Quantisation of Black Hole Horizon Area and Multipartite Entanglement of Black Hole Information Subsystems

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Declaration

It is hereby declared that

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2. The thesis does not contain material previously published or written by a third party, except where this is appropriately cited through full and accurate referencing.
3. The thesis does not contain material which has been accepted, or submitted, for any other degree or diploma at a university or other institution.
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Abstract

We formulate a model consisting of multipartite entanglement that helps provide a physical significance to the Bekenstein-Hawking entropy obtained after quantising the horizon area of a black hole as proposed by Jacob D. Bekenstein in "Spectroscopy of the quantum black hole". We propose an entanglement between the black hole information and the information of the Hawking pairs, giving rise to such entanglement entropies. We calculate the entanglement entropy of a black hole using the W state for $k \geqslant 3$ qubits and attempt to obtain the Page curve.

Keywords: Bekenstein-Hawking Entropy; Entanglement Entropy; Qubit; Multipartite Entanglement; W State; Quantum Gravity; Page Curve

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Chapter 1

Introduction

This paper gives an insight into the information entanglements inside black holes in order to open doors to understanding black hole constituents and properties. This field has been a growing and emerging field in the Physics community, venturing to gain more knowledge and peak into the unknown mysteries of black holes.

Ever since black holes were understood, it was described as a region of space with such a high field of gravitation that no particle or electromagnetic wave, i.e. light, could escape once it had fallen inside. It was presumed that once a black hole assimilates a particle, the black hole horizon area increases, and the black hole gets bigger.

However, Stephen Hawking led the way to newer findings in his 1975 paper [6]. In quantum field theory for curved spacetime, it is predicted that event horizons expel Hawking radiation, with the exact spectrum as that of a black body of temperature that is inversely proportional to its mass. So, we then understood that black holes emit some sort of thermal radiation. Moreover, the black holes eventually evaporate over time due to this very Hawking radiation. Thus, we came to the conclusion that black holes evaporate eventually, and Hawking radiation is one of the main causes.

Nevertheless, the mystery surrounding black holes never disappeared. As it is known, there are quantum conservation laws. One of them being the conservation of quantum information. However, culminating to black hole properties, any particle that is pulled beyond the event horizon of the black hole is lost, along with its quantum information. This violates the law of conservation of information, leading to a paradox. This is what we now call the Black Hole Information Paradox.

1.1 A Guide of the Thesis

In order to answer the big questions, this paper paves through tricks and mathematical tools addressing the problem.

Moving onto Chapter 2, it focuses on the preliminaries to understanding quantum systems and how they are denoted with respect to their nature. Chapter 2 lays a brief introduction to pure and mixed states of a quantum system and how their properties would be determined using density operators and density matrices.

The main objective of Chapter 3 is to explain the whole idea of quantum entanglement. It is a grave topic in quantum physics, and is crucial if one wants to understand black hole entropy. This chapter provides minimal explanation of the monogamy property of entangled pairs, the Einstein-Podolsky-Rosen paradox which states how quantum physics fails to paint the true picture of reality due to its probabilistic nature. It also explains Bell's inequality and the entropy associated with the entanglement of particle pairs.

Chapter 4 is crucial for this paper, and is one of the most important derivations till date. It was originally published by Stephen Hawking, and was later derived by many others. Chapter 4 is about Hawking radiation that leads to the evaporation of black holes. An important take from this is the derivation of Hawking temperature that provides us with the unique idea that Hawking radiation is, in fact, a thermal radiation. it is believed that Hawking radiation is due to the particle pair-production near the event horizon. Due to the heavy pull of gravity, immediately in the moment of emergence, one from the pair is pulled in, and to conserve momentum, the other flies off. The particles that are thrown off the event horizon is deemed as Hawking radiation and they radiate a thermal spectrum. This paper uses the WKB approximation to derive the Hawking temperature.

The Information Paradox, as discussed earlier, is elaborately explained in Chapter 5. It is explicitly inspired by an excellent paper published by Samir D. Mathur [19] and it encompasses his take on the paradox.

As black hole entropy is in discussion, one must mention the Bekenstein-Hawking entropy. It is a thermal entropy and the derivation is done through quantisation of space. It is shown in Chapter 6, how taking the maximised entropy leads to an expression that is very similar to the Bekenstein-Hawking entropy.

Heavily inspired by the papers of J. Bekenstein, V. Mukhanov [8] and Shahar Hod [13], Chapter 7 wraps the idea of quantisation of the horizon area leading to changes in how the Bekenstein-Hawking entropy is expressed. This chapter is concluded with a new establishment made by Mr. Shahar Hod in respect to the quantisation of black hole horizon area.

Chapter 8 is the main soul to this thesis paper. It constitutes of the work that has been put for this thesis. In this chapter, we propose an idea of multipartite entanglements between the Hawking pairs and the black hole system. Assisted by the Information Theory, we devise calculations of the entanglement entropies between the Hawking pair information subsystems with that of the black hole information by treating these systems as quantum bits. We also present a toy model explaining ways in which the information of the black hole may be retrieved.

Chapter 2

Pure and Mixed States

2.1 Quantum Systems

In terms of quantum physics, a quantum sate is one that describes the state of an integrated quantum system. It give the probability distribution for the magnitude of each observable *i.e.* for the result of each probable measurement on the system.

The state of a system can be called a pure state, if the system can be described using only one wave function, i.e. one ket-vector, $|\psi\rangle$.

A mixed state, however, is a state that cannot be described by solely one wave-function. In order to describe a mixed state, multiple wavefunctions are needed.

For example, a complicated case is given by the singlet state,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(\,|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\,\big) \tag{2.1}$$

which describes a superposition of connected spin states for two particles with spin-$\frac{1}{2}$. This is a pure state.

A mixed quantum state refers to a probable mixture of pure states; however, differing distributions of pure states can give rise to equal mixed states. Mixed states can be expressed using density matrices.

Since mixed states are arrays of pure states, pure states are a subset of mixed states. Therefore, the density operator can be used to identify if a state is pure or mixed.

2.2 Density Matrices

Density matrices are used to describe the probabilistic state of a quantum system. The density matrix is a portrayal of the density operator. It is retrieved from the density operator by basis selection in space. The operator is self-adjoint(or Hermitian) in nature, positive semi-definite, of *trace* one, and may be infinite dimensional.

For example, taking a collection of particles that consist of several smaller groups of particles. There are N groups of particles, each group corresponding to a different pure state and containing a fraction of the particles in the collection.

A fraction with a relative population A_i is in the pure state c. The states $|\alpha^i\rangle$ are normalized but not necessarily orthogonal to each other. They are expressed in terms of the basis vectors $|\eta\rangle$, where $\eta \in \{e_1, e_2, ...\}$ of the Hilbert space.

$$|\alpha^i\rangle = \sum_n c_n |\eta\rangle \tag{2.2}$$

The coefficients can be calculated,

$$c_n^{(i)} = \langle \eta \mid \alpha^{(i)} \rangle$$

and

$$c_n^{i\,*} = \langle \alpha^{(i)} \mid \eta' \rangle$$

The density matrix of a state will reflect the fraction of particles that are in each pure state in terms of the probabilities A_i, where A_i are real, $0 \leq A_i \leq 1$ and $\sum_i A_i = 1$.
The definition of the density operator is,

$$\rho = \sum_{i=1}^{N} |\alpha_{(i)}\rangle A_i \langle \alpha_{(i)}| \tag{2.3}$$

The density operator is Hermitian, as in, $\rho = \rho^\dagger$, and Hermitian matrices are diagonalisable.

A represenation of pure and mixed states can be expressed by light polrization. Photons are bound to having two perpendicular quantum states, $|R\rangle$ which refers to right circular polarization, and $|L\rangle$ which refers to left circular polarization. They can also be in a superposition state of both, $\frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)$.

For the example of unpolarised light, the density operator equals,

$$\rho = \frac{1}{2}\big(|R\rangle \langle R| + |L\rangle \langle L| \big)$$

where it showcases both of the orthogonal quantum states for photons.

Chapter 3

Quantum Entanglement

Quantum entanglement is a physical phenomenon that occurs among a pair or group of particles in a way such that there is an interdependence of properties between them, even at space-like separation, leading to correlations among them. Consider a pair of entangled particles that are generated such that their total spin is zero. If a measurement of spin is taken of one of the particles along the z-axis, resulting in a clockwise spin, then a spin measurement of the complement particle along the same axis results in an anti-clockwise spin. Measurements of various other physical properties such as position, momentum and polarisation on entangled particles also show similar correlations.

Thus, particles exhibiting the phenomenon of entanglement can only be described by a single non-separable wavefunction, as opposed to non-entangled particles that can be described by a state consisting of multiple separate wavefunctions describing each subsystems. As explained in [27], consider a system of N particles. If there are no correlations present between them, the system of particles can be described by a tensor product of separate pure states of each subsystem $|\psi_1\rangle, |\psi_2\rangle, ..., |\psi_N\rangle$,

$$|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes ... \otimes |\psi_N\rangle, \tag{3.1}$$

whose Hilbert space is,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_N, \tag{3.2}$$

where $|\psi_1\rangle \in \mathcal{H}_1, |\psi_2\rangle \in \mathcal{H}_2, ..., |\psi_N\rangle \in \mathcal{H}_N$. This is not the case for a system of entangled particles.

3.1 Monogamy of Entanglement

Monogamy of entanglement is one fundamental property of quantum entanglement where maximal entanglement can only occur between a pair of particles. Consider 3 particles A, B and C. If particles A and B are maximally quantumly correlated, then neither of the particles have any correlation with particle C. This is clearly shown in Coffman-Kundu-Wooters (CKW) inequality in [14],

$$\tau_{AB} + \tau_{AC} \leqslant \tau_{A(BC)}, \tag{3.3}$$

where τ_{AB} and τ_{AC} are a measure of entanglement known as the "tangle", between pairs AB and AC respectively, which is related to the entanglement of formation. If particle A has a certain amount of entanglement with the pair BC, then that amount bounds A's entanglement with particles B and its entanglement with particle C. The amount of entanglement that A has to particle B is not available to particle C. The above inequality can be generalised for the case of n particles.

From the above inequality, we can see that the tangle between pairs AB plus the tangle between AC cannot be greater than the tangle between A and the pair BC. Therefore, there is a trade-off between A's entanglement with B and its entanglement with C.

3.2 The EPR Paradox

In 1935, Einstein, Podolsky and Rosen published a paper which is now famously known as the EPR Paradox [1] where it was proposed that the theory of quantum mechanics was an incomplete description of physical reality. We cannot define all classical physical observables of a system simultaneously with complete precision, such as position and momentum. This was due to the fact that quantum mechanics is probabilistic, as opposed to previous theories that were deterministic. There must be hidden variables along with wavefunctions that would completely characterise the state of a system in order to get a deterministic theory. The paper stated that there is an element that corresponds to each of the elements of reality in a complete theory. Here, the elements of reality are all the physical properties that exist in a system.

Referring to David Bohm's version of the EPR Paradox in [2] and [26], consider a neutral pi meson having 0 spin at rest that decays into a positron and electron,

$$\pi^0 \rightarrow e^- + e^+.$$

To preserve conservation of angular momentum, the positron and electron fly off in opposite directions and are in the singlet configuration,

$$\frac{1}{\sqrt{2}} \left(\uparrow_- \downarrow_+ - \downarrow_- \uparrow_+ \right).$$

If the electron has an up spin, then the positron must have a down spin. Therefore, the above state is an entangled state. Now, if a measurement is taken after there is a separation of an arbitrarily large distance between the particles, e.g. 10 light years, and we get an up spin for the electron, then we automatically know the spin of the positron. Before the measurement, the particles did not have a definite spin until the measurement caused the wavefunction to collapse, which instantaneously produces the positron's spin. However, this violates the principle of locality as it insinuates that information would have to travel faster than light speed in order to give the positron its down spin. Therefore, Einstein rejected the "orthodox" position and believed that both particles always had definite spins even before any measurements were made and the implementation of hidden variables would solve this paradox.

Yet, there are no such violations. Experiments have shown that the collapse of the wavefunction is instantaneous and the spins have perfect correlations.

3.3 Bell's Inequality

In 1964, John Bell published a paper [3] that would vindicate quantum mechanics by proving that the use of local hidden variables would not be valid.
Once again, we consider the above thought experiment where we have an electron and positron in the singlet entangled state,

$$\frac{1}{\sqrt{2}}\left(\uparrow_{-}\downarrow_{+} - \downarrow_{-}\uparrow_{+} \right). \tag{3.4}$$

However, this time the spins of 2 particles are measured in different directions by having the detectors oriented at angles that are different with respect to the z-axis. The spin of the electron is measured in the direction of a unit vector u whereas the positron spin is measured in the direction of the unit vector v. The values +1 and -1 are given by each of the detectors for spin up and spin down respectively. The possible combinations of spins and their products are,

positron	electron	product
+1	+1	+1
+1	-1	-1
-1	+1	-1
-1	-1	-1

For a set of detector orientations, let $P(u, v)$ represent the average value of the product of the spins. If our detectors were parallel, $u = v$ and,

$$P(u, u) = -1 \tag{3.5}$$

as the product will always be -1, due to an up spin and down spin of the electron and positron respectively. For anti-parallel orientations of the detectors,

$$P(u, -u) = +1 \quad . \tag{3.6}$$

For the rest possible spin combinations,

$$P(u, v) = -u \cdot v, \tag{3.7}$$

If we consider a hidden variable or variables denoted by λ that characterise the electron positron system, then $U(u, \lambda)$ and $V(v, \lambda)$ are some functions that give the result of an electron measurement and positron measurement respectively.

$$U(u, \lambda) = \pm 1; \quad V(v, \lambda) = \pm 1 \tag{3.8}$$

The idea is that $U(u, \lambda)$ should be completely independent of the orientation of the positron detector. If the detectors are aligned, then the results are perfectly correlated for all λ under the assumption of locality,

$$U(u, \lambda) = -V(u, \lambda) \tag{3.9}$$

The average of the product of the measurements is given by,

$$P(u, v) = \int \rho(\lambda) U(u, \lambda) V(v, \lambda) d\lambda \tag{3.10}$$

$\rho(\lambda)$ represents the probability density for the hidden variable and is nonnegative. From (3.9),

$$P(u, v) = -\int \rho(\lambda) U(u, \lambda) U(v, \lambda) d\lambda \tag{3.11}$$

Taking w to be any other unit vector,

$$P(u, v) - P(u, w) = -\int \rho(\lambda)[U(u, \lambda)U(v, \lambda) - U(u, \lambda)U(w, \lambda)] d\lambda \tag{3.12}$$

As $[U(v, \lambda)]^2 = 1$,

$$P(u, v) - P(u, w) = -\int \rho(\lambda)[1 - U(v, \lambda)U(w, \lambda)]U(u, \lambda)U(v, \lambda) d\lambda \tag{3.13}$$

From (3.8), it follows that

$$-1 \leqslant U(u, \lambda)U(v, \lambda) \leqslant +1$$

and

$$\rho(\lambda)[1 - U(v, \lambda)U(w, \lambda)] \geqslant 0$$

Thus,

$$|P(u, v) - P(u, w)| \leqslant \int \rho(\lambda)[1 - U(v, \lambda)U(w, \lambda)] d\lambda \tag{3.14}$$

which leads to the Bell's inequality,

$$|P(u, v) - P(u, w)| \leqslant 1 + P(v, w) \tag{3.15}$$

However, we can show that this inequality simply does not hold for (3.7). Let's assume that all 3 of our vectors u, v and w lie on the same plane where u and v are perpendicular to each other and w is in between, creating a 45° angle with both. Hence,

$$P(u, v) = -u \cdot v = 0; \quad P(u, w) = P(v, w) = -uw \cos\frac{\pi}{4} = -0.7071067812$$

$$| P(u, v) - P(u, w) | \leqslant 1 + P(v, w) = -0.7071067812$$

$$1 + P(v, w) = 1 - 0.7071067812 = 0.2928932188$$

Substituting these values into our Bell's inequality,

$$-0.7071067812 \nleqslant 0.2928932188$$

This inequality clearly does not hold at all for such a case, Thus, using hidden variables does not hold or exonerate quantum mechanics from the violations of locality and causality as proposed by the 1935 paper of the EPR Paradox. Many experiments have since proven that this inequality indeed does not hold and that quantum mechanics is complete as it is and the properties of quantum particles are not predetermined before measurements are made.

3.4 Entanglement Entropy

If we have a quantum mechanical system which can be represented by a density matrix ρ, we can calculate its entropy by,

$$S(\rho) = -\mathrm{tr}(\rho \ln \rho). \tag{3.16}$$

This is called the Von Neumann entropy, whose units are in qubits. A qubit is the most basic unit of quantum information and is described by a 2 state quantum mechanical system such as,

$$|\psi\rangle = a_1 |0\rangle + a_2 |1\rangle ,$$

where $|0\rangle$ and $|1\rangle$ are the basis states. A qubit can either possess the state $|0\rangle$ or $|1\rangle$ or a superposition of both. We can also write ρ as the sum of all possible states ρ_{x_i},

$$\rho = \sum_i p(x_i)\rho_{x_i},$$

where $p(x_i)$ is the probability of the ensemble being in the state given by ρ_{x_i}. If all these states are pure and orthogonal, then there is only a single element of unity on one of the diagonal elements of the density matrix. In such cases, our Von Neumann entropy starts to resemble the Shannon entropy,

$$S(\rho) = -\sum_i p(x_i) \ln p(x_i). \tag{3.17}$$

As there is only a single diagonal element of unity in the density matrix, the Von Neumann entropy is equal to 0. Further along this paper, we shall use the form of (3.16) to calculate the entanglement entropies for the cases of multiple qubits. Many of the above topics in this chapter have been discussed in this paper with the help of [15]

Chapter 4

Hawking Radiation

4.1 The WKB Approximation

The WKB approximation is incorporated in order to obtain general solutions of linear differential equations. The WKB approximation comes in handy in the differential equation solutions that have either constants or slowly varying coefficients. This can also be used to find approximate solutions to the Schrodinger equation. It is specifically helpful when finding solutions concerning tunneling rates through potential barriers.

We [25] eliminate the time-dependence of the wavefunction because the space-like involvement to the tunneling event happens quite rapidly. We can write the wavefunction as,

$$\psi(x) = Ae^{\frac{iS(x)}{\hbar}}. \tag{4.1}$$

Here, S is the classical action.

The time-independent Schrodinger equation is,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x). \tag{4.2}$$

This explains a probability distribution of the wavefunction in an environment with a potential distribution.

The solutions to the Schrodinger equation has the form,

$$\psi \sim e^{\frac{iS(x)}{\hbar}}. \tag{4.3}$$

From this equation, we get

$$\psi' = \frac{i}{\hbar}S'\psi, \tag{4.4}$$

$$\psi'' = \left(\frac{i}{\hbar}S'' - \frac{i}{\hbar^2}S'^2\right)\psi. \tag{4.5}$$

We, now, substitute the derivatives into the time-independent Schrodinger equation, we get

$$-\frac{\hbar^2}{2m}\left(\frac{1}{\hbar}S'' - \frac{i}{\hbar^2}S'^2\right) - [E - V(x)] = 0. \tag{4.6}$$

The system has a total energy of $E = \frac{p^2}{2m} + V(x)$ that can be shuffled to get $p^2 = 2m(E - V(x))$. We incorporate this into the previous equation and get

$$i\hbar S'' - S'^2 - p^2 = 0 \tag{4.7}$$

From this we take a semi-classical WKB approximation by Taylor expanding the classical action in powers of $\hbar$ and dropping terms that exceed linear order.

The Taylor expansion of the classical action is as follows,

$$S(x) = S_0(x) + S_1(x)\hbar + S_2(x)\hbar^2 + ... \ . \tag{4.8}$$

We differentiate this equation and put it in (4.7),

$$i\hbar\left(S_0'' + S_1''\hbar + S_2''\hbar^2 + ...\right) - \left(S_0 + S_1\hbar + S_2\hbar^2 + ...\right) - p^2 = 0 \ , \tag{4.9}$$

This simplifies to,

$$-\left(p^2 + S_0'^2\right) + \left(iS_0'' - 2S_0'S_1'\right)\hbar + \left(iS_1'' - S_1'^2 - 2S_0'S_2'\right)\hbar^2 + ... = 0 \quad . \tag{4.10}$$

Since the right hand side equals to zero, we take the zeroth order term in the equation that gives us,

$$p^2 = -S_0'^2 \quad \longrightarrow \quad S_0(x) = \pm\int_{x_0}^{x} p(x)dx \tag{4.11}$$

The first order term,

$$\frac{i}{2}S_0'' = S_0'S_1' \ ,$$

$$\frac{i}{2}p' = pS_1' \quad ,$$

and

$$\frac{i}{2}\int\frac{dp}{p} = \int dS_1 \tag{4.12}$$

Therefore,

$$S_1(x) = \frac{i}{2}\ln|p|. \tag{4.13}$$

We can use these equations to find the semi-classical approximation for ψ,

$$\psi(x) = \exp\left[\frac{i}{\hbar}S(x)\right].$$

Expanding this further will give us,

$$\psi(x) \approx \exp\frac{i}{\hbar}\left(S_0(x) + S_1(x)\hbar\right).$$

And finally,

$$\psi(x) = \exp\left[\pm\frac{i}{\hbar}\int_{x_0}^{x} p(x)dx - \frac{1}{2}\ln|p|\right]. \quad (4.14)$$

After taking the second term as a coefficient, this leads us to approximate a general solution to the time-independent Schrodinger equation,

$$\psi(x) \approx \frac{1}{\sqrt{|p(\dot{x})|}}\left[C + e^{+\frac{i}{\hbar}\int_{x_0}^{x} p(x)dx} + Ce^{-\frac{i}{\hbar}\int_{x_0}^{x} p(x)dx}\right] \quad (4.15)$$

Here,

$$p(\dot{x}) = \sqrt{2m(E - V(x))}.$$

This is the WKB approximation for $\psi(x)$.

4.2 WKB Approximation at the Event Horizon

General plane wave solutions to the time-independent Schrodinger equation can be approximated as,

$$\psi(x) = \frac{C_{\pm}}{\sqrt{|p(\dot{x})|}} e^{\pm\frac{i}{\hbar}\int_{x_0}^{x} p(x)dx} \quad (4.16)$$

where,

$$p(\dot{x}) = \sqrt{2m(E - V(x))},$$

Here, m is the mass, E is the total energy and p is the classical momentum of the tunneling particle. $V(x)$ acts as the potential barrier that the particle must cross in order to tunnel through. The potential barrier at the event horizon can be thought of as the gravitational potential barrier that the particle has to cross to reach the outside of the black hole. The particle must, however, tunnel through a potential barrier that is directly related by the particle's own total energy. From the equation, we can see that the momentum and energy would be conserved if the $V(x)$ is equal to the particle's own total energy.

The integral in the expression for $\psi(x)$ is the classical action,

$$S(x) = \int_{x_0}^{x} p(x')dx'$$

The expression for the classical action is real-valued when the particle is in a classically viable region where $E > V(x)$, establishing that $p(\dot{x})$ is real. $S(x)$ is, however, imaginary where the region follows $E < V(x)$, which implies $p(\dot{x})$ is imaginary.

If the particle has negligible mass, the momentum should be expressed in a fashion where it does not heavily depend on the particle's mass. This can be organised through a Hamilton formalism. This implies that $p(\dot{x})$ can be generally referred to as the particle's canonical momentum, where

$$p(\dot{x}) = \frac{\partial \mathcal{L}}{\partial \dot{x}} \ .$$

Here, $\mathcal{L}$ is the Lagrangian density. We can see that the equation does not explicitly depend on the particle's mass.

A particle, if it were to cross the event horizon, would cross along a radial coordinate axis. Therefore, assuming ($x' \longrightarrow r$) into the classical action equation and replacing the momentum with a component of a momentum 4-vector,

$$S = \int_{r_f}^{r_i} p_r dr \ .$$

This equation can be introduced while obtaining the tunneling rates of Hawking radiation.

4.3 Surface Gravity

A stagnant subject in the Schwarzschild geometry has proper acceleration that diverges near the event horizon. Surface Gravity can be predicted to be the acceleration occuring due to gravity that is observed by a particle which is very close to the surface of the gravitational source, given the condition that the particle has negligible mass. Here, the gravity has been normalised by gravitational redshifting. Due to this normalisation, the surface gravity near the event horizon of a black hole is finite.

Assuming a particle with a finite and unit mass is situated at r. It is accelerated in the up direction by a force F to a distance δr. The work done to carry this out would be,

$$\delta W = F(r)\delta r = ma(r)\delta r \Big|_{m=1}$$

$$a(r)\delta r$$

The equation for 4-acceleration is,

$$a(r) = \frac{M}{r^2\sqrt{1-\frac{2M}{r}}} \ .$$

This can also be thought as the observer's proper acceleration that is needed for it to stay stationary in the Schwarzschild geometry. Incorporating the value of the proper acceleration in to the equation for work done,

$$\delta W = \frac{M}{r^2\sqrt{1-\frac{2M}{r}}}\delta r \ . \tag{4.17}$$

If work done by the observer were to be converted to a signal radiation, i.e. a high energy photon, to be transmitted to another observer at infinity, the angular frequency of this high energy photon would be $\delta\omega$. If we assume the conversion has an efficiency of 100%, we can write,

$$\delta\omega = \frac{M}{r^2\sqrt{1-\frac{2M}{r}}}\delta r \ . \tag{4.18}$$

The other observer that receives this high energy photon would intercept it at a much lower energy than that with which it was sent. The photon would be gravitationally redshifted. We can calculated the degree of the gravitational redshift.

We consider the Schwarzschild metric,

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{4.19}$$

We consider $dr = d\Omega = 0$ in the metric and take $d\tau^2 = -ds^2$. This gives

$$d\tau = \sqrt{1-\frac{2M}{r}}dt \ . \tag{4.20}$$

From this equation, we can observe that the proper time of an observer at spatial infinity, and one that is in rest, is equal to the time coordinate of the Schwarzschild metric, i.e. $d\tau_\infty = dt$. Therefore, we can write

$$d\tau_r = \sqrt{1-\frac{2M}{r}}d\tau_\infty \ , \tag{4.21}$$

where τ_r is the proper time of a stationary observer at a spatial distance r.

Now, by the general relation, we can consider the proper frequency of the photon to be,

$$\delta\omega = \frac{1}{\delta\tau} \ . \tag{4.22}$$

Then the proper frequency of the photon at r can be written in relation to the proper frequency of the photon at infinity as,

$$\delta\omega_\infty = \sqrt{1-\frac{2M}{r}}\delta\omega_r \ . \tag{4.23}$$

Putting the value of $\delta\omega_r$ from equation (4.17) into equation (4.22), we get

$$\delta\omega_\infty = \frac{M}{r^2}\delta r \ . \tag{4.24}$$

We can see that the terms in this equation have units of energy, as in, units of [force]x[distance].

Therefore, we get an equation for force by dividing both sides by δr

$$\delta F = \frac{M}{r^2} \ , \tag{4.25}$$

Recalling that we considered the mass of the observer in question as unitary, therefore, this force can also be considered to be acceleration. Hence, we write

$$\kappa(r) = \frac{M}{r^2} \ . \tag{4.26}$$

This way of calculating acceleration can be deemed as surface gravity. Particularly, it can be considered as the acceleration needed to keep an observer situated near the event horizon from falling into the black hole. In the field of black holes, the surface gravity is measured at the event horizon. For a Schwarzschild black hole, the event horizon is at $r = 2M$. Therefore,

$$\kappa(r = 2M) = \frac{1}{4M} \tag{4.27}$$

4.4 Considering Near Horizon Approximation for Calculation of Hawking Temperature

In this section, considering the coordinates of near horizon approximation we calculate the Hawking temperature using gravitational WKB approximation by acknowledging the relation between the Schwarzschild geometry and Rindler space, as shown in [25]. We will first derive the near horizon approximation by using the Schwarzschild metric. We can relate Rindler space to a local set of Minkowski coordinates as Rindler space is conformally flat, allowing us to work with a new set of local Rindler coordinates. We will then use these coordinates to calculate the Hawking temperature.

4.5 Taking Near Horizon Approximation

The Schwarzschild geometry in the local inertial frame of an observer near the horizon is described by the near horizon approximation. The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \ , \tag{4.28}$$

We set $dt = d\Omega = 0$ in order to take an infinitesimal element of proper radial distance (as it would be measured by an observer at $r > 2M$). Taking the square root of both sides of the metric above gives us

$$ds = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} dr \ . \tag{4.29}$$

Integrating (4.28), after a long and lengthy calculation, gives us

$$\rho(r) = \int_0^\rho ds$$

$$= \int_{2M}^{r} \left(1 - \frac{2M}{r'}\right)^{-\frac{1}{2}} dr' = 2M \sinh^{-1}\left(\sqrt{\frac{r}{2M} - 1}\right) + \sqrt{r(r-2M)} \ , \qquad (4.30)$$

where ρ is the proper radial distance from r to $2M$ ($r > 2M$). As $d\rho = ds$

$$d\rho^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \ . \qquad (4.31)$$

For $r > 2M$, we can now state the Schwarzschild metric as a function of proper radial distance to the event horizon

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + d\rho^2 + r^2 d\Omega^2 \ . \qquad (4.32)$$

Here, $r = r(p)$. As r approaches $2M$, $\sinh^{-1}\left(\sqrt{\frac{r}{2M} - 1}\right)$ tends to zero. Doing a first order Taylor expansion of $\sinh^{-1}\left(\sqrt{\frac{r}{2M} - 1}\right)$ about $r = 2M$ gives us

$$\sinh^{-1}\left(\sqrt{\frac{r}{2M} - 1}\right) \approx 2M\sqrt{\frac{r}{2M} - 1} - \frac{M}{3}\left(\frac{r}{2M} - 1\right)^{\frac{3}{2}} + ...$$

$$= \sqrt{2M(r-2M)} \ . \qquad (4.33)$$

We consider the region near the event horizon of the black hole for (4.32). Similarly, Taylor expanding the second term of (4.29) at $r = 2M$

$$r(r-2M)\ |_{r=2M} \approx 2M(r-2M) \ . \qquad (4.34)$$

So we can approximate the second term as

$$\sqrt{r(r-2M)} \approx \sqrt{2M(r-2M)} \ . \qquad (4.35)$$

We can now write (4.29) as

$$\rho(r) = 2\sqrt{2M(r-2M)} \qquad (4.36)$$

in the neighbourhood of $r = 2M$. The inverse of $\rho(r)$ is as follows

$$r(\rho) = \frac{\rho^2}{8M} + 2M \ .$$

Substituting this into the time component $g_{tt}(\rho)$ of the metric (4.31) gives

$$g_{tt}(\rho) = -\left(1 - \frac{2M}{\left(\frac{\rho^2}{8M} + 2M\right)}\right) dt^2 \ ,$$

$$= -\left(\frac{\frac{\rho^2+16M^2}{8M} - 2M}{\frac{\rho^2+16M^2}{8M}}\right) dt^2 \ ,$$

$$= -\left(\frac{\rho^2}{\rho^2 + 16M^2}\right) dt^2 \ ,$$

$$
\begin{aligned}
&= -\rho^2 \left(\frac{1}{\left(\frac{\rho}{4M}\right)^2 + 1} \right) \left(\frac{dt}{4M} \right)^2 \ , \\
&\approx -\rho^2 \left(\frac{dt}{4M} \right)^2 \ . \qquad (4.37)
\end{aligned}
$$

We assume that $\rho \ll 4M$, therefore $\left(\frac{\rho}{4M}\right)^2 \ll 1$ as we consider the observer to be very close to the horizon. We can now write (4.31) using (4.36) as

$$ds^2 = -\rho^2 \left(\frac{dt}{4M} \right)^2 + d\rho^2 + r^2 d\Omega^2 \ . \qquad (4.38)$$

We now define

$$\omega \equiv \frac{t}{4M} \ , \qquad (4.39)$$

in order to re-scale the time coordinate. Here, t is the Schwarzschild time. In terms of ω, (4.37) can now be written as

$$ds^2 = -\rho^2 d\omega^2 + d\rho^2 + r^2 d\Omega^2 \ . \qquad (4.40)$$

As our coordinate frame is in the neighbourhood of $r = 2M$ due to the near horizon approximation, we can write the angular component of the metric as

$$r^2 d\Omega^2 \approx (2M)^2 d\Omega^2 = (2M)^2 (d\theta^2 + \sin^2\theta d\phi^2) \ , \qquad (4.41)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is an infinitesimal displacement on the surface of a sphere. We will use the relation between Cartesian and spherical coordinates

$$x = \sin\theta\cos\phi \qquad (4.42)$$

$$\text{and} \quad y = \sin\theta\sin\phi \qquad (4.43)$$

whose squared differentials can be written as

$$
\begin{aligned}
dx^2 + dy^2 &= \cos^2\theta d\theta^2 + \sin^2\theta d\phi^2 \\
&\neq d\theta^2 + \sin^2\theta d\phi^2 = d\Omega^2 \ . \qquad (4.44)
\end{aligned}
$$

We can then centre the coordinate frame at $\theta = 0$ as the geometry of th Schwarzschild black hole displays rotational invariance. However, centring removes one degree of freedom of Ω from $\Omega(\theta, \phi) \rightarrow \Omega(\phi)$. We require two degrees of freedom in order to approximate the two-dimensional spatial component of Rindler space, which is orthogonal to the acceleration. To get around this issue, we can let θ vary around zero by small amounts. Hence,

$$dx^2 + dy^2 \approx d\theta^2 + \theta^2 d\phi^2 \approx d\Omega^2 \ . \qquad (4.45)$$

So we can now write (4.39) as

$$ds^2 \approx -\rho^2 d\omega^2 + d\rho^2 + dx^2 + dy^2 \qquad (4.46)$$

From this we can see that there is an equivalence between the local geometry at the event horizon of a Schwarzschild black hole and the hyperbolic spacetime of a uniformly accelerating observer in Minkowski space. We can now say that the metric is locally hyperbolic for near horizon approximation. It is conformally flat in its time-radial component.

4.6 Near Horizon Coordinates with a (1+1) Minkowski Spacetime Metric

We can now define a local Minkowski space from the hyperbolic coordinates since the hyperbolic angle and local radial coordinates in flatspace are equivalent to ρ and ω [25]. Thus,

$$T \ = \rho \sinh \omega \ , \tag{4.47}$$

$$X = \rho \cosh \ , \tag{4.48}$$

and,

$$Y = y \ , \tag{4.49}$$

$$Z = z \ . \tag{4.50}$$

For the Schwarzschild metric, the surface gravity is

$$\kappa(r) = \frac{M}{r^2} \ . \tag{4.51}$$

About $r = r_0$, we can take its Taylor expansion using

$$\kappa(r = 2M) = \frac{1}{4M} \ . \tag{4.52}$$

Now, we will consider (4.50) to be the surface gravity of its black hole. From (4.38), $\omega = \kappa t$ and (4.46) and (4.47) becomes

$$T = \rho \sinh(\kappa t) \tag{4.53}$$

and

$$X = \rho \cosh(\kappa t) \ . \tag{4.54}$$

For an observer near the event horizon, T and X are the local (1+1)-dimensional Minkowski space coordinates. Here, κ and t are quantities that an observer can measure at infinity, just like the Hawking temperature of the black hole. Hence, these coordinates need to be re-scaled by some conformal factor so that they can be defined as distances that are measured using coordinates of an observer also at infinity.

The gravitational redshift normalisation factor is given by

$$f(r) = \frac{1}{\sqrt{\left(1 - \frac{2M}{r}\right)}}$$

We now multiply (4.35) by this factor, giving us

$$\begin{aligned} \rho(r) f(r) &= \frac{2\sqrt{2M(r-2M)}}{\sqrt{1 - \frac{2M}{r}}} \\ &= 2\sqrt{\frac{2M(r-2M)r}{(r-2M)}} \end{aligned}$$

$$= 2\sqrt{2Mr} \tag{4.55}$$

If we consider the region near $r = 2M$, we get

$$\begin{aligned}\rho(r \approx 2M)f(r \approx 2M) &\approx 2\sqrt{2M \cdot 2M} \\ &= 4M \\ &= \frac{1}{\kappa}\end{aligned} \tag{4.56}$$

Thus, near the event horizon $\rho \approx \frac{1}{\kappa}$. We can rewrite our local (1+1)-dimensional Minkowski coordinate as

$$T \approx \frac{1}{\kappa}\sinh(\kappa t) \tag{4.57}$$

and

$$X \approx \frac{1}{\kappa}\cosh(\kappa t) \tag{4.58}$$

An observer in a locally inertial rest frame S near the horizon can be described by the coordinates T and X. If a second observer in a reference frame S' is dropped into the black hole from the origin of S, S' will have some acceleration a with respect to S. This acceleration can be normalised to κ with respect to the coordinates of an observer at infinity.

The following set of basis vectors will be used for a frame that is comoving instantaneously with S' as the it starts falling into the black hole

$$e_{\mu'} = e_{\mu'}(\tau)$$

The 4-velocity vector of the comoving observer $e_{0'} = u$ and its local time axis are aligned if we we consider the observer to be in their own rest frame. $e_{1'}$ can be aligned with the acceleration of the observer, which coincides with the X axis of S. Spatial rotations of the basis vectors can be neglected by choosing

$$e_{2'} = e_2; \quad e_{3'} = e_3$$

We can relate the S basis vectors with the comoving basis vectors by a proper-time-dependent Lorentz transformation

$$e_{\mu'}(\tau) = \Lambda^{\nu}{}_{\mu'}(\tau)e_{\nu} \tag{4.59}$$

where

$$\Lambda^{\nu}{}_{\mu'} = \begin{pmatrix} \gamma & -\nu\gamma & 0 & 0 \\ -\nu\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.60}$$

$\nu(\tau)$ is a proper-time-dependent velocity boost along the X axis. It is the velocity required to bring the comoving frame up to the velocity of S'. Using $\nu = \tanh^{-1}(\kappa\tau)$ (4.59) can be written as

$$\Lambda^{\nu}{}_{\mu'}(\tau) = \begin{pmatrix} \cosh(\kappa\tau) & -\sinh(\kappa\tau) & 0 & 0 \\ -\sinh(\kappa\tau) & \cosh(\kappa\tau) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.61}$$

Using the inverse of the above Lorentz transformation matrix $\Lambda_\nu{}^{\mu'}$, we can obtain the local tangent space basis vectors

$$[e_{0'}(\tau)]^\mu = \Lambda_\nu{}^\mu(\tau)\,[e_0]^\nu = \begin{pmatrix} \cosh(\kappa\tau) & \sinh(\kappa\tau) & 0 & 0 \\ \sinh(\kappa\tau) & \cosh(\kappa\tau) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\kappa\tau) \\ \sinh(\kappa\tau) \\ 0 \\ 0 \end{pmatrix} \tag{4.62}$$

Similarly, for $[e_{i'}]^\mu$

$$[e_{1'}]^\mu = \begin{pmatrix} \cosh(\kappa\tau) \\ \sinh(\kappa\tau) \\ 0 \\ 0 \end{pmatrix} ;\ [e_{2'}]^\mu = \begin{pmatrix} \cosh(\kappa\tau) \\ \sinh(\kappa\tau) \\ 0 \\ 0 \end{pmatrix} ;\ [e_{2'}]^\mu = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} ;\ [e_{3'}]^\mu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.63}$$

The origin of S' at proper time τ is given by $A(\tau)$ where a spacelike hypersurface exists at each moment of proper time defined by the spacelike vectors in (4.62) and $A(\tau)$. Consider $x_0(\tau)$ and $x'(\tau)$ to be the position vector of $A(\tau)$ and the spacelike separation vector between some point $B(\tau)$ and $A(\tau)$ with respect to the origin of S'. $B(\tau)$ is described by the coordinate of S'

$$\xi^\mu = (\xi^0, \xi^1, \xi^2, \xi^3) = (\tau, \xi^i)$$

Therefore, the spacelike separation vector is given by

$$x'(\tau) = \xi^{i'} e_{i'}(\tau)$$

We can now define the position vector $x(\tau)$ of $B(\tau)$

$$x(\tau) = x_0(\tau) + x'(\tau) = x_0(\tau) + \xi^i e_{i'}(\tau) \tag{4.64}$$

The corresponding position 4-vector is

$$x^\mu(\tau) = x_0(\tau) + \xi^{i'}\,[e_{i'}(\tau)]^\mu \tag{4.65}$$

The position 4-vector of the local tangent space origin is

$$x_0^\mu(\tau) = \left[\frac{1}{\kappa}\sinh(\kappa\tau), \frac{1}{\kappa}\cosh(\kappa\tau), 0, 0\right] \tag{4.66}$$

From the second term of (4.64), the 4-vector is

$$\xi^{i'}\,[e_{i'}(\tau)]^0 = (0, \xi^{1'}, \xi^{2'}, \xi^{3'}) \cdot [\cosh(\kappa\tau), \sinh(\kappa\tau), 0, 0] = \xi^{1'}\sinh(\kappa\tau) \tag{4.67}$$

$$\xi^{i'}\,[e_{i'}(\tau)]^1 = (0, \xi^{1'}, \xi^{2'}, \xi^{3'}) \cdot [\sinh(\kappa\tau), \cosh(\kappa\tau), 0, 0] = \xi^{1'}\cosh(\kappa\tau) \tag{4.68}$$

and

$$\xi^{i'}\left[e_{i'}(\tau)\right]^2 = (0, \xi^{1'}, \xi^{2'}, \xi^{3'}) \cdot [0,0,1,0] = \xi^{2'} \tag{4.69}$$

$$\xi^{i'}\left[e_{i'}(\tau)\right]^3 = (0, \xi^{1'}, \xi^{2'}, \xi^{3'}) \cdot [0,0,0,1] = \xi^{3'} \tag{4.70}$$

Using the above 4-vector components and (4.65), we get

$$T(\xi^{\mu'}) = \left(\frac{1}{\kappa} + \xi^{1'}\right)\sinh(\kappa\xi^{0'}) \tag{4.71}$$

$$X(\xi^{\mu'}) = \left(\frac{1}{\kappa} + \xi^{1'}\right)\cosh(\kappa\xi^{0'}) \tag{4.72}$$

$$Y(\xi^{\mu'}) = \xi^{2'} \tag{4.73}$$

$$Z(\xi^{\mu'}) = \xi^{3'} \tag{4.74}$$

These are just the Minkowski coordinates in terms of the local coordinates of the comoving frame. The line element for S' is

$$ds^2 = -(1+\kappa\xi^{1'})^2(d\xi^{0'})^2 + (d\xi^{1'})^2 + (d\xi^{2'})^2 + (d\xi^{3'})^2 \tag{4.75}$$

4.7 Calculation of Hawking Temperature via Gravitational WKB Approximation

Following [25], from the line element of S', the (1+1)-dimensional component is

$$ds^2 = -(1+\kappa\xi^{1'})^2(d\xi^{0'})^2 + (d\xi^{1'})^2 \tag{4.76}$$

Let there be a coordinate transformation $\xi^{\mu'} \to q^\mu$ such as

$$\xi^{0'} \to q^0; \qquad \xi^{1'} \to \frac{1}{\kappa}\left(\sqrt{|1+2\kappa q^1|} - 1\right) \tag{4.77}$$

Taking its differentials and then substituting into the (1+1)-dimensional component of the line element for S' gives us

$$\xi^{0'} = dq^0 \tag{4.78}$$

$$d\xi^{1'} = \frac{dq^1}{\sqrt{|1+2\kappa q^1|}} \tag{4.79}$$

$$\begin{aligned} ds^2 &= -(1+\kappa q^1)^2(dq^0)^2 + (dq^1)^2 \\ &= -(1+2\kappa q^1)(dq^0)^2 + \frac{(dq^1)^2}{(1+2\kappa q^1)} \end{aligned} \tag{4.80}$$

In the q^μ coordinate frame, the Rindler horizon is at $q^1 = -\frac{1}{2\kappa}$ and at this point the metric is singular

$$\xi^{1'}(q^1 = -\frac{1}{2\kappa}) = \frac{1}{\kappa}\left[\sqrt{|1+2\kappa\left(-\frac{1}{2\kappa}\right)|} - 1\right] = \frac{1}{\kappa} \tag{4.81}$$

The Minkowski coordinates can be rewritten as

$$T = \frac{1}{\kappa}\sqrt{1+2\kappa q^1}\sinh(\kappa q^0) \tag{4.82}$$

$$X = \frac{1}{\kappa}\sqrt{1+2\kappa q^1}\cosh(\kappa q^0) \tag{4.83}$$

For a scalar field of mass m, the Hamilton-Jacobi equation are given as

$$g^{\mu\nu}\partial_\mu S \partial_\nu S + m^2 = 0 \tag{4.84}$$

Here, S is the classical action which we can separate into time and space components

$$S(q^0, q^1) = \omega q^0 + \overline{S}(q^1) \tag{4.85}$$

ω represents the energy of a tunneling particle and the second term comes from the spatial part of the classical action.

Using the (1+1)-dimensional Rindler spacetime metric in (4.79), we can work out the inverse metric components and so our Hamilton-Jacobi equations become

$$\begin{aligned} 0 &= g^{00}(\partial_0 S)^2 g^{11}(\partial_1 S)^2 + m^2 \\ &= -\frac{(\partial_0 S)^2}{1+2\kappa q^1} + (1+2\kappa q^1)(\partial_1 S)^2 + m^2 \end{aligned} \tag{4.86}$$

From (4.84), taking partial derivatives

$$\partial_0 S(q^0, q^1) = \frac{\partial S(q^0, q^1)}{\partial q^0} = \omega \tag{4.87}$$

$$\partial_1 S(q^0, q^1) = \frac{\partial S(q^0, q^1)}{\partial q^1} = \partial_1 \overline{S}(q^1) \tag{4.88}$$

Substituting the above 2 equations into (4.85)

$$-\frac{\omega^2}{1+2\kappa q^1} + (1+2\kappa q^1)\left[\partial_1 \overline{S}(q^1)\right]^2 + m^2 = 0 \tag{4.89}$$

From which we solve for $\overline{S}(q^1)$ as follows

$$\left[\partial_1 \overline{S}(q^1)\right]^2 = \left(\frac{\omega^2}{1+2\kappa q^1} - m^2\right)\left(\frac{1}{1+2\kappa q^1}\right)\frac{\omega^2 - m^2(1+2\kappa q^1)}{(2\kappa)^2\left[q^1 + (2\kappa)^{-1}\right]^2} \tag{4.90}$$

then,

$$\begin{aligned} \overline{S}(q^1) &= \int_\infty^\infty \partial_1 \overline{S}(q^1) dq^1 \\ &= \pm\frac{1}{2\kappa}\int_\infty^\infty \frac{\sqrt{\omega^2 - (1+2\kappa q^1)m^2}}{q^1 + (2\kappa)^{-1}} dq^1 \end{aligned} \tag{4.91}$$

The physical significance of the plus and minus signs are due to ingoing and outgoing particles tunneling at the Rindler horizon in the positive and negative q^1 directions respectively.

Let

$$\epsilon e^{i\phi} = q^1 + \frac{1}{2\kappa}; \quad dq^1 = i\epsilon e^{i\phi} d\phi \tag{4.92}$$

Considering ingoing particles that possess positive energy

$$\overline{S} = \lim_{\epsilon \leftarrow 0} \frac{1}{2\kappa} \int_{\pi}^{2\pi} \frac{\sqrt{\omega^2 - \left[1 + 2\kappa(\epsilon e^{i\phi} - (2\kappa)^{-1}\right] m^2}}{\epsilon e^{i\phi}} (i\epsilon e^{i\phi} d\phi)$$

$$= \frac{i\omega}{2\kappa} \int_{\pi}^{2\pi} d\phi = \frac{i\pi\omega}{2\kappa} \tag{4.93}$$

Doing a similar calculation for outgoing particles also give us the same value for $\overline{S}$. Therefore, we can write the total spatial contribution to the action as

$$S_{0,\text{total}} = \frac{i\pi\omega}{\kappa} \tag{4.94}$$

by summing both the terms for ingoing and outgoing particles. Now, we will focus on the time component of the classical action.

When $X = \frac{1}{\kappa}$, there is a rotation of coordinate axes into the complex plane which we can see from the invariant interval

$$X^2 - T^2 = \kappa^2$$

$$\Longrightarrow \pm\sqrt{X^2 - \frac{1}{\kappa^2}} \tag{4.95}$$

During this rotation, the amount of time occurring is defined as

$$T \equiv T_0 e^{i\phi} \tag{4.96}$$

Taking differentials on both sides gives us $dT = iT_0 e^{i\phi} d\phi$. Defining $X = \frac{1}{\kappa}$ to be the radius at which the axes rotate, then $T_0 = \frac{1}{\kappa}$. An imaginary time translation $\xi^{0'} \leftarrow \xi^{0'} - \frac{i\pi}{2}$ causes the axes to rotate

$$\omega \Delta T = \omega \int dT = \frac{i\omega}{\kappa} \int_{\frac{\pi}{2}}^{0} d\phi = -\frac{i\pi\omega}{2\kappa} \tag{4.97}$$

The coordinates T and X individually add a factor of $-\frac{i\pi\omega}{2\kappa}$ to the rate of tunneling across the Rindler horizon. Therefore, the total time component contribution to the classical action is

$$\omega \Delta T = \frac{i\pi\omega}{\kappa} \tag{4.98}$$

From the gravitational WKB approximation, the tunneling rate is

$$\Gamma \sim e^{-\frac{1}{\hbar}\left[\mathrm{Im}\left(\oint p_x dx\right) - \omega \mathrm{Im}(\Delta t)\right]} \tag{4.99}$$

The tunneling rate Γ is similar to a Boltzmann distribution of energy states

$$\Gamma \sim e^{-\frac{\omega}{T_H}} \tag{4.100}$$

where ω is the energy states and T_H is the temperature of the thermal flux. Hence,

$$\frac{\omega}{T_H} = \frac{1}{\hbar}\left[\mathrm{Im}\left(\oint p_x dx\right) - \omega \mathrm{Im}(\Delta t)\right] \tag{4.101}$$

Using natural units $\hbar = 1$

$$T_H = \frac{\omega}{\text{Im}\left(\oint p_x dx\right) - \omega \text{Im}(\Delta t)} \tag{4.102}$$

This is the Unruh temperature. Now,

$$\text{Im}\left(\oint p_x dx\right) = \text{Im}(\Delta t) = \frac{\pi\omega}{\kappa} \tag{4.103}$$

Therefore, (4.102) becomes

$$T_H = \frac{\kappa}{2\pi} \tag{4.104}$$

This is the Hawking temperature. We can treat κ as a constant by choosing a certain value of r at which we calculate the tunneling rates. Thus, we can write (4.103) as

$$T_H(r) = \frac{\kappa(r)}{2\pi} = \frac{M}{2\pi r^2} \tag{4.105}$$

Chapter 5

The Information Paradox

In this chapter, we will take a look at the information paradox issue as proposed by Hawking, and the consequences that arise as a result. We will ignore any of the possible effects that quantum gravity can affect, as we will be working within the confines of a limit where these effects become small where we will describe this limit as "solar system physics".

The solar system physics consider various conditions known as niceness conditions N such that physics is described more highly accurately as a small parameter ϵ is taken to be arbitrarily less than 1. From [19], the niceness conditions for local evolution are stated as follows

(N1) We introduce our spacelike slice, where we define our quantum state to be on, to have an intrinsic curvature ${}^{(3)}R$ *that should be a lot more smaller than the planck scale:* ${}^{(3)}R \ll \frac{1}{l_p^2}$.

(N2) The spacelike slice is embedded in a 4-dimensional spacetime. We define the extrinsic curvature of the slice to be K*, which should be small:* $K \ll \frac{1}{l_p^2}$.

(N3) In the neighbourhood of the spacelike slice, the 4-curvature should be small: ${}^{(4)}R \ll \frac{1}{l_p^2}$

(N4) All quanta that are present on the spacelike slice should have wavelengths much larger than planck length ($\lambda \gg l_p$)*. The energy and momentum densities* U *and* P *should be small on the slice compared to the planck density:* $U \ll l_p^{-4}$*. All usual energy conditions such as the dominant energy condition should be satisfied by any matter present on the slice. Therefore, any matter on the slice can be referred to as 'good'.*

(N5) As we evolve the spacelike slices to later slices, all the latter slices including the initial one should also be 'good'. The lapse and shift vectors, which are required to specify the evolution of the spacelike slices, should be changing smoothly with position: $\frac{dN^i}{ds} \ll \frac{1}{l_p}$, $\frac{dN}{ds} \ll \frac{1}{l_p}$

If we incorporate locality with these niceness conditions, we see that the evolution of our spacelike slices lead to many major issues which we will discuss in this chapter.

5.1 Hawking Pair Production

We will now introduce a set of slices in a vacuum state that satisfy the niceness conditions N. From [19], in the figure below, we show several initial slices that are evolved in the right hand region of the slice. In the middle region of the slice where it is evolved, there is a distortion in intrinsic geometry, leading to pair production of particles in that region. This is due to the fact that the later evolved slice is not in the natural vacuum state as in the initial slice. The very bottom part of the figure shows matter that is immensely far away from the region of pair production. Due to locality, there should be very weak correlation between the matter and the particle pairs.

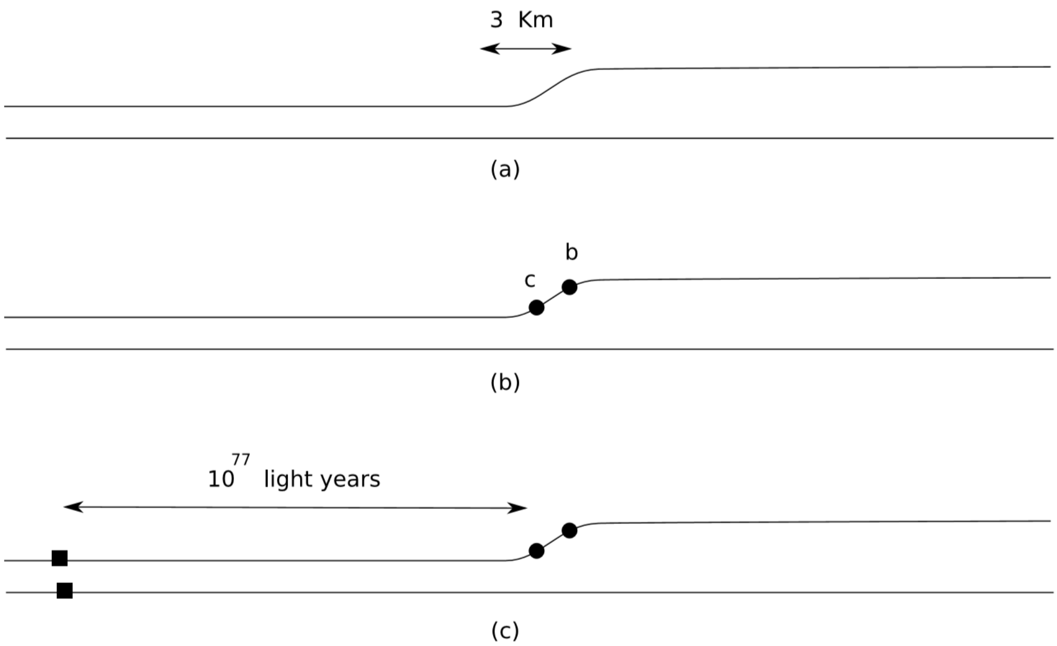

Figure 5.1: Evolution of Spacelike Slices [19]

The deformation of the geometry will give rise to particle-antiparticle creation. The time and length scale denoted by L characterises the geometry in the region of deformation. The particle pairs that are created in this region have wavelength $\lambda \sim L$. We will consider $L \sim 3$ since at the horizon of a black hole of solar mass, this is the length scale of curvature. The particle pair is denoted by c and b quanta. The particle pair is in a state such as

$$|\Psi\rangle_{\text{pair}} = \frac{1}{\sqrt{2}}(|0\rangle_c |0\rangle_b + |1\rangle_c |1\rangle_b) \tag{5.1}$$

On our previously introduced spacetime slice, there will be some kind of matter present in the $|\Psi\rangle_M$ state. However, the separation of the matter from our region of pair production is of the order $L^{'} \sim 10^{77}$ light years. Here, $L^{'} \geqslant L$, where L is the length and time scale describing the geometry at the deformation region.

Our state on the spacelike slice now becomes

$$|\Psi\rangle \approx |\Psi\rangle_M \otimes \frac{1}{\sqrt{2}} \left(|0\rangle_c |0\rangle_b + |1\rangle_c |1\rangle_b \right) \tag{5.2}$$

Despite such large separation distances, the state $|\Psi\rangle_M$ still has some effect on the Hawking pairs. We can write our matter state $|\Psi\rangle_M$ having a single up and down spin as

$$|\Psi\rangle_M = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle + |\downarrow\rangle \right)$$

If we consider $|\Psi\rangle_M$ to have no effect on the correlated pairs, the state on the spacelike slice becomes

$$|\Psi\rangle \approx \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_M + |\downarrow\rangle_M \right) \otimes \frac{1}{\sqrt{2}} \left(|0\rangle_c |0\rangle_b + |1\rangle_c |1\rangle_b \right) \tag{5.3}$$

Since locality allows some small deviations

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle + |\downarrow\rangle \right) \otimes \left(\frac{1}{\sqrt{2}} + \epsilon \right) |0\rangle_c |0\rangle_b + \left(\frac{1}{\sqrt{2}} + \epsilon \right) |1\rangle_c |1\rangle_b ; \quad \epsilon \leqslant 1 \tag{5.4}$$

However, we cannot write a state that is drastically different such as

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_M |0\rangle_c + |\downarrow\rangle_M |1\rangle_c \right) \otimes \frac{1}{\sqrt{2}} \left(|0\rangle_b + |1\rangle_b \right) \tag{5.5}$$

5.2 Quantification of Locality

From (5.3), there is only entanglement between the creation pairs b and c. They have no correlations with M. Hence, our entanglement entropy is

$$S_{ent} = -\mathrm{tr}(\rho \ln \rho) = \ln 2 \tag{5.6}$$

Similarly for (5.4)

$$S_{\mathrm{ent}} = -\mathrm{tr}(\rho \ln \rho) = \ln 2 - \epsilon^2 (6 - 2 \ln 2) \approx \ln 2 \tag{5.7}$$

However, (5.5) gives

$$S_{ent} = 0 \tag{5.8}$$

Consider the following limits

$$\frac{L}{l_p} \geqslant 1, \qquad \frac{L'}{l_p} \geqslant 1, \qquad \frac{L'}{L} \geqslant 1 \tag{5.9}$$

where l_p is the planck length. From the first 2 inequalities, we see that length scales are much greater than the planck length. The third inequality is due to the fact that the matter M on the spacelike slice is drastically separated from the regions of pair creation, which have length and time scales in the order of $\sim L$.

As stated earlier, the niceness conditions give us 'solar system physics'. Therefore, from (5.9)

$$\frac{S_{ent}}{\ln 2} - 1 \leqslant 1 \tag{5.10}$$

5.3 Traditional Black Hole

The black hole can be described by the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad (5.11)$$

where the coordinate singularity at $r = 2M$ is the horizon. The curvature in the region of the horizon is of the order $R \sim \frac{1}{M^2}$. We can call this a traditional black hole as there is no information about the black hole present in the region of the horizon. We get a time-independent black hole geometry from (5.11). We must make a set of spacelike slices such that it does not become intersected by the curvature singularity at $r = 0$, otherwise the niceness conditions N would no longer be valid everywhere on the slice.

5.4 Spacelike Slices

We shall now proceed to make a spacelike slice that satisfy the niceness conditions. For regions $r > 4M$, the slice is constant at $t = t_1$. The slices inside the region $r < 2M$ become $r = r_1 =$ constant where $\frac{M}{2} < r_1 < \frac{3M}{2}$. Thus, the slice is not close to the coordinate singularity $= 2M$.

These 2 parts of the slices can be connected together while still satisfying the niceness conditions with a connector segment C. We shall focus on black holes that begin with a flat space consisting of a shell of mass M converging to the origin where $r = 0$. With a collapsing black hole, we can easily follow the $r = r_1$ part of the slice to early times and eventually extend it to $r = 0$ at which point the singularity has not yet formed. This as a whole can now be considered a spacelike slice.

These spacelike slices will now be evolved over time at a point where a new entangled pair is produced as the previous pair increase in separation distance. For the evolved slice, at $r > 4M$, $t = t_1 + \Delta$ and $r = r_1 + \delta$ where $\delta_1 \leqslant M$ for the part of the slice where $r =$ constant. δ_1 is considered to be very small. Once again, the constant part of r and t is joined by a connector segment and $r =$ constant is brought down to $r = 0$ at early times of the black hole.

As the spacelike slice S_1 evolves, there will be no change in the intrinsic geometry of the region where $t =$ constant. We can describe the evolution of this part of the slice with the lapse function $N = \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}$.

Similarly, taking the limit $\delta_1 \longrightarrow 0$, there is also no change in intrinsic geometry of the constant part of r as it evolves from the initial slice to the evolved slice. As the initial slice S_1 evolves to later slices S_{n+1}, the connector segment between the 2 constant parts become more and more stretched, which can be interpreted as the fourier modes present near the horizon becoming stretched to longer wavelengths, leading to pair production. The constant parts are pushed away further over each evolution. Physically, the connector region has dimensions of order M

(3km). These slices all satisfy the niceness conditions. The figure below provides a visual representation of the spacelike slice being evolved to a later one, as introduced in [19].

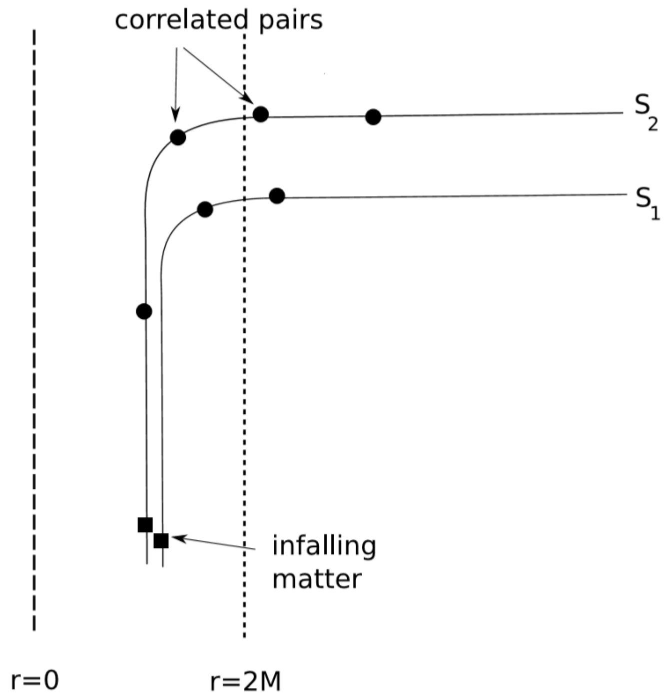

Figure 5.2: Schematic Set of Coordinates for the Black Hole [19]

5.5 Leading Order Hawking State

We will now focus on the leading order Hawking state of entangled pairs and analyse the consequences of taking small and order unity corrections. Clarifying the process of pair production near the black hole horizon, as the initial spacelike slice evolves to a later one, the connector region stretches and the initial pair of quanta c_1, b_1 and the matter $|\Psi\rangle_M$ move away from the region of pair creation, resulting in the creation of a new pair of quanta c_2, b_2 via a Schwinger process. There is no correlation whatsoever between the matter or previously produced quanta pair with the newly produced pair. There is only entanglement within the pairs of quanta. As more pairs of quanta are produced, the entanglement continues to increase despite the introduction of small corrections to the leading order state. The state on the spacelike slice after a new correlated pair of quanta is produced due to stretching of the connector region is given by

$$|\Psi\rangle \approx |\Psi\rangle_M \otimes \frac{1}{\sqrt{2}} \left(|0\rangle_{c_1} |0\rangle_{b_1} + |1\rangle_{c_1} |1\rangle_{b_1} \right) \tag{5.12}$$

Calculating the entanglement entropy of b_1 with M, c_1

$$S_{ent} = \ln 2 \tag{5.13}$$

As we evolve the spacelike slices, $|\Psi\rangle_M$ undergoes negligible change as this part of the slice does not have any evolution. After c_2, b_2 are produced due to the stretching of the middle part of the slice, the new state on the spacelike slice is

$$|\Psi\rangle \approx |\Psi\rangle_M \otimes \frac{1}{\sqrt{2}}\left(|0\rangle_{c_1}|0\rangle_{b_1} + |1\rangle_{c_1}|1\rangle_{b_1}\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle_{c_2}|0\rangle_{b_2} + |1\rangle_{c_2}|1\rangle_{b_2}\right) \tag{5.14}$$

The entanglement entropy of the outgoing set of quanta $\{b_1, b_2\}$ with the in-going quanta and matter $\{c_1, c_2, M\}$ is

$$S_{\text{ent}} = 2\ln 2 \tag{5.15}$$

Generalising this for N steps, our state on the spacelike slice becomes

$$\begin{aligned} |\Psi\rangle \approx |\Psi\rangle_M \otimes \frac{1}{\sqrt{2}}\left(|0\rangle_{c_1}|0\rangle_{b_1} + |1\rangle_{c_1}|1\rangle_{b_1}\right) \\ \otimes \frac{1}{\sqrt{2}}\left(|0\rangle_{c_2}|0\rangle_{b_2} + |1\rangle_{c_2}|1\rangle_{b_2}\right) ... \\ \otimes \frac{1}{\sqrt{2}}\left(|0\rangle_{c_N}|0\rangle_{b_N} + |1\rangle_{c_N}|1\rangle_{b_N}\right) \end{aligned} \tag{5.16}$$

Whose entanglement between $\{b_i\}$ and M, $\{c_i\}$ is

$$S_{\text{ent}} = N\ln 2 \tag{5.17}$$

The black hole mass M_{hole} will continuously decrease due to its evaporation, which is the result of Hawking pairs being produced over the evolution of the spacelike slices. If we reach a point where $M_{hole} \sim m_{pl}$, then our niceness conditions N will not all be satisfied as $R \leqslant l_p^{-2}$ no longer holds. Thus, we will not evolve the slices any further.

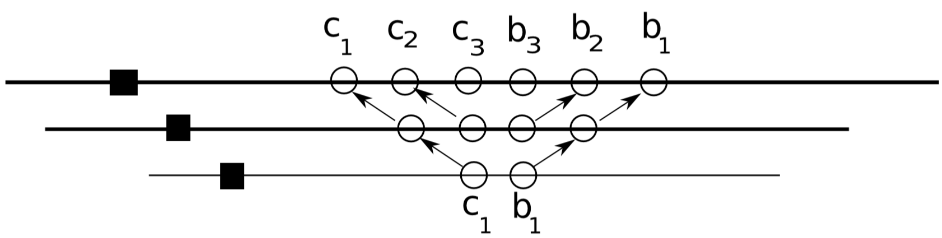

Figure 5.3: Evolution of Spacelike Slices at Region of Pair Production [19]

If a black hole does not evaporate completely and leaves something behind, we refer to that as a remnant. From [19], it is stated that such a thing referred to as remnants exist if we have a set of objects that have mass and size less than the bounds provided below exist

$$m < m_{remnant}, \quad l < l_{remnant} \tag{5.18}$$

and yet these remnants can have an entanglement that can be arbitrarily high with systems that are far away from the object. An object that can have a number of possible states similar to or greater than N, then it can have an entanglement like (5.17) with another system. Therefore, remnants will possess energy and size that satisfy the bounds stated above, but yet can have unbounded degeneracy.

We shall now discuss the few possibilities that the concept of applying spacelike slices to describe the activities of the traditional black hole near the horizon entails, and what problems that arise from such possibilities.

One possibility is that the final state that we are left with after complete evaporation of the black hole is a mixed state. This is the result of the set of all emitted quanta $\{b_i\}$ having an entanglement of $S_{ent} \sim N \ln 2 \neq 0$ even though they have nothing to be entangled with. This leads to the violation of unitarity as we get a mixed state after evolution of the spacelike slices even though we initially started off with a pure state.

The second possibility is the idea of remnants actually existing when the black hole stops evaporating after a certain point where $M_{hole} \sim m_{remnant}$, which is what $\{b_i\}$ will have entanglement with. However, the existence of remnants strays from the expected behaviour of quantum systems and it can lead to loop divergences.

So far, we have shown that as our spacelike slices evolve, pair creation occurs and our entanglement entropy increases by a factor of $\ln 2$ each time. However, at the endpoint of the black hole evaporation there can be 2 possibilities, both individually leading to some violations. We cannot assume which possibility will actually occur as the niceness conditions N are violated at this endpoint. The evaporation of the black hole also has a major distinction from the radiation of a normal hot body. The stretching of spacelike slices for black holes are not applicable to that of a hot body as the radiation from a hot body is directly leaving from the atoms in the body and thus, there is a separation of zero between the matter in the body and the radiation. Whereas for black holes, the matter making the black hole $|\Psi\rangle_M$ after half the evolution is separated from the region of pair production by a distance of order

$$L \sim M \left(\frac{M}{m_p}\right)^2 \sim 10^{77} \; light \; years \tag{5.19}$$

5.6 Stability of the Hawking State

We have not yet imposed any small corrections to the state (5.16), which should exist. Therefore, what we have covered so far is just an initial outline of the Hawking argument. We shall eventually introduce these small corrections which can arise from small interactions between consecutively produced pairs or instanton effects leading to $|\Psi\rangle_M$ having some effects on the pairs

$$A_{instanton} \sim e^{-S_{instanton}}, \quad S_{instanton} \approx GM^2 \tag{5.20}$$

where these effects are exponentially small. Here, the action of the standard instanton is used that is present in black hole physics

$$ds^2 = -\left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \tag{5.21}$$

where $\tau = \tau + 8\pi GM$. When we consider small corrections into the state (5.16), the entanglement occurring among the set $\{b_i\}$ and M, $\{c_i\}$ still remains and the paradox is not solved. The conclusion that is reached in the original Hawking argument does not change. This fact is known as the "stability of entanglement of the Hawking state".

We will now denote the state of the matter shell that fell into the black hole, as well as all the in-falling quanta c before the time step t_n as $\Psi_{M,c}$. The set of all outgoing quanta produced before this time step is ψ_b. $|\Psi_{M,c}, psi_b(t_n)\rangle$ represents the state of the modes at t_n. Evolving the spacelike slice to the step t_{n+1}

$$|\Psi_{M,c}, \psi_b(t_n)\rangle \rightarrow |\Psi_{M,c}, \psi_b(t_n)\rangle \frac{1}{\sqrt{2}} \left[|0\rangle_{c_{n+1}} |0\rangle_{b_{n+1}} + |1\rangle_{c_{n+1}} |1\rangle_{b_{n+1}} \right] \tag{5.22}$$

Here, the term in the box brackets represent the state of the newly created pair. We will show that imposing small corrections does not circumvent around the possibilities of having mixed states or remnants at the endpoint of black hole evaporation.

As our spacelike slice evolves, we will get a pair creation in one mode at each step and the mode can only have an occupation number of 0 or 1. The new region that is created due to the stretching of the connector part is spanned by 2 vectors with the basis states

$$\begin{aligned} S^{(1)} &= \frac{1}{\sqrt{2}} \left(|0\rangle_{c_{n+1}} |0\rangle_{b_{n+1}} + |1\rangle_{c_{n+1}} |1\rangle_{b_{n+1}} \right) \\ S^{(2)} &= \frac{1}{\sqrt{2}} \left(|0\rangle_{c_{n+1}} |0\rangle_{b_{n+1}} - |1\rangle_{c_{n+1}} |1\rangle_{b_{n+1}} \right) \end{aligned} \tag{5.23}$$

At time t_n, we have the state $|\Psi_{M,c}, \psi_b(t_n)\rangle$ where there is entanglement between the emitted quanta b_i and the shell of matter inside the black hole M, and all in-going quanta c_i up to t_n. Now, for the quanta inside the hole we can choose a basis of orthonormal state ψ_n and similarly for b_i we can choose χ_n to be its orthonormal basis. Hence,

$$|\Psi_{M,c}, \psi_b(t_n)\rangle = \sum_{m,n} C_{mn} \psi_m \chi_n \tag{5.24}$$

We can make unitary transformations on ψ_i and χ_j

$$|\Psi_{M,c}, \psi_b(t_n)\rangle = \sum_i C_i \psi_i \chi_i \tag{5.25}$$

The reduced density matrix describing b_i is given by

$$\rho_{ij} = |C_i|^2 \delta_{ij} \tag{5.26}$$

Thus, at time step t_n, the entanglement entropy is

$$S_{ent}(t_n) = -\sum_i |C_i|^2 \ln |C_i|^2 \tag{5.27}$$

All the b_i quanta that have been produced up to this time step t_n is left unaffected as we evolve the spacelike slice to the next time step t_{n+1}. Otherwise, locality would be violated since the emitted quanta are too largely separated from the matter inside the black hole or the in-going quanta. Thus, at t_{n+1}

$$\chi_i \to \chi_i \tag{5.28}$$

$$\psi_i \to \psi_i S^{(1)} + \psi_i^{(2)} S^{(2)} \tag{5.29}$$

ψ_i evolves to a tensor product to ψ_i that describes (M, c_i) and $S^{(i)}$. This evolved state represents the state of the new pair that is created. From unitarity of evolution

$$||\psi_i^{(1)}||^2 + ||\psi_i^{(2)}||^2 = 1 \tag{5.30}$$

For the leading order evolution

$$\psi_i^{(1)} = \psi_i, \quad \psi_i^{(2)} = 0 \tag{5.31}$$

Substituting (5.29) into the state (5.25) at the time step t_{n+1}

$$|\Psi_{M,c}, \psi_b(t_{n+1})\rangle = \sum_i C_i \left[\psi_i S^{(1)} + \psi_i^{(2)} S^{(2)}\right] \chi_i \tag{5.32}$$

We will now compute the entanglement $S_{ent}(t_{n+1})$ for this state at t_{n+1} between the b quanta (which includes all quanta up to the one produced at t_{n+1}) and (M, c) using (5.27). We can write the state (5.32) as

$$\begin{aligned} |\Psi_{M,c}, \psi_b(t_{n+1})\rangle &= S^{(1)} \left[\sum_i C_i \psi_i^{(1)} \chi_i\right] + S^{(1)} \left[\sum_i C_i \psi_i^{(2)} \chi_i\right] \\ &\equiv S^{(1)} \Lambda^{(1)} + S^{(2)} \Lambda^{(2)} \end{aligned} \tag{5.33}$$

where

$$\Lambda^{(1)} = \sum_i C_i \psi_i^{(1)} \chi_i, \quad \Lambda^{(2)} = \sum_i C_i \psi_i^{(2)} \chi_i \tag{5.34}$$

From normalisation of the state (5.33) consisting of the basis vectors $S^{(1)}, S^{(2)}$ that span the newly created connector region, we get

$$||\Lambda^{(1)}||^2 + ||\Lambda^{(2)}||^2 = 1 \tag{5.35}$$

We will now define corrections to be small if

$$||\Lambda^{(2)}||^2 < \epsilon, \quad \epsilon \ll 1 \tag{5.36}$$

If this bound is not satisfied, the corrections imposed will be of the order unity. Since we are working with the traditional black hole, we know that the effect of the

b quanta on our spacelike slice is small under the niceness conditions N. Thus, we can find the $S^{(1)}$ state when a pair is created and the probability for finding the $S^{(2)}$ state will be a lot less than unity. The above condition for small conditions still allows the amplitude for $S^{(2)}$ to be large for ψ_i if C_i is small enough. Hence, if we have a probability that is close to unity for the newly created pair to be in the $S^{(1)}$ state, then we can have small corrections consistent with the above bound.

5.7 Entropy Bounds

We shall now show that even after imposing small corrections to the state (5.33), our entanglement entropy will continue to increase after every time step as we evolve our spacelike slice. We will call our entanglement entropy at t_n to be S_0. We will prove that the increase in entropy after each step should be close to the value $\ln 2 - 2e$, considering the bound (5.36) holds.

Let's consider a system in a pure state that consists of smaller subsystems denoted by A, B, and C. Then $S(A) \equiv -\text{tr}\rho_A \ln \rho_A$ gives the entanglement entropy of subsystem A with the rest of the subsystems B,C. Here, ρ_A is the density matrix for the subsystem, A. Likewise, $S(A+B)$ is the entanglement entropy of the system $(A+B)$ with C.

We shall now work with 3 subsystems. The set b consist of all the outgoing quanta that are produced up to and including the step t_{n+1}, which we consider to be one subsystem. The entropy at this step is denoted as S_0. The 2nd subsystem is (M, c) containing the shell of matter in the black hole and all the complement quanta of the set b that have entered the black hole. The pair created in the next step can have some weak interactions with this subsystem, leading to entanglements that we never considered in the Hawking state of leading order. Finally, the 3rd subsystem is the pair $p \equiv (c_{n+1}, b_{n+1})$ created in the next step t_{n+1}. We assume that the set of quanta b does not have any impact on future pair creation and thus, at t_{n+1} we still have

$$S_b = S_0 \tag{5.37}$$

We eventually see that $S(b, b_{n+1}) > S_n - 2e$, which shows that we still have an increasing entanglement entropy over time despite imposing small corrections. We shall now introduce 3 lemmas in order to show this and prove the stability of entanglement of the Hawking state of leading order. The proof for the following lemmas can be found in [19].

Lemma 1: Considering the bound (5.36) be satisfied, the entanglement of the new pair p with the rest of the system is bounded as

$$S(c_{n+1}, b_{n+1}) \equiv -\text{tr}\rho_{(c_{n+1},b_{n+1})} \ln \rho_{(c_{n+1},b_{n+1})} < \epsilon \tag{5.38}$$

Lemma 2:

$$S(b+p) \geqslant S_0 - \epsilon \tag{5.39}$$

Lemma 3:

$$S_{c_{n+1}} > \ln 2 - \epsilon \tag{5.40}$$

As stated before, if at step t_n we have entanglement entropy S_0, in the next step t_{n+1} the Hawking state of leading order can change by a very small amount less than $\epsilon \ll 1$, satisfying our bound (5.36). If so, then the entanglement entropy at this step of $b_1, ..., b_{n+1}$ is

$$S(b + b_{n+1}) > S_0 + \ln 2 - 2e \tag{5.41}$$

Therefore, considering the bound is satisfied, we will have an increase in entropy after each step during the evolution. To prove this, we will apply the theorem of strong subadditivity of entropy for 3 systems

$$S(A + B) + S(B + C) \geqslant S(A) + S(C) \tag{5.42}$$

Setting $A = b$, $B = b_{n+1}$ and $C = c_{n+1}$, we get

$$S(b + b_{n+1}) + S(p) \geqslant S(b) + S(c_{n+1}) \tag{5.43}$$

Using (5.37), (5.40) and $S(p) < \epsilon$ which comes from the proof of lemma 1, we get

$$S(b + b_{n+1}) > S_0 + \ln 2 - 2e \tag{5.44}$$

From this, we have proven that considering that the bound holds, our entanglement entropy after each step increases by $\ln 2 - 2e$. Therefore, entanglement does not go down but continues to increase and we still get the issues that arise from the possibilities of getting remnants and mixed states at the endpoint of black hole evaporation unless it is possible for us to have order unity corrections.

Chapter 6

Bekenstein-Hawking Entropy

6.1 Quantisation of Space

The simple hypothesis of non-commutative spatial dimension gives rise to a spectrum that is very similar to that of quantum loop gravity and it resembles a black hole spectrum. We follow from paper [30], that the quantisation of space has lead to a discrete spectrum of the black hole area. The discrete spectra is expressed in terms of $2\pi\hbar\Theta$, and it takes the form,

$$A_n = 2\pi\hbar\Theta\left(n + \frac{1}{2}\right) \tag{6.1}$$

Here, n can be assumed to be the number of quantities of space. However, n is speculated to be rather the idea of state number instead of space quantities. The concept of n could be explained but the harmonic oscillator having different energy states with a zero energy state at the ground.

Assuming the introduction of mass or energy into the space in question would enable excitation of its quanta. Hence, we can imply that there is a direct relationship between the energy absorbed by the space and the discrete area spectrum mentioned in (6.1). This relationship is described to be

$$E_n = \alpha A_n \tag{6.2}$$

where $\alpha = k\Omega$ and k is an unknown dimensionless constant, and Ω is introduced in order to set the dimension.

The average area of space quanta is expressed in the form,

$$\langle A \rangle = \frac{\sum_{n=0}^{\infty} A_n e^{-\beta A_n}}{\sum_{n=0}^{\infty} e^{-\beta A_n}} \tag{6.3}$$

The dimension of A_n is of area and the argument $(-\beta A_n)$ is dimensionless, therefore, the dimension of β should be the inverse of area.

This ensures that we define β as $\beta = \frac{\alpha}{k_B T}$ where k_B and T have their usual meanings, k_B is the Boltzmann constant and T is the system temperature. Also, equation (6.2) is expressed using the partition function that is $\sum_{n=0}^{\infty} e^{-\beta A_n}$.

With the incorporation of all of that and simplification, we can express the average energy equation as,

$$\langle A \rangle = \hbar\Theta \left(1 + \frac{2}{e^{2\beta(\hbar\Theta)} - 1}\right) \tag{6.4}$$

Furthermore, we can write the relationship between the total area of N quanta and the average area of each quanta, assuming there exists no interaction between the space quanta. It will take the form,

$$A_0 = N\langle A \rangle \tag{6.5}$$

Since we began with the assumption that the space quanta do not have interactions among themselves, the partition function of the system can be expressed as,

$$Z = (Z)^N = \left(\frac{e^{-\beta\hbar\Theta}}{1 - e^{-2\beta\hbar\Theta}}\right)^N \tag{6.6}$$

6.2 Entropy Calculation

Now, we calculate the entropy of the system using the partition function;

$$S = \left(\frac{k_B c^3 A_0}{\hbar G}\right)\gamma(T) \tag{6.7}$$

where $\gamma(T)$ is defined as

$$\gamma(T) = \left(\frac{b}{T} - \Delta\right)$$

with $b = kT_p$ and $T_p =$ Planck Temperature

$$\Delta = \left(\frac{e^{\frac{b}{T}} - 1}{e^{\frac{b}{T}} + 1}\right) \ln\left(e^{\frac{b}{T}} - e^{\frac{b}{T}}\right)$$

Now, for specific cases, the entropy equation will show certain characteristics. If we set the temperature to zero ($T \longrightarrow 0$), entropy tends to go to zero ($S \longrightarrow 0$). The entropy we defined is a function of temperature and area. On the contrary, however, Bekenstein-Hawking entropy is a function of the area only. Our task is to conjure up a connection between these two entropies.

6.3 Maximum Entropy

Maximising the entropy equation for temperature, we get

$$\frac{\partial S}{\partial T} = 0 \quad \Longrightarrow \quad T = \frac{KT_p}{\ln\left(\frac{\sqrt{5}+1}{2}\right)}$$

Putting the temperature value to the entropy equation will get us the maximum entropy,

$$S_{\max} = \left(\frac{k_B c^3 A_0}{\hbar G}\right) \ln \frac{\sqrt{5}+1}{2} \tag{6.8}$$

where the entropy is a function of the surface area only.

The entropy obtained has similarities with the Bekenstein-Hawking entropy except the coefficient, which is slightly different. The coefficient for the maximum entropy is,

$$\ln\left(\frac{\sqrt{5}+1}{2}\right) \approx \frac{1}{2}$$

This gives us an entropy,

$$S_{max} \approx \left(\frac{k_B c^3 A_0}{2\hbar G}\right) \tag{6.9}$$

6.4 Origin of the Entropy

If the maximised entropy is used to explain the entropy of black holes, then it can be said that the origin of this entropy is due to the excitation of space quanta. Energy existing in space is spread out evenly among the space quanta, but since the quanta of space is in an excited state, the energy may be dividing in different ways.

The source of the black hole entropy is the result of our lack of information about the possible number of states that space quanta can acquire in the presence of some external energy.

This is in regard of when, for example, a finite energy enters the black hole's horizon. The space quanta will receive the energy and become excited. However, we do not know the extent of this excitation. This lack of information leads us to having an entropy of the black hole and the space-time.

From the above equations, we can see that we arrive with an expression for the entropy by maximising the temperature. This gives us a maximum entropy. Furthermore, we know from thermodynamics and statistical mechanics that a system is in equilibrium when it is at its maximum entropy. So, therefore, this maximised entropy suggests that this system is in thermodynamic equilibrium.

6.5 Bekenstein-Hawking Entropy

In a paper [9], published by Jacobson in 1995, he showed that Einstein's gravity equation can be obtained from thermodynamic equilibrium by incorporating Claussius equation. Jacobson used the Bekenstein-Hawking entropy and incorporated it into the Claussius equation,

$$S_{BH} = \left(\frac{k_B c^3 A_0}{4}\right) \longrightarrow \quad \partial S = \frac{\partial Q}{T} \tag{6.10}$$

where A_0 is the area of the horizon.

$$\partial A_0 = -\int \lambda R_{ab} k^a k^b d\lambda dA$$

and

$$\partial Q = -\int \lambda T_{ab} k^a k^b d\lambda dA$$

By carrying out a series of calculations, the Einstein Field equation was obtained.

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab} \tag{6.11}$$

However, if we place the entropy equation that we obtained in equation (6.9), we formulate the Einstein Field equation,

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = \frac{4\pi G}{c^4} T_{ab} \tag{6.12}$$

If we expand this equation in the case for weak fields, we get the equation for Newtonian gravity,

$$F = \frac{GMm}{2r^2}$$

Except there is a 2-factor in the denominator. We derived the Bekenstein-Hawking entropy considering maximum entropy, and it shows that if we take a factor of $2\langle A\rangle$ instead of $\langle A\rangle$, the coefficient issue gets resolved. So, we take

$$A_0 = 2N\langle A\rangle$$

This leads us to have the equation for Newtonian gravity as follows,

$$F = \frac{GMm}{r^2}$$

Now, for the maximum entropy equation, we have

$$S_{\max} = \frac{k_B c^3 A_0}{2\hbar G} \ln\left(\frac{\sqrt{5}+1}{2}\right)$$

where the constant numerical value corresponds to,

$$\ln\left(\frac{\sqrt{5}+1}{2}\right) \approx \frac{1}{4}$$

Furthermore, the entropy that we obtained is very similar to the Bekenstein-Hawking entropy.

$$\text{error} = \frac{S_{\max} - S_{BH}}{S_{BH}} = 0.037$$

Now, since we get the error to be very minute, we can finally analyse that the entropy we obtained is equal to the Bekenstein-Hawking entropy. By quantising space, and maximising entropy for thermodynamic equilibrium, we obtain the Bekenstein-Hawking entropy

$$S_{\max} = S_{BH}$$

This paves the way to understanding that gravity is nothing but the excitations of space quanta.

Chapter 7

Spectroscopy of Quantum Black Holes

7.1 Quantisation of a Black Hole

In [8], Bekenstein and Mukhanov show that in quantum gravity, for various series of quanta that are produced near the black hole horizon, they all display characteristics of thermal radiation, which is due to the degeneracy of near levels for quantum transitions, and that a black hole should give us a discrete mass spectrum. The spectral lines from the line emission should be broad and separate.

In [8], it is stated that quantum systems that have a finite size usually display a discrete energy spectrum. Therefore, a black hole should also give us a similar mass spectrum. The smallest black holes can be compared to elementary particles and thus, can also be described by some quantum numbers such as mass, charge, spin, etc. Hence, the horizon area of a black hole can be quantised in integers. As the black hole gets larger, it becomes more classical. The black hole horizon area is

$$A_n = \alpha\hbar\text{n}; \qquad \text{n} = 1{,}2...,; \qquad (G = c = 1) \tag{7.1}$$

where α is a pure number. (7.1) implies that if we assume the commutativity of the mass and area operators that is insinuated by the classical relation $A = 16\pi M^2$, we should get a discrete mass spectrum for a non-rotating neutral black hole. The multiplicity (or degeneracy) of an energy level, n, can be denoted by $g(n)$. We can then indentify the Bekenstein-Hawking entropy of the black hole in the nth level with the natural logarithm of the multiplicity

$$\frac{\alpha n}{4} + C = \ln g(n)$$

We naturally assume that at the non-degenerate ground state, $S_{BH}(n = 1) = 0$ and $g(1) = 0$, which leads us to choose

$$g(n) = \exp\frac{\alpha(n-1)}{4}$$

$$\Longrightarrow g(n) = e^{S_{BH}}$$

However, $g(n)$ has to be integral so

$$\alpha = 4\ln k; \qquad k = 2, 3, 4...$$

and

$$g(n) = e^{4\ln k\frac{(n-1)}{4}} = k^{n-1} \tag{7.2}$$

Our horizon area can then be written as

$$A_n = 4n\hbar\ln(k)$$

A value of $k = 2$ is chosen as it is seen to be best fitting for several reasons. For this value, our entropy spacing for consecutive energy levels becomes 1 bit and our multiplicity leads to $g(n) = 2^{n-1}$. Interestingly, this also exactly represents the number of ways a black hole starting from a horizon area of $A = 0$ can be raised up the ladder of levels n in all possible ways. Similarly, a black hole at the level n can also decay in such number of steps to a horizon area of $A = 0$ where we cease to have a black hole. Hence, we get

$$S_{BH} = (n-1)\ln 2; \qquad \alpha = 4\ln 2$$

For zero charges and spin, mass spectrum is of the form

$$M \propto \sqrt{n}; \qquad n = 1, 2, ...$$

Therefore, the fundamental frequency for the energy spacing between consecutive energy levels $n \Longrightarrow n-1$ in the case of $M \gg \hbar$ is

$$\bar{\omega} = \frac{dM}{\hbar} = \frac{\ln 2}{8\pi M} \tag{7.3}$$

which agrees with Bohr's correspondence principle, stating that at larger quantum numbers, we should get oscillation frequencies that are more classical. Our line emissions should consist of lines concentrated at integer multiples of $\bar{\omega}$. However, below this fundamental frequency we should get little to no radiation at all, otherwise we should get effects of quantum gravity above the Planck scale. Also, below this frequency, the quantum black hole cannot absorb a single quantum.

7.2 De-excitation Probabilities

In order to learn more about the properties of the emissions near the horizon of our quantum black hole, we shall look at some de-excitation probabilities, as explained in [8]. We define Δt to be some interval of time corresponding to a sequence of integers $\{n_1, n_2, ...n_j\}$. Here, n_j denotes the number of levels jumped down by the black hole in the jth jump. After each jump, a particle of some species having energy $n_k\hbar\bar{\omega}$ $(k = 1, 2, ..., j)$ is emitted. For a sequence having zero length $\{0\}$ $(j = 0)$, the black hole does not decay at all in Δt.

For a length j, we can have a corresponding conditional probability $P_{\Delta t}(\{n_1, n_2, ...n_j\} \mid j)$. For normalisation

$$\sum_{\{n_1, n_2, ...n_j\}} P_{\Delta t}(\{n_1, n_2, ...n_j\} \mid j) = 1 \tag{7.4}$$

where all n_k are non-zero. $P_{\Delta t}(j)$ represents the probability that in the duration of Δt, there are exactly j jumps occurring. Thus j quanta are also emitted over this duration. If we can consider $P_{\Delta t}(j)$ to be small enough, then the decay over this interval is very little, leaving the black hole to have almost the same mass. Therefore, we can consider the immediate next interval Δt to also be equal to the previous. Hence

$$P_{2\Delta t}(1) = P_{\Delta t}(0)P_{\Delta t}(1) + P_{\Delta t}(1)P_{\Delta t}(0) = 2P_{\Delta t}(0)P_{\Delta t}(1) \tag{7.5}$$

The above simply means that in the interval of $2\Delta t$, we may either have no jumps in the first half of the interval and a single jump in the second half, or the reverse. We can generalise (7.5) to any odd j

$$\begin{aligned} P_{2\Delta t}(j) = 2P_{\Delta t}(0)P_{\Delta t}(j) + 2P_{\Delta t}(1)P_{\Delta t}(j-1) + ... \\ ... + 2P_{\Delta t}(j/2 + 1/2)P_{\Delta t}(j/2 - 1/2) \end{aligned} \tag{7.6}$$

Similarly, we can do the same for even j

$$P_{2\Delta t}(j) = 2P_{\Delta t}(0)P_{\Delta t}(j) + 2P_{\Delta t}(1)P_{\Delta t}(j-1) + ... + [P_{\Delta t}(j/2)]^2 \tag{7.7}$$

Following from (7.5), for $j = 0$

$$P_{2\Delta t}(0) = P_{\Delta t}(0)P_{\Delta t}(0) = [P_{\Delta t}(0)]^2 \tag{7.8}$$

This is the survival probability of the black hole over the interval $2\Delta t$ in a given level. The solution to (7.8) is given by

$$P_{\Delta t}(0) = e^{-\frac{\Delta t}{\tau}} \tag{7.9}$$

Here, τ is introduced as a survival timescale that is later defined to be the mean time between quantum leaps. Substituting the above solution into (7.5)

$$P_{2\Delta t}(1) = 2e^{-\frac{\Delta t}{\tau}} P_{\Delta t}(1)$$

which gives us a solution for $P_{\Delta t}(1)$

$$P_{\Delta t}(1) = \left(\frac{\Delta t}{\tau^*}\right) e^{-\frac{\Delta t}{\tau}} \tag{7.10}$$

Now, for $P_{\Delta t}(2)$, we can similarly get a functional equation by substitution of (7.9) and (7.10) into (7.7)

$$P_{2\Delta t}(j) = 2e^{-\frac{\Delta t}{\tau}} P_{\Delta t}(j) + 2\left(\frac{\Delta t}{\tau^*}\right) e^{-\frac{\Delta t}{\tau}} P_{\Delta t}(j-1) + ... + [P_{\Delta t}(0)]^2$$

which we can solve to get

$$P_{\Delta t}(2) = \left(\frac{1}{2}\right)\left(\frac{\Delta t}{\tau^*}\right)^2 e^{-\frac{\Delta t}{\tau}} \tag{7.11}$$

From the above solutions, we can assume that for any j

$$P_{\Delta t}(j) = \left(\frac{1}{j!}\right)\left(\frac{\Delta t}{\tau^*}\right)^j e^{-\frac{\Delta t}{\tau}} \tag{7.12}$$

In order to verify this, we substitute (7.9) and (7.12) into (7.6)

$$P_{2\Delta t}(j) = 2e^{-\frac{\Delta t}{\tau}} \left(\frac{1}{j!}\right) \left(\frac{\Delta t}{\tau^*}\right)^j e^{-\frac{\Delta t}{\tau}} + 2\left(\frac{\Delta t}{\tau^*}\right) e^{-\frac{\Delta t}{\tau}} \frac{1}{(j-1)!} \left(\frac{\Delta t}{\tau^*}\right)^{j-1} e^{-\frac{\Delta t}{\tau}}$$

$$+ ... + 2\frac{1}{(j/2+1/2)!} \left(\frac{\Delta t}{\tau^*}\right)^{j/2+1/2} e^{-\frac{\Delta t}{\tau}} \frac{1}{(j/2-1/2)!} \left(\frac{\Delta t}{\tau^*}\right)^{j/2-1/2} e^{-\frac{\Delta t}{\tau}}$$

$$\Longrightarrow P_{2\Delta t}(j) = 2\left(\frac{\Delta t}{\tau^*}\right)^j e^{-\frac{2\Delta t}{\tau}} \left[\frac{1}{0!j!} + \frac{1}{1!(j-1)!} + \frac{1}{2!(j-2)!} + ... + \frac{1}{(j/2+1/2)!(j/2-1/2)!}\right]$$

Multiplying by $j!$ we get,

$$(j!)P_{2\Delta t}(j) = 2\left(\frac{\Delta t}{\tau^*}\right)^j e^{-\frac{2\Delta t}{\tau}} \left[1 + j + \frac{j(j-1)}{2!} + \frac{j(j-1)(j-2)}{3!} + ... + \frac{j(j-1)...(j/2+1/2)}{2(j/2)!}\right]$$

$$\Longrightarrow (j!)P_{2\Delta t}(j) = 2\left(\frac{\Delta t}{\tau^*}\right)^j e^{-\frac{2\Delta t}{\tau}} [2^{j-1}]$$

$$2^{j-1} - 1 = j + \frac{j(j-1)}{2!} + \frac{j(j-1)(j-2)}{3!} + ... + \frac{j(j-1)...(j/2+1/2)}{2(j/2)!} \tag{7.13}$$

which is just the binomial expansion of $(1+1)^j$. Therefore, (7.12) is correct for all odd j. In a similar fashion, we can also verify (7.12) for even j. Now, checking the normalisation for $P_{\Delta t}(j)$

$$\sum_j P_{\Delta t}(j) = \sum_j \frac{(\frac{\Delta t}{\tau^*})^j}{j!} \exp\left(-\frac{\Delta t}{\tau}\right)$$

$$\Longrightarrow \sum_j \exp\left(\frac{\Delta t}{\tau^*} - \frac{\Delta t}{\tau}\right) \tag{7.14}$$

For normalisation, we must set $\tau^* = \tau$

$$P_{\Delta t}(j) = \left(\frac{1}{j!}\right) \left(\frac{\Delta t}{\tau}\right)^j e^{-\frac{\Delta t}{\tau}} \tag{7.15}$$

This clearly resembles Poisson's probability distribution. Hence, we have proven that the number of quanta j that are emitted during the interval $P_{\Delta t}$ follows a Poisson probability distribution.

As shown in [8], we shall now take a look at one series of quanta and focus on its probability distribution. $P_{\Delta t}(k \mid n_k\bar{\omega})$ refers to the probability that in the interval Δt, there will be an emission of k quanta, each of which possess a frequency of

$n_k\bar{\omega}$. If there are j jumps occurring over this interval, the many ways that we can choose k quanta out of j is given by

$$C_k^j = \frac{j!}{k!(j-k)!}$$

where $2^{-(n_1+n_2+\ldots+n_{j-k})}(2^{-n_k})^k$ gives the probability for each one of the selections. Without delving into too much explicit details

$$P_{\Delta t}(k \mid n_k\bar{\omega}, j) = \frac{j!}{k!(j-k)!}\left(1 - \frac{1}{2^{n_k}}\right)^j \left(\frac{1}{2^{n_k}}\right)^k \tag{7.16}$$

Multiplying by $P_{\Delta t}(j)$ and summing over all $j \geqslant k$,

$$P_{\Delta t}(j)P_{\Delta t}(k \mid n_k\bar{\omega}, j) = \sum_{j=k}^{\infty} \frac{(\Delta t/\tau)^j}{(j-k)!} e^{-\frac{\Delta t}{\tau}} \left(\frac{1}{k!}\right) [1 - 2^{-n_k}]^j [2^{n_k} - 1]^{-k}$$

$$\implies P_{\Delta t}(k \mid n_k\bar{\omega}) = \left(\frac{1}{k!}\right)(x_{n_k})^k e^{-x_{n_k}}; \qquad x_n = \left(\frac{\Delta t}{\tau}\right) 2^{-n} \tag{7.17}$$

Once again, we have arrived at a Poisson probability distribution. However, so far we have assumed that we should get a thermal distribution. Thus, we shall try to show that our Poisson distributions can be consistent with that of a thermal one.

We consider a random sub-volume consisting of a certain amount of quanta in a black-body cavity that is held at temperature, T. We should get a Boltzmann distribution for every quantum that is extracted from this sub-volume. For a single quantum that is taken out possessing frequency ω has the probability $Ae^{-\frac{\hbar\omega}{T}}$. The probability of taking out k quanta from a series of j drawings of frequency ω_k is given by

$$C_k^j A^k e^{-\frac{k\hbar\omega_k}{T}} A^{j-k} \prod_{i=1}^{j-k} e^{-\frac{\hbar\omega_i}{T}}$$

This is summed over all ω_i that are distinct from ω_k. Due to normalisation

$$A \sum_{\omega} e^{-\frac{\hbar\omega}{\tau}} = 1$$

If ω_k is not present in the summation, then there will be a corresponding factor of

$$1 - Ae^{-\frac{\hbar\omega_k}{\tau}}$$

in the product. Therefore, we will get a distribution very similar to that of (7.16) for k, except $Ae^{-\frac{\hbar\omega_k}{\tau}}$ replaces the term 2^{-n_k}.

From this, we can see that (7.17) can indeed be consistent with thermal radiation. Various other series of quanta are looked at in greater detail in [8], which all give distributions that can be interpreted as thermal radiation.

7.3 Area Spectrum of Quantum Black Holes

The whole notion of quantising black holes was made due to the inevitable observation that the horizon area of nonextremal black holes show classical characteristics. It behaves as though a classical adiabatic invariant. In reference to the Ehrenfest principle, an adiabatic invariant can be explained as a quantum subject with a discrete spectrum.

We know from Christodoulou's paper [4], when a nonextremal black hole absorbs an uncharged particle with negligible radius, the action can be reversed if the particle is inserted about the radial turning point of its motion. In this regard, the black hole area remains unchanged and changes in other black hole attributes can be reversed using other reversible processes. However, the particle will be subjected to disagreements with the Heisenberg uncertainty principle, as it can not be at the horizon and at a radial turning point of its motion. This way the particle's momentum and location will both be known, which clearly is a violation.

From [5], we know that the injection of a neutral particle will definitely contribute to an increase in the black hole's horizon area. However, this increase can be minimalised if the particle is in such a state where the centre of mass of the particle is at a finite distance a away from the event horizon.

In that case,

$$\Delta A_{min} = 8\pi\mu a \tag{7.18}$$

where A = black hole surface area, μ = rest mass of the particle.

From this we can see that for a point particle which has $a = 0$, the change in horizon area would also be $\Delta A_{min} = 0$, recalling Christodoulou's reversible processes. Regardless, a quantum particle will inevitably be subjected to quantum uncertainty, therefore, a(radius of the particle) cannot be smaller than $\frac{\hbar}{\mu}$ which is its Compton wavelength.

This would produce a minimal boundary on the increase of horizon area due to the absorption of a neutral particle. Administering the lower bound on the minimalised area equation, we get

$$\Delta A_{min} = 8\pi\mu \left(\frac{\hbar}{\mu}\right)$$

So, we can write,

$$\Delta A_{min} = 8\pi l_p^2 \tag{7.19}$$

where l_p is the Planck length. The Planck length is $l_p = (\frac{G}{c^3})^{\frac{1}{2}}\hbar^{\frac{1}{2}}$, and in natural units $G = c = 1$.

This lower bound is applicable only for nonextremal black holes, hence, there is a universal minimum increase in the horizon area for such black holes as soon as quantum effects are introduced to the equation.

Apart from neutral particles, a similar lower bound equation was found for charged particles in [17],

$$A_{min} = 4l_p^2 \tag{7.20}$$

The fundamental physics that excludes the possibility of a complete reversible process is the Heisenberg uncertainty principle. For charged particles, however, another physical mechanism must be added. This mechanism is the Schwinger discharge(vacuum polarisation) of the black hole. In QFT, and especially QED, vacuum polarization is a processs that describes the production of virtual electron-positron pairs due to an electromagnetic field in the background. These pairs disrupt the original electromagnetic field by changing the distribution of the charges and currents. In this case, however, we can sense that the Schwinger discharge mechanism is the production of paired particles-antiparticles at the event horizon.

The universal lower bound is a clear beckon in favour of a uniformly spaced area spectrum for quantum black holes. The quantisation condition for the area spectrum should be of the form,

$$A_n = \gamma l_p^2 n \qquad n = 1, 2, 3.... \tag{7.21}$$

where γ is a dimensionless constant.

7.4 Black Hole Background

The black hole perturbations were taken under a wave analysis and it was noted that, at later times, all perturbations seem to decrease in frequency like that of a bell that ceases to ring in time.

Therefore, quasinormal modes were introduced. Quasinormal modes would be called normal modes if the perturbations were to ring forever, but amplitude of oscillation decays in time. These quasinormal mode frequencies are characteristics of the black hole itself.

Outside of the black hole horizon, the perturbation fields can follow a Schrodinger-like wave equation which has some time dependence,

$$\frac{\partial^2 \psi}{\partial r_*^2} + [\omega^2 - V(r)]\psi = 0 \tag{7.22}$$

where r_* is the tortoise radial coordinate that can be related with the spatial r by,

$$dr_* = \frac{dr}{(1 - \frac{2M}{r})}$$

and the effective potential is,

$$V(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} + \frac{\sigma}{r^3}\right] \tag{7.23}$$

with μ = black hole mass, l = multipole moment index, $\sigma = 2, 0, 6$ for scalar, electromagnetic, and gravitational perturbations, respectively.

The quasinormal modes which are the black hole's unperturbed oscillations are the solutions to the wave equation but with the assigning of some physical boundary conditions. These conditions have completely emitting waves at spatial infinity($r \longrightarrow +\infty$) and completely absorbing waves crossing the event horizon($r \longrightarrow -\infty$).

Pole singularities in the scattering amplitude of the background give rise to quasinormal mode frequencies. The ringing oscillations are located at a complex plane characterised by $Im(\omega) < 0$,

$$\omega = \omega' + i\omega''$$

So, for a given l, there exists a number of quasinormal normal modes, giving way to decreasing relaxation time.

However, the real part of the frequency tends to become a constant value as n becomes greater. Bohr's Correspondence principle states, "transition frequencies at large quantum numbers should equal to classical oscillation frequencies." Hence, the asymptotic behaviour, when $n \longrightarrow \infty$, of the ringing frequencies is of much importance.

Taking the ringing frequencies,

$$\omega = \omega_R - i\omega_I$$

then $\tau = \omega^{-1}$ is the proper time for the black hole to return to an inactive state. Hence, this relaxation time is generally very small when $n \longrightarrow \infty$.

Hans Peter Nollert, in his paper [7] found the ringing frequencies of a Schwarzschild black hole to be,

$$M\omega_n = 0.0437123 - \frac{i}{4}\left(n + \frac{1}{2}\right) + O\left[\left(n + \frac{1}{2}\right)^{-\frac{1}{2}}\right] \tag{7.24}$$

The ringing frequencies seem to be mass dependent only. It stays consistent with the speculation of the damped frequencies are properties of the black hole, independent of l, σ.

We observe that the limit $Re(\omega_n \longrightarrow 0.0437123M^{-1})$ matches with the expression $\frac{\ln 3}{8\pi}$. This identification is backed up by statistical and thermodynamic physics arguments. Using the relations, $A = 16\pi M^2$ and $dM = E = \hbar\omega$, we find

$$\frac{dA}{dM} = 32\pi M$$

$$dA = 32\pi M dM$$

$$dA = 32\pi M(\hbar\omega)$$

$$dA = 32\pi M(\hbar)\left(\frac{\ln 3}{8\pi}\right)M^{-1}$$

and

$$\Delta A = 4\hbar \ln 3 \tag{7.25}$$

Thus, we can make a conclusion that the dimensionless constant, γ from $A_n = \gamma l_p^2 n$ is $\gamma = 4\ln 3$, and the area spectrum of the quantum Schwarzschild black hole is $A_n = 4\hbar \ln 3$.

In essence of the Boltzmann-Einstein formula, the degeneracy, $g(n) = exp(S_{BH}(n))$ can be thought of as the amount of microstates referring to a complementing macrostate. In other words, $g(n) =$ degeneracy of the n^{th} area eigenvalue. As mentioned before, the known area-entropy relation is sought to be,

$$\gamma = 4\ln k \qquad k = 1, 2, ..$$

In [13], it provides the first independent derivation of the value of k. The relation $\gamma = 4\ln 3$ is competent with the area-entropy relation of the black hole, statistical physics arguments and Bohr's correspondence principle. Hence, we deduce that the value of k in the area-entropy relation would be 3, instead of 2, as devised in [13]

7.5 Physical Significance

We have a model which might describe the concept of $\ln 3$ in the area spectroscopy of quantum black holes. Consider the first infalling matter, generated by pair production near the event horizon.The infalling matter will contribute to the evaporation of the black hole. As soon as the infalling particle enters the horizon, it will decrease the black hole's energy, which will eventually decrease black hole mass.

Now, treating them as information subsystems, The Hawking quanta that gets radiated away will carry information of the infalling matter, and the black hole. The following particles that are produced will also contribute to the decrease of the black hole mass and hence, they will carry information of the old radiation (black hole information), and their corresponding Hawking partner.

Conventionally, we devised the entropy of the black hole as $N\ln 2$, where N accounts for the number of Hawking pairs each time step near the event horizon. The value of $k = 2$, in the degeneracy $g(n) = exp\left(\frac{\gamma(n-1)}{4}\right)$ where $\gamma = 4\ln k$, is derived due to the two-way entanglements that were initially thought. However, from [13] we independently derived the uniform area spacing constant k, where the relation $\gamma = 4\ln 3$ is consistent with area-entropy relations, statistical physics and Bohr's correspondence principle.

With this model, we explain the physical significance of $\ln 3$ in the entropy. This is an outcome of the tripartite entanglement between the Hawking pair information subsystems and the information subsystem of the black hole.

Chapter 8

Multipartite Entanglement

8.1 Locally Maximally Entangled States

In this chapter of the thesis, we will introduce the main bulk of our own work. We discuss the tripartite entanglement of qubits and extend it to the entanglement of multiple qubits using a locally maximally entangled state which we use to describe the state of our entangled qubits. We will then introduce a toy model in an attempt to show how the entanglement entropy of a black hole with multiple qubits can lead to an exponentially decreasing entropy over time.

If each element subsystem is maximally entangled with its complement, we call this a locally maximally entangled state. Considering a multipart system $\mathcal{H}_i$ having dimensions d_i. If we take the trace over any one of the subsystems in $\mathcal{H}$ in our system, the reduced density matrix that we obtain after taking this trace will just be a multiple $\frac{1}{d_i}$ of the identity operator. If this is the case, the state describing the multipart system is a locally maximally entangled state.

$$S_{LME} = \left\{ |\psi\rangle \in \mathcal{H} \;\middle|\; \rho_i \equiv tr_i \, |\psi\rangle \langle\psi| = \frac{1}{d_i}\mathbb{I} \right\} \tag{8.1}$$

Some common examples of LME states are

- Bell states:

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle \otimes |i\rangle \in \mathcal{H}_d \otimes \mathcal{H}_d \tag{8.2}$$

- The Greenberger-Horne-Zeilinger state:

$$|GHZ\rangle = \frac{1}{\sqrt{d}} \sum_{i=0} |i\rangle \otimes .... |i\rangle \tag{8.3}$$

The 3-qubit GHZ state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}} \left(|000\rangle + |111\rangle\right) \tag{8.4}$$

One property of the GHZ state is that when we trace over one of the three systems, we get

$$Tr_3\left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)\left(\frac{\langle 000| + \langle 111|}{\sqrt{2}}\right)$$

which gives us,

$$\left(\frac{|00\rangle\langle 00| + |11\rangle\langle 11|}{2}\right) \tag{8.5}$$

which is an unentangled mixed state. Therefore, taking the measurement of one qubit system in the GHZ state leaves the remaining 2-qubit state separable hence they lose entanglement.

- W state:

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right) \tag{8.6}$$

The W state is not unique to a bi-separable state property. It is one of the two existing states, the other being the GHZ state. However, these two cannot be turned to one another, even by certain local quantum operations.
In the W state, if the information of one qubit is lost, the other two qubits stay maximally entangled with each other unlike what is present in the GHZ state. This rigidity of entanglement is the unique characteristic of the W state.

8.2 Tripartite Entanglement

Now, according to the model we presented, there happens to be a speculated tripartite entanglement between three subsystems which would explain the multiple of $\ln 3$ in the Bekenstein-Hawking entropy independently calculated in Shahar Hod's 1998 paper. We think there could be an entanglement entropy present between the black hole and hawking pairs produced at the event horizon. We argue that the three subsystems be treated as three information subsystems where subsystem A is the black hole information, subsystem B is the information of the b quanta that radiated away, and subsystem C is the information of the c quanta that fell into the black hole.

We choose the W state to represent this system because, as assumed, the black hole information is lost, the b quanta and c quanta still remain maximally entangled. This is in contrast to the GHZ state, where as per this information loss, b quanta and c quanta would lose entanglement, which is not the case.

We define the density matrix to be of the form,

$$\rho = |W\rangle\langle W|$$

$$= \frac{1}{3}\left(|001\rangle + |010\rangle + |100\rangle\right)\left(\langle 001| + |010\rangle + |100\rangle\right)$$

$$= \frac{1}{3}(|001\rangle\langle 001| + |010\rangle|010\rangle + |100\rangle\langle 100| + |001\rangle\langle 010| + |001\rangle\langle 100|$$

$$+ |010\rangle \langle 001| + |010\rangle \langle 100| + |100\rangle \langle 001| + |100\rangle \langle 010|)$$

We define our basis states to be,

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The $|001\rangle$ vector is just the tensor product of 3 ket vectors,

$$|001\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Similarly,

$$\langle 001| = \begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Calculating all the outer product terms,

$$|001\rangle \langle 001| = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|010\rangle \langle 010| = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|100\rangle \langle 100| = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|001\rangle \langle 010| = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|001\rangle\langle100| = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$|010\rangle\langle001| = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$|010\rangle\langle100| = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$|100\rangle\langle001| = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

$$|100\rangle\langle010| = \begin{pmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}$$

Calculating our density matrix,

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{8.7}$$

After diagonalisation,

$$\rho_{diag} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{8.8}$$

Our entanglement entropy is defined as,

$$S_{ent} = -\mathrm{tr}(\rho \ln \rho) = \ln 1 = 0 \tag{8.9}$$

Since our state is a pure state, our total entanglement entropy is 0. In order to obtain the entanglement entropy of the subsystem A with B and C, we take the partial trace over A,

$$\rho_A = \mathrm{tr}_A(\rho_{ABC}) \tag{8.10}$$

Which gives us,

$$\rho_A = \left(\frac{1}{3}\right) (|01\rangle_{BC} \langle 01|_{BC} \langle 0| |0\rangle_A + |10\rangle_{BC} \langle 10|_{BC} \langle 0| |0\rangle_A + |00\rangle_{BC} \langle 00|_{BC} \langle 1| |1\rangle_A$$

$$+ |01\rangle_{BC} \langle 10|_{BC} \langle 0| |0\rangle_A + |10\rangle_{BC} \langle 01|_{BC} \langle 0| |0\rangle_A)$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}$$

Diagonalising our reduced density matrix then gives

$$\rho_{diag} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}$$

From this, we can thus calculate the entanglement entropy

$$S_{ent} = -\left(\frac{2}{3}\ln\frac{2}{3} + \frac{1}{3}\ln\frac{1}{3}\right) = \ln 3 - \frac{2}{3}\ln 2 \tag{8.11}$$

8.3 Multipartite Entanglement

Consider a 4 qubit system consisting of subsystems A, B, C and D represented by a state,

$$|\Psi\rangle = \frac{1}{\sqrt{4}}\left(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle\right) \tag{8.12}$$

Our density matrix has the form,

$$\rho_{ABCD} = |\Psi\rangle\langle\Psi|$$

$$\begin{aligned}
= \left(\frac{1}{4}\right)(&|0001\rangle\langle 0001| + |0010\rangle\langle 0010| + |0100\rangle\langle 0100| + |1000\rangle\langle 1000| \\
&+ \langle 0001|\,|0010\rangle + |0001\rangle\langle 0100| + |0001\rangle\langle 1000| + |0010\rangle\langle 0001| \\
&+ |0010\rangle\langle 0100| + |0010\rangle\langle 1000| + |0100\rangle\langle 0001| + |0100\rangle\langle 0010| \\
&+ |0100\rangle\langle 1000| + |1000\rangle\langle 0001| + |1000\rangle\langle 0010| + |1000\rangle\langle 0100|)
\end{aligned}$$

Taking the trace over B, C, D using the same basis states,

$$\rho_A = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} \tag{8.13}$$

Calculating the entanglement entropy,

$$S_{ent} = \frac{1}{4}\ln 4 + \frac{3}{4}\ln\frac{4}{3} = \ln 4 - \frac{3}{4}\ln 3 \tag{8.14}$$

Similarly for 5 qubits,

$$|\Psi\rangle = \frac{1}{\sqrt{5}}\left(|00001\rangle + |00010\rangle + |00100\rangle + |01000\rangle + |10000\rangle\right) \tag{8.15}$$

Taking the partial trace over subsystems B,C,D,E, we get the density matrix,

$$\rho_A = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{4}{5} \end{pmatrix}$$

Which gives us the entanglement entropy,

$$S_{ent} = \ln 5 - \frac{4}{5}\ln 4 \tag{8.16}$$

Therefore, for generalization if we have a k number of qubits, we get,

$$S_{ent} = \ln(k) - \frac{k-1}{k}\ln(k-1); \quad k > 1 \tag{8.17}$$

We only consider $k > 1$ since no entanglement can exist when $k = 1$ or less.

8.4 Information Loss

As assumed before, the $\ln k$ term comes from the entanglement between k number of qubits. So, evidently total entanglement entropy is supposed to be $\ln k$. However, we get an extra term of $\frac{k-1}{k} \ln (k-1)$. We propose that since we take a partial trace over the rest $k-1$ number of qubits (for example, for k = 4, taking partial trace over subsystems B, C, D) and treat them as one subsystem while operating an inner product through them, we are avoiding the entanglement that is present between the individual qubits in the rest $k-1$ number of qubits. Hence, what we get is not wholly an entanglement entropy of $\ln k$, but a part of it which includes the entanglement that we avoided of the individual qubits subtracted from the total, and therefore, there is an information loss.

Furthermore, the information loss itself is a matter of intrigue. From what we can assume is happening, the partial trace should have given us a subtraction of $\ln (k-1)$ arising from the ignored entanglement of the rest of the individual qubits. However, what we get is partially that, with a missing component of $\frac{1}{k} \ln (k-1)$.

Additionally, the missing information seems to decrease with respect to the increase in the number of qubits.

For $k = 3$;

$$\frac{1}{3} \ln 2 = 0.2310$$

For $k = 4$;

$$\frac{1}{4} \ln 3 = 0.2747$$

For $k = 5$;

$$\frac{1}{5} \ln 4 = 0.2773$$

For $k = 6$;

$$\frac{1}{6} \ln 5 = 0.2682$$

Until it converges to zero, For $k = 10000$;

$$\frac{1}{10000} \ln 9999 = 0.0009$$

and so on.

As we keep increasing the number of entangled qubits, we get a graph plot of the information loss against the number of qubits as follows:

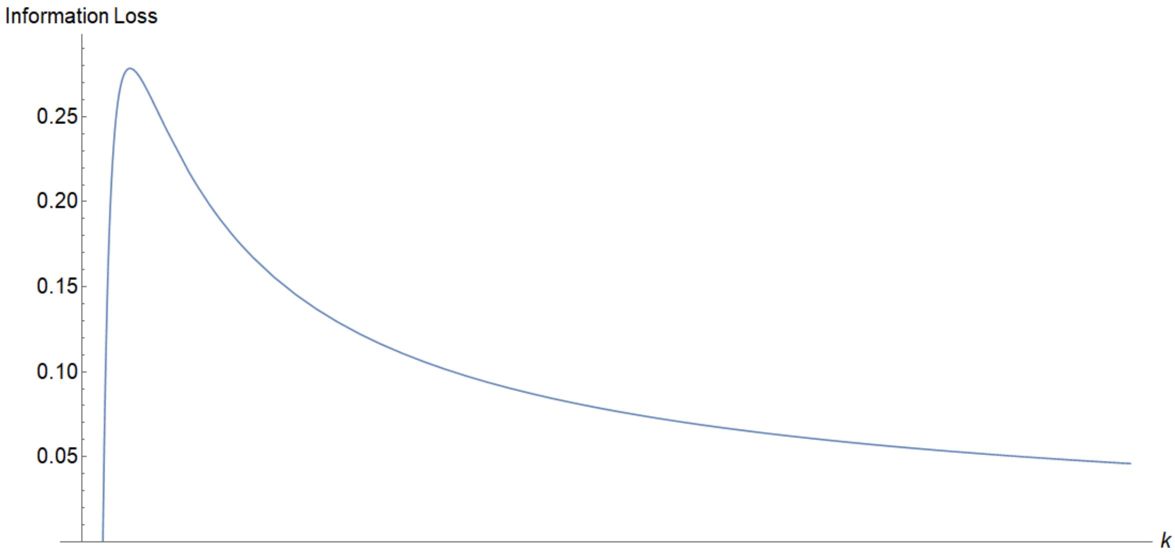

Figure 8.1: Graph of information loss against number of entangled qubits k

8.5 The Box and Triangle Model

We present a model where we assume the references as follows:

- The triangles represent the b and c quanta pair produced near the event horizon.
- The box represents the black hole information subsystem.

At step 1, considering the initial pair production there is an entanglement between the information subsystems of the b_1 and c_1 quanta with the black hole information after c_1 falls into the black hole and b_1 is radiated away.

At step 2, a new Hawking pair is produced (b_2, c_2) near the event horizon. As there is already an entanglement between the new Hawking pair, the new in-falling c_2 particle gives rise to an entanglement between the new pair and the black hole information. However, the black hole information was already entangled with the prior particles that were produced in step 1. Hence, at the end of step 2, we have an entanglement between the black hole information, b_2, c_2, and b_1, c_1.

For simplicity, let's replace the Hawking pairs and the black hole information subsystem with our references mentioned above. We assign Δ_{b_1} as information of b_1, Δ_{c_1} as information of c_1, $\Box_{BH}$ as information of the black hole information for k number of quanta produced.

At step 3, the same procedure occurs for the newly produced pair $\Delta_{b_3}, \Delta_{c_3}$. The radiated Δ_{b_3} carries information of Δ_{c_3}, $\Delta_{b_2}, \Delta_{c_2}$, and $\Box_{BH}$. This occurs because the black hole information subsystem remains entangled with the prior subsystems, therefore when the new Hawking particle is radiated away, it will carry information of its infallen partner and information of the prior entangled particles.

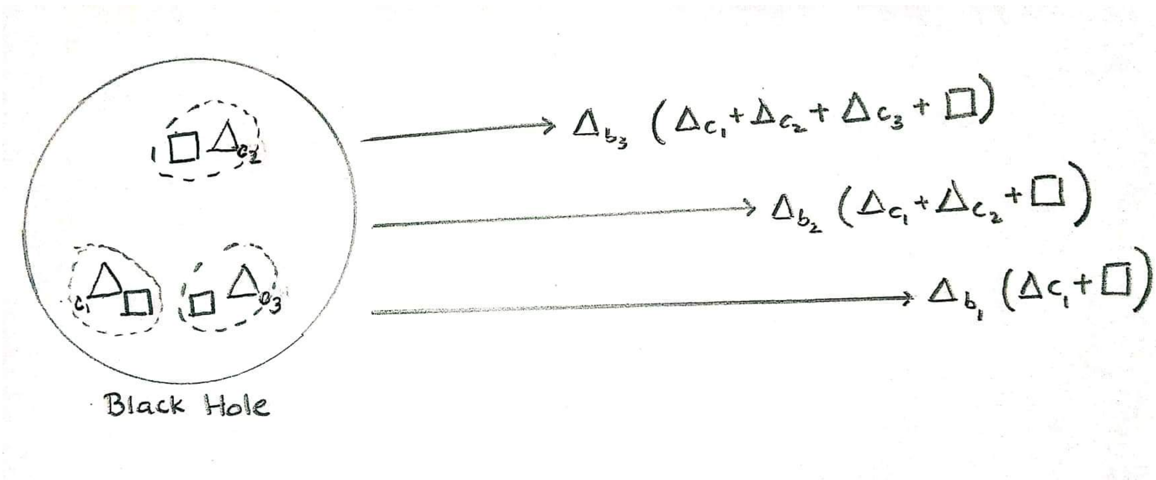

Figure 8.2: Entanglement of black hole with multiple qubits after each consecutive step.

After N steps, this draws to a k number of qubits entangled with the black hole subsystem, giving rise to multipartite entanglements. This paves an abstract way of information escaping from the black hole.
We consider there to be N steps over a certain duration of time of black hole evaporation. Using (17), we can now calculate S_{ent} of the black hole at each step. Initially, we start off with 3 subsystems, one representing the black hole information and the other 2 representing Hawking pairs.

$N = 1$,

$$k = 3, \quad S_{ent} = \ln 3 - \frac{2}{3}\ln 2 = 0.6365141683$$

$N = 2$,

$$k = 5, \quad S_{ent} = \ln 5 - \frac{4}{5}\ln 4 = 0.5004024235$$

$N = 3$,

$$k = 7, \quad S_{ent} = \ln 7 - \frac{6}{7}\ln 6 = 0.4101163183$$

$N = 7$,

$$k = 15, \quad S_{ent} = \ln 15 - \frac{14}{15}\ln 14 = 0.2449300268$$

$N = 17$,

$$k = 35, \quad S_{ent} = \ln 35 - \frac{34}{35}\ln 34 = 0.1297406947$$

$N = 49999$,

$$k = 99999, \quad S_{ent} = \ln 99999 - \frac{99998}{99999}\ln 99998 = 0.000125130356$$

and so on.
After each consecutive step, there is an increase of total entangled subsystems, k by 2 due to each newly produced Hawking pair. At $N = 1$, there is a sudden rise of S_{ent}. Evidently, as N increases, S_{ent} decreases exponentially right after $N = 1$.

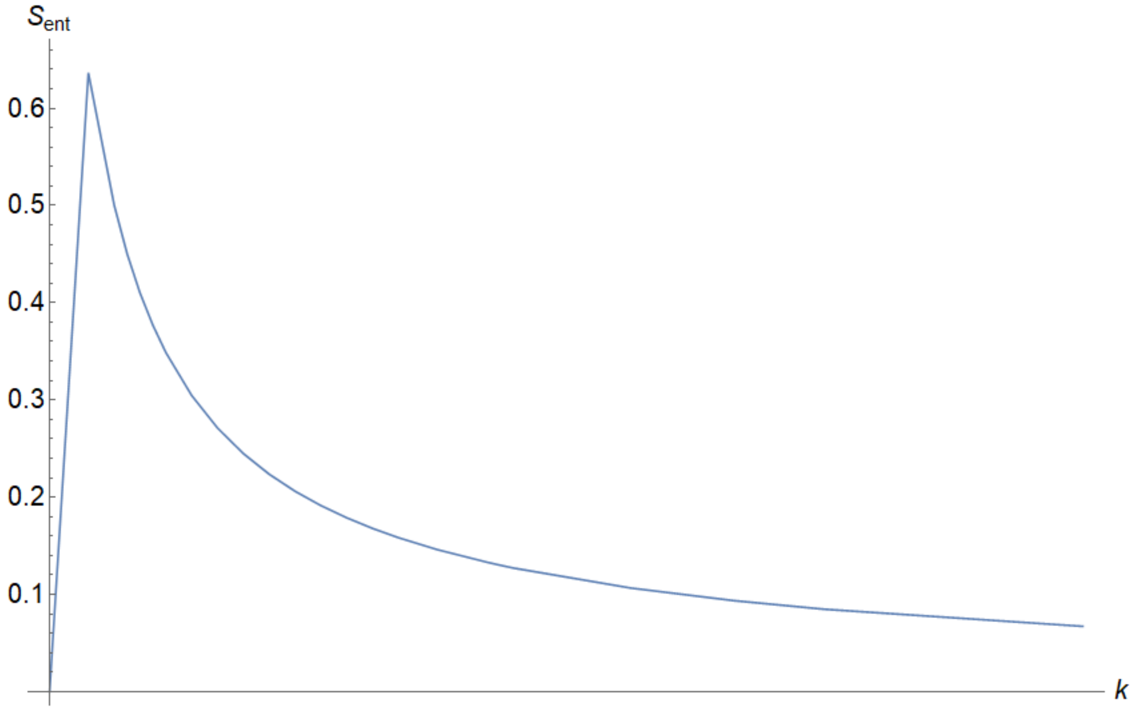

Figure 8.3: Graph of entanglement entropy against k qubits over time, showing a decreasing exponential entropy as number of entangled subsystems increase.

8.6 Subadditivity of Entropies

As mentioned in [16] ,consider a multipartite state of N subsystems, represented by $\rho_{1...N} \equiv tr_{2...N}(\rho_{1...N})$, whose corresponding entropy is given by $S_{1...N} \equiv S(\rho_{1...N}) \equiv -tr[\rho_{1...N}\log(\rho_{1...N})]$. In the case of 2 subsystems, according to the subadditivity of entropy, $S_{12} \leqslant S_1 + S_2$. This can be further generalised for $N \geqslant 2$ subsystems where,

$$S_{1...N} \leqslant \sum_{n=1}^{N} S_n \tag{8.18}$$

The equality holds $\iff \rho_{1...N} = \left(\prod^{\otimes}\right)_{n=1}^{N} \rho_n,$ *i.e.*, if all subsystems are uncorrelated.

From (8.10),

$$\rho_A \equiv \rho_B \equiv \rho_C = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$$

Applying the subadditivity of entropy for subsystems where $k = 3$,

$$S_A \equiv S_B \equiv S_C = -\mathrm{tr}[\rho_A \log(\rho_A)] = \log 3 - \frac{2}{3}\log 2 \tag{8.19}$$

$$\begin{aligned} S_A + S_B + S_C &= 3\left(\log 3 - \frac{2}{3}\log 2\right) \\ &= 1.909542505 \end{aligned} \tag{8.20}$$

As our W state is a pure state, our total entropy,

$$S_{ABC} \equiv S(\rho_{ABC}) \equiv [\rho_{ABC} \log(\rho_{ABC})] = 0$$

as shown in (8.9).

Evidently,

$$S_{ABC} \leqslant S_A + S_B + S_C \tag{8.21}$$

which satisfies the subadditivity of entropy.

For higher values of k,

$$k = 5, \quad S_1 + S_2 + S_3 + S_4 + S_5 = 5\left(\ln 5 - \frac{4}{5}\ln 4\right)$$
$$= 2.502012118$$

$$k = 7, \quad S_1 + ... + S_7 = 7\left(\ln 7 - \frac{6}{7}\ln 6\right)$$

$$= 2.870814228$$

$$k = 15, \quad S_1 + ... + S_{15} = 15\left(\ln 15 - \frac{14}{15}\ln 14\right)$$
$$= 3.673950402$$

$$k = 35, \quad S_1 + ... + S_{35} = 35\left(\ln 35 - \frac{34}{35}\ln 34\right)$$
$$= 4.540924315$$

$$k = 99999, \quad S_1 + ... + S_{99999} = 99999\left(\ln 99999 - \frac{99998}{99999}\ln 99998\right)$$
$$= 12.51291047$$

and so on.

The value of the sum of the individual entanglement entropies of each subsystem with the rest of the subsystems continues to increase with k. As the total entropy $S_{1...N}$ for any value of k is always 0 due to our pure state, (8.18) is always satisfied for all values of k.

Chapter 9

Conclusion

This paper has established a new take on viewing entanglement entropies of black holes. This has been possible because of the review of the preliminaries in the prior chapters.

However, there are still many gaps that could be filled in this thesis and make improvements. The surfacing of the information loss is still a matter of question and can be interpreted in various ways. It may be because of polygamy of the entropies, leading to a loss in entanglement and eventually information. Also, we notice a sharp fall in the entanglement entropy from the very first emission of quanta, which is not what we expected. Nevertheless, the Black Hole Information Paradox is still a field that is yet to be exercised and solved.

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