Non-Local Gravitational Interactions and the Black hole Information Paradox

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in partial fulfillment of the requirements for the degree of
B.Sc. in Physics

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Declaration

It is hereby declared that

1. The thesis submitted is my/our own original work while completing degree at Brac University.
2. The thesis does not contain material previously published or written by a third party, except where this is appropriately cited through full and accurate referencing.
3. The thesis does not contain material which has been accepted, or submitted, for any other degree or diploma at a university or other institution.
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Abstract

This thesis looks into Non-local gravitational interactions that hawking radiated quanta's from Black holes might possess. We shall first explain Hawking's original calculation in [1], and how such a process would lead to the catastrophic information paradox. Then we shall look into other physical models that try to resolve such a paradox, and the various pitfalls that these models face. Subsequently, the latter part of this thesis looks into how Non-local gravitational dynamics is a suitable resolve to the information paradox; furthermore, we show that our calculations are not bounded in the same way as Mathur's is, in [3]. We then show how Non-local correlations would manifest itself in the wave function of the pair produced particles, and in turn, we show how the entangled entropy decreases analytically, thus preserving unitarity.

Keywords: Black hole; Information Paradox; Non-locality; Physical model; Entropy bounds

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Chapter 1

Introduction to Black holes and the Schwarzschild solution

The implications of Einstein's gravitational field equations are far more deeper than it looked when it first came out. Starting from gravitational wave to Black holes, the magical nature of these equations are yet to be captured in any experiment. It was only recently gravitational wave was observed although it was predicted almost a century ago. Life is much more sad for Black holes and Stephen Hawking. Hawking radiation was never been observed and Stephen Hawking was denied his Nobel prize. The problem that such a process leads to is called the Black hole information paradox and is probably one of the most studied problems in all of theoretical physics. In this thesis, we first introduce the concept of Black holes by talking about Schwarzschild solution and introducing other coordinate systems. We then introduce Hawking radiation from two different viewpoints and then present a discussion regarding the information paradox. Lastly, we present a general toy model by considering non-locality and subsequently bypassing the entropy bounds formulated in [3] and [4].

1.1 Schwarzschild Solution

The Schwarzschild solution is the first exact solution of Einstein's equation. It was proposed by Karl Schwarschild in 1916 while in the trenches on the Eastern Front during the First World War, but sadly he did not survive the conflict. Nevertheless, it is one of the simplest metrics out there, but it reveals a great deal about the amazing features of the Einstein equation.

Some features of the Schwarzschild metric are:

1. It is a static and isotropic metric. Isotropy means it looks the same in every direction from a particular point. On the other hand, static means there is no motion in time. A river with no current(not even constant current) is a good example of the static scenario.

2. It represents the space-time geometry outside a spherically symmetric matter distribution.

3. Although this point is a restatement of the static space-time property, this will help to understand the Schwarzschild geometry better. The metric components $g_{\mu\nu}$ are independent of x^0 and the line element is invariant under the time reversal ($x^0 \rightarrow -x^0$).

The Schwarzschild metric can be written in the form,

$$ds^2 = c^2(1 - \frac{2GM}{c^2 r})dt^2 - (1 - \frac{2GM}{c^2 r})^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2 \phi^2 \tag{1.1}$$

There is an important theorem regarding the Schwarzschild geometry which makes the solution even more powerful. Birkhoff's theorem [7] states that the space-time geometry outside a spherically symmetric matter distribution is that of the Schwarzschild geometry. If we notice carefully, then we can see that the metric has two singularities: one at $r = \frac{2GM}{c^2}$ and another at r = 0. The $r = \frac{2GM}{c^2}$ is a coordinate singularity and can be removed by a good choice of a coordinate system. One must reiterate that the Space-Time singularity cannot be changed from $r = 0$. Nevertheless, we will now explore two beautiful coordinates and see how the singularity of $r = \frac{2GM}{c^2}$ can easily be removed.The first one is Eddington-Finkelstein coordinates and the second one is the Kruskal-Szekers coordinates. These two coordinates has been studied extensively and is used to predict two more fascinating and yet to be observed things i.e the White hole and the Worm hole. The Eddington-Finkelstein uses ingoing and outgoing light rays as coordinates hence the following picture from [6] will provide a basis for understanding these coordinates.

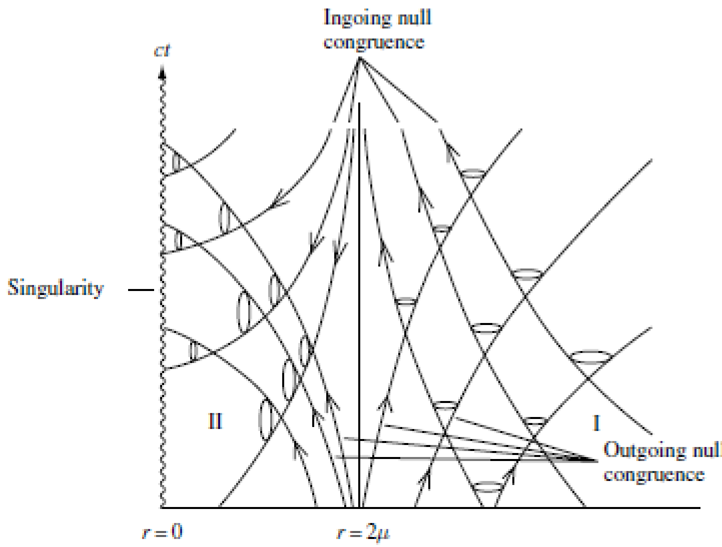

Figure 1.1: Light rays in Eddington-Finkelstein coordinates [6]

1.2 Eddington-Finkelstein Coordinates

In order to arrive at the Eddington-Finkelstein coordinates, we first need to go through the behaviour of incoming and outgoing photons. Lets see how geodesics are in the Schwarzschild geometry and then look at the null/lightlike geodesics of the geometry. One way to do it is to use the geodesic equation,

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\sigma}\frac{dx^\rho}{d\sigma} = 0 \tag{1.2}$$

We present a procedure presented in [6]. They use a Lagrangian,

$$L = g_{\mu\nu}x^\mu x^\nu$$

where, $x^\mu = (x^0, x^1, x^2, x^3)$. So, for

$$\mu = 0, \qquad \frac{dx^\mu}{d\sigma} = \frac{dx^0}{d\sigma} = t,$$

$$\mu = r, \qquad \frac{dx^\mu}{d\sigma} = \frac{dr}{d\sigma} = r,$$

$$\mu = \theta, \qquad \frac{dx^\mu}{d\sigma} = \frac{d\theta}{d\sigma} = \theta$$

,

$$\mu = \phi, \qquad \frac{dx^\mu}{d\sigma} = \frac{d\phi}{d\sigma} = \phi.$$

Using the Schwarzschild metric, the Lagrangian can be written in the form:

$$L = c^2(1 - \frac{2\mu}{r})t^2 - (1 - \frac{2\mu}{r})^{-1}r^2 - r^2(\theta^2 + \sin^2\theta\phi^2) \tag{1.3}$$

where, $\mu = \frac{GM}{c^2}$

The Eular-Lagrange equation can be written as

$$\frac{d}{d\sigma}(\frac{\delta L}{\delta x_\mu}) - \frac{\delta L}{\delta x^\mu} = 0 \tag{1.4}$$

Using the E-L equations, we can obtain the Geodesic equations.
Now, when $\mu = 0$,

$$\frac{\delta L}{\delta x^0} = 2c^2(1-\frac{2\mu}{r})t$$
$$\frac{\delta L}{\delta x^0} = 0$$
$$\frac{d}{d\sigma}(\frac{\delta L}{\delta x^0}) = \frac{\delta L}{\delta x^0}$$
$$= \frac{d}{d\sigma}[2c^2(1-\frac{2\mu}{r})t] = 0$$
$$(1-\frac{2\mu}{r})t = k,$$

where k is a constant and after integrating both sides by σ we obtain four geodesic equations which are:

$$(1-\frac{2\mu}{r})\dot{t} = k, \tag{1.5}$$
$$(1-\frac{2\mu}{r})^{-1}\ddot{r} + \frac{\mu c^2}{r^2}\dot{t}^2 - (1-\frac{2\mu}{r})^{-2}\frac{\mu}{r^2}\dot{r}^2 - r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = 0, \tag{1.6}$$
$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\phi^2 = 0, \tag{1.7}$$
$$\text{and} \quad r^2\sin^2\theta\ddot{\phi} = h \tag{1.8}$$

We can simplify our equations by considering $\theta = \frac{\pi}{2}$ [equation (1.7) clearly satisfies it]. Since it is a spherical symmetry, we can confine our attention to motion of particles in the $\theta = \frac{\pi}{2}$ with no loss of generality. The equations can simplified further. As we want photon geodesics, we can not take proper time as a parameter but can take any affine parameter along the geodesic.An affine parameter is a parameter related to any previously considered parameter by the equation $\zeta = a\rho + b$. Here the previously considered parameter is ρ and the new parameter called the affine parameter is ζ and "a" and "b" are constants. Besides, we can further simplify the radial equation by considering the condition,

$$g_{\mu\nu}x^{\mu}x^{\nu} = 0$$

The four equations are now reduced to three and can be written in the following simplified form:

$$(1-\frac{2\mu}{r})\dot{t} = k, \tag{1.9}$$

$$c^2(1-\frac{2\mu}{r})\dot{t}^2 - (1-\frac{2\mu}{r})^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0,, \tag{1.10}$$

$$\text{and} \quad r^2\dot{\phi}^2 = h, \tag{1.11}$$

Since, we are interested in the radial motion of photons (i.e. the second equation we can put $\dot{\phi} = 0$ in the second equation and thus it becomes,

$$\begin{gathered} \frac{d}{d\sigma}(\frac{\delta L}{\delta x_\mu}) - \frac{\delta L}{\delta x^\mu} = 0 \\ c^2(1-\frac{2\mu}{r})\dot{t}^2 = (1-\frac{2\mu}{r})^{-1}\dot{r}^2 \end{gathered} \tag{1.12}$$

which can be reduced to obtain,

$$\frac{dr}{dt} = \pm c(1-\frac{2\mu}{r}) \tag{1.13}$$

The two solutions upon integrating is,

$$ct = r + 2\mu\ \ln(\frac{r}{2\mu} - 1) + \text{constant} \tag{1.14}$$

and,

$$ct = -r - 2\mu\ \ln(\frac{r}{2\mu} - 1) + \text{constant} \tag{1.15}$$

The first equation represents an outgoing photon and the second equation represents an incoming photon. At the end of previous section, we stated that the singularity at $r = 2\mu$ is a coordinate singularity and occurs due to a bad choice of coordinates. The idea behind removing the singularity is to use outgoing and incoming photon geodesics as coordinates.

Let's first look into a radially ingoing photon:

$$ct = -r - 2\mu \ln(\frac{r}{2\mu} - 1) + \text{constant} \tag{1.16}$$

The idea is to use the integration constant as new coordinate and to show the general mass how smart physicists are. Denote the new coordinate by P

$$\begin{aligned} p &= ct + r + 2\mu \ln(\frac{r}{2\mu} - 1), \\ dp &= cdt + \frac{r}{r - 2\mu}dr, \\ \text{and} \quad cdt &= dp - \frac{r}{r - 2\mu}dr \end{aligned} \tag{1.17}$$

Substituting for dt in the Schwarzschild metric we get,

$$ds^2 = (1 - \frac{2\mu}{r})dp^2 - 2dpdr - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.18}$$

Even though life is unfair, one might point out the unfairness in choosing a null coordinate because it is "intuitively unfamiliar". Instead a related timelike coordinate $\tilde{t}$ can be used such that,

$$c\tilde{t} = p - r = ct + 2\mu \ln(\frac{r}{2\mu} - 1) \tag{1.19}$$

and the line element thus takes the form:

$$ds^2 = c^2(1 - \frac{2\mu}{r})d\hat{t}^2 - \frac{4\mu c}{r}d\tilde{t}dr - (1 + \frac{2\mu}{r})dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1.20}$$

The coordinates (t', r, θ, ϕ) are called advanced Eddington-Finkelstein coordinates. The above figure gives us the lightcone structure in Eddington-Finkelstein coordinates.

The figure (1.2) is more convincing than any other explanation. The radial trajectories of an infalling photon (can be generalized to any infalling particle) is now continuous at $r = 2\mu$ in advanced Eddington-Finkelstein coordinate. But $r = 2\mu$ still retains an interesting feature. It is clear from the lightcone structure of a particle after crossing $r = 2\mu$ that it's fate is decided at the singularity r = 0.

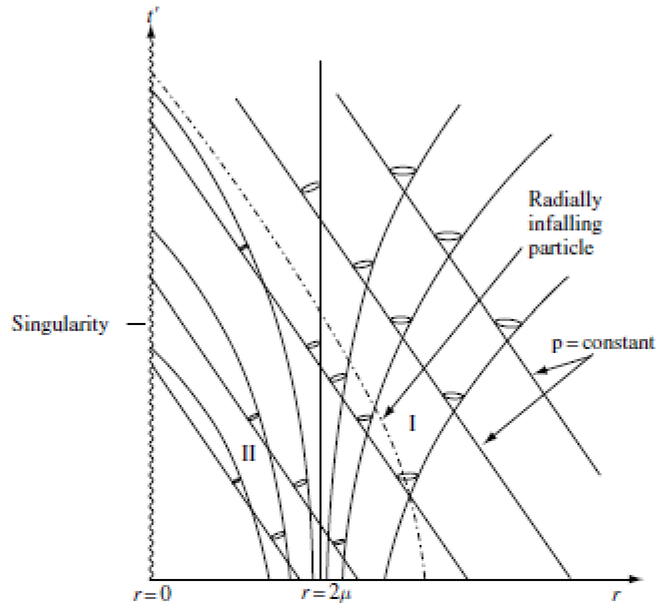

Figure 1.2: Advanced Eddington-Finkelstein Coordinates from [6]

Any motion inside $r = 2\mu$ even at light speed, would lead it to the singularity. To actually cross the event horizon ($r = 2\mu$) one needs to violate causality or actually travel faster than light. The event horizon ($r = 2\mu$) actually acts as a one-way membrane i.e allowing stuffs to only go in. An object with an event horizon is nicely termed as a "Black Hole". If one considers radially outgoing photon geodesic as a new coordinate, the new coordinates are called retarded Eddington-Finkelstein and they lead to an opposite thing of a black hole i.e. the "White Hole". In the white hole case, $r = 2\mu$ acts as a one way street i.e. allowing only stuffs to come out of $r = 2\mu$ and stop things penetrating $r = 2\mu$.

1.3 Kruskal Coordinates:

Ingoing and outgoing null geodesics provide two beautiful entities i.e. black holes and White hole. The advanced Eddington-Finkelstein and retarted Eddington-Finkelstein are two separate coordinates and the search for a single coordinate without singularity at r=2m continues. In natural units the Schwarzchild line element is

$$\begin{aligned} ds^2 &= -\left(1-\frac{2m}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} \\ &= -\left(1-\frac{2m}{r}\right)\left(dt - \frac{dr}{1-\frac{2m}{r}}\right)\left(dt + \frac{dr}{1-\frac{2m}{r}}\right) \end{aligned} \tag{1.21}$$

Defining,

$$\begin{aligned} du &= dt - \frac{dr}{1-\frac{2m}{r}}, \\ u &= t - \int \frac{dr}{1-\frac{2m}{r}}, \\ dv &= dt + \frac{dr}{1-\frac{2m}{r}}, \\ \text{and } v &= t + \int \frac{dr}{1-\frac{2m}{r}}. \end{aligned} \tag{1.22}$$

We calculate,

$$v - u = 2\int \frac{dr}{1-2m/r}, \tag{1.23}$$

$$\begin{aligned} \frac{v-u}{2} &= \int \frac{dr}{1-2m/r}, \\ &= \int \frac{1}{r-2m/r}dr, \\ &= \int \frac{r}{r-2m}dr \\ &= \int \frac{r-2m+2m}{r-2m}dr, \\ &= \int \left(1+\frac{2m}{r-2m}\right)dr, \\ &= \int \left(1+\frac{1}{\frac{r}{2m}-1}\right)dr, \\ \frac{v-u}{2} &= r + 2m\ln(\frac{r}{2m}-1). \end{aligned} \tag{1.24}$$

The equation relating coordinates u and v is badly behaved at r=2m. The best way to fix the ugly behaviour of natural log is to exponentiate it. Thus,

$$\exp\left(\frac{v-u}{4m}\right) = \exp\left(\frac{r}{2m}\right)\Big(\frac{r}{2m}-1\Big) \tag{1.25}$$

$$\left(1-\frac{2m}{r}\right) = \frac{2m}{r}\exp\left(\frac{v-u}{4m}\right)\exp\left(\frac{-r}{2m}\right) \tag{1.26}$$

Plugging equation (1.14) in equation (1.09),

$$\begin{aligned} ds^2 &= -\left(1-\frac{2m}{r}\right)\left(dt-\frac{dr}{1-\frac{2m}{r}}\right)\left(dt+\frac{dr}{1-\frac{2m}{r}}\right) \\ &= \frac{-2m}{r}\exp\left(\frac{v-u}{4m}\right)\exp\left(\frac{-r}{2m}\right)dudv \\ ds^2 &= \frac{-32m}{r}\exp\left(\frac{v}{4m}\right)\frac{dv}{4m}4m\exp\left(\frac{-u}{4m}\right)\frac{du}{4m}4m \end{aligned} \tag{1.27}$$

Let,

$$\begin{aligned} dU &= \exp\left(\frac{-u}{4m}\frac{du}{4m}\right), \\ U &= -\exp\left(\frac{-u}{4m}\right., \\ dV &= \exp\left(\frac{v}{4m}\right)\frac{dv}{4m}, \\ V &= -\exp\left(\frac{v}{4m}\right), \\ \text{and } UV &= -\exp\left(\frac{v-u}{4m}\right) \\ &= \exp\left(\frac{r}{2m}\right)(1-\frac{r}{2m}) \end{aligned} \tag{1.28}$$

Therefore, the line element takes the form

$$ds^2 = \frac{-32m}{r}\exp(\frac{r}{2m})(1-\frac{r}{2m})dUdV \tag{1.29}$$

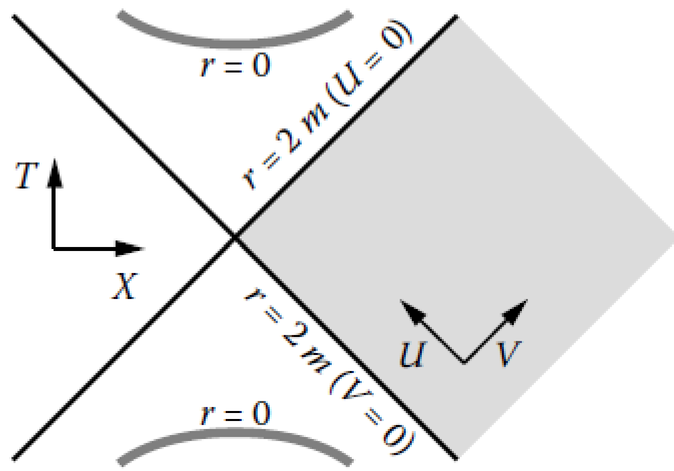

Figure 1.3: Kruskal coordinates [30]

This coordinates (u,v,θ, ϕ) are known as Kruskal-Szekeres coordinates. We can also express them in term of orthogonal coordinates T and O.

$$\begin{aligned} U &= T - O \\ V &= T + O \end{aligned} \tag{1.30}$$

The Kruskal coordinates extends the Schwarzchild geometry very nicely. The important thing to notice is that there is no singularity at r=2m and the two coordinates are thus combined. Kruskal geometry inherits the features of both the coordinates. As r=2m corresponds to either u=0 and v=0, the original Schwarzschild geometry corresponds to region I. Since T points upward, any particle in region II (which corresponds to black hole interior) must hit the singularity at r=0. Coming out of the black hole would lead to a violation of causality. Region IV corresponds to a white hole (a time reversed of a black hole) and anything inside a white hole is ejected to region III which is similar to the Schwarzschild region on the right. Region I and region III are causally disconnected regions. Region I and III are often referred to as "Us" and "Them" giving a green signal to science fiction writers.

1.4 Penrose Diagrams

Penrose diagrams, which Roger Penrose called Conformal diagrams, are beautiful diagrams to represent the metric and structure of spacetime. Penrose diagrams maybe defined as as two dimensional spacetime diagrams representing paths at infinity.Some of the features of Penrose diagrams are:

1. The main reason behind the introduction of Penrose diagrams is to include points at infinity. The spacetime whose Penrose diagram we are considering should be flat assymptotically or in other words it must look like Minkowski space far away.

2. The angular coordinates are suppressed with radial and time coordinates receiving the sole attention.

3. Since in a conformal transformation the angles are preserved, light rays which used to be at 45 degrees from space and time axes would still remain at 45 degrees. Therefore the holy path of light is preserved.

4. To include points at infinity we use the following conformal factor $\Omega = \frac{1}{r}$. The point at infinity corresponds to $r = 0$ and thus adds a conformal boundary to the diagram of the space-time itself.

Penrose diagrams are beautiful ways to represent the structure of spacetime. The idea of Penrose diagrams is captured beautifully in [30] whose diagrams we present.

1.4.1 Penrose Diagram Of Minkowski space and Kruskal Coordinates

An interesting question that everyone should ask is in how many ways can you go to infinity? One? Two? The answer is you can reach infinity in 5 ways. One is spatial infinity i.e going infinitely far away in spatial directions (denoted by i^0 then two time infinities i.e going infinitely far from the present to the future and going infinitely away to the past(denoted by $i^{+,-}$.) There are other two paths of infinity which only light rays can traverse in Lorentzian signature. These rays add two conformal boundaries which represent incoming and outgoing radiation. They are called past and future null infinities. They are denoted by $\mathscr{I}^{+,-}$. The

Penrose diagram of Minkowski space is:

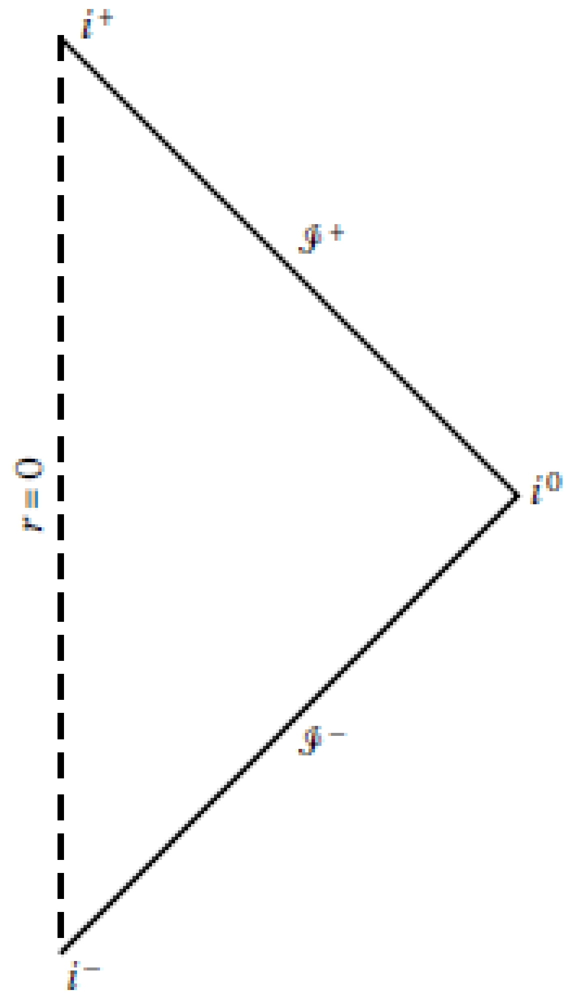

Figure 1.4: Penrose diagram of Minkowski space [30]

The Penrose diagram for Kruskal Geometry is:

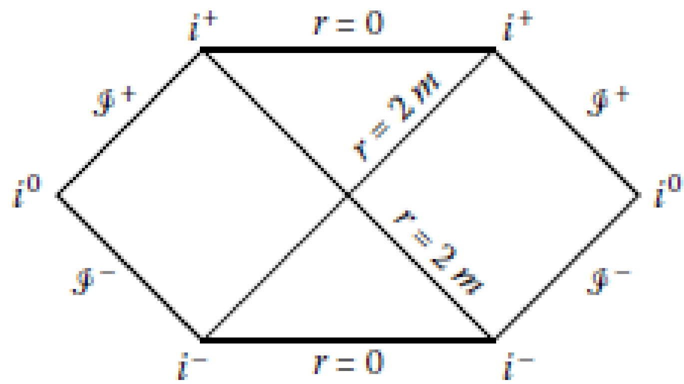

Figure 1.5: Penrose diagram of Kruskal Geometry [30]

This needs some explanation. Since Kruskal geometry is the maximally extended Schwarzschild solution, the Penrose diagram representing Kruskal Geometry must incorporate both the black hole structure and White hole structure. As can be seen from the diagram,the Black hole and White hole horizon ($r = 2m$) is represented in the diagram by thin lines whereas the thicker line represents the singularity $r = 0$. The horizons divide the spacetime into four parts. The right quadrant represents the original Schwarzschild region where the upper quadrant represents the Black hole interior. The left quadrant represents a causally disconnected part from the Schwarzschild region on the right. This is very similar to the Schwarzschild region but is causally disconnected from it. The bottom quadrant represents a White hole.

Chapter 2

Hawking Radiation

Due to the advent of Quantum Field Theory, we know that particles are constantly popping in and out of existence. A popular particle production method in vacuum was called the Unruh effect, where an accelerating observer, observes a thermal flux of particles. Stephen Hawking in [1], extended this idea to in the case of Black holes, he showed that vacuum fluctuations caused by the Black hole would cause particle production. One of the two particles being produced would succumb to the Black hole's immense gravity and thus be pulled into it and the other would escape and would cause an outside observer to feel a thermal flux of particles, as if it were being emanated by the Black hole.
In the following section we shall try and find the temperature term for Hawking radiation using the work done in [32].

2.1 Quantization in Flat Minkowski Space

We begin first, by looking at how canonical quantization of a field in normal Minkowski spacetime , and then we shall make the transition into what form it takes in curved space time, where Hawking effect comes into play.
We look at the simple example of a massless scalar field f, for which it's action looks like

$$S = -\frac{1}{2}\int d^4x \partial_\mu f \partial^\mu f, \tag{2.1}$$

and the equation of motion that the Klein-Gordon equation gives

$$\partial_\mu \partial^\mu f = 0, \tag{2.2}$$

Quantizing this field f, and splitting it into positive and negative frequency solutions we get

$$f(t, \overrightarrow{x}) = \sum_i \left[a_i u_i(t, \overrightarrow{x}) + a_i^\dagger u_i^*(t, \overrightarrow{x}) \right], \tag{2.3}$$

With respect to t, which is the global inertial time of the Minkowski space. Which implies

$$\frac{\partial}{\partial t} u_j(t, \overrightarrow{x}) = -i w_j u_i(t, \overrightarrow{x}), \quad w_j > 0. \tag{2.4}$$

The solution that we set to form our set of solutions for field f, are chosen such that they form a complete orthonormal basis, with respect to our Klein-Gordon product

$$(f_1, f_2) = i \int d^3 \overrightarrow{x} (f_1 \partial_t f_1^* - f_2^* \partial_t f_1). \tag{2.5}$$

On the positive frequency solutions case the scalar product happens to be positive definite, allowing to construct one particle Hilbert space, and subsequently a many particle space called Fock space. Such is described by the commutation relationship

$$[a_i, a_j^\dagger] = (u_i, u_j^\dagger)\hbar = \delta_{ij}\hbar, \tag{2.6}$$

and the operators also follow suit to the following relationship

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0. \tag{2.7}$$

Promoting the classical to field operator and transitioning it into a quantum field operator and abiding by the equal time commutation relationship, it takes the form,

$$[f(t, \overrightarrow{x}), \pi(t, \overrightarrow{x}')] = i\hbar \delta^3(\overrightarrow{x} - \overrightarrow{x}'). \tag{2.8}$$

In similar fashion to (2.7) the operators obey the following relationship

$$[f(t, \overrightarrow{x}), f(t, \overrightarrow{x}')] = [\pi(t, \overrightarrow{x}), \pi(t, \overrightarrow{x}')] = 0. \tag{2.9}$$

Where $\pi = \partial_f f$ is the canonical conjugate variable of f. We construct the Fock space out of the vacuum state $|0\rangle$ with a_i to be the annihilation operator such that,

$$a_i |0\rangle = 0. \tag{2.10}$$

And, $a_j^\dagger$ acting as the creation operator such that,

$$a_i^\dagger |0\rangle = \sqrt{\hbar}\, |1_i\rangle \tag{2.11}$$

$$|1_i\rangle = \hbar^{-\frac{1}{2}} a_i^\dagger |0\rangle\,, \tag{2.12}$$

which spans the one particle Hilbert space, and in similar fashion we construct the multi particle Fock space

$$\begin{aligned} |n^1_{i_1}, n^2_{i_2}, ..., n^k_{i_k}\rangle =& (n^{(1)}!, n^{(2)}!, .., n^{(k)}!)^{-\frac{1}{2}} (\hbar^{-\frac{1}{2}} a^\dagger_{i_1})^{n(1)} \\ & (\hbar^{-\frac{1}{2}} a^\dagger_{i_2})^{n(2)} ... (\hbar^{-\frac{1}{2}} a^\dagger_{i_k})^{n(k)} |0\rangle\,. \end{aligned} \tag{2.13}$$

Where i_1, i_2, etc represent different one particle states.
The standard choice for the orthonormal basis is given by the following plane wave solution

$$u_{\overrightarrow{k}} \equiv \frac{1}{\sqrt{16\pi^3\omega}} \exp(-i\omega t + i\,\overrightarrow{k}\,\overrightarrow{x}), \tag{2.14}$$

where the index i is replaced with the $\overrightarrow{k}$ vector, and the frequency ω having the dispersion relationship $\omega = |\overrightarrow{k}|$. The commutation relationship for the ladder operators now take the form

$$\left[a_{\overrightarrow{\mathrm{k}}}, a^{\dagger}_{\overrightarrow{\mathrm{k}}}\right] = \hbar\delta^3\left(\overrightarrow{\mathrm{k}} - \overrightarrow{\mathrm{k}}'\right). \tag{2.15}$$

Similarly

$$\left[a_{\overrightarrow{\mathrm{k}}}, a_{\overrightarrow{\mathrm{k}}'}\right] = \left[a^{\dagger}_{\overrightarrow{\mathrm{k}}}, a^{\dagger}_{\overrightarrow{\mathrm{k}}'}\right] = 0 \tag{2.16}$$

Here the quantization enjoys the property of it being independent of inertial time t. Any other choice of t related via the *Poincare* transformation leads to similar positive frequency solutions. Leading us to the conclusion that the splitting between positive and negative frequency modes in the expansion is invariant. Furthermore, one can also conclude that the vacuum state to the whole Fock space is invariant.

2.2 Quantization in Curved Spacetime

To understand particle creation by Black holes, one must generalize the steps in the last section and extend it's ideas in curved spacetime.
In flat spacetime one can generalize the solution of the wave equation to be

$$\nabla_\mu \nabla^\mu f = 0. \tag{2.17}$$

The formalism of the Klein-Gordon product in Minkowski flat spacetime can be extended in curved spacetime as well albeit with certain changes.

$$(f_1, f_2) = -i \int_\zeta d\zeta^\mu (f_1 \partial_\mu f_2^* - f_2^* \partial_\mu f_1). \tag{2.18}$$

Here, the ζ is a Cauchy hyper surface and acts as an initial data term and $d\zeta^\mu = d\zeta^\mu n^\mu$. Here $d\zeta$ is the volume element and n^μ is the orthonormally directed unit vector to our initial data term. Using Gauss's theorem, we can also show that taking the inner product is independent of the choice of a hypersurface.

In curved spacetime, unlike flat space we lack a unique definition of a vacuum state. Different choices in positive frequency solutions will lead to different definitions of such a vacuum state and the corresponding Fock space that we are trying to build up. One, can construct a space consisting of positive frequency solutions, such a situation would come bout if our curved spacetime is stationary. This would imply that the we get timelike vector field ξ^μ which leaves the spacetime metric be invariant.

The above mentioned positive frequency modes thus take the form

$$\xi \nabla_\mu u_j = -i w_j u_j. \tag{2.19}$$

The standard definition of the Fock space in flat Minkowski spacetime can have it's interpretation stretched here as well, in the case of curved spacetime.

When the background spacetime is not stationary, like in our case where the background houses gravitational collapse. Such an implication has us lose the general particle definition that we would would generally get for stationary situations.

Nonetheless, one can gauge an appropriate interpretation, thinking of the spacetime have past and future regions that act asymptotically stationary.

The orthonormal modes thus represent

$u_i^{in} \longrightarrow$ Positive frequency modes in the past.

$u_i^{out} \longrightarrow$ Positive frequency modes in the future.

To achieve such stationary solutions we do a Bogoliubov transformation and thus understand the effects cause by dynamical background gravity, which we shall look into the next section.

2.3 Bogoliubov Transformations

The field f which can be expressed in terms of positive frequency modes u_i^{in} in the region where it is initially stationary.

$$f = \sum_i \left[a_i^{in} u_i^{in} + a_i^{in^\dagger} u_i^{in^*}\right]. \tag{2.20}$$

In the same fashion f can also be expressed in terms of positive frequency modes u_i^{out}.

$$f = \sum_i \left[a_i^{out} u_i^{out} + a_i^{out^\dagger} u_i^{out^*}\right]. \tag{2.21}$$

The modes in (2.20) hold the following orthonormal conditions

$$(u_i^{in}, u_j^{in}) = \delta_{ij}, \quad (u_i^{in^*}, u_j^{in^*}) = -\delta_{ij}, \quad (u_i^{in}, u_j^{in^*}) = 0 \tag{2.22}$$

and the creation and annihilation operators obey the following commutation relationship.

$$[a_i^{in}, a_j^{in^\dagger}] = \hbar\delta_{ij}, \tag{2.23}$$

$$\text{and} \quad \left[a_i^{in}, a_j^{in}\right] = \left[a_i^{in^\dagger} a_j^{in\dagger}\right] = 0. \tag{2.24}$$

Similarly, one gets similar relations for the negative frequency modes.
We have now a complete set of modes, if expanded we can figure out a relation between the modes u_i^{out} and u_i^{in}. We cannot say beforehand that u_i^{out} simply consists of positive frequency solutions, so we write in terms of both,

$$u_j^{out} = \sum_i (\alpha_{ji} u_i^{in} + \beta_{ji} u_j^{in^*}). \tag{2.25}$$

These are known as called Bogoliubov transformations and α_{ji} and β_{ji} are called the Bogoliubov coefficients,given by.

$$\alpha_{ij} = (u_i^{out}, u_j^{in}), \quad \text{and} \quad \beta_{ij} = -(u_i^{out}, u_j^{in*}). \tag{2.26}$$

(2.22) leads to the following relations

$$\sum_k = (\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^* = \delta_{ij}), \tag{2.27}$$

$$\text{and,} \quad \sum_k = (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0, \tag{2.28}$$

and (2.25) implies that we can write

$$u_i^{in} = \sum_j (\alpha_{ji}^* u_j^{out} + \beta_{ji}^* u_j^{out*}). \tag{2.29}$$

We can also now show what form the creation and annihilation operators take

$$a_i^{in} = \sum_j (\alpha_{ji} a_j^{out} + \beta_{ji}^* a_j^{out\dagger}) \tag{2.30}$$

$$a_i^{out} = \sum_j (\alpha_{ij} a_j^{in} + \beta_{ij}^* a_j^{in\dagger}) \tag{2.31}$$

If β_{ij} does not annihilate the vacuum state totally, defined as,

$$a_i^{in} \left|in\right\rangle = 0, \tag{2.32}$$

$$\text{and} a_i^{out} \left|out\right\rangle = 0. \tag{2.33}$$

We can construct a number operator out of the ladder operators and therefore finds its expectation value, to which it takes the form,

$$N_i^{out} \equiv \hbar^{-1} a_i^{out\dagger} a_i^{out} \tag{2.34}$$

$$\begin{aligned} \langle in|N_i^{out}|in\rangle &= \hbar^{-1} \langle in|a_i^{out\dagger} a_i^{out}|out\rangle \\ &= \sum_j |\beta_{ij}|^2. \end{aligned} \tag{2.35}$$

Also, if all the β_{ij} coefficients were to vanish it would imply,

$$\sum_k \alpha_{ik}\alpha_{jk}^* = \delta_{ij}. \tag{2.36}$$

Thus, we an conclude that the positive frequency modes in the in and out basis have a unitary transformation between them and the matrix α_{ij}, implying $|in\rangle = |out\rangle$

To figure out the content of the particles created in the $|in\rangle$ state in the asymptotically stationary region, we need a full definition of it in out "out" Fock space. We can achieve such a thing by simply applying the creation operator on $|in\rangle$ state.

$$\sum_j (\alpha_{ji} a_j^{out} + \beta_{ji}^* a_j^{out\dagger} |in\rangle = 0. \tag{2.37}$$

and,

$$(a_k^{out} + \sum_{ij} \beta_{ji}^* a_{ik}^{-1} a_j^{out\dagger} |in\rangle = 0. \tag{2.38}$$

$$\text{Taking} \quad V_{jk} \equiv -\sum_i \beta_{ji}^* \alpha_{ik}^{-1}. \tag{2.39}$$

We can write the $|in\rangle$ state as

$$|in\rangle = \langle out|in\rangle \exp\left(\frac{1}{2\hbar}\sum_{ij} V_{ij} a_i^{out\dagger} a_j^{out\dagger}\right)|out\rangle\,. \tag{2.40}$$

The above equation just goes to show that one cannot produce an odd number of particles

$$\langle 1_{j_1}, 1_{j_2}, ..., 1_{j_n}|in\rangle = 0 \quad n \quad odd \tag{2.41}$$

The particles being created are therefore being produced in pairs

$$\langle 1_{j_1}, 1_{j_2}|in\rangle \tag{2.42}$$

Since our out state is normalizable we get the following fact

$$\sum_{ij} |\beta_{ij}|^2 \;<\; +\infty. \tag{2.43}$$

Which shows that the number of particles being produced, are in fact of a finite amount.

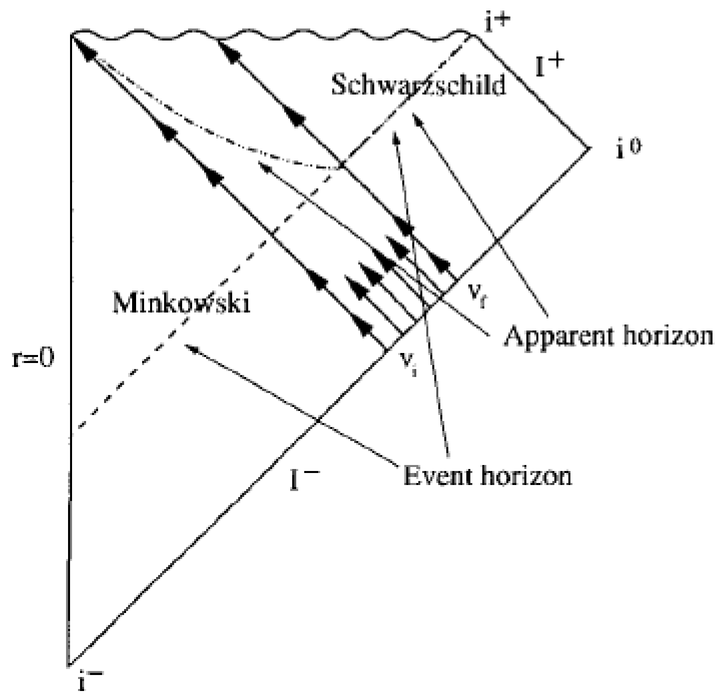

Figure 2.1: A Black hole being made [32]

2.4 Vaidya Spacetime

We shall now try to look into the derivation of Hawking radiation in Vaidya Spacetime, in which the metric takes the form

$$ds^2 = -(1 - \frac{2M(v)}{r}dv^2) + 2dvdr + r^2 d\Omega^2. \tag{2.44}$$

Where the stress-energy tensor looks like

$$T_{vv} = \frac{L(v)}{4^2}, \tag{2.45}$$

and

$$\frac{dM(v)}{dv} = L(v). \tag{2.46}$$

Taking v_i to be the instant when we start seeing particle production, and then v_f the instant when it stops.

At $v = v_0$ we see that L and M take the form

$$L(v) = M\delta(v - v_0), \tag{2.47}$$

and

$$M(v) = M\theta(v - v_0). \tag{2.48}$$

We know from before that mass less Klein-Gordon equation for field f takes the form

$$\nabla_\mu \nabla^\mu f = 0. \tag{2.49}$$

We take our background to be spherically symmetric and thus can state f as the expansion below

$$f(x^\mu) = \sum_{l,m} \frac{f_1(t,r)}{r} \mathrm{Y}_{lm}(\theta, \phi). \tag{2.50}$$

Here, the above equation has been just been generalized for two dimensions in the vicinity of $v < v_0$, the solutions would not be any different if we had worked in 4 dimensions.

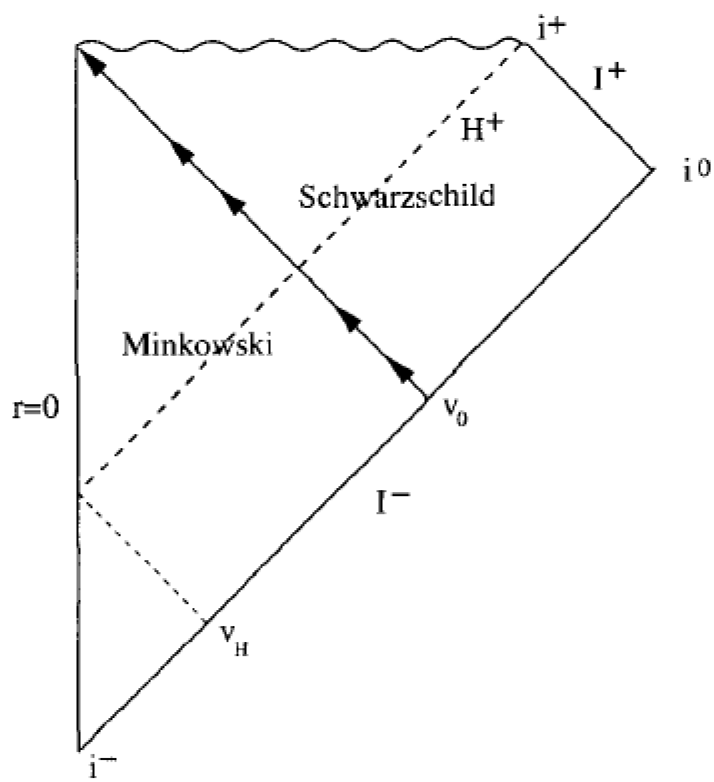

Figure 2.2: $v = v_0$ causing the production of Schwarzchild Black hole [32]

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2} \right) f_1(t,r) = 0, \tag{2.51}$$

and in the vicinity of $v > v_0$ our two dimensional equation takes the form

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} - V_l(r) \right) f_1(t,r) = 0. \tag{2.52}$$

Where the potential V_l takes the form

$$V_l(r) = \left(1 - \frac{2M}{r}\right) \left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3} \right]. \tag{2.53}$$

At $r = 2M$, we see that at this boundary term, which is the horizon the of the Black hole, the potential term vanishes and since we theorize that particle productions are being produced the near the horizon of the Black hole

Thus for the $v < v_0$ we have the equation for f to be

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} \right) f(t,r) = 0. \tag{2.54}$$

With the following equation serving as a regularity condition, as we need the filed to go to 0 at $r = 0$

$$f(t, r = 0) = 0, \tag{2.55}$$

and subsequently for the $v > v_0$ region our equation takes the form

$$\left(- \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^{*2}} \right) f(t,r) = 0. \tag{2.56}$$

Imposing the particles to be of positive frequency we that f can be broken into

$$f(t,r) = \exp(-i\omega t) f(r). \tag{2.57}$$

and thus we can get a solution for the equation of f(r)

$$\frac{d^2 f(r)}{dr^2} + \omega^2 f(r) = 0., \tag{2.58}$$

and

$$\frac{d^2 f(r)}{dr{*}^2} + \omega^2 f(r) = 0. \tag{2.59}$$

Thus, we can write out our metric for the region $v < v0$ which is the Minkowski metric

$$ds^2 = -du_{in} dv + r_{in}^2 d\omega^2. \tag{2.60}$$

where, $u_{in} = t_{in} - r_{in}$ and $v_{in} = t_{in} + r^*_{in}$

For the region where $v > v_0$ we write out the Schwarzchild metric to be

$$ds^2 = -\left(1 - \frac{2M}{r_{out}}\right)du_{out}dv + r_{out}^2 d\Omega^2. \tag{2.61}$$

where, $u_{out} = t_{out} - r^*_{out}$ and $v = t_{out} + r^*_{out}$

We have the wave solutions be subdivided into $e^{-i\omega v}$ (ingoing waves), $e^{-i\omega u_{out}}$ in the Schwarzchild region and in the flat Minkowski region $e^{-i\omega u_{in}}$.

At I^- we see the positive frequency modes take the form

$$u_\omega^{in} = \frac{1}{4\pi\sqrt{\omega}}\frac{\exp(-i\omega v)}{r}. \tag{2.62}$$

and

$$(u_\omega^{in}, u_{\omega'}^{in}) = -i\int_{I^-} dvr^2 d\Omega(u_\omega^{in}\partial_v u_{\omega'}^{in} - u_{\omega'}^{in}\partial_v u_\omega^{in}) = \delta(\omega - \omega'). \tag{2.63}$$

Doing the same for the positive frequency modes as above at I^+

$$u_\omega^{out} = \frac{1}{4\pi\sqrt{\omega}}\frac{\exp(-i\omega u_{out})}{r}, \tag{2.64}$$

and,

$$(u_\omega^{out}, u_{\omega'}^{out}) = -i\int_{I^+} du_{out}r^2 d\Omega(u_\omega^{out}\partial_v u_{\omega'}^{out} - u_{\omega'}^{out}\partial_v u_\omega^{out}) = \delta(\omega - \omega') \tag{2.65}$$

Here I^- is a Cauchy surface for which we can construct the Bogololiubov coefficient $\beta_{\omega\omega'}$ take the form

$$\beta_{\omega\omega'} = (u_\omega^{out}, u_{\omega'}^{in*}) = i\int_{I^-} dvr^2 d\Omega(u_\omega^{out}\partial_v u_{\omega'}^{in} - u_{\omega'}^{in}\partial_v u_\omega^{out}). \tag{2.66}$$

To figure out the modes at I^+ in the Minkoswki region one needs to first adhere to two sets of conditions.

One of which is that at the very beginning, right as the shockwave is forming the mode u_{out} looks like

$$u_\omega^{out} = \frac{1}{4\pi\sqrt{\omega}}\frac{\exp(-i\omega u_{out}(u_{in}))}{r}. \tag{2.67}$$

Which implies,

$$r(v_0, u_{in}) = r(v_0, u_{out}). \tag{2.68}$$

and,

$$r(v_0, u_{in}) = \frac{v_0 - u_{in}}{2}. \tag{2.69}$$

$$r(v_0, u_{out}) + 2M\ln\left(\frac{r(v_0, u_{out})}{2M} - 1\right) = \frac{v_0 - u_{out}}{2}. \tag{2.70}$$

Which implies

$$u_{out} = u_{in} - 4M\ln\left(\frac{v_0 - 4M - u_{in}}{4M}\right). \tag{2.71}$$

$$u_{\omega}^{out} = \frac{1}{4\pi\sqrt{\omega}}\left(\frac{\exp(-i_{out}(u_{in})}{r} - \frac{\exp(-i_{out}(v)}{r}\theta(v_H - v)\right), \tag{2.72}$$

and, imposing the second condition that states that $r = 0$ which implies

$$u_{out}(v) = u_{out}(u_{in} \leftrightarrow v) \tag{2.73}$$

,

where

$$u_{\omega}^{out} \approx -\frac{1}{4\pi\sqrt{\omega}}\frac{\exp(-i\omega v)}{r}. \tag{2.74}$$

Also defining $v_H = v_0 - 4M$, where v_H is the location where the horizon exists and where the hypothesized particle production occurs.

$$u_{out}(u_{in}) = v_H - 4M\ln\frac{v_H - v_{in}}{4M} \tag{2.75}$$

and,

$$u_{out}(v) = v_H - 4M\ln\frac{v_H - v}{4M}. \tag{2.76}$$

Finally getting us the positive frequency modes at the cauchy surface I^-, near v_H

$$u_\omega^{out} \equiv \frac{1}{4\pi\sqrt{\omega}}\frac{\exp(-i\omega(v_H - 4M\ln\frac{v_H-v}{4M}))}{r}\theta(v_H - v) \tag{2.77}$$

$$\langle in|N_\omega^{out}|in\rangle = \int_0^\infty d\omega'|\beta_{\omega\omega'}|^2. \tag{2.78}$$

$$u_{jn}^{out} = \frac{1}{\sqrt{2}}\int_{j\epsilon}^{(j+1)\epsilon} d\omega\exp(2\pi i\omega n/\epsilon)u_\omega^{out}. \tag{2.79}$$

2.5 Wave Packets

As we stated in the the previous section

$$\left\langle in\left|N_w^{out}\right|in\right\rangle = \int_0^\infty dw'\left|\beta_{ww'}\right|^2. \tag{2.80}$$

The above equation makes use of the bogoliubov coefficients constructed in the above section and thus gives us the average number of particles being produced at I^+

The following equation represents the basis of orthonormal modes of the wave packets

$$u_{jn}^{out} = \frac{1}{\sqrt{\epsilon}}\int_{j\epsilon}^{(j+1)\epsilon} d\omega\exp(2\pi i\omega n/\epsilon)u_\omega^{out}, \tag{2.81}$$

But to complete (2.80) one must compute $\beta_{jn,w'}$

$$\beta_{jn,w'} = -\left(u_{jn}^{out}, u_{w'}^{in*}\right) = i\int_{I^-} dvr^2 d\Omega \left(u_{jn}^{out}\partial_v u_{\omega'}^{in} - u_{\omega'}^{in}\partial_v u_{jn}^{out}\right), \tag{2.82}$$

Computing the integration and ignoring the boundary terms gives us

$$\beta_{jn,w'} = 2i\int_{I^-} dvr^2 d\Omega u_{jn}^{out}\partial_v u_{\omega'}^{in}. \tag{2.83}$$

Computing for late times

$$\beta_{jn,\omega'} = \frac{-1}{2\pi\sqrt{\epsilon}}\int_{-\infty}^{v_H} dv \int_{j\epsilon}^{(j+1)\epsilon} d\omega \exp(2\pi i\omega n/\epsilon)\sqrt{\frac{\omega'}{w}}\exp(-i\omega\left(v_H - 4M\ln\frac{v_H - v}{4M}\right) - i\omega' v). \tag{2.84}$$

We must also get a generalized expression for $\alpha_{jn,\omega'}$ as we did for $\beta_{jn,\omega'}$

$$\begin{aligned}\alpha_{jn,\omega'} =& \left(u_{jn}^{\text{out}}, u_{\omega'}^{in}\right) = -2i\int_{I^-} dvr^2 d\Omega u_{jn}^{\text{out}}\, \partial_v u_{\omega'}^{in*} \\ =& \frac{-1}{2\pi\sqrt{\epsilon}}\int_{-\infty}^{v_H} dv \int_{je}^{(j+1)\epsilon} d\omega \exp(2\pi i\omega n/\epsilon) \\ & \times \sqrt{\frac{\omega'}{w}}\exp(-i\omega\left(v_H - 4M\ln\frac{v_H - v}{4M}\right) + i\omega' v).\end{aligned} \tag{2.85}$$

Taking $x = v_H - v$

$$\begin{aligned}\beta_{jn,\omega'} =& \frac{-e^{-i\omega' v_H}}{2\pi\sqrt{\epsilon}}\int_0^{+\infty} dx \int_{j\epsilon}^{(j+1)\epsilon} d\omega \exp(2\pi i\omega n/\epsilon) \\ & \times \sqrt{\frac{\omega'}{\omega}}\exp(-i\omega\left(v_N - 4M\ln\frac{z}{4M}\right) + i\omega' x).\end{aligned} \tag{2.86}$$

Taking the integral and also keeping in mind that frequency ω varies minutely in the integral

$$\beta_{jn,\omega'} = \frac{-\exp(-i\left(\omega_j + \omega'\right)v_H}{)}\pi\sqrt{\epsilon}\sqrt{\frac{\omega'}{\omega_j}}\int_0^{+\infty} dx\exp(i\omega' x)\frac{\sin \epsilon L/2}{L}\exp(iLw_j), \tag{2.87}$$

where $w = j\epsilon \approx (j + \frac{1}{2}\epsilon)$

$$L = \frac{2\pi n}{\epsilon} + 4M\ln\frac{x}{4M} \tag{2.88}$$

and also in similar fashion

$$\alpha_{jn,\omega'} = \frac{-\exp(-i\left(\omega_j - \omega'\right)v_H}{)}\pi\sqrt{\epsilon}\sqrt{\frac{\omega'}{\omega_j}}\int_0^{+\infty} dx\exp(-i\omega' x)\frac{\sin \epsilon L/2}{L}\exp(iL\omega_j). \tag{2.89}$$

Resulting in

$$I\left(\omega'\right) = \int_0^{+\infty} dx\exp(-i\omega' x)\frac{\sin \epsilon L/2}{L}\exp(iL\omega_j). \tag{2.90}$$

Here the logarithmic function is analytic in the lower part or the negative part of the real axis when $\omega' > 0$

$$I\left(\omega' > 0\right) = -i\int_0^{+\infty} dy\exp(-\omega' y)\frac{\sin \epsilon L_y/2}{L_y}e^{iL_y w_j}, \tag{2.91}$$

where

$$\begin{aligned} L_y &= \frac{2\pi n}{\epsilon} + 4M\ln\left(-\frac{iy}{4M}\right) \\ &= \frac{2\pi n}{\epsilon} + 4M\left(-\frac{i\pi}{2} + \ln\frac{y}{4M}\right) \end{aligned} \tag{2.92}$$

Giving us

$$I\left(\omega' > 0\right) = -ie^{2M\pi\omega_j}\exp(2\pi i n\omega_j/\epsilon)\int_0^{+\infty} dy\exp(-\omega' y)\frac{\sin \epsilon L_y/2}{L_y}e^{i4M\ln(y/4M)\omega_j}. \tag{2.93}$$

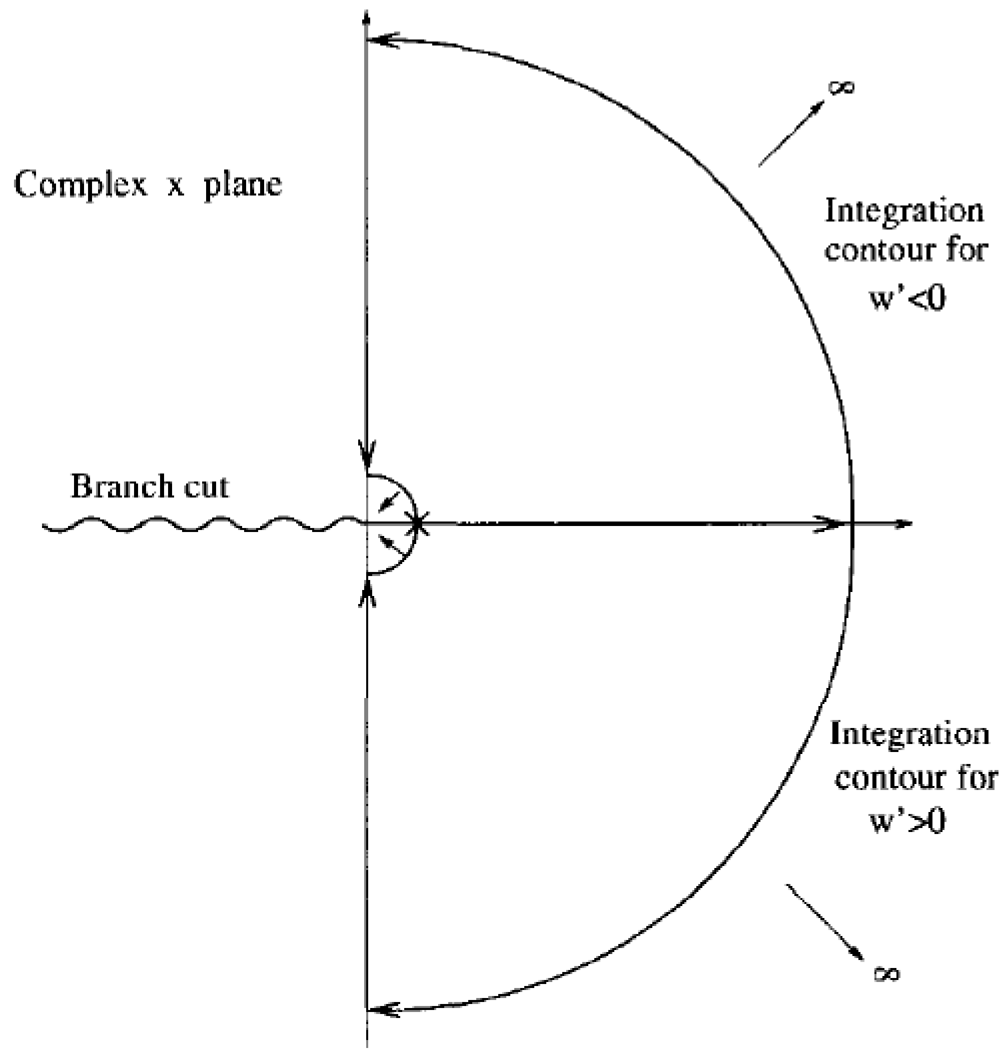

Figure 2.3: Contour Integrals for computing the Bogoliubov coefficients [32]

Here the opposite is true if the case of where it is analytic when $\omega' > 0$

$$I\left(\omega' < 0\right) = i\int_{0}^{+\infty} dz \exp(\omega') z \frac{\sin \epsilon L_z/2}{L_z} \exp(iL_z\omega_j). \tag{2.94}$$

The above integrals then give the following relation

$$I\left(\omega' > 0\right) = -\exp(4\pi M\omega_2) I\left(\omega' < 0\right). \tag{2.95}$$

Thus

$$\alpha_{jn,\omega'} = -\exp(4\pi M w_j) \exp 2i\omega' v_H \beta_{jn,\omega'}. \tag{2.96}$$

Giving us the important relation

$$\left|\alpha_{jn,t\omega'}\right| = \exp(4\pi M\omega_j) \left|\beta_{jn,\omega'}\right|. \tag{2.97}$$

2.6 Hawking Temperature

Using the relation calculated in (2.97) we get finally solve for $\langle in \left| N_w^{out} \right| in \rangle$

$$\int_{0}^{+\infty} d\omega' \left(\alpha_{jn,\omega'} \alpha^{*}_{j'n',\omega'} - \beta_{jn,\omega'} \beta^{*}_{j'n',\omega'}\right) = \delta_{jj'} \delta_{nn'}. \tag{2.98}$$

$$\int_{0}^{+\infty} d\omega' \left(\left|\alpha_{jn,\omega'}\right|^2 - \left|\beta_{jn,\omega'}\right|^2\right) = 1. \tag{2.99}$$

Using the relationship calculated in (2.97) we get

$$(\exp(8\pi M w_j) - 1) \int_0^{+\infty} d\omega' \left|\beta_{jn,\omega'}\right|^2 = 1. \tag{2.100}$$

Thus we get the Black body spectrum for the Hawking radiated particles

$$\left\langle in \left| N_{jn}^{out} \right| in \right\rangle = \int_0^{+\infty} d\omega' \left|\beta_{jn,\omega'}\right|^2 = \frac{1}{\exp(8\pi M \omega_j - 1)}. \tag{2.101}$$

Comparing with the equation for a Black body

$$\frac{1}{e^{\hbar\omega_j/k_B T} - 1}, \tag{2.102}$$

thus by comparing (2.101) and (2.102) we get the corresponding Hawking temperature

$$T_H = \frac{\hbar}{8\pi k_B M}. \tag{2.103}$$

Chapter 3

Nice slices of the Black hole Geometry

Quantum field theory is a local theory. As we saw in the previous chapter, considering Local Quantum field theory outside a black hole leads to to modes being radiated away and modes falling inside the black hole. We call this the Hawking process. The most complete theory, which has been experimentally verified to a great extent, is the Standard Model. The problem is that it doesn't include gravity. When we try to Quantize gravity we run in to the issue of non-renormalizability. Therefore, there exists no complete theory of gravity which has great support from experiments. So while doing physics we have to be careful so that we do not pick up Quantum effects of gravity. Mathur in [3] introduced something called the "Solar System Physics" or the "Solar System Limit". It is nothing but the limit upto which we can neglect the effects of Quantum Gravity - a theory of everything. As we do not have a Quantum theory of Gravity (well string theory is a candidate but it runs into some controversies regarding experimental verification and not having a horizon in case of a black hole), we have to know a limit up to which we can neglect Quantum Gravity effects and do our day to day physics. We also expect Quantum Gravity effects in the "Solar System Physics" to appear but it is expected to be negligibly small. Mathur [3] goes on to present a "Niceness Condition" N which keeps us in solar system limit ignoring Quantum Gravity effects. He also presents the evolution of such slices and introduces the notion of pair production due to the distortion of such slices. The pair production occuring at the horizon leads to the notion of Hawking radiation. With the assumptions N (Niceness Condition), he presents the famous information paradox as the "Hawking Theorem".

As our work is closely related to the conclusion reached by Mathur, we present the Niceness Conditions and nice slices of the Black hole geometry, the Hawking theorem of [3]. We also try to get to the core issue of the information paradox in the next chapter.

3.1 The Nice Slices of Mathur and the case of Black hole Slices

Sadly as Quantum Field Theory fails at the Planck scale, and we expect the Quantum Gravity effects to be very dominant then. So the nice slices should have properties which would let us avoid Planck scale physics efficiently. The following properties keeps us in the semi-classical domain (little to no Quantum Gravity effects):

1. We take a space like slice and define our quantum state on that slice. The intrinsic and extrinsic curvature of the slices should be very much less than Planck scale ($\ll \frac{1}{l_p^2}$) where l_p is the Planck length. Also, the 4-curvature of the spacetime near the slices should be small compared to Planck scale.

2. The matter on the slices should not have ultra-Planckian properties. The wavelength of any quanta on the slices should have wavelength longer than Planck length ($\lambda \ll l_p^2$). Also energy density should be very small compared to Planck scale ($\ll l_p^{-4}$).

3. Evolution to future slices should also fulfill the criteria i.e. the matter on the slices and the slices themselves should have properties which are in the domain of semi-classical physics.

The evolution of the slices will lead to some interesting properties. As the vacuum state on one slice won't necessarily be vacuum on the next slices, the previous slices will see pairs pop-out in the next slices. The state of the created pairs are of the form,

$$|\Psi\rangle_{pair} = Ce^{r\hat{c}^+\hat{b}^+} |0\rangle_c |0\rangle_b \,, \tag{3.1}$$

where, in case of Black holes, b and c quanta denote hawking radiated partner and infalling partner. Here, r is a number of order unity. The equation above can also be written in the form,

$$|\Psi\rangle_{pair} = \frac{1}{\sqrt{2}}(|0\rangle_c |0\rangle_b + |1\rangle_c |1\rangle_b). \tag{3.2}$$

which we will use throughout our work because it captures the essence of entanglement beautifully.

In the next section, we will talk about slicing the Black hole geometry and evolution of the nice slices in the Black hole case.

3.2 Nice Slices of the black hole geometry

The full quantum state of each slices in case of black holes in written as

$$|\Psi\rangle = |\Psi\rangle_M \otimes \frac{1}{\sqrt{2}}(|0\rangle_c |0\rangle_b + |1\rangle_c |1\rangle_b). \tag{3.3}$$

As described in the first section the Schwarzschild metric is of the form:

$$ds^2 = -(-1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2dr^2. \tag{3.4}$$

It can easily be noticed that as we go from r>2M to r<2M spatial part and temporal part which switch signs. So, we have to be careful about this when we slice the black hole geometry. Since, black holes have a singularity at r=0 and if this singularity intersects our slices then our niceness conditions would be violated. Our slices should all follow the niceness condition so that we can get to the conclusion of Hawking radiation as derived in the previous chapter. Now coming to the issue of slices, we divide our slices in 3 parts.

Part 1: As we want a spacelike slice, for r>4M, we consider our slices to be t = constant.If time is oriented along the y-axis and space along the x-axis, our spacelike slice would imply a straight line equation of the form $y = constant$.

Part 2: When we are inside the black hole i.e. r<2M, in order to make the slices spacelike, we have to take r = constant piece rather than t = constant. If time is y-axis and space x-axis then this would imply a straight line equation of the form $x = constant$.The spacelike slices inside a black hole is opposite to the of the outside. This is due to switching of character of space and time inside the horizon of a black hole. Another problem that pops up inside a Black hole is the singularity issue. Our slices cannot intersect or come close to r=0. Hence, to avoid the singularity, we can restrict our slices to be in the bound $\frac{M}{2} < r < \frac{3M}{2}$ and thus avoid the signature problem at the horizon as well.

Part 3: We join the slices of part 1 and 2 by the connector region. The connector also satisfied the niceness condition.

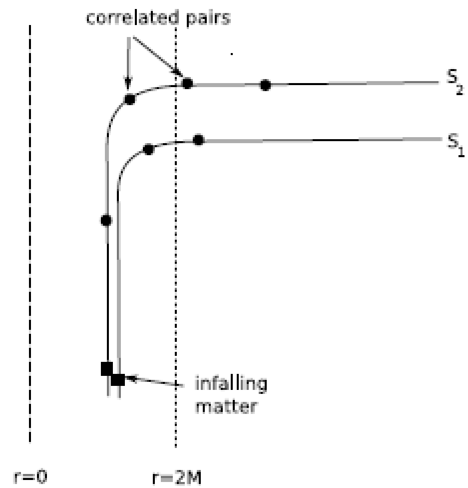

Figure 3.1: Evolution of slices in black hole gemetry [3]

3.3 Evolution of the slices and Hawking radiation

The three parts described in the previous section evolve as follows:

Part 1: The evolution of r>4M part is straight forward. As this is a space like slice, we evolve the slice in time. So, this part evolves like $t = t_1 + \Delta$. An example of this could be the evolution of our straight line from $y = constant$ to some y greater than the constant.

Part 2: The r = constant parts' evolution is something we should be careful about as it is evolving inside the black hole. The time evolution of this piece is in the decreasing r direction because something inside the black hole must hit the singularity, otherwise violation of causality would occur. It is evolved as $r = r_1 + \Delta$ where $S_1 \ll M$. This is strictly because we don't want to be in the vicinity of r = 0.

Part 3: Part 1 and 2 are joined by the connector again. New pairs are being produced in the connector region and the previous pairs on the slices are pushed further away on the slices as seen from the figure. This beautifully captures the essence of Hawking radiation of the previous chapter in terms of particles.

Moreover, the figure (3.1) also gives us an intuitive picture of Hawking radiation. The quantas that are labelled as "b" are the Hawking radiated partners. The quanta labelled "c" are the infalling partners. The b quantas constitute the Hawking radiation and has the temperature and properties as found out from the original Hawking calculation presented in the previous chapter. The famous "Information Paradox" which occurs due to Hawking radiation is presented in the following chapter.

Chapter 4

The Information Paradox

Unresolved for long 45 years, the Information Paradox could offer us a deep insight into every physicists dreams theory "The Theory of Quantum Gravity". Hawking radiation described in the previous chapters lead to a paradox and provides another basis for the conflict between Einstein's General Relativity and Quantum Mechanics.Since, considering classical gravitational properties in a quantum framework leads to a violation of quantum mechanics, this is a tricky paradox to solve.

The first thing we need to do is to understand what the paradox actually is. Unitarity of Quantum Mechanics is one of the beautiful features of the theory. But when we consider the evolution of quantum fields outside a black hole, we observe a subtle violation of Unitarity of Quantum mechanics. Unitarity says that a pure state must evolve to a pure state which does not occur in the case of black holes. We explore the problem in detail at the end of chapter. There is also the issue of information loss which is closely related to unitarity. In a very sloppy language the information which falls into the black hole is "lost" forever and that is basically the information paradox. One might point out that information loss is a basic feature of the universe ranging from burning papers to human life itself. Take the example of a burning paper. The issue that would be raised is that when a paper is completely burnt the information is completely lost. Another issue that may be raised in a more formal language is that a burning paper burns its atoms and it becomes "entangled" with the radiation or radiated photons so how does the entanglement entropy goes down as the paper is completely burnt. So is unitarity violated when a paper burns? The answer is a big NO. The answer to the question of information loss would be that the information about the paper is encoded in the radiated photons and the ash of the paper. The answer to the second question is that as the paper burns initially, the entanglement entropy goes up (Page nicely showed this in [12]) and after certain time (Page time), the entanglement entropy decreases to zero. This happens as the radiated products overall form a pure state and has all the information about the original paper that made them exist. Thus a pure state evolves to a pure state and unitarity is saved in everyday circumstances. Black holes love to contradict the statement in semi-classical picture and thus force us to develop a quantum theory of gravity.

4.1 Entanglement entropy and the essence of the paradox

Entanglement entropy tells us how systems or states are correlated with each other or how much entangled they are. The entanglement entropy is an important tool because we want unitary evolution of Quantum states and hence entanglement entropy will let us see which system evolved into pure and mixed states and saved unitarity. In order to calculate the entanglement entropy we need to first find something called the density matrix.We will explore these concepts in this section and calculate the density matrix and entanglement entropy of a bell pair representing a vacuum state outside the event horizon of a black hole. As mentioned earlier bell pair state offers us amazing look into the black hole information paradox, we represent the vacuum state outside the event horizon by bell pair states.The quantum state of a black hole can be written as

$$|\Psi\rangle_L = |\Psi\rangle_M \otimes \frac{1}{\sqrt{2}}(|0\rangle_{c_1} |0\rangle_{b_1} + |1\rangle_{c_1} |1\rangle_{b_1}).$$

Now, in order to calculate the entanglement entropy of the b quantas with (M, C_1) we first need to compute the density matrix and take partial trace over the b quanta. Taking partial would tell us how b quantas are correlated with (M, C_1). A subscript L is given as we want to denote it as the leading order Hawking state or the result we would obtain from Hawking's original calculation.

The density matrix ρ is defined as

$$\rho = |\Psi\rangle\langle\Psi| \,. \tag{4.1}$$

$$\begin{aligned} S_{entanglement} &= -tr_b(\rho \ln \rho) \\ &= -tr_b(|\Psi\rangle_L \langle\Psi|_L \ln |\Psi\rangle_L \langle\Psi|_L). \end{aligned} \tag{4.2}$$

Let's denote $|0\rangle$ and $|1\rangle$ state by,

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|0\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{aligned} Sent_{(i)} &= -tr(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \ln(\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|) \\ &= -tr[\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \ln\frac{1}{2} & 0 \\ 0 & \ln\frac{1}{2} \end{pmatrix}] \\ &= -tr \begin{pmatrix} \frac{1}{2}\ln\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\ln\frac{1}{2} \end{pmatrix} \\ &= -\ln\frac{1}{2} \\ &= \ln 2 \end{aligned}$$

The entanglement entropy after second pair emission (the state would be something like):

$$|\Psi\rangle_L = |\Psi\rangle_M \otimes \frac{1}{\sqrt{2}}(|0\rangle_{c_1}|0\rangle_{b_1} + |1\rangle_{c_1}|1\rangle_{b_1}) \otimes \frac{1}{\sqrt{2}}(|0\rangle_{c_1}|0\rangle_{b_1} + |1\rangle_{c_1}|1\rangle_{b_1})$$

.

After N such steps the entanglement entropy thus stands

$$S_{ent(N)} = N \ln 2$$

.

Now, if the black hole, for example, has evaporated after N steps, then the b quantas are highly entangled with basically nothing. But we run into difficulty much earlier. As particles fall into black hole, the Black hole loses its mass and after enough particles have fallen inside the black hole, it becomes planck sized and our niceness condition would not hold and we can not evolve the slice further. Thus, if we consider the black hole has evaporated completely, then the nice slices leads us to the violation of quantum mechanics and if we consider the black hole lives being planck sized then we are lead to the "Remnant Scenario". We will explain both scenarios which will help to understand what the information paradox actually is.

4.2 Unitarity issue and violation of Quantum Mechanics

Operators in Quantum Mechanics are unitary. So, for Quantum Mechanics to hold true we would require the states to be unitary. Time evolution operator in Quantum mechanics requires a state $|\psi\rangle$ to evolve to a pure state if $|\psi\rangle$ cause a pure state initially.A pure state must evolve to a pure state and the same is true for mixed states as well.

Now, coming to the case of Hawking radiation, we notice a subtle violation of unitarrity. The state at the horizon is

$|\Psi\rangle_L = |\Psi\rangle_M \otimes \frac{1}{\sqrt{2}}(|0\rangle_{c_1} |0\rangle_{b_1} + |1\rangle_{c_1} |1\rangle_{b_1}),$
which is a pure state. Once the "c" quantas fall into the Black hole, all we are left with are the "b" quantas. The "b" quantas are highly entangled with the "c" quantas and thus are mixed states. Therefore, the time evolution of $|\psi\rangle$ would give us a mixed state even though $|\psi\rangle$ was a pure state initially. If the Black hole evaporates in N steps we would have "N" mixed states. Thus saving the sacred life of Quantum Mechanics is a necessity by solving the Black hole information paradox.

Another thing that needs to be mentioned is the "missing information" problem. Conservation of Quantum information is a fundamental requirement of Quantum mechanics. A good example of information loss can be understood by considering 4 states $|W\rangle$,$|X\rangle$, $|Y\rangle$, $|Z\rangle$. If $|W\rangle$ and $|X\rangle$ leads to $|Y\rangle$ and $|Z\rangle$, then we have no information loss. But if states $|X\rangle$, $|Y\rangle$ both evolve to state $|Z\rangle$, we do not have certain information which state ($|X\rangle$ or $|Y\rangle$) is its past. The future state is deterministic but not the past. Here we have information loss. Same occurs for Black holes. The whole state was $|\psi\rangle_M$ and $|\psi\rangle_{b,c}$ which after Black hole has evaporated evolves to a state composed of only b quanta. Hence, we lose information of the initial Black hole state $|\psi\rangle_M$ or the vacuum state $|\psi\rangle_{b,c}$.

4.3 Remnants

The nice slices we considered above stops evolving when the Black hole becomes Planck sized. The Black hole, when Planck sized, is called a remnant. Two definition of a Black hole remnant is presented here. The first one is given by Mathur in [2].

<u>Definition 1</u>: Remnants are objects whose mass and and size are less that given bounds ($m < m_{remnant}$ and $l < l_{remnant}$) but are highly entangled with systems far away from the object.

The second definition is from [18] and is as follows :

<u>Definition 2</u>: A remnant is a localized late stage of a black hole under Hawking evaporation, which is either (i) absolutely stable, or (ii) long-lived.

Remnants,unlike the violation of unitarity, is not a violation of Quantum Mechanics. Remnants are actually "unwanted stuffs" because we expect a finite number of states in a finitely bound mass and energy state. Remnants also leads to loop divergences because of the infinite number of possible objects circulating in the loop.The cases of remnants aren't really popular with people dealing with the information paradox. Notice that in the second definition remnants are considered either stable or long lived. Long lived remnants can also be considered stable as the evaporation process of such remnants are much longer than the evaporation time of a black hole. Again, the notion of space and time isn't really clear at that scale and thus a horizon may not be present in case of remnant. Many proposals in favour of remnants are presented in [18].Some popular proposals presented in [19],[20],[21],[22] suggest remnants to be some sort of elementary particles.

The penrose diagram below shows that the remnant is completely stable and has an infinite lifetime:

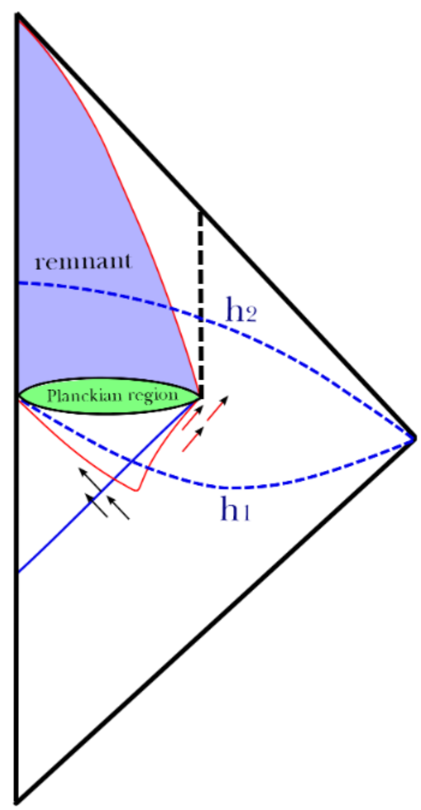

Figure 4.1: Penrose diagram for long lived remnants [18]

4.4 Possible resolutions and the Page Curve

The widely accepted theory of Quantum gravity we have is String theory. String theorists have an issue with the leading order hawking computations[11]. The solution that pops out of String theory is the "Fuzzball solution". Fuzzball solution do not have regular horizons. It basically consists of energy microstates and counting these microstate would give us the Bekenstein entropy. The absence of a horizon is what the main criticism comes up. Even though you can count microstates but still you run into the "infall problem" [9], [10].

In 1997 Juan Maldacena released his infamous paper [33] laying the ground works for the AdS/CFT correspondence principle. He showed that the formulation of quantum gravity in String Theory and M-Theory is dual to various Quantum Field Theories or Conformal Fied Theories. In other words, in AdS/CFT the quantum gravity theory/ bulk is mapped to conformal field theories that exist in the boundaries. Black holes in AdS should be unitary as it is dual to a CFT. But this is not the case. A nice argument in [3] shows why the Ads/CFT correspondence cannot get away with the paradox.

The main problem of the information paradox is that the initial wavefunction disappears from the final configurations of the system. This is troubling as we want unitary evolution and this surely violates that. For unitary evolution, we would expect a behaviour similar to the burning paper where the entanglement entropy goes up initially but reduces to zero after Page time. Page presented some interesting ideas about this Page time and Page Curve in [12],[13],[14],[15]. Unitary evolution requires a development like the following figure:

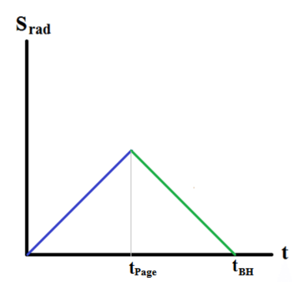

Figure 4.2: Page curve

Most of the physicists today more or less agree on the fact that unitarity must be saved and in order to do so information must come out non-locally. In the next section we deal with non-locality and see how a possible resolution can be formulated.

Chapter 5

Non-Locality

If we want the Hawking radiated quantity carry information about the black hole then information must be carried off non-locally. The leading contender for information coming out of black holes is non-locality. To understand what non-locality information transfer actually is we must clearly understand the concept of locality.

Locality in special relativity is simple to understand. Consider two points that are casually disconnected(information transmission be faster than light speed). If they are to make any contact they must send information faster than the speed of light. We see in the figure below that two points A and B are causally disconnected and to actually make a contact information must be transmitted faster that the speed of light thus violating causality.

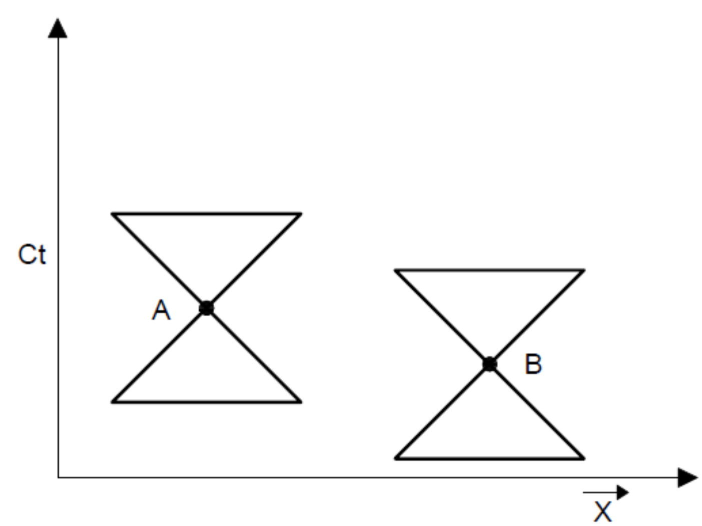

Figure 5.1: Two Causally disconnected points A and B

Locality is a key feature of Einstein's relativistic theories. It is a key requirement from causality that locality holds. In quantum field theory operators at two different distant points in a field commute because one point does not have effect on other point. Two distant points cannot influence each other because they are causally disconnected. Now, let us turn to the case of black holes. If a pair production occurs at the horizon one particle falls inside the horizon and the other partner is radiated away. This is basically what Hawking radiation actually is. The pairs that are radiated away carry information only about their infalling partners. This is the leading order Hawking state and is a result Hawking obtained(Chapter 2). This is where locality is preserved. But if we have a case where a Hawking radiated quanta or b quanta carries some information about the black hole mass state then this is a non-local information transfer. We call this non-local because of a very specific reason we know that time and space switch character inside a black hole. So once inside a black hole,your future is directed towards the singularity r=0. If anyone wants to come out of a black hole, he/she needs to violate causality and travel faster than the speed of light. Therefore, if a b quanta did carry information about the mass state of the black hole, information must have been provided violating locality. This is a non-local information transfer. A simple non-local game will make us a little more confused about non-locality and causality. Consider three observers Mishaal(denoted by M), Tasnuva(denoted by T) and Mehdi(denoted by M'). They are all equipped with lasers. Mishaal is causally connected to Tasnuva and Tasnuva is causally connected to Mehdi but Mishaal and Mehdi are causally disconnected.

Even though Mishaal and Mehdi are causally disconnected, they can in principle communicate through Tasnuva. Thus causality is saved ! Something similar could happen to the black hole case and information retreival might be possible. Even if a b quanta carries information about previous infalling pairs c, it's still a non-local information transfer because the previous c quantas are already inside the hope. This non-local information transfer can not decrease the entanglement entropy which we will see in the next chapter.

Giddings in [23],[24],[25],[26],[27],[28] suggested that non-locality is simply an intrinsic feature of strong gravity. Giddings, who has by far done the most work on non-locality explains the non-locality effect beautifully in [23]. Locality in QFT can be captured by understanding the fact that if a spacelike slice is divided into two non-overlapping regions of space then we can say that observables commute at spacelike separations.

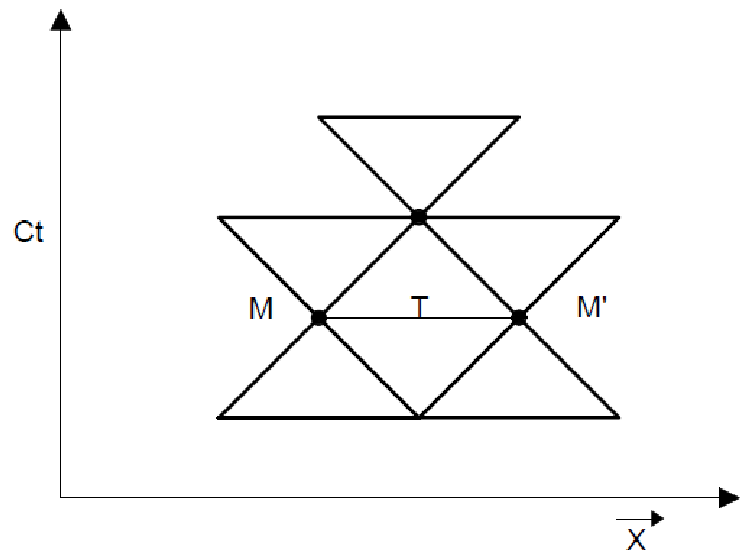

Figure 5.2: Lightcone of M, $M^{'}$ and T

Since, QFT fails at Planck scale locality could be violated at these circumstances. We know that in QFT particles are created or annihilated and if one tries to measure the two particles of large energies is a small region then the back reaction of this deforms metric and also the causal structure of that small region. In such cases, a clear description of locality and Giddings and his collaborators advocates that there is no local description of that situation. He also investigated the roles of gravitational and string non-localities in high energy scattering [29]. Giddings also presented a generalized uncertainty principle where certain circumstances can be found where classical variables must not hold and Quantum wave fuction is the only hole. The locality bound in D-dimensions is given by

$$|x - y|^{D-3} \geq G_D |p + q|, \tag{5.1}$$

where positions are denoted by x and y and p, q denotes momenta.

5.1 Small non-local corrections to the leading order Hawking state

The leading order Hawking state is

$$S^{(1)} = \frac{1}{\sqrt{2}}(|0\rangle_{c_{n+1}} |0\rangle_{b_{n+1}} + |1\rangle_{c_{n+1}} |1\rangle_{b_{n+1}}). \tag{5.2}$$

We saw that if particles are produced in $S^{(1)}$ state only then the entanglement entropy after N steps is Nln2. The question now stands if small correction to the $S^{(1)}$ state above could fix the paradox and cause the entanglement entropy to go down. Therefore, can we accumulate large number of small corrections and thus cause the entanglement entropy to be 0 essentially solving the paradox? Lets try a bit. Considering a second state of the form

$$S^{(2)} = \frac{1}{\sqrt{2}}(|0\rangle_{c_{n+1}} |0\rangle_{b_{n+1}} - |1\rangle_{c_{n+1}} |1\rangle_{b_{n+1}}). \tag{5.3}$$

At timestep t_n the complete state of the system can be denoted by

$$|\Psi_{M,c}, \psi_b(t_n)\rangle = \sum_{m,n} C_{mn}\psi_m\chi_n. \tag{5.4}$$

Here,

- ψ_n is the basis of orthonormal states for the mass of the black hole M and the infalling Hawking pairs i.e the c quanta.
- χ_n is the basis of Hawking radiated quantas b.

By a unitary transformation the equation (5.4) can be written as

$$|\Psi_{M,c}, \psi_b(t_n)\rangle = \sum_i C_i \psi_i \chi_i. \tag{5.5}$$

Mathur considered his b quantas to be free of any further interactions. This is considered because once the b quantas have left the vicinity of the black hole, they cannot be influenced by the matter inside the hole without violation of locality. Thus any corrections to the leading order Hawking state can be termed as non-local information transfer as we see in this section.

Therefore,

$$\chi_i \longrightarrow \chi_i \tag{5.6}$$

$$\psi_i \longrightarrow \psi_i^{(1)} S^{(1)} + \psi_i^{(2)} S^{(2)}. \tag{5.7}$$

Here, $S^{(i)}$ denotes the state of the newly created pairs and ψ_i denotes the state of the M and c_i quantas. Unitary evolition requires

$$||\psi_i^{(1)}||^2 + ||\psi_i^{(2)}||^2 = 1. \tag{5.8}$$

This comes from normalization i.e $\langle\psi|\psi\rangle = 1$ and since the states $S^{(1)}$ and since the states $S^{(1)}$ and $S^{(2)}$ are orthonormal. In the leading order Hawking state we had

$$\psi_i^{(1)} = \psi_i, \psi_i^{(2)} = 0.$$

Since we are considering evolution in both the $S^{(1)}$ and $S^{(2)}$ states i.e new pairs are being created in both $S^{(1)}$ and $S^{(2)}$ state. At t_{n+1} the state can be written as

$$\begin{aligned}|\Psi_{M,c},\psi_b(t_n)\rangle &= \sum_i C_i[\psi_i^{(1)}S^{(1)} + \psi_i^{(2)}S^{(2)}]\chi_i \\ &= \sum_i C_i\psi_i^{(1)}S^{(1)}\chi_i + \sum_i C_i\psi_i^{(2)}S^{(2)}\chi_i \\ &= S^{(1)}\Lambda^{(1)} + S^{(2)}\Lambda^{(2)}.\end{aligned} \tag{5.9}$$

Here,$\Lambda^{(1)}$ and $\Lambda^{(2)}$ are defined by

$$\Lambda^{(1)} = \sum iC_i\psi_i^{(1)}\chi_i. \tag{5.10}$$

$$\Lambda^{(2)} = \sum_i C_i\psi_i^{(2)}\chi_i. \tag{5.11}$$

Now,

$$\begin{aligned}\langle\Psi_{M,c},\psi_b(t_n)|\Psi_{M,c},\psi_b(t_n)\rangle = 1 \\ ||\Lambda^{(1)}||^2 + ||\Lambda^{(2)}||^2 = 1.\end{aligned} \tag{5.12}$$

Mathur in [3] called the corrections small if

$$||\Lambda^{(2)}|| < \epsilon, \epsilon << 1. \tag{5.13}$$

He considered this because the new pairs are mostly created in $S^{(1)}$ i.e in the leading order Hawking state. This is exactly what small corrections would imply. If there is no such bound then the corrections are called order unity. Now comes the case of non-locality. Mathur developed some non-local models having small corrections as above in [9] and [10]. This corrections are non-local corrections to the leading order state or in Mathur's wording "Space-time somehow remembers the previous emissions" which is a non-local corrections as remembering previous emission would imply the violation of locality as "remembering" would have to be done by information provided from inside the hole.

5.2 Entropy bounds

Mathur[3] presented a bound on the entanglement entropy which was further generalized by Mahbub and others in their paper [4]. Giddings in his paper [27] also put forward the idea that information can not escape via bell pair states(Appendix A). In the first section, we present Mathur's bound which was put forward in [2] and in subsequent sub-sections discuss the other two bounds and present our proposal for avoiding such bound.

5.2.1 Mathur's bound

Before going into the discussion on the bounds discussing some basic concepts of subsystems and entanglement between subsystems will help us understand the approach to the paradox better. Let's consider three subsystems A, B and C. A, B and C subsystems are part of a system S. The entanglement entropy of any system is given by

$$S \equiv -tr(\rho \ln \rho). \tag{5.14}$$

Now,$S(A) = -tr_A(\rho_A \ln \rho_A)$ would give us the entanglement of a A with combined subsystem B and C (BUC). Here, ρ_A is the density matrix representing subsystem A. Again $S(A+C) = -tr_{AC}(\rho_{AC} \ln \rho_{AC})$ gives the entanglement of subsystem (A+C) with B.

The section on small correlations to the leading order Hawking state mentioned that the small correlations do not solve the paradox and the entanglement entropy still rises.

Mathur in [2] proved using three lemmas and a theorem that despite small correlations the entanglement entropy rises by $\ln 2 - 2$ in each step. In this section we will look into the three lemmas and the theorem of [2]. Before going into the lemmas and theorem, lets look into the subsystem of the black hole system:

1. The first subsystem that Mathur considered is the subsystem (M,c). This is basically the subsystem denoting the interior of a black hole. It basically consists of the mass M of the hole and infalling Hawking pair c. Mathur considered that new pairs that are being created at time step t_{n+1} can weakly interact with (M,c).

2. The second subsystem is the one which has the set of all Hawking radiated quantas i.e. the b quantas. He considered that the b quantas emitted at earlier steps do not influence the pair production.

3. The third subsystem is the new pair that will be created at timestep t_{n+1}. We will use the notation mathur used throughout our work i.e. $p \equiv (c_{n+1}, b_{n+1})$.

Important points to note here is that the b quantas do not influence the pair production further. This has an important implication while deriving Mathur's bound and in our later works. Here the entanglement entropy after n steps is $S_b = S_0$.

Now, we look into the three lemmas of [3].

<u>Lemma 1:</u> If the correlations to the leading order Hawking state is considered small then,

$$S_P \equiv -tr(\rho_P \ln \rho_P)., \tag{5.15}$$

where $P \equiv (c_{n+1}, b_{n+1})$ and ϵ is a small number i.e $\epsilon << 1$. This has an important implication. It says that the new pairs are almost maximally entangled.

<u>Proof:</u> The density matrix P_p can be written as

$$\rho_P = \begin{pmatrix} \langle \Lambda^{(1)} | \Lambda^{(1)} \rangle & \langle \Lambda^{(1)} | \Lambda^{(2)} \rangle \\ \langle \Lambda^{(2)} | \Lambda^{(1)} \rangle & \langle \Lambda^{(2)} | \Lambda^{(2)} \rangle \end{pmatrix}. \tag{5.16}$$

Here since correlations are small,

$$
\begin{aligned}
||\Lambda^{(2)}||^2 &= \langle\Lambda^{(2)}|\Lambda^{(2)}\rangle \\
&\equiv \epsilon_1^2 < \epsilon.
\end{aligned} \tag{5.17}
$$

$$
\begin{aligned}
||\Lambda^{(1)}||^2 &= \langle\Lambda^{(1)}|\Lambda^{(1)}\rangle \\
&= 1 - \epsilon_1^2 < \epsilon.
\end{aligned} \tag{5.18}
$$

$$
\langle\Lambda^{(1)}|\Lambda^{(2)}\rangle = \epsilon_2 < \epsilon. \tag{5.19}
$$

$$
\rho_P = \begin{pmatrix} 1-\epsilon_1^2 & \epsilon_2 \\ \epsilon_2 & \epsilon_1^2 \end{pmatrix}. \tag{5.20}
$$

$$
\begin{aligned}
S_P &= -tr(\rho_P \ln \rho_P) \\
&= -tr\left[\begin{pmatrix} 1-\epsilon_1^2 & \epsilon_2 \\ \epsilon_2 & \epsilon_1^2 \end{pmatrix} \begin{pmatrix} \ln(1-\epsilon_1^2) & \ln(\epsilon_2) \\ \ln(\epsilon_2) & \ln(\epsilon_1^2) \end{pmatrix} \right] \\
&= -tr\left[\begin{pmatrix} (1-\epsilon_1^2)\ln(1-\epsilon_1^2) + \epsilon_2\ln(\epsilon_2) & (1-\epsilon_1^2)ln(\epsilon_2) + \epsilon_2\ln(\epsilon_1^2) \\ \epsilon_2\ln(\epsilon_2) & \epsilon_1^2\ln(\epsilon_1^2) \end{pmatrix} \right].
\end{aligned} \tag{5.21}
$$

Here, $\epsilon_1^2 < \epsilon$ and $\epsilon_2 < \epsilon$ and the ln(1+x) expansion is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... \tag{5.22}$$

So, if x is ϵ and is a very small term then the higher order terms are far neglible and far smaller. Since, trace is a conserved quantity(diagonalization also preserves trace) and each term of the matrix formed by $(\rho_p \ln \rho_p)$ is less than ϵ i.e $S_p < \epsilon$.

Lemma 2: The second lemma states,

$$S(b+p) \geq S_0 - \epsilon. \tag{5.23}$$

This means the entanglement of the union of subsystem b and P and the (M,c) quantas is greater than the entanglement entropy at timestep t_n minus some small number ϵ.
Proof: This can be easily proved using the sub-additivity theorem of the entropy of two systems A, B i.e.

$$S(A+B) \geq |S(A) - S(B)|. \tag{5.24}$$

Here,A=b and B=p implies

$$S(b+p) \geq S_0 - \epsilon. \tag{5.25}$$

Lemma 3: The third Lemma is

$$S(c_{n+1}) > \ln 2 - \epsilon. \tag{5.26}$$

This implies that small corrections to the leading order Hawking state decreases the entanglement entropy by a small amount ϵ.

Proof: The complete state with leading order Hawking state and a second state as mentioned in the previous section can be written as

$$|\Psi_{M,c},\psi_b(t_{n+1})\rangle = \left[|0\rangle_{c_{n+1}}|0\rangle_{b_{n+1}}\frac{1}{\sqrt{2}}\left(\Lambda^{(1)}+\Lambda^{(2)}\right)\right] + \left[|1\rangle_{c_{n+1}}|1\rangle_{b_{n+1}}\frac{1}{\sqrt{2}}\left(\Lambda^{(1)}-\Lambda^{(2)}\right)\right]. \tag{5.27}$$

The density Matrix of c_{n+1} therefore becomes

$$\rho_{c_{n+1}} = \begin{pmatrix} \frac{1}{2}\left\langle\left(\Lambda^{(1)}+\Lambda^{(2)}\right)|\left(\Lambda^{(1)}+\Lambda^{(2)}\right)\right\rangle & 0 \\ 0 & \frac{1}{2}\left\langle\left(\Lambda^{(1)}-\Lambda^{(2)}\right)|\left(\Lambda^{(1)}-\Lambda^{(2)}\right)\right\rangle \end{pmatrix}. \tag{5.28}$$

Using $\left\langle\Lambda^{(2)}|\Lambda^{(2)}\right\rangle = \epsilon_2^2$, $\left\langle\Lambda^{(1)}|\Lambda^{(1)}\right\rangle = 1-\epsilon_2^2$, we get

$$\rho_{c_{n+1}} = \frac{1}{2}I + \begin{pmatrix} \mathrm{Re}\left\langle\Lambda^{(1)}|\Lambda^{(2)}\right\rangle & 0 \\ 0 & -\mathrm{Re}\left\langle\Lambda^{(1)}|\Lambda^{(2)}\right\rangle \end{pmatrix} + O\left(\epsilon^2\right). \tag{5.29}$$

With the density matrix at our disposal we can calculate the entanglement entropy and thus prove the lemma.

$$S\left(c_{n+1}\right) = \mathrm{ln}2 - 2\left[\mathrm{Re}\left(\left\langle\Lambda^{(1)}|\Lambda^{(2)}\right\rangle\right)\right]^2 \geq \mathrm{ln}2 - 2\epsilon^2 + O\left(\epsilon^3\right) > \mathrm{ln}2 - \epsilon. \tag{5.30}$$

The three lemmas will now enable us to prove an important theorem.

Theorem: The entanglement entropy of the b quanta with the hole is S_0. At any timestep t_{n+1} the new pairs deviate from the leading order Hawking state by an amount ϵ which is very small. If this small correlation holds then the entanglement entropy of the emitted quantas b and b_{n+1} at timestep t_{n+1} will satisfy $S(b+b_{n+1}) > S_0 + \ln 2 - 2\epsilon$ This implies that the entanglement entropy rises each emission even if small deviation of the leading order Hawking state are considered.

Proof:

The proof of the theorem is simple and can be derived from the strong sub-additivity theorem which relates the entropy of three systems. The strong sub-additivity theorem of three systems A, B and C can be written as

$$S(A+B) + S(B+C) >= S(A) + S(C). \tag{5.31}$$

Setting $A = b$, $B = b_{n+1}$ and $C = C_{n+1}$ gives us

$$S(b + b_{n+1}) + S(b_{n+1} + c_{n+1}) \geq S(b) + S(C_{n+1}). \tag{5.32}$$

We defined $Sb = S_0$ and from the lemmas we had $S_p < \epsilon$ and $S(c_{n+1}) > \ln 2 - \epsilon$. From this we get

$$S(b + b_{n+1}) > S_0 + \ln 2 - 2. \tag{5.33}$$

This is the important result that small correlations leads us to. If we assume the nice slices hold true and a smooth horizon then small correlations do not fix the paradox.

5.2.2 Generalized Mathurs' bound

Mahbub and others in [4] generalized Mathur's bound which was $\Delta S \geq \ln 2 - 2\epsilon$. They considered a state of the form

$$|\Psi\rangle_{n+1} = \sum_{i=0}^{2^n-1} a_i |i\rangle_b |i\rangle_c \otimes \frac{1}{\sqrt{2}}(\exp(S_{i,n,0}) |0\rangle_b |0\rangle_c + \exp(S_{i,n,1}) |1\rangle_b |1\rangle_c \tag{5.34}$$

Here $|i\rangle_c$ and $|i\rangle_b$ denote the state of the n ingoing and n outgoing quanta respectively. The term $\frac{1}{\sqrt{2}}\exp(S_{i,n,j})$ in front of the $|0\rangle$ and $|1\rangle$ states give us the amplitude to observe the new pair in the state $|j\rangle_b |j\rangle_c$ if the previous pairs we given by the states $|i\rangle_b |i\rangle_c$. If we consider small correlations like the previous section then $|S_{i,n,j}|$ is a small number. They showed that even large correlations to this model can not lead to entropy decrease and presented a lower bound in addition to Mathur's upper bound which is

$$0 \leq \Delta \leq \log 2. \tag{5.35}$$

Mahbub and others [4] also went on to generalize the three lemmas and the theorem of Mathur in [2] for $\epsilon \sim 1$ order correlations as well.So, how does our model get pass the bound and solve the paradox? We look into this in the next chapter and see that only a small simple assumption can get us pass paradox even without having $\epsilon \sim 1$ order corrections in the early days of the black holes lifetime. Another similar bound was also presented by Giddings in [27]. [4] also presented a modified page curve which is

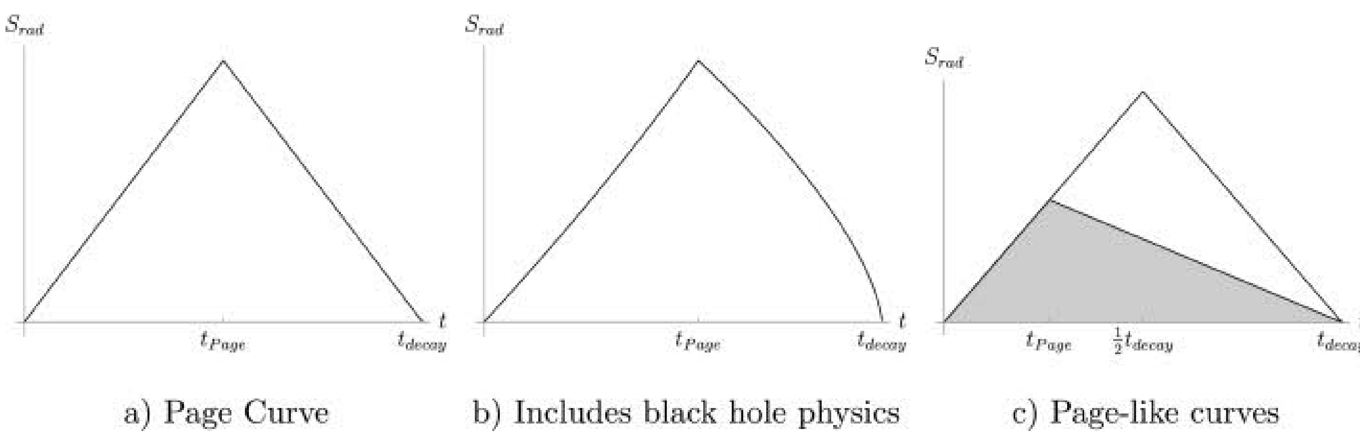

Figure 5.3: Modified Page curve [4]

Chapter 6

Our proposal

The bound presented in [4] which bounds the entanglement entropy by $0 \leq S_{ent} \leq \ln 2$ thus putting a worrying constraint regarding the entanglement entropy between particles created by Black holes, that cannot be negative. Such a constraint tells us that Bell pair states cannot solve the paradox. The bound was derived from Mathur's bound which says the entanglement entropy rises by ln2 at each successive step. The basic assumption used for deriving the bound was that the Hawking radiated pairs i.e the set of b quantas are independent of the quantas that have been radiated in the past.As seen from the previous chapter the assumption was $S_b = S_0$ which means radiated pairs do not effect the pair production. In this model, we assume that the new pair produced at the horizon will have non-local gravitational effects from the Black hole mass state(s) and also the other radiated set of b quantas. We expect the effects that the radiated quantas have on each other to also reduce with the distance from the horizon. Moreover, we want the information about the initial mass states of the Black hole to come out of .Thus giving us precedent that the dynamical mass states of the Black hole play a key role in the state of the quantas produced on the horizon.

The wave function of an entangled state without any of the correlations that we talked above would look like this:

$$|\Psi\rangle_{n+1} = \frac{1}{\sqrt{2}}(|0\rangle_b |0\rangle_c + |1\rangle_b |1\rangle_c) \tag{6.1}$$

And our proposed wave function which carries the non-local interaction terms that we talked about would look like this:

$$|\Psi\rangle_{n+1} = \frac{\exp(\mu MG - \frac{\alpha \Sigma_n I(n)}{f(r)})}{\sqrt{2}}(|0\rangle_b |0\rangle_c + |1\rangle_b |1\rangle_c). \tag{6.2}$$

Explanation of the first term: The first term in the exponential (i.e. μMG) gives us the non-local gravitational interactions between the mass states inside the Black hole and the new pairs that are being produced at the horizon. Thus b quantas have captured information non-locally from the black hole. $'M'$ is the fixed mass state term that the Black hole begins with. The pre-factor μ is the dynamical term that changes with each particle emission. It can be also thought of as the rate at which the Black hole mass states are decreasing. [1] shows us that particle creation by Black holes will cause it's internal mass states to decrease as a result of this process.

Explanation of the second term: The second term in the exponential $\Big($i.e. $\sum \alpha \frac{I(n)}{f(r)}\Big)$, gives us the non-local gravitational effects between hawking radiated particles that produced at each successive time steps. The radiated quantas have some effect on the new pair production. $\sum I(n)$ is a function that outputs binary values (1 for particle creation or 0 for no particle creation). This is because if no particle emission occurs i.e a 0 state then we have no reason to expect that these will have effect on the new pairs as a function of distance r. The term $f(r) \sim 1/r$ represents the effect of the radiated quanta on the new pairs and we expect these interactions to occur among all of the radiated particles $\Big($thus the $\sum i$n the front of the function), and that the interactions to decrease with r. Moreover, we kept a function $f(r) \sim 1/r$ solely because if $f(r) \sim 1/r$ then the wavefunction is not normalizable (we see this later in the next sections).

The following figure depicts our model and labels r as function of distance.

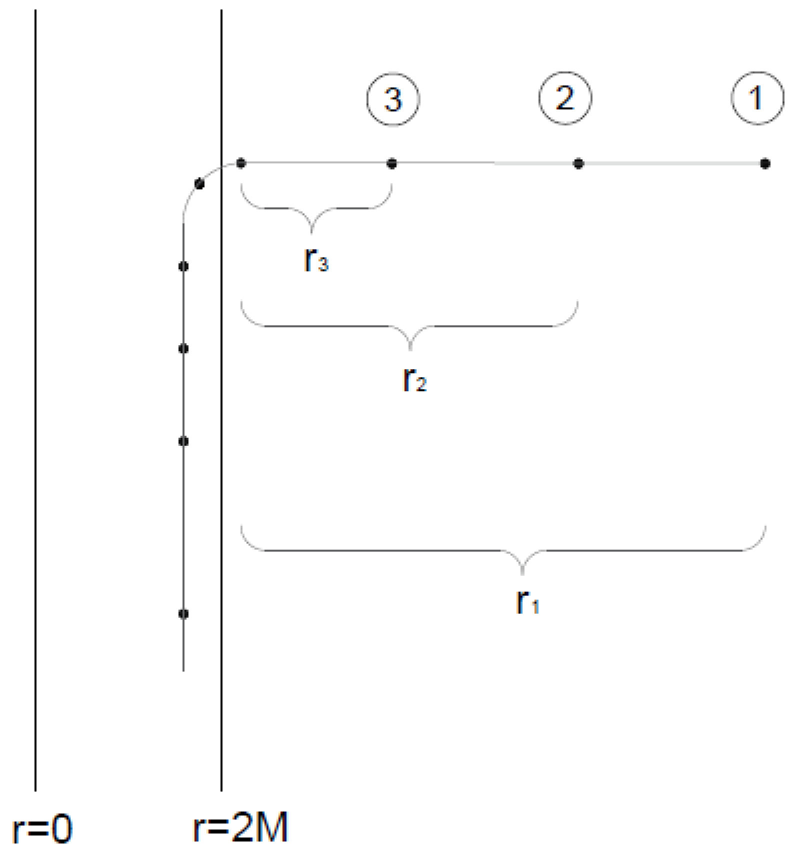

Figure 6.1: Our model

6.1 Dimensional analysis

Dimension of μ:

$$\begin{aligned}\mu &= \frac{MT^2}{L^3} * M^{-1} \\ &= \frac{T^3}{T^4} \\ &= \frac{1}{T}.\end{aligned} \tag{6.3}$$

As we can see μ acts like the mass decrease rate term, as the dimension analysis reveals.

Dimension of α:

$$\begin{aligned}\alpha &= \frac{f(r)}{G} \\ &= \frac{MT^2}{L^3}L \\ &= \frac{T^2}{T^3} \\ &= \frac{1}{T}.\end{aligned} \tag{6.4}$$

We can also see here that α acts like the Hawking particle emission rate term as the dimension analysis reveals.

6.2 Entropy Bounds

The fundamental assumption that Mathur considered, while deriving the entropy bounds is that the Hawking radiated pairs are independent of any future interactions i.e they cannot influence the pair production further once radiated. But in our model we are considering that they can influence the new pairs and we explore what happens to the entropy bounds that was put forward in [3] and [4].

Lets, denote the entropy of the b pairs at time step n as

$$S_b = S_0. \tag{6.5}$$

Lets denote the pair p that will be created at time step t_{n+1} by S_p. In Hawking's original calculation the newly created pairs where considered to be maximally entangled. But we are considering a correction to leading order Hawking state and hence the pairs aren't maximally entangled.

$$
\begin{aligned}
S_{(p)} &= -tr(\rho \ln \rho) \\
&= -tr(|\psi\rangle_n \langle\psi|_n \ln(|\psi\rangle_n \langle\psi|_n) \\
&= -\frac{1}{2} tr(\exp(2A_n) \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2A \end{pmatrix} \\
&= -tr(\begin{pmatrix} 0 & 0 \\ 0 & \exp 2A \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2A \end{pmatrix}) \\
S_{(p)} &= -2A \exp 2A.
\end{aligned} \tag{6.6}
$$

This leads to an important result. Depending on the time step $S_{(p)}$ can be weakly entangled or highly entangled with the other parts of the system.This is due the fact that the term "A" which stands for $A = \mu MG - \alpha \Sigma_n I(n) f(r)$. A is negative when the second term dominates i.e the radiation has more information about the previous radiated pairs and A is positive when there is more information about the initial mass states of the hole. The first case implies S_p is positive and entanglement between the pairs increases. The second case implies S_p to be negative which would account for the entanglement to decrease which we would notice in the refutation of Mathur's theorem below.This is in contrast to the Lemma 1 of [3] which proves $S_{(p)}$ is weakly entangled with the system i.e $S_{(p)} < \epsilon$ where ϵ is a small number.

Now, lets look into the lemma 3 of [3] presented in the previous chapter. It says

$$
S\left(c_{n+1}\right) > \ln 2 - \epsilon. \tag{6.7}
$$

This means the entanglement entropy must increase by $\ln 2 - \epsilon$ in each emission and thus the entanglement goes on increasing. Lets see what happens in our case.

$$\rho = \begin{pmatrix} \frac{\exp 2A}{2} & 0 \\ 0 & \frac{\exp 2A}{2} \end{pmatrix}$$
$$\ln\rho = \begin{pmatrix} \ln\frac{\exp 2A}{2} & 0 \\ 0 & \ln\frac{\exp 2A}{2} \end{pmatrix} \tag{6.8}$$
$$= \begin{pmatrix} 2A - \ln 2 & 0 \\ 0 & 2A - \ln 2. \end{pmatrix}$$

$$S_{c_{n+1}} = -tr_{c_{n+1}}(\rho \ln\rho)$$
$$S_{c_{n+1}} = -tr_{c_{n+1}}(\begin{pmatrix} \frac{\exp 2A_{n+1}}{2} & 0 \\ 0 & \frac{\exp 2A_{n+1}}{2} \end{pmatrix} \begin{pmatrix} 2A_{n+1} - \ln 2 & 0 \\ 0 & 2A_{n+1} - \ln 2 \end{pmatrix} \tag{6.9}$$
$$S_{c_{n+1}} = \exp 2A_{n+1} \ln 2 - 2A_{n+1} \exp 2A_{n+1}.$$

Here, the term A_{n+1} captures the exponential term at (n+1)th step i.e

$$\mathrm{A}_{n+1} = \exp\left(\mu_{n+1} MG - \frac{\alpha \sum_{i=0}^{n+1} I(i)}{f(r)}\right).$$

Now, lets look at the entropy inequality theorem that mathur proposed in [3] and which was presented in the previous chapter. The theorem says $S(b + b_{n+1}) > S_0 + \ln 2 - 2\epsilon$.

In our case,

Denoting $A = b$, $B = b_{n+1}$,$C = c_{n+1}$, we get from the entropy theorem of three subsystems

$$\begin{aligned}
S(A+B)+S(B+C) &\geq S(A)+S(C)\\
S(b+b_{n+1})+S(b_{n+1}+c_{n+1}) &\geq S(b)+S(C_{n+1})\\
S(b+b_{n+1})+S_p &\geq S(b)+S(C_{n+1})\\
S(b+b_{n+1}) &\geq S(b)+S(C_{n+1})-S_p\\
S(b+b_{n+1}) &\geq S(b)+\text{exp}2A_{n+1}\text{ln}2-2A_{n+1}\text{exp}2A_{n+1}+2A\text{exp}2A.
\end{aligned} \tag{6.10}$$

$S(b+b_{n+1})$ gives the correlations of b and b_{n+1} with c, c_{n+1} and the Black hole mass M. Since S_b can be positive and negative depending on the time step. Hence, there is no such bound on the entanglement entropy. If our μ's and α's are finely tuned, the entanglement entropy can actually go down as the black hole counts its last days. The important thing to notice here is that unlike the Mathur's bound or the bound of [4], the correction to the leading order state comes to the ln2 term as well. In the bounds of Mathur and [4] the correction comes as a second term to the leading order state i.e the bound as mentioned previously is $S(b+b_{n+1}) > S_0 + \text{ln}2 - 2\epsilon$. But in our case, the entanglement entropy at the first step only is ln2(we see this in the upcoming sections) otherwise we have a corrected term which would help the entanglement entropy to go down in the last days of the black hole's lifetime.The next question that could be thrown is that does our entanglement entropy follow a page-curve like path? The answer is that we have not developed a numerical model testing our model because of time constraints but we do realize from our calculations that the entanglement entropy must go down at one stage. The next step would to be to run a simulation of maybe 20-qubits and see if the model exibits page curvelike behaviour. In order to do this we need a fine tuned relationship between our rates μ and α.

6.3 Test for Convergence

We know from the previous section our wave function takes the form

$$|\Psi\rangle = \frac{\exp[G(\mu M - \sum \alpha \frac{I(n)}{f(r)})]}{\sqrt{2}}[|0\rangle_b |0\rangle_c + |1\rangle_b |1\rangle_c].$$

Which as we saw before is our wave function, we shall try and figure out whether the inner product of the wave function, whether it converges and what normalization constant it gives. To keep calculations simple we take inner product of Ψ for only two particle emissions. Where one particle has been emitted in the past and the second one is just being created in horizon.

For the next part of the calculation and for the following sections which include utilizing our model we shall have $|\psi\rangle_n$ representing the wave function of entangled particle being formed in the n^{th} step and $|\Psi\rangle_n$ representing the whole tensor product combining all the Hilbert spaces in the n^{th} step.

The wave function $|\psi\rangle_2$ takes the form

$$|\psi\rangle_2 = \frac{1}{\sqrt{2}} \exp(\mu_2 MG - G\alpha_2 r_1)\Big[\, |0\rangle_{b_2} |0\rangle_{c_2} + |1\rangle_{b_2} |1\rangle_{c_2} \Big]. \tag{6.11}$$

For the sake of simplicity we shall take $|a\rangle_n = \frac{1}{\sqrt{2}}\Big[\, |0\rangle_{b_n} |0\rangle_{c_n} + |1\rangle_{b_n} |1\rangle_{c_n} \Big]$.

Since $\mu_1 = 0$ and the fact that only one particle emission will have zero correlations. The first particle emission will act as the leading order hawking state. Therefore,

$$|\Psi\rangle_1 = |\psi\rangle_1 = \frac{1}{\sqrt{2}}(|0\rangle_{b_1} |0\rangle_{c_1} + |1\rangle_{b_1} |1\rangle_{b_1}) = |a\rangle_1 . \tag{6.12}$$

So for two particle emissions we can see our wave function takes the form

$$|\Psi\rangle_2 = \exp(\mu_2 MG - \alpha_2 r_1 G)\,|a\rangle_2 \otimes |a\rangle_1\,. \tag{6.13}$$

$$\langle\Psi|\Psi\rangle_2 = 4\pi \exp(2\mu_2 MG)\int_0^\infty r_1^2 \exp(-2G\alpha_2 r_1)dr_1. \tag{6.14}$$

Taking

$$\zeta = 2G\alpha_2 r_1, \quad d\zeta = 2G\alpha_2 dr_1.$$

$$\begin{aligned} \langle\Psi|\Psi\rangle_2 &= \frac{4\pi\exp(2\mu_2 MG)\times 2!}{(2G\alpha_2)^3} \\ &= \frac{\pi\exp(2\mu_2 MG)}{(G\alpha_2)^3}. \end{aligned} \tag{6.15}$$

We see that our wave function is a well behaved function, as in it converges when applying the above integral . It is quite understandable now that even if we take multiple particle emissions it would result in the same outcome, as in the inner product would not diverge.

6.4 Relation between α and μ

In the case of 1 particle emission as we stated in the above section $\mu_1 = 0$ and the state acts like the leading order Hawking state does, that lacks any correlations.

In the case of 2 particle emissions we have seen in (6.15), the inner product of the wave-function takes the form,

$$\langle\Psi|\Psi\rangle_2 = \frac{\pi \exp(2\mu_2 MG)}{(G\alpha_2)^3}.$$

Imposing the normalization condition for our wave function

$$\frac{\pi \exp(2\mu_2 MG)}{(G\alpha_2)^3} = 1 \tag{6.16}$$

and, therefore

$$2MG\mu_2 = \ln\left[\frac{(G\alpha_2)^3}{\pi}\right]. \tag{6.17}$$

Taking $\frac{\ln[A]}{2MG} = \overline{\ln}[A]$

$\mu_2 = \overline{\ln}\left[\frac{(G\alpha_2)^3}{\pi}\right]$.(6.18)

For 3 particle emissions, the wave function looks like

$|\psi\rangle_3 = \exp(\mu_3 MG - \alpha_3 G r_2)\,|a\rangle_3$

$$|\Psi\rangle_3 = \exp(\mu_3 MG + \mu_3 MG - \alpha_3 Gr_2 - \alpha_2 Gr_1)\, |a\rangle_3 \otimes |a\rangle_2 \otimes |a\rangle_1 . \ (6.19)$$

Taking it's inner product,

$$\begin{aligned}\langle\Psi|\Psi\rangle_3 &= (4\pi)^2 \exp\left(\mu_3 MG + \mu_3 MG\right) \int_0^\infty \int_0^\infty r_1^2 r_2^2 \exp\left(-\alpha_3 Gr_2 - \alpha_2 Gr_1\right) dr_1 dr_2 \\ &= \frac{(4\pi^2) \exp(2MG(\mu_2 + \mu_2))}{(2G\alpha_2)^3 \times (2G\alpha_3)^3 \times (2!)}\end{aligned} \tag{6.20}$$

Computing in the same fashion as to how we got the relation in (6.18), we get

$$\mu_3 = \overline{\ln}\left[\frac{(G\alpha_2)^3 (G\alpha_3)^3}{\pi^2}\right] - \overline{\ln}\left[\frac{(G\alpha_2)^3}{\pi}\right]. \tag{6.21}$$

In the same fashion for 4 particle emission our wavefunction looks like,

$$|\psi\rangle_3 = \exp(\mu_4 MG - \alpha_4 Gr_3)\, |a\rangle_4$$

$$|\Psi\rangle_3 = \exp(\mu_4 MG + \mu_3 MG + \mu_3 MG - \alpha_4 Gr_3 - \alpha_3 Gr_2 - \alpha_2 Gr_1)\, |a\rangle_4 \otimes |a\rangle_3 \otimes |a\rangle_2 \otimes |a\rangle_1 .$$

Computing in the same fashion as (**??**), we get the relation of μ_4

$$\mu_4 = \overline{\ln}\left[\frac{(G\alpha_2)^3 (G\alpha_3)^3 (G\alpha_4)^3}{\pi^3}\right] - \overline{\ln}\left[\frac{(G\alpha_2)^3 (G\alpha_3)^3}{\pi^2}\right. \tag{6.22}$$

We can see from these following relationships we can easily get a relation between μ_n and α_n. Where as we mentioned above is the rate it which the internal mass states decrease, or the rate at which the information regarding the internal mass states are coming out and α_n is the rate at which particle production is occurring. Both of these terms are dynamical since the rate at which evaporation occurs is depended on the size of the Black hole, and since evaporation causes the size of it to vary, both the rates happen to be dynamical.

μ is inherently a negative number and we saw above it is directly linked with, to an extent preceding α's. α will simply reach 0 right when the Black hole has completely evaporated and we can simply see with the relation they have μ will also be ≈ 0. Such an implication is necessary since we need to lose all notions of any correlations when the Black hole has completely evaporated. Otherwise it would seem that information is being created out of nowhere.

After 'n' many emissions our wave function looks like

$$|\Psi\rangle_n = |\psi\rangle_1 \otimes |\psi\rangle_2 \otimes .... \otimes |\psi\rangle_n \, . \tag{6.23}$$

Calculating the density matrix will give us the total entanglement entropy after n many time steps. We expect that the Black hole to be evaporated after 'n' many steps, thus we expect that the S_{ent} to be 0. This can be done for example busing a simple simulation of 20 qubits by which the black hole would evaporate.

6.5 A General model and its implications

This model can serve as a general model for future qubit models for some specific reasons. The main one being the assumption of [3] regarding the entanglement entropy being fixed i.e $S_b = S_0$. Therefore, only escape route from [3] and [4]'s entropy bounds is to consider the entropy to be dynamic as presented in the previous sections. Moreover, we considered information coming out of the black hole non-locally through some Quantum gravity effects. Information of the mass states of the black hole and the information of previous Hawking pairs is carried off by new pairs. In our toy model we considered α and μ as rates, anyone who wants to have different non-local effects can consider these α and μ differently and the only thing they have to do is some dimension analysis.

As seen from the figure (6.1), the calculations were done keeping in mind the distance from the horizon. Another model that could escape the bound is defining r to be the distance between pairs rather than the distance from the horizon. The following diagram embraces this idea:

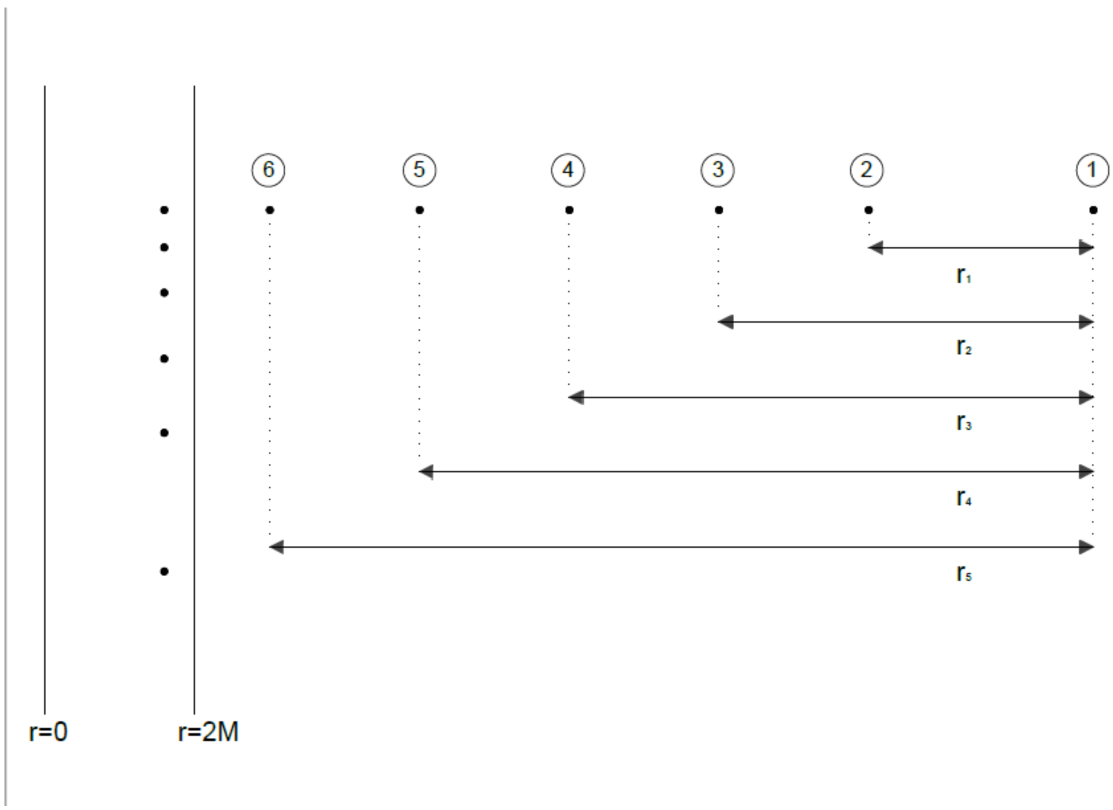

Figure 6.2: An Alternate model

As we see from the figure, the radiated pairs have induced effect on the new quanta being produced and thus escaping the entropy bounds as the bounds were calculated keeping in mind that once radiated the radiated pairs have no effect on the new quanta being produced. According to the figure above, the quanta labelled 6 is part of the new pair produced at the horizon. We see from the figure above that r_1 is the distance between 1 and 2 and r_2 the distance between 1 and 3 and so on. Besides their are interactions between 2 and 3 and others as well.All particles 1,2,3,4 and 5 have effect on the new quantas labelled 6 and thus on pair production.

Just to show how different this model is to the one above, we present the wave function for a few particle emission

$$|\psi\rangle_1 = \exp(\mu_1 MG) \, |a\rangle_1 \, .$$

Since here μ_1 is 0 as well it leads to the leading order hawking term

Computing now for 2 particles, now the r term will spring into action and into our wave function

$$|\psi\rangle_2 = \exp(\mu_2 MG - G\alpha_2 r_1) \, |a\rangle_2 \, .$$

and

$$|\Psi\rangle_2 = \exp(\mu_2 MG - G\alpha_2 r_1) \, |a\rangle_2 \otimes |a\rangle_1 \, .$$

Computing for 3 particle emission our wave function looks like

$$|\psi\rangle_3 = \exp(\mu_3 MG + \mu_2 MG - G\alpha_3 r_2 - G\alpha_2 r_1) \, |a\rangle_3 \, .$$

and subsequently

$$|\Psi\rangle_3 = \exp(\mu_3 MG + 2\mu_2 MG - G\alpha_3 r_2 - 2G\alpha_2 r_1) \, |a\rangle_3 \otimes |a\rangle_2 \otimes |a\rangle_1 \, .$$

One can clearly see that this manifests to another model for Black hole evaporation with Non-local correlations.

Chapter 7

Conclusion

Writing conclusions in theoretical science papers is really hard because every small research provides a scope of further analysis and refutations. The black hole information paradox is one of the longest unsolved problems in all of physics. Steve Giddings and the brilliant Gerrard T'Hooft rightfully compared the black hole information paradox to the classical problem of ultraviolet catastrophe which brought the majestic Quantum Mechanics into the picture. The paradox needs new physics in order to be resolved which is quite clear from the fact that all the three big guns c,$\hbar$,G are involved in black holes. The black hole information paradox requires Quantum Gravity effects to be taken into account and Quantum Gravity is something that we know almost nothing about. There are still no sensitive experiments which can pick up the effect of Quantum Gravity. The reason behind this is quite simple. The Standard Model forces have relative strengths(compared to gravity) of 10^{38} (Strong Force), 10^{36} (EM Force), 10^{25} (Weak Force) if we consider the strength of Gravity to be 1. Its clear that our instruments have to be capable of detecting very very high energy collisions in order to pick up Quantum Gravity effects. A problem that we run into when we try to Quantize Gravity is that gravity is sadly not renormalizable and unitarity is violated. So, our Quantum Field theory gets a major setback in this regard. Ofcourse there is String theory where Gravity arises automatically and there is no issue of non-renormalizability in String theory. The issue here is the scale of energies we need to go to actually detect Stringy effects is truely beyond the scope of current experimental setups. Theoretical physicists are too impatient to sit back and wait for experimentalists to catch up and so they deviced various mathematical setups to actually test String theory and Quantum Gravity effects.

The two fundamental founding pillars of Quantum Field Theory are Unitarity and Locality. Causality is also a founding pillar but that can be implied from locality. The issue we run into when we consider locality and try to Quantize Gravity is that we have to give up Unitarity because the probability amplitude grows with Energy squared and at high energies the probability is more than one which is absolute nonsense. Therefore, unitarity isn't something we can give up on. We must resort to giving up locality or relaxing the locality condition a bit and see what pops up. Many theorists agree that Quantum Gravity doesn't look very loving towards locality. Some might advocate for sacrificing Unitarity but violation of probability pushes us towards unavoidable issues. One cannot give up the issue of probability conservation that easily. The sacrifice thus has to be made by locality. Locality violation or non-local information transfer could also explain how information could come out of a black hole.

Our proposal came from two important observations. The first one came from the conclusions about the entropy bounds put forward in [3] and [4]. Mathur [3] proposed an upper bound on the entanglement entropy of each emission which was expanded by proposing a lower bound in [4]. The important thing we observed that both the bounds were derived by the assumption that the previously emitted hawking quanta has no effect in the new pair production at the horizon. Since we already have two such important bounds on the entanglement entropy, we tried to see what happens when we consider nonlocal interactions with the previous hawking quanta. The result on the entropy bounds which we found where that no such bound holds if we consider that previously radiated quanta have effect on the new pairs. Thus anyone who tries to solve the information paradox by bell pair states they need to consider the Hawking radiated quanta to have effects on the new pairs otherwise they would fall under the bound of [4]. The second observation came from [14]. Although Page did not explicitly mention the entropy bounds of [4], they proposed a non-local information transfer and a Unitary transformation of information transfer which makes use of the hawking radiated Quanta. We therefore wanted to construct a General toy model which could be used by anyone who wants to transfer information from inside a black hole by bell pair states. As analyzed in chapter 6 our model would actually account for decrease in Entanglement entropy and thus help resolve the paradox.

Our analysis of the problem is trivial in the sense we used qubit model to actually try and provide a possible resolution. One can also try to develop a field perspective of such a model because that would provide a nice step in understanding how non-locality can be a feature of space-time fields as well. Black hole information paradox can actually serve as a filter for future Quantum Gravity theories and thus provide a great step in the direction of a theory of everything. Therefore, solving the black hole information paradox will help us unlock the features of Quantum Gravity and as Hawking duly said "know the mind of God" in [31].

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