51}$$

giving us,

$$l^{\mp} = -\frac{k^{\mp}(M^{\mp}b^{\mp} - 1)}{(M^{\mp}a^{\mp} - 1)} \tag{8.52}$$

then we do $l^{\pm}$ integration and write

$$B^{\mp} = -\frac{(M^{\mp}b'^{\mp} - 1)}{(M^{\mp}a^{\mp} - 1)} \tag{8.53}$$

so we get

$$\begin{aligned} A = \int \frac{d^2k}{2\pi}\frac{1}{-2B^+B^-k^+k^- + b^2}\frac{1}{-2k^+k^- + b^2} \\ \frac{1}{-2M^+M^-(k^-B^-a^- + b'^-k^-)(a^+k^+B^+ + b'^+k^+) + b^2}. \end{aligned} \tag{8.54}$$

For convenience we write

$$
\begin{aligned}
M^2 &= M^+M^- \\
B^2 &= B^+B^- \\
k^+k^- &= k^2
\end{aligned}
$$

and this choice gives us

$$
A = \int \frac{d^2k}{(2\pi)^2} \frac{1}{-2k^2B^2 + b^2} \frac{1}{-2k^2 + b^2} \frac{1}{-2M^2k^2(a^+B^+ + b'^+)(a^-B^- + b'^-) + b^2} \tag{8.55}
$$

then we choose to write

$$
\gamma = -2M^2(a^+B^+ + b'^+)(a^-B^- + b'^-) \tag{8.56}
$$

so (8.56) becomes

$$
\begin{aligned}
A &= \int \frac{d^2k}{(2\pi)^2} \frac{1}{(k^2\gamma + b^2)} \frac{1}{(4B^2k^4 - 2B^2k^2b^2 - 2k^2b^2 + b^4)} \\
&= \int \frac{d^2k}{(2\pi)^2} d\alpha_1\, d\alpha_2 \exp\left[-\alpha_1(4B^2k^4 - 2B^2k^2b^2 - 2k^2b^2 + b^4)\right] \exp\left[-\alpha_2(k^2\gamma + b^2)\right] \\
&= \int \frac{d^2k}{(2\pi)^2} d\alpha_1\, d\alpha_2 \exp\left(-\left[4B^2k^4\alpha_1 - k^2(2B^2b^2\alpha_1 + 2b^2\alpha_1 - \alpha_2\gamma) + (\alpha_1 b^4 + \alpha_2 b^2)\right]\right)
\end{aligned} \tag{8.57}
$$

then choose

$$
\begin{aligned}
B' &= 2B^2b^2\alpha_1 + 2b^2\alpha_1 - \alpha_2\gamma \\
C &= \alpha_1 b^4 + \alpha_2 b^2.
\end{aligned} \tag{8.58}
$$

Now we want to complete the square for k in (8.58). Completing the square for k gives us the term that will be in the power of the exponential appearing in (8.58).

So after we complete the square , we get

$$
4B^2\alpha_1 \left(k^2 - \frac{B'}{8B^2\alpha_1}\right)^2 + \left(C - \frac{B'}{16B^2\alpha_1}\right). \tag{8.59}
$$

Putting this back in the integral we get

$$\int \frac{d^2k}{(2\pi)^2}\, d\alpha_1\, \alpha_2 \, \exp\left[-\left\{4B^2\alpha_1\left(k^2 - \frac{B'}{8B^2\alpha_1}\right)^2 + \left(C - \frac{B'}{16B^2\alpha_1}\right)\right\}\right] \tag{8.60}$$

we wick rotate, that is we do $k^0 \;\to -ik^0$, and thus we integrate over the angular direction and shift the momentum as

$$k'^2 \;=\; k^2 - \frac{B'}{8B^2\alpha_1} \tag{8.61}$$

so we get $k'\, dk' \;=\; k\, dk$. As a result , our integral becomes

$$\frac{1}{2\pi} \int dk'\, k' d\alpha_1\, d\alpha_2 \, \exp\left[-\left\{4B^2\alpha_1 k'^4 + \left(C - \frac{B'^2}{16B^2\alpha_1}\right)\right\}\right] \tag{8.62}$$

To see things in a simplified manner , we will do the following

$$\begin{aligned}
\frac{(B')^2}{16B^2\alpha_1} &= \frac{(2B^2b^2\alpha_1 + 2b^2\alpha_1 - \alpha_2\gamma)^2}{16B^2\alpha_1} \\
&= \frac{\{\alpha_1(2B^2b^2 + 2b^2) - \alpha_2\gamma\}^2}{16B^2\alpha_1} \\
&= \frac{(\alpha_1\delta - \alpha_2\gamma)^2}{16B^2\alpha_1} \\
&= \frac{\alpha_1^2\delta^2 + \alpha_2^2\gamma^2 - 2\alpha_1\alpha_2\delta\gamma}{16B^2\alpha_1}
\end{aligned}$$

where we introduced

$$\delta \;=\; b^2(2B^2 + 2). \tag{8.63}$$

Now we explicitly calculate the term $C - \frac{B'^2}{16B^2\alpha_1}$ as follows

$$\begin{aligned}
C - \frac{B'^2}{16B^2\alpha_1} &= \frac{16B^2\alpha_1 C - B'^2}{16B^2\alpha_1} \\
&= \frac{1}{16B^2\alpha_1}\left(16B^2\alpha_1^2 b^4 + 16B^2\alpha_1\alpha_2 b^2 - \alpha_1^2\delta^2 + \alpha_2\gamma^2 - 2\alpha_1\alpha_2\gamma\delta\right).
\end{aligned}$$

Then we want to complete the square for α_2 from

$$16B^2\alpha_1^2 b^4 + 16B^2\alpha_1\alpha_2 b^2 - \alpha_1^2\delta^2 + \alpha_2\gamma^2 - 2\alpha_1\alpha_2\gamma\delta \tag{8.64}$$

and completing the square gives us

$$\gamma^2\left[\alpha_2 + \frac{\alpha_1(8B^2b^2 - \gamma\delta)}{\gamma^2}\right]^2 + \left[\alpha_1^2(16B^2b^4 - \delta^2) - \alpha_1^2\left(\frac{(8B^2b^2 - \gamma\delta)^2}{\gamma^2}\right)\right] \tag{8.65}$$

we then shift $\alpha_2 \to \alpha_2'$ in the following manner

$$\alpha_2' \;=\; \alpha_2 + \frac{\alpha_1(8B^2b^2 - \gamma\delta)}{\gamma^2} \tag{8.66}$$

so we have $d\alpha_2 \;=\; d\alpha_2'$. Now putting all this into (8.63) we get

$$\begin{aligned}
&\frac{1}{2\pi}\int dk'\, d\alpha_1 d\alpha_2'\, k' \;\exp(-4B^2\alpha_1 k'^4)\;\exp\left(-\frac{\gamma^2\alpha_2'^2}{16B^2\alpha_1}\right)\\
&\qquad \exp\left(-\frac{\alpha_1^2}{16B^2\alpha_1}\left[(16B^2b^4-\delta^2) - \frac{(8B^2b^2-\gamma\delta)^2}{2\gamma^2}\right]\right)\\
&= \frac{1}{2\pi}\int dk'\, d\alpha_1 d\alpha_2'\, k' \;\exp(-4B^2\alpha_1 k'^4)\;\exp\left(-\frac{\gamma^2\alpha_2'^2}{16B^2\alpha_1}\right)\\
&\qquad \exp\left(-\frac{\alpha_1}{16B^2}\left[(16B^2b^4-\delta^2) - \frac{(8B^2b^2-\gamma\delta)^2}{2\gamma^2}\right]\right)\\
&= \int \frac{dk'\,k'}{2\pi} d\alpha_1 d\alpha_2'\,\exp(-\alpha_1\mu)\;\exp\left(-\frac{\gamma^2\alpha_2^2}{16B^2\alpha_1}\right)
\end{aligned} \tag{8.67}$$

where we have written μ as

$$\mu \;=\; 4B^2k'^4 + \frac{1}{16B^2}\left[(16B^2b^4-\delta^2) - \frac{(8B^2b^2-\gamma\delta)^2}{2\gamma^2}\right]. \tag{8.68}$$

We can now compute the Gaussian integral of α_2' and we will have

$$\int \frac{dk'\,k'}{2\pi} d\alpha_1\,\frac{1}{2}\sqrt{\frac{16\pi^2B^2}{\gamma^2}}\;\alpha_2^{\frac{1}{2}}\;e^{-\alpha_1\mu} \tag{8.69}$$

then computing α_1 integral gives us

$$\int \frac{dk'\,k'}{2\pi}\frac{2\sqrt{\pi}B}{\gamma}\Gamma\left(\frac{3}{2}\right)\mu^{-\frac{3}{2}} \tag{8.70}$$

giving us the final form of the effective potential

$$\int dr\, V(r) \;=\; \frac{2\sqrt{\pi}B}{\gamma}\Gamma\left(\frac{3}{2}\right)\int dr\,dk'\,\frac{e^{-ik'r}}{(4B^2k'^2+\Omega)^{\frac{3}{2}}} \tag{8.71}$$

where we wrote

$$\Omega \;=\; \frac{1}{16B^2}\left[(16B^2b^4-\delta^2) - \frac{(8B^2b^2-\gamma\delta)^2}{2\gamma^2}\right]. \tag{8.72}$$

The integral over k' in (8.72) was evaluated with mathematica and the result is

$$\int dk' \, \frac{e^{-ik'r}k'}{(4B^2k'^4+\Omega)^{\frac{3}{2}}} = \frac{G^{5,1}_{1,5}\left(\begin{array}{c} \frac{1}{2} \\ 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \end{array} \middle| \frac{\Omega r^4}{1024B^2} \right)}{4\sqrt{2}\pi^2\Omega^{3/2}\sqrt{\frac{B^2}{\Omega}}}. \tag{8.73}$$

The code of this computation will be attached in the Appendix. Now, we will look in to asymptotic behaviour of the solution. Firstly we will look into the late time behaviour of the potential and then we will look into the early time behaviour .

In the late time regime, our solution, MeijerG function has the following asymptotic behaviour

$$\begin{gathered} G_3 \approx \frac{\sqrt{\pi\, r}\sqrt{\frac{B^2}{\Omega}}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{8}}}{\sqrt{\Omega}B^2} \; \exp\left(-\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right) \\ \left\{\exp\left(\frac{r\,i}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)\left(\frac{\left(\frac{1}{4}-\frac{i}{4}\right)(-1)^{\frac{1}{8}}}{2^{\frac{3}{4}}}\right) + \exp\left(-\frac{r\,i}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)\left(\frac{(-1)^{\frac{1}{8}}}{4 \cdot 2^{\frac{1}{4}}}\right)\right\}. \end{gathered} \tag{8.74}$$

Now we will compute the numerical values of the complex term in this expansion. We do this using the following code in mathematica.

```
In[ ]:=  N[ (-1)^(1/8) / (4 × 2^(1/4)) ]

Out[ ]=  0.194222 + 0.0804493 i

In[6]:=  N[ ((1/4 - i/4) (-1)^(1/8)) / 2^(3/4) ]

Out[6]=  0.194222 - 0.0804493 i
```

Figure 8.7: Numerical computation of the imaginary terms using mathematica.

Here we notice that the numerical values of the real part are same and the imaginary part picks up a minus sign. So the terms in the second bracket takes the following form

$$e^{i\theta}(a+ib) + e^{-i\theta}(a-ib) \tag{8.75}$$

after a little algebra this gives us

$$2\left\{a\cos\theta - b\sin\theta\right\} \tag{8.76}$$

here we compare a and b with the real and imaginary part of the quantities evaluated above. In our case we have $a = 0.19$ and $b = 0.08$.

The θ is easily recognized as $\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}$. So our effective potential for this interaction (without the summation over spins , which we will add later) will look like

$$\begin{aligned}\int V_3^{late} \approx \int dr\, \frac{2\sqrt{\pi\, r}\,\sqrt{\frac{B^2}{\Omega}}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{8}}}{\sqrt{\Omega}B^2}\ \exp\left(-\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)\\ \left\{0.19\cos\left(\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right) - 0.08\sin\left(\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)\right\}.\end{aligned} \tag{8.77}$$

We then put back the values of Ω and B as we defined earlier. Simplifying them is a long task and requires careful handling of algebraic quantities. Since this is very long but not complicated , we will not write the full details of it here , rather we will present the result here[1].

The value for B^2 becomes

$$B^2 = \frac{\sinh^2\left(\pi\ell_s Q(q+m)\right)}{\sinh^2\left(\pi\ell_s Q(q+n)\right)} \tag{8.78}$$

Also we can simplify Ω as follows

$$\Omega = b^4\left\{\frac{7}{8} + \frac{1}{16B^2} + \left[B^2\left(\frac{4}{\gamma} - 1\right) - 1\right]^2\right\} \tag{8.79}$$

and putting the values of B^2 and the values of γ gives us

$$\begin{aligned}\Omega = b^4\left(\frac{7}{8} + \frac{\sinh^2\left(\pi\ell_s Q(q+n)\right)}{16\sinh^2\left(\pi\ell_s Q(q+m)\right)}\right)\\ + b^4\left(\frac{\sinh^2\left(\pi\ell_s Q(q+m)\right)}{\sinh^2\left(\pi\ell_s Q(q+n)\right)}\left(\frac{2\,\sinh^2\left(\pi\ell_s Q(q+n)\right)}{\cosh^2\left(\pi\ell_s Q(n-m)\right) - \cosh^2\left(\pi\ell_s Q(q+n)\right)} - 1\right) - 1\right)^2\end{aligned} \tag{8.80}$$

[1]The way to compute Ω and B is to using relations from (8.49) and then putting them in. Simple, yet lengthy procedure of substitution.

So as one can see this expressions are very complicated. So for a graphical representation lets consider the following function

$$f(r) \;=\; \sqrt{r}\, e^{-\frac{r}{2}} \left(0.19 \cos\left(\frac{r}{2}\right) - 0.08 \sin\left(\frac{r}{2}\right)\right) \tag{8.81}$$

and plotting with mathematica we can see the following behaviour

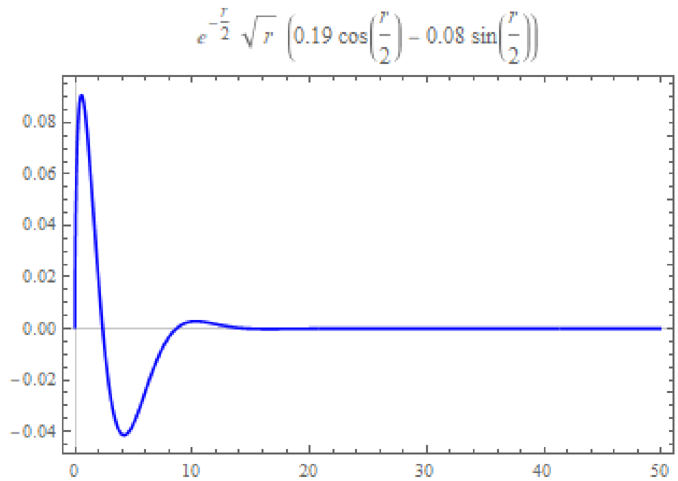

Figure 8.8: Behaviour of effective potential for the Boson-Fermion interaction in late time regime.

We can easily see that the potential dies really fast as there is exponential dying factor.

So the late time effective potential takes its fullest from as the following

$$\begin{aligned}\int V_3^{late} \approx \sum_{n,m,q} \int dr\, 2\pi \frac{2\sqrt{\pi r} B}{\gamma} \Gamma\left(\frac{3}{2}\right) \sinh^4\left(\frac{\pi Q \ell_s n}{2}\right) \sinh^4\left(\frac{\pi Q \ell_s m}{2}\right) \sinh^4\left(\frac{\pi Q \ell_s q}{2}\right) \\ \sqrt{\pi} \frac{1}{\Omega B} \left(\frac{\Omega}{B^2}\right)^{\frac{1}{8}} \exp\left(-\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right) \left\{0.19 \cos\left(\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right) - 0.08 \sin\left(\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)\right\}.\end{aligned} \tag{8.82}$$

Here Ω and B^2 are as mentioned in equation (8.81) and (8.79) respectively. Moreover the term γ takes the following form [2]

$$\gamma \;=\; -2 \sum_{n,q,m} \left\{\frac{\cosh^2\left(\pi \ell_s Q(q+n)\right) - \cosh^2\left(\pi \ell_s Q(n-m)\right)}{\sinh^2\left(\pi \ell_s Q(q+n)\right)}\right\} \tag{8.83}$$

[2]Computation of γ can be done in the same way as we computed B^2 and Ω.

and also

$$B = \sum_{n,q,m} \frac{\sinh\left(\pi \ell_s Q(q+m)\right)}{\sinh\left(\pi \ell_s Q(q+n)\right)}. \tag{8.84}$$

Now comes the time for finding the values of n, m, q so that we get the dominant term in our potential. Dominant terms will come from $n = \pm 9$ $m = \pm 10$, $q = \pm 10$.[3] Now we will look at the early time behaviour of the potential . The solution we have has the asymptotic expansion as follows in early times

$$G_3 \approx -\frac{r\left(\pi^{3/2}\sqrt{\frac{B^2}{\Omega}}\sqrt[4]{\frac{\Omega}{B^2}}\right)}{8\left(\sqrt{2}B^2\sqrt{\Omega}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)} \tag{8.85}$$

Again we have ignored the constant piece that comes from it. Also notice that there is a minus sign that signals for an attractive potential near the big bang . So in the fullest form , the potential in early times is

$$\int V_3^{early} \approx -\sum_{n,m,q} \int dr \,, 2\pi \frac{2\sqrt{\pi}B}{\gamma} \Gamma\left(\frac{3}{2}\right) \sinh^4\left(\frac{\pi Q \ell_s n}{2}\right) \sinh^4\left(\frac{\pi Q \ell_s m}{2}\right) \sinh^4\left(\frac{\pi Q \ell_s q}{2}\right) \\ \frac{r\left(\pi^{3/2}\sqrt{\frac{B^2}{\Omega}}\sqrt[4]{\frac{\Omega}{B^2}}\right)}{8\left(\sqrt{2}B^2\sqrt{\Omega}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)} \tag{8.86}$$

where γ and Ω are same as in (8.85). Again the dominant terms will come from $n = \pm 9$ $m = \pm 10$, $q = \pm 10$.

So far we have only just computed the two loop effective potentials coming from the boson -fermion interaction. We found that that in late time this potential is vanishing very rapidly. And , we are getting a regular commutative spacetime. However , we haven't really explained why this should be so. We will make some comments about this in the conclusion. Also we have not taken into account the gauge interaction that rises from our action of the matrix big bang. In the next section , we will write down the gauge interaction terms explicitly and specify their interactions.

8.4 Gauge Interactions

So far in our work , we have only considered Boson-Fermion interactions and a four point interaction in the effective action. But we can notice that the covariant derivatives in the

[3]There is an issue here. For my work , I used Python program. But there might exist some other way that will give more precise result

action is actually

$$D_\mu X^i \;=\; \partial_\mu X^i + i[A_\mu, X^i]. \tag{8.87}$$

So we have

$$\begin{aligned}(D_\mu X^i)^2 \;&=\; \Big(\partial_\mu X^i + i[A_\mu, X^i]\Big)\,\Big(\partial^\mu X^i + i[A^\mu, X^i]\Big)\\ &=\; \partial_\mu X^i \partial^\mu X^i + i\partial_\mu X^i [A^\mu, X^i] + i[A_\mu, X^i]\partial^\mu X^i - [A^\mu, X^i][A_\mu, X^i]\end{aligned} \tag{8.88}$$

From here one can see clearly that there is interaction between scalar and gauge fields. To compute this propagations one needs the gauge field propagator in this Milne Orbifold.

The term $F_{\mu\nu}F^{\mu\nu}$ should give us the kinetic operator (by choosing a suitable gauge) for the gauge fields. We can do an expansion for this term using

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu].$$

So,

$$\begin{aligned}F_{\mu\nu}F^{\mu\nu} \;=\;& \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu + i\partial_\mu A_\nu [A^\mu, A^\nu] - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu - i\partial_\nu A_\mu [A^\mu, A^\nu]\\ &+ i[A_\mu, A_\nu]\partial^\mu A^\nu - i[A_\mu, A_\nu]\partial^\nu A^\mu - [A_\mu, A_\nu][A^\mu, A^\nu].\end{aligned} \tag{8.89}$$

Now , at this point one needs to find a suitable gauge. We will work in the Background Field Method . A review can be found in [1]. First step is to write the gauge fields as $A_\mu = B_\mu + Y_\mu$ where B_μ represents the background field and Y_μ denotes the quantum corrections. But a thing to notice is that our background is time dependent , so we will choose the gauge as [17]

$$G \;\equiv\; \partial_\mu Y^\mu - ie^{2Q\tau}[B_\mu, Y^\mu] \tag{8.90}$$

where B_μ is our background field and Y_μ are the quantum correction to the Gauge FiledA_μ.In the standard gauge fixing procedure one inserts the following into the action

$$1 = \Delta_{fp} \int [d\xi]\delta(G - f(t)g(t)) \tag{8.91}$$

where $f(t) = e^{Q\tau}$ and $g(t)$ is any function. Following [17] we get the gauge fixing term as

$$S_{gf} \;=\; e^{-Q\tau} G^2. \tag{8.92}$$

Then we take the background fields to be zero. This means going to the co-moving frame. And thus our gauge choice finally becomes

$$\partial_\mu Y^\mu = 0 \tag{8.93}$$

now this looks exactly like the Lorentz gauge. Now also notice that since we moved to co-moving frame as we tune the background fields to zero , our terms in the action for the Gauge fields is same as (8.90). So the action for the gauge fields will be

$$\begin{aligned} S_g = \int \mathrm{Tr}\,(&\partial_\mu Y_\nu \partial^\mu Y^\nu - \partial_\mu Y_\nu \partial^\nu Y^\mu + i\partial_\mu Y_\nu [Y^\mu, Y^\nu] - \partial_\nu Y_\mu \partial^\mu Y^\nu + \partial_\nu Y_\mu \partial^\nu Y^\mu - i\partial_\nu Y_\mu [Y^\mu, Y^\nu] \\ &+ i[Y_\mu, Y_\nu]\partial^\mu Y^\nu - i[Y_\mu, Y_\nu]\partial^\nu Y^\mu - [Y_\mu, Y_\nu][Y^\mu, Y^\nu]). \end{aligned} \tag{8.94}$$

Doing integration by parts and using the gauge condition one can get the equation of motion of the gauge fields as

$$\partial_\mu \partial^\mu Y^\nu = 0 \tag{8.95}$$

and since σ is periodically identified , and we are considering a time regime where the **Non-Abelian** modes are much heavier than **KK-modes**, derivatives on the σ direction will be ignored. Like [9] we an go to $\xi^\pm$ co ordinates and we have

$$\frac{\partial^2 Y^\nu}{\partial \xi^+ \partial \xi^-} = 0 \tag{8.96}$$

So , the propagator will be given by the Minkowski space propagator with a summation over images. It is

$$G_1^{\mu\nu}(\xi, \xi') = \sum_n \int \eta^{\mu\nu} \frac{dp^+ dp^-}{(2\pi)^2} \frac{\exp\left(-ip^-\left(\xi^+ - e^{2\pi Q\ell_s n}\xi'^+\right) - ip^+\left(\xi^- - e^{-2\pi Q\ell_s n}\xi'^-\right) + 2\pi Q\ell_s n\right)}{p^+ p^-} \tag{8.97}$$

where the metric $\eta_{\mu\nu}$ is coming from $ds^2 = -2d\xi^+ d\xi^-$. Now before we move any further, lets notice something about propagation of massless fields. It is known for any dimension d

$$\int \frac{d^d p}{p^2} = 0 \tag{8.98}$$

and this has an interpretation [4] of the cancellation between UV and IR divergences. A simple way to to see this is to consider the following integral

$$\lim_{m\to 0} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + m^2}. \tag{8.99}$$

Then we evaluate the integral and take limit $m \to 0$ in the end. So we get,

$$
\begin{aligned}
\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2+m^2} &= -\frac{1}{\Gamma\left(\frac{d}{2}\right)} \frac{1}{(4\pi)^{\frac{d}{2}}} \int_0^\infty dk\, k^{d-1} \frac{1}{k^2+m^2} \\
&= -\frac{1}{\Gamma\left(\frac{d}{2}\right)} \frac{1}{(4\pi)^{\frac{d}{2}}} \int_0^\infty d\alpha \int_0^\infty dk\, k^{d-1} e^{-\alpha\left(k^2+m^2\right)} \\
&= \frac{1}{2\Gamma\left(\frac{d}{2}\right)} \frac{1}{(4\pi)^{\frac{4}{2}}} \int_0^\infty d\alpha e^{-\alpha m^2} \int_0^\infty du\, u^{\frac{d}{2}-1} e^{-\alpha u} \\
&= \frac{1}{2} \frac{1}{(4\pi)^{\frac{4}{2}}} \int_0^\infty d\alpha e^{-\alpha m^2} \frac{1}{\alpha^{\frac{d}{2}}} \\
&= \frac{1}{2} \frac{1}{(4\pi)^{\frac{d}{2}}} \Gamma\left(1-\frac{d}{2}\right) \left(m^2\right)^{\frac{d}{2}-1}.
\end{aligned} \tag{8.100}
$$

Now taking the limit $m \to 0$ we get our result from (8.99).

From this result it is evident that the massless fields do not contribute to the two loop diagrams. Also for the interaction term

$$
\mathrm{Tr}\left([A^\mu, X^i][A_\mu, X^i]\right) = -\frac{1}{2} f^{abc} f^{def} A^{\mu,a} X^{i,b} A^d_\mu X^{i,e}. \tag{8.101}
$$

This interaction gives rise to similar figure of eight diagram at two loop and since these two loop momenta are independent , because of the massless loop momentum , this whole thing goes to zero. Also note that we have used A_μ instead of Y_μ. The reason is that we went to the comoving frame and turned of the background fields.

We then take the following interaction term into account,

$$
\mathrm{Tr}\left(i\partial_\mu X^i [A^\mu, X^i]\right). \tag{8.102}
$$

After taking the trace this becomes

$$
\mathrm{Tr}\left(i\partial_\mu X^i [A^\mu, X^i]\right) = -\frac{i}{8} f^{bca} \partial_\mu X^i_a A^\mu_b X^i_c \tag{8.103}
$$

and a similar term arises for

$$
\mathrm{Tr}\left([A_\mu, X^i] \partial^\mu X^i\right). \tag{8.104}
$$

Now we will compute this propagations using our proposed gauge propagator. Notice that we will only take into the non zero contribution coming from the gauge propagator, that is we will only take into account the components of the metric where $\eta_{\mu\nu} \neq 0$. The relevant diagram is a fish diagram with two bosonic propagator and one massless gauge propagator. The loop integral then becomes

$$
\begin{aligned}
&\sum_{q,n,m} \int d\xi^+ \, d\xi^- d\xi'^+ d\xi'^- \frac{d^2k \, d^2l \, d^2p}{(2\pi)^6} \frac{1}{-2k^+k^- + b^2} \frac{1}{p^+p^-} \frac{1}{-2l^+l^- + b^2} \\
&\exp\left(-il^-\left(\xi^+ - e^{2\pi\ell_s Qn}\xi'^+\right) - il^+\left(\xi^- - e^{2\pi\ell_s Qn}\xi'^-\right)\right) \\
&\exp\left(-ik^-\left(\xi^+ - e^{2\pi\ell_s Qn}\xi'^+\right) - ik^+\left(\xi^- - e^{2\pi\ell_s Qn}\xi'^-\right)\right) \\
&\exp\left(-ip^-\left(\xi^+ - e^{2\pi\ell_s Qn}\xi'^+\right) - ip^+\left(\xi^- - e^{2\pi\ell_s Qn}\xi'^-\right)\right).
\end{aligned}
\tag{8.105}
$$

Again we have not written down the sum over the spin terms explicitly here since they are not relevant for computing the integrals. We will add them at the end of the calculation. We then proceed by starting integrating over the $\xi'^{\pm}$ which will give us a delta function just as we had done for the Yukawa interaction. Then using that delta function we can integrate over $p^{\pm}$. The delta function is

$$
\delta^{\pm}\left(p^{\mp}e^{\pm 2\pi\ell_s Qq} + l^{\mp}e^{\pm 2\pi\ell_s Qn} + k^{\mp}e^{\pm 2\pi\ell_s Qm}\right) \tag{8.106}
$$

and this gives us

$$
p^{\pm} \;=\; M^{\mp}\left(a^{\pm}l^{\pm} + b'^{\pm}k^{\pm}\right) \tag{8.107}
$$

where we wrote as before

$$
\begin{aligned}
M^{\mp} &= e^{\mp 2\pi\ell_s Qq} \\
a^{\pm} &= e^{\pm 2\pi\ell_s Qn} \\
b'^{\pm} &= e^{\pm 2\pi\ell_s Qm}
\end{aligned}
\tag{8.108}
$$

and integrating over $p^{\pm}$ gives us

$$
\begin{aligned}
&\int \frac{d^2l d^2k}{(2\pi)^4} d^2\xi \frac{1}{-2l^+l^- + b^2} \frac{1}{-2k^+k^- + b^2} \frac{1}{(a^-l^- + b'^-)(a^+l^+ + b'^+)} \\
&\exp\left(-il^-\xi^+ - il^+\xi^-\right)\exp\left(-ik^-\xi^+ - ik^+\xi^-\right) \\
&\exp\left(-i\xi^-M^+\left(l^+a^- + k^+b'^-\right) + i\xi^+M^-\left(a^-l^- + k^-b'^-\right)\right).
\end{aligned}
\tag{8.109}
$$

Then doing $\xi^{\pm}$ integration, gives us another delta function. The integral is,

$$
\int d\xi^{\pm} \exp\left(i\xi^{\pm}\left(M^{\mp}\left(a^{\mp}l^{\mp} + b'^{\mp}k^{\mp}\right) - l^{\mp} - k^{\mp}\right)\right) \tag{8.110}
$$

So, we have

$$
\delta^{\pm}\left\{M^{\mp}\left(l^{\mp}a^{\mp} + k^{\mp}b'^{\mp}\right) - \left(k^{\mp} + l^{\mp}\right)\right\} \tag{8.111}
$$

and

$$
l^{\mp} = -\frac{k^{\mp}\left(M^{\mp}b^{\mp} - 1\right)}{\left(M^{\mp}a^{\mp} - 1\right)}. \tag{8.112}
$$

After this we do $l^{\pm}$ integration. After doing the integration we get

$$\int \frac{d^2k}{2\pi} \frac{1}{-2B^+B^-k^+k^- + b^2} \frac{1}{-2k^+k^- + b^2} \frac{1}{(k^-B^-a^- + b^-k^-)\,(a^+k^+B^+ + b^+k^+)}. \tag{8.113}$$

Now we write

$$\gamma \;=\; (a^+B^+ + b'^+)\,(a^-B^- + b'^-) \tag{8.114}$$

and writing $k^+k^- = k^2$ and $B^+B^- = B^2$ the integral becomes

$$\begin{aligned}
&\int \frac{d^2k}{(2\pi)^2} \frac{1}{(-2B^2k^2 + b^2)\,(-2k^2 + b^2)} \frac{1}{k^2\gamma} \\
&= \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2\gamma} \frac{1}{(4B^2k^4 - 2B^2k^2b^2 - 2k^2b^2 + b^4)} \\
&= \int \frac{d^2k}{(2\pi)^2} d\alpha_1\, d\alpha_2 \exp\left[-\alpha_1\left(4B^2k^4 - 2B^2k^2b^2 - 2k^2b^2 + b^4\right)\right] \exp\left[-\alpha_2 k^2\gamma\right] \\
&= \int \frac{d^2k}{(2\pi)^2} d\alpha_1\, d\alpha_2\, \exp\left[-\left\{4B^2k^4\alpha_1 - k^2(2B^2b^2\alpha_1 + 2b^2\alpha_1 - \gamma\alpha_2) + \alpha_1 b^4\right\}\right].
\end{aligned} \tag{8.115}$$

We complete square for k and then we shift k as k' where we took

$$k'^2 \;=\; k^2 - \frac{B'^2}{8B^2\alpha_1} \tag{8.116}$$

and completing the square gives us

$$4B^2\alpha_1 \left(k^2 - \frac{B'^2}{8B^2\alpha_1}\right)^2 + \left(C - \frac{B'^2}{16B^2\alpha_1}\right) \tag{8.117}$$

we notice that

$$k'\,dk' \;=\; kdk. \tag{8.118}$$

Here we took

$$\begin{aligned}
B'^2 &\;=\; 2B^2b^2\alpha_1 + 2b^2\alpha_1 - \gamma\alpha_2 \\
C &\;=\; \alpha_1 b^4.
\end{aligned} \tag{8.119}$$

Putting all this in the integral we get

$$\frac{1}{2\pi} \int dk'\, k' d\alpha_1 d\alpha_2\, \exp\left[-\left(4B^2\alpha_1 k^4 + \left(C - \frac{B'^2}{16B^2\alpha_1}\right)\right)\right] \tag{8.120}$$

here we have wick rotated and carried out the integration in the angular direction. Now we do the following

$$C - \frac{B'^2}{16B^2\alpha_1} \;=\; \frac{1}{16B^2\alpha_1}\,\left(16\alpha_1^2 b^4 B^2 - \alpha_1\delta^2 - \alpha_2^2\gamma^2 + 2\alpha_1\alpha_2\gamma\delta\right) \tag{8.121}$$

here we took

$$\delta \;=\; b^2(2B^2+2) \tag{8.122}$$

then we complete the square for α_2 and shift as we have done for momentum k before. Here we will have

$$\begin{aligned}\alpha_2' \;&=\; \alpha_2 - \frac{\alpha_1\delta}{\gamma}\\ d\alpha_2' \;&=\; d\alpha_2,\end{aligned}$$

also, we get from completing the square the following

$$\alpha_2' \;=\; \gamma^2\left(\alpha_2 - \frac{\alpha_1\delta}{\gamma}\right)^2 - \alpha_1^2\delta^2\gamma. \tag{8.123}$$

Putting all this into the integral we have and simplifying things gives us

$$\frac{1}{2\pi}\int dk'\,k'\,d\alpha_1\,d\alpha_2'\,\exp\left(-4B^2\alpha_1 k'^4\right)\,\exp\left\{-\alpha_1 A + \alpha_1 c\right\}\,\exp\left(-\frac{\alpha_2^2}{16B^2\alpha_1}\right) \tag{8.124}$$

where we took

$$\begin{aligned}A \;&=\; b^4 - \frac{\delta^2}{16B^2}\\ c \;&=\; \frac{\delta^2\gamma^2}{16B^2}.\end{aligned} \tag{8.125}$$

Then doing the α_2 integration we will get the following

$$\frac{1}{2\pi}\sqrt{16B^2\pi}\int dk'\,k'd\alpha_1\,\alpha_1^{\frac{1}{2}}\,\exp\left(-\alpha_1\mu\right) \tag{8.126}$$

where we wrote

$$\mu \;=\; 4B^2k'^4 + (A-c), \tag{8.127}$$

finally doing this α_1 integration results in

$$\int dk'\,k'\frac{B}{\sqrt{\pi}}\,\Gamma\left(\frac{3}{2}\right)\mu^{-\frac{3}{2}} \tag{8.128}$$

and in the fullest form this integral looks like

$$\frac{B}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right)\int dk'\,k'\frac{1}{(4B^2k'^4+\Omega)^{\frac{3}{2}}} \tag{8.129}$$

where we wrote like before $\Omega = A-C$. To get the potential we do a fourier transformation and we need to compute

$$V \approx \int dk'\,\frac{k'\,e^{-ik'r}}{(4B^2k'^4+\Omega)^{\frac{3}{2}}}. \tag{8.130}$$

We have already encountered with integral like this in the Yukawa interaction. The result is

$$\int dk'\,\frac{e^{-ik'r}k'}{(4B^2k'^4+\Omega)^{\frac{3}{2}}} = \frac{G^{5,1}_{1,5}\left(\begin{array}{c}\frac{1}{2}\\ 0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1\end{array}\middle|\frac{\Omega r^4}{1024B^2}\right)}{4\sqrt{2}\pi^2\Omega^{3/2}\sqrt{\frac{B^2}{\Omega}}}. \tag{8.131}$$

We notice that this is similar to the effective potential we found fro the Yukawa interaction. So, in the late time regime we will have

$$\begin{aligned} G_3 \approx & \frac{\sqrt{\pi\,r}\,\sqrt{\frac{B^2}{\Omega}}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{8}}}{\sqrt{\Omega}B^2}\;\exp\left(-\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)\\ & \left\{\exp\left(\frac{r\,i}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)\left(\frac{\left(\frac{1}{4}-\frac{i}{4}\right)(-1)^{\frac{1}{8}}}{2^{\frac{3}{4}}}\right)+\exp\left(-\frac{r\,i}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)\left(\frac{(-1)^{\frac{1}{8}}}{4\,2^{\frac{1}{4}}}\right)\right\}. \end{aligned} \tag{8.132}$$

and as we worked out before this expression simplifies to give the potential as

$$\begin{aligned} \int V_3^{late} \approx & \int dr\,\frac{B}{\sqrt{\pi}}\frac{2\sqrt{\pi\,r}\,\sqrt{\frac{B^2}{\Omega}}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{8}}}{\sqrt{\Omega}B^2}\;\exp\left(-\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)\\ & \left\{0.19\cos\left(\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)-0.08\sin\left(\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)\right\}. \end{aligned} \tag{8.133}$$

and one can get

$$\begin{aligned} \int V_{s-g}^{late} \approx & \int dr\,\frac{2\sqrt{r}\,\left(\frac{\Omega}{B^2}\right)^{\frac{1}{8}}}{\Omega}\;\exp\left(-\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)\\ & \left\{0.19\cos\left(\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)-0.08\sin\left(\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)\right\}. \end{aligned} \tag{8.134}$$

and we use subscript $s-g$ to denote the interaction between scalar and gauge fields. Now we write down the explicit expression for Ω and B^2.

In this case we denoted $\Omega = A - c$ and we wrote

$$\begin{aligned}
A &= b^4 - \frac{\delta^2}{16B^2} \\
c &= \frac{\delta^2\gamma^2}{16B^2} \\
\delta &= b^2(2B^2+2) \\
\gamma &= (a^+B^+ + b'^+)\,(a^-B^- + b'^-).
\end{aligned}$$

Further simplification gives us,

$$A = b^4\left[1 - \frac{(B^2+1)^2}{4B^2}\right] \tag{8.135}$$

$$c = b^4\left[\frac{\gamma^2(B^2+1)^2}{4B^2}\right], \tag{8.136}$$

using these expressions we get

$$\Omega = b^4\left[1 - \left\{\frac{(B^2+1)^2}{4B^2}\left(1+\gamma^2\right)\right\}\right] \tag{8.137}$$

and the explicit expression B^2 turns out to be the same as we have found in the Yukawa interaction. It is the following

$$B^2 = \sum_{q,m,n} \frac{\sinh^2\left(\pi\ell_s Q(q+m)\right)}{\sinh^2\left(\pi\ell_s Q(q+n)\right)} \tag{8.138}$$

and for γ^2 we will have

$$\gamma = \sum_{q,m,n}\left\{\frac{\cosh^2\left(\pi\ell_s Q(q+n)\right) - \cosh^2\left(\pi\ell_s Q(n-m)\right)}{\sinh^2\left(\pi\ell_s Q(q+n)\right)}\right\}. \tag{8.139}$$

So the full form of the potential will be

$$\begin{aligned}
\int V_{s-g}^{late} \approx \int \sum_{n,q,m} dr\, \sinh^4\left(\frac{\pi Q\ell_s n}{2}\right)\sinh^4\left(\frac{\pi Q\ell_s m}{2}\right)\, e^{2\pi\ell_s Qq}\frac{2\sqrt{r}\,\left(\frac{\Omega}{B^2}\right)^{\frac{1}{8}}}{\Omega} \\
\exp\left(-\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)\left\{0.19\cos\left(\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right) - 0.08\sin\left(\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right)\right\}.
\end{aligned} \tag{8.140}$$

Again we have found an effective potential that turns off really fast at late time. This is essential for the dynamical emergence of spacetime.

We then look into the early time behaviour of this potential. In early times the potential looks like the following, which comes from the asymptotic expansion of the MeijerG function. We get,

$$\begin{aligned}\int V_{s-g}^{early} \approx & -\int \sum_{n,q,m} dr \, \sinh^4\left(\frac{\pi Q \ell_s n}{2}\right) \sinh^4\left(\frac{\pi Q \ell_s m}{2}\right) \, e^{2\pi \ell_s Q q} \\ & \frac{r\left(\pi^{3/2}\sqrt{\frac{B^2}{\Omega}}\sqrt[4]{\frac{\Omega}{B^2}}\right)}{8\left(\sqrt{2}B^2\sqrt{\Omega}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)\right)}.\end{aligned} \tag{8.141}$$

Here we again ignored the constant piece that comes along with asymptotic expansion.The dominant terms will come from$n = \pm 9, m = \pm 10$ and $q = 9$. And with this we have calculated all the two loop diagrams that arises in the Matrix Big Bang model. The ghost fields wont contribute as they are massless here and don't interact with the scalar fields or the gauge fields.

Chapter 9

Conclusions and Further Directions

In this thesis we have computed the two loop effective potentials to the Matrix Big Bang model proposed in [10]. This two loop calculation was proposed as an open problem in [9, 8]. We found that in early time we get attractive potential, which dies very rapidly at late times. This is essential for a spacetime description. We have also found out the interaction terms in the effective action[9]. From the nature of this potential one can conclude to a decision that, despite strong coupling, the potential vanishes because of multi-D-brane contribution as was mentioned in [9].

From our earlier discussions, we saw that general relativity breaks down near cosmological singularities. We also briefly discussed why resolving these singularities is important to find a consistent theory of quantum gravity. Working in a time dependent background in string theory and solving lightlike and spacelike singularities has been notoriously difficult. Lightlike singularities have been studied in the context of AdS/CFT in [7, 11, 18]. This Matrix Big Bang model is an attempt to give a non-perturbative description of lightlike singularity. It has also been generalized in various directions as was mentioned in [8].

We followed the footsteps of [9] for our calculation. From our computation we have found the effective potential to be vanishing in late time which is essential for emergence of spacetime. The authors of [9] calculated the time dependent potential at the one loop level. A similar one loop calculation was done in [17]. There they found some instability in the potential. But according to [8], the potential that was calculated in [17] was the time average potential.

In this work we propose that the r term in the effective potential be compared with the string coupling constant in the following way

$$r \approx \frac{1}{g_s}$$

where the string coupling in the light like linear dilation background is given as $g_s = e^{-Q\tau}$.

Now lets look at what happens if we consider this proposal. First we take our late time potential that arises from the Yukawa Interaction. Without writing all the constant terms explicitly (we will only include terms with b and r), we get

$$V_{late}^{2loop} \approx \frac{\Omega^{\frac{1}{8}}}{\Omega} \sqrt{r}\, \exp\left(-\frac{r}{2}\left(\frac{\Omega}{B^2}\right)^{\frac{1}{4}}\right).$$

One can recall that Ω had the form of

$$\Omega = Cb^4 \tag{9.1}$$

where C is a constant. So if we take $r \approx \frac{1}{g_s}$ and we put the value of Ω in the potential equation we get

$$\begin{aligned} V_{late}^{2loop} &\approx \frac{1}{b^2}\sqrt{\frac{b}{g_s}}\exp\left(-\frac{b\,C'}{g_s}\right) \\ &\approx \sqrt{\frac{b}{g_s}}\exp\left(-\frac{b\,C'}{g_s}\right) \end{aligned}$$

which indicates that this is indeed coming from multi-D-brane contributions. Notice that we have absorbed B inside C' for the sake of simplicity.

The reason we compared r with $\frac{1}{g_s}$ actually was inspired from [9]. The authors in [9] introduced

$$\xi^{\pm} = \frac{1}{\sqrt{2}Q}\, e^{Q(\tau\pm\sigma)}.$$

If we recall equation 7.47 , we can see that the integration measure is actually a one dimensional measure in the disguise of a two dimensional measure. It is easy to see that this is indeed a time dependent potential. In our two loop computation, we also find the same thing.

Though it was mentioned in [10, 9] that a late time vanishing potential is essential for the dynamical emergence of spacetime, there was no explanation as to why that should be the case. To get some insight, we would like to focus on the time dependent dilaton here. The dilaton couples to gravity and in early time, near the big bang, the coupling is strong. Due to high curvature and strong coupling noncomutative structure may arise. At this point, we would like to mention a feature of the result we got at two loop level. From our graph 8.3 of the potential we can see that there is a minimum. It would be interesting to see this minimum from a physical point of view and having a precise interpretation of it.

We want to conclude by stating some more open questions regarding this model. One of them is to go beyond the static potential along the lines of [13]. The v^2 expansion might give some insight into what replaces the cosmological singularity. Also, one can go on and ask what are the derivative terms in the effective action are [8]. A very fascinating question to ask is, whether or not time can be extended beyond the big bang singularity. Some relevant work on a toy model was done in [19] for FLRW spacetimes. Also works regarding Milne singularity can be found in [15].

Finally, we would like emphasize that this model is a great approach towards answering these questions. But at the same time this arena of research is still at very early stage. To get more insight, one needs to solve these problems. If this model can filter through all these questions, it might be the proper key to look into the nature of singularities.

Appendix A

Python & Mathematica Codes for finding Dominating Terms and Evaluating Integrals

```
import math
import numpy as np
M=[]
N=[]
G=[]
f=(2*(np.sinh(n)**2))/(np.sinh(m)**2)
for m in range(-100,100):
    for n in range(-100,100):
        g=(2*(np.sinh(n))**2)/(np.sinh(m)**2)
        if(g<1):
            G.append(g)

        else:
            continue
print(max(G))
value=max(G)
for m in range(-100,100):
    for n in range(-100,100):
        f=(2*(np.sinh(n))**2)/(np.sinh(m)**2)
        if(f==value):
            print(n,m)
        else:
            continue


import numpy as np
```

```
M=[]
N=[]
Q=[]
G=[]
er=float('+inf')
for m in range(-10,10):
    for n in range(-10,10):
        for q in range(-10,10):
            g=((np.sinh(q+n)**2))/(16*(np.sinh(q+m))**2)+((7/8)
            +(((np.sinh(q+m))**2)/(np.sinh(q+n))**2)
            *(((2*(np.sinh(q+n))**2)
            /(-(np.cosh(q+n))**2+(np.cosh(n-m))**2))-1)-1)**2

            if(g!=er):
                G.append(g)
            else:
                continue
print(max(G))
value=max(G)
for m in range(-10,10):
    for n in range(-10,10):
        for q in range(-10,10):
            g=((np.sinh(q+n)**2))/(16*(np.sinh(q+m))**2)+((7/8)
            +(((np.sinh(q+m))**2)/(np.sinh(q+n))**2)*
            (((2*(np.sinh(q+n))**2)
            /(-(np.cosh(q+n))**2+(np.cosh(n-m))**2))-1)-1)**2

            if(g==value):
                print(n,m,q)
            else:
                continue
```

In[ ]:= $$\mathtt{Integrate}\left[\frac{\left(k * e^{-i k r}\right)}{\left(4\,B\char`^2\,k\char`^4+\Omega\right)^{\frac{3}{2}}},\ \{k,\ 0,\ \infty\}\right]$$

Out[ ]= $$\mathtt{ConditionalExpression}\left[\frac{\mathtt{MeijerG}\left[\left\{\left\{\frac{1}{2}\right\},\ \{\}\right\},\ \left\{\left\{0,\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4},\ 1\right\},\ \{\}\right\},\ \frac{r^4\,\Omega}{1024\,B^2}\right]}{4\sqrt{2}\,\pi^2\sqrt{\frac{B^2}{\Omega}}\,\Omega^{3/2}},\right.$$
$$\left.-\frac{(1-i)\,\Omega^{1/4}}{\sqrt{B}}\notin\mathbb{R}\ \&\&\ \frac{(1+i)\,\Omega^{1/4}}{\sqrt{B}}\notin\mathbb{R}\ \&\&\ \mathrm{Re}[\Omega]\geq 0\ \&\&\ \mathrm{Im}[r]\leq 0\right]$$

In[ ]:= $$\mathtt{Series}\left[\frac{\mathtt{MeijerG}\left[\left\{\left\{\frac{1}{2}\right\},\ \{\}\right\},\ \left\{\left\{0,\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4},\ 1\right\},\ \{\}\right\},\ \frac{r^4\,\Omega}{1024\,B^2}\right]}{4\sqrt{2}\,\pi^2\sqrt{\frac{B^2}{\Omega}}\,\Omega^{3/2}},\ \{r,\ 0,\ 1\}\right]$$

Out[ ]= $$(-1)^{\mathtt{Floor}\left[\frac{\pi-4\,\mathrm{Arg}[r]-\mathrm{Arg}\left[\frac{\Omega}{B^2}\right]}{2\pi}\right]}\,O[r]^2+e^{\frac{3}{2}i\pi\,\mathtt{Floor}\left[\frac{\pi-4\,\mathrm{Arg}[r]-\mathrm{Arg}\left[\frac{\Omega}{B^2}\right]}{2\pi}\right]}\,O[r]^3+$$
$$\mathtt{Floor}\left[\frac{\pi-4\,\mathrm{Arg}[r]-\mathrm{Arg}\left[\frac{\Omega}{B^2}\right]}{2\pi}\right]O[r]^4+\left(\frac{\sqrt{\frac{B^2}{\Omega}}\,\mathrm{Gamma}\left[\frac{1}{4}\right]\mathrm{Gamma}\left[\frac{3}{4}\right]}{4\sqrt{2}\,B^2\,\pi\sqrt{\Omega}}+O[r]^2\right)+$$
$$e^{\frac{1}{2}i\pi\,\mathtt{Floor}\left[\frac{\pi-4\,\mathrm{Arg}[r]-\mathrm{Arg}\left[\frac{\Omega}{B^2}\right]}{2\pi}\right]}\left(-\frac{\left(\pi^{3/2}\sqrt{\frac{B^2}{\Omega}}\left(\frac{\Omega}{B^2}\right)^{1/4}\right)r}{8\left(\sqrt{2}\,B^2\sqrt{\Omega}\,\mathrm{Gamma}\left[\frac{1}{4}\right]\mathrm{Gamma}\left[\frac{5}{4}\right]\right)}+O[r]^2\right)$$

In[ ]:= $$\mathtt{Series}\left[\frac{\mathtt{MeijerG}\left[\left\{\left\{\frac{1}{2}\right\},\ \{\}\right\},\ \left\{\left\{0,\ \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4},\ 1\right\},\ \{\}\right\},\ \frac{r^4\,\Omega}{1024\,B^2}\right]}{4\sqrt{2}\,\pi^2\sqrt{\frac{B^2}{\Omega}}\,\Omega^{3/2}},\ \{r,\ \infty,\ 0\}\right]$$

Out[ ]= $$O\left[\frac{1}{r}\right]^2+e^{-\frac{\left(\frac{1}{2}-\frac{i}{2}\right)\left(\frac{r^4\,\Omega}{B^2}\right)^{1/4}r}{r}+O\left[\frac{1}{r}\right]^2}\left(\frac{\left(\frac{1}{4}-\frac{i}{4}\right)(-1)^{1/8}\sqrt{\pi}\sqrt{\frac{B^2}{\Omega}}\left(\frac{\Omega}{B^2}\right)^{1/8}\sqrt{r}}{2^{3/4}\,B^2\sqrt{\Omega}}+\sqrt{O\left[\frac{1}{r}\right]}\right)+$$
$$e^{-\frac{\left(\frac{1}{2}+\frac{i}{2}\right)\left(\frac{r^4\,\Omega}{B^2}\right)^{1/4}r}{r}+O\left[\frac{1}{r}\right]^2}\left(\frac{(-1)^{1/8}\sqrt{\pi}\sqrt{\frac{B^2}{\Omega}}\left(\frac{\Omega}{B^2}\right)^{1/8}\sqrt{r}}{4\times 2^{1/4}\,B^2\sqrt{\Omega}}+\sqrt{O\left[\frac{1}{r}\right]}\right)$$

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