Perturbations on Time Dependent Orbifold Singularities

by

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Declaration

It is hereby declared that

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2. The thesis does not contain material previously published or written by a third party, except where this is appropriately cited through full and accurate referencing.

3. The thesis does not contain material which has been accepted, or submitted, for any other degree or diploma at a university or other institution.

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Abstract

We start with the standard time-dependent backgrounds such as the geometry and dynamics of **Freidmann-Robertson-Walker**(FRW) cosmologies. First, we discuss the dynamics and geometry of FRW cosmologies. Then we introduce perturbations on this FRW cosmologies. We then study more exotic orbifold spacetimes and examine their symmetries. We examine how those symmetries determine the partition functions on such symmetrical spacetimes.

Keywords: FRW cosmologies; Orbifold Singularities; Tree-level Amplitudes; Cosmological perturbation Theory

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Chapter 1

Introduction

Orbifolds are the generalizations of manifolds. In conformal field theory(CFT), we can create a new CFT from the old one which is invariant under the action of a discrete group Γ by, (1) adding a twisted sector which is

$$\phi(\sigma^1 + 2\pi) = h\phi(\sigma^1)$$

and (2) restricting to Γ-invariant states. This new theory is called an orbifold of the previous CFT. An example of orbifold is the toroidal compactification, in which first we break down the D-dimensional Minkowski space to $SO(D-2,1) \times U(1)$. After that, we add winding strings to this and restrict the spectrum by quantizing the momentum along the compactification direction. In string theory, we can solve the space-time singularity problem by studying the string orbifolds. Therefore, understanding the geometry of orbifolds has a great importance if we truly want to understand the quantum gravity phenomenon.

Before going deep into the subject matter of time-dependent orbifolds, first we start with the basic geometry of FRW cosmologies. We go through the basic dynamics and geometry of FRW cosmologies by introducing the metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)dl^2 \tag{1.1}$$

This allows us to define the Freidmann equation. We calculate the Freidmann equation from Newtonian gravity and also introduce the parameters such as the Hubble parameter(H) and the scale factor(a). We then try to find out how the universe corresponds to this scale factor.

In the third chapter, we introduce inhomogeneities and consider small perturbations on our FRW metric. We talk about the gauge problem and how to fix this problem. Then we go through matter perturbations and their linearised equations of motion. We then compute the perturbed Einstein equations and conclude the chapter with different types of perturbations i.e. the adiabatic perturbation and the curvature perturbation.

In the forth chapter, we introduce the topic of time-dependent orbifolds in three-dimensional Minkowski space. We classify different types of orbifolds and their geometry. After that, we examine the single particle wave functions on these time-dependent orbifolds.

In the last chapter of this dissertation, We start with the particle interactions and discuss about the formation of large black holes. Then, we talk about the back-reaction in three-dimension. From that, we want to know that whether there is particle production in time-dependent orbifolds because the geometry varies with time. And to answer this, we calculate the amplitudes from the single particle wave functions which is known as the Tree-level amplitudes. In this chapter we focus only on the three-point and the four point amplitudes. We conclude our thesis with the discussion about Eikonal resummation and the calculation of one-loop amplitudes.

Chapter 2

Geometry and Dynamics

In cosmological studies, first we made some basic assumptions. These assumptions are, on large scales, the universe is isotropic and homogeneous and the General Relativity gives an appropriate description of the universe. Cosmic Microwave Background or CMB backs up the assumption of isotropy and homogeneity and when we introduce the concepts of "Dark Matter" and "Dark Energy", General relativity gives an appropriate explanation of the development of the universe.

2.1 Metric

The cosmological assumptions help us to build a metric in the form [12]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) dl^2 \tag{2.1}$$

In equation 2.1, dl^2 is the constant curvature 3-metric, which is

$$dl^2 = \gamma_{ij} dx^i dx^j \tag{2.2}$$

This constant curvature 3-metric can be

1. Positive curvature (Spherical, $\mathbb{S}^3$)
   In this case,
   $dl^2 = d\mathbf{x}^2 + du^2$
   Where, $|\mathbf{x}|^2 + u^2 = \mathbf{R}^2$
2. Zero curvature (Euclidean, $\mathbb{E}^3$), $dl^2 = d\mathbf{x}^2$
3. Negative curvature (Hyperbolic, $\mathbb{H}^3$), $dl^2 = d\mathbf{x}^2 - du^2$ where, $|x|^2 - u^2 = -\mathbf{R}^2$

Rescaling the coordinates from $\mathbf{x} \to \mathbf{R}\mathbf{x}$ and $u \to \mathbf{R}u$ gives

$$dl^2 = \mathbf{R}^2 (d\mathbf{x}^2 \pm du^2) \tag{2.3}$$

Where, $\mathbf{x}^2 \pm u^2 = \pm 1$. Now, considering the constrain over this we can get $udu = \mp \mathbf{x} d\mathbf{x}$ and substituting this in equation 2.3

$$\begin{aligned} dl^2 &= \mathbf{R}^2 \left(d\mathbf{x}^2 + K \frac{(\mathbf{x}.d\mathbf{x})^2}{1 - k\mathbf{x}^2} \right) \\ &= \mathbf{R}^2 \left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right) \end{aligned} \tag{2.4}$$

In equation 2.4, K is the comoving curvature which has values: 1 for positive curvature, 0 for zero curvature and -1 for negative curvature. Finally, substituting equation 2.4 in equation 2.1, we get the **Freidmann-Robertson-Walker** or **FRW** metric,

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{2.5}$$

2.2 Kinematics and Momentum

When we are considering the only acting force on a particle is gravity then the path taken by this particle is a geodesic. Let's assume a particle with mass m, the 4-velocity of this particle is

$$U^\mu \equiv \frac{dX^\mu}{ds} \tag{2.6}$$

The equation of motion for this particle is the geodesic equation

$$\frac{dU^\mu}{ds} + \Gamma^\mu_{\alpha\beta} U^\alpha U^\beta = 0 \tag{2.7}$$

Here, the Christoffel symbol

$$\Gamma^\mu_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\lambda}\left(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}\right) \tag{2.8}$$

From equation 2.6,

$$\frac{dU^\mu}{ds} = \frac{dX^\alpha}{ds}\frac{dU^\mu}{dX^\alpha} = U^\alpha \partial_\alpha U^\mu \tag{2.9}$$

Substituting this value in equation 2.7,

$$\begin{aligned} U^\alpha \partial_\alpha U^\mu + \Gamma^\mu_{\alpha\beta} U^\alpha U^\beta &= U^\alpha \left(\partial_\alpha U^\mu + \Gamma^\mu_{\alpha\beta} U^\beta\right) \\ &= U^\alpha \nabla_\alpha U^\mu \\ &= 0 \end{aligned} \tag{2.10}$$

Here, ∇_α is the covariant derivative and we can write equation 2.10 in terms of the momentum, $P^\mu = mU^\mu$ as

$$P^\alpha \partial_\alpha P^\mu = -\Gamma^\mu_{\alpha\beta} P^\alpha P^\beta \tag{2.11}$$

Equation 2.11, is also valid for massless particle. Homogeneous and isotropic conditions make, $\partial_i P^\mu = 0$. Therefore equation 2.11 becomes

$$\begin{aligned} P^0 \frac{P^\mu}{dt} &= -\Gamma^\mu_{\alpha\beta} P^\alpha P^\beta \\ &= -\left(2\Gamma^\mu_{0j} P^0 + \Gamma^\mu_{ij} P^i\right) P^j \end{aligned} \tag{2.12}$$

When, P^i is equal to zero, $\partial_t P^i$ also becomes zero and the particle is at rest. If we consider $\mu = 0$ then, $E\frac{dE}{dt} = -\frac{\dot{a}}{a}P^2$. From $-m^2 = -E^2 + P^2$, we got $EdE = PdP$, which leads to

$$\frac{\dot{P}}{P} = -\frac{\dot{a}}{a}$$
$$P(t) \propto \frac{1}{a(t)}. \tag{2.13}$$

Equation 2.13 shows that with the expansion of the Universe, the momentum decays.

2.3 Dynamics

Einstein's famous equation is the building block of Cosmology. This equation states as

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.14}$$

Here, $T_{\mu\nu}$ is the Energy-Momentum tensor.

$$T_{\mu\nu} = \begin{bmatrix} T_{00} & T_{0i} \\ T_{j0} & T_{ij} \end{bmatrix}$$

Due to isotropy and homogeneity, the most general form $T_{\mu\nu}$ can take is

$$T_{00} = \rho(t)$$
$$T_{0i} = 0$$
$$T_{ij} = P(t) g_{ij} \tag{2.15}$$

From equation 2.15, we can see that for a perfect fluid it is the Energy-Momentum tensor and this can be written in the following form

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + P g_{\mu\nu} \tag{2.16}$$

From the law of conservation, we can write equation 2.15 as

$$\nabla_\mu T^\mu_\nu = \partial\mu T^\mu_\nu + \Gamma^\mu_{\mu\lambda} T^\lambda_\nu - \Gamma^\lambda_{\mu\nu} T^\mu_\lambda = 0 \tag{2.17}$$

Again, for the fluid the law of conservation implies

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{2.18}$$

If we consider, $\omega = \frac{P}{\rho}$; equation 2.18 becomes

$$\frac{\dot{\rho}}{\rho} = -3(1+\omega)\frac{\dot{a}}{a}$$
$$\Rightarrow \rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+\omega)} \tag{2.19}$$

Every known particles in the universe can take one of the three equations of state and based on this we have created a Cosmic inventory [12].

Cosmic Inventory

Matter (m)	-Cold Dark Matter (c) -Baryons (b)	$\omega = 0$	$\rho \propto a^{-3}$
Radiation (m)	-Photons(γ) -Neutrinos (ν) -Gravitons (g)	$\omega = 1/3$	$\rho \propto a^{-4}$
Dark Energy (Λ)	Vaccum Energy or Modified Gravity or ????	$\omega = -1$	$\rho \propto a^0 =$ const.

Figure 2.1: Cosmic Inventory

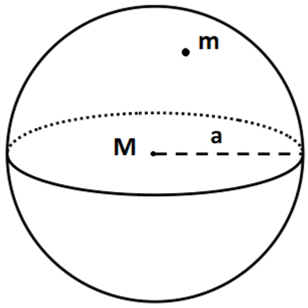

Figure 2.2: A particle of mass m on a sphere with radius a and mass M.

Now, we introduce **Friedmann equation** from Newtonian gravity. Let's consider a test particle of mass m on a surface of a sphere, which has radius a and mass M. By Gauss's law, we know that gravity is determined only by the interior matter. Now, from Newtonian Gravity we know that

$$\begin{aligned} m\ddot{a} &= -\frac{GMm}{a^2} \\ \Rightarrow \ddot{a} &= -\frac{GM}{a^2} \\ \Rightarrow \dot{a}\ddot{a} &= -\frac{GM\dot{a}}{a^2} \\ \Rightarrow \dot{a}\ddot{a} + \frac{GM\dot{a}}{a^2} &= 0 \\ \Rightarrow \frac{d}{dt}\left(\frac{\dot{a}^2}{2}\right) + GM\left(-\frac{d}{dt}\left(\frac{1}{a}\right)\right) &= 0 \\ \Rightarrow \frac{d}{dt}\left(\frac{\dot{a}^2}{2} - \frac{GM}{a}\right) &= 0 \end{aligned} \tag{2.20}$$

Let's assume, $\frac{\dot{a}^2}{2} - \frac{GM}{a} = -\frac{k}{2}$; where k is a constant term. We know that, $\rho \sim \frac{M}{V}$.

Which leads to, $M = \frac{4}{3}\pi a^3 \rho$. Therefore, equation 2.20 becomes

$$\begin{aligned} \frac{\dot{a}^2}{2} - \frac{G(\frac{4}{3}\pi a^3 \rho)}{a} &= -\frac{k}{2} \\ \Rightarrow \frac{\dot{a}^2}{2a^2} - \frac{4}{3}\pi G\rho &= -\frac{k}{2a^2} \qquad \text{[Both sides divided by } a^2\text{]} \\ \Rightarrow \frac{\dot{a}^2}{a^2} - \frac{8\pi G\rho}{3} &= -\frac{k}{a^2} \end{aligned} \tag{2.21}$$

Equation 2.21 is the Friedmann equation where, $\left(\frac{\dot{a}^2}{a^2}\right)$ is the kinetic term and $\left(\frac{8\pi G\rho}{3}\right)$ is the potential term. Here, $\left(\frac{\dot{a}}{a}\right)$ is defined as the Hubble parameter. The Hubble parameter, $H \equiv \frac{\dot{a}(t_0)}{a(t_0)}$; where, t_0 is the time today. Therefore, the Friedmann equation in terms of Hubble parameter is

$$H^2 - \frac{8\pi G\rho}{3} = -\frac{k}{a^2} \tag{2.22}$$

Depending on the value of k the universe can be open, closed or flat. If $k > 0$, the universe is closed. If $k < 0$, the universe is open. If $k = 0$, the universe is flat. Our universe is considered as a flat universe. For flat universe, equation 2.22 becomes

$$H^2 = \frac{8\pi G\rho}{3} \tag{2.23}$$

From the cosmic inventory (figure 2.1), when $\rho \propto a^{-3}$ equation 2.23 becomes

$$\begin{aligned} \frac{\dot{a}^2}{a^2} &= \frac{8\pi G}{3}\frac{1}{a^3} \\ \Rightarrow \dot{a}^2 &= \frac{8\pi G}{3}\frac{1}{a} \\ \Rightarrow \dot{a} &= \sqrt{\frac{8\pi G}{3}}\frac{1}{\sqrt{a}} \\ \Rightarrow \dot{a} &= \frac{\alpha}{\sqrt{a}} \\ \Rightarrow \sqrt{a}\frac{da}{dt} &= \alpha \\ \Rightarrow \int \sqrt{a}da &= \int \alpha dt \\ \Rightarrow \frac{2}{3}a^{3/2} &= \alpha t \\ \Rightarrow a^{3/2} &= \left(\frac{3}{2}\alpha\right)t \\ \Rightarrow a &= \left(\frac{3}{2}\alpha\right)^{2/3} t^{2/3} \\ \Rightarrow a &\propto t^{2/3} \end{aligned} \tag{2.24}$$

Here, a is called the scale factor. When, $a(t) \propto t^{2/3}$; the universe is considered to be matter dominated(MD). Again, from the cosmic inventory (figure 2.1), when $\rho \propto a^{-4}$, equation 2.23 becomes

$$\begin{aligned}
\frac{\dot{a}^2}{a^2} &= \frac{8\pi G}{3}\frac{1}{a^4} \\
\Rightarrow \dot{a}^2 &= \frac{8\pi G}{3}\frac{1}{a^2} \\
\Rightarrow \dot{a} &= \sqrt{\frac{8\pi G}{3}}\frac{1}{a} \\
\Rightarrow a\frac{da}{dt} &= \alpha \\
\Rightarrow \int ada &= \int \alpha dt
\end{aligned} \tag{2.25}$$

Then, we can write this as

$$\begin{aligned}
a^2 &= 2\alpha t \\
\Rightarrow a &\propto t^{1/2}
\end{aligned} \tag{2.26}$$

When, $a(t) \propto t^{1/2}$, the universe is considered to be radiation dominated(RD). Again, for $\rho \propto a^0 = constant$, we get

$$\begin{aligned}
\frac{\dot{a}^2}{a^2} &= \frac{8\pi G a^0}{3} \\
\Rightarrow \dot{a} &= \sqrt{\frac{8\pi G a^0}{3}}a
\end{aligned}$$

This is an exponential equation and we can write this as

$$\begin{aligned}
a &= a(0)e^{\sqrt{\frac{8\pi G a^0}{3}}t} \\
\Rightarrow a &\propto e^{H_0 t}
\end{aligned} \tag{2.27}$$

When, $a \propto e^{H_0 t}$ the universe is considered to be dark energy dominated(ΛD).

Chapter 3

Cosmological Perturbation Theory

Cosmological perturbation theory introduces inhomogeneities, which is important for the formation of large-scale structures. In this chapter, we start with introducing small perturbation on the FRW metric. We see that based on our coordinate selection a problem i.e. gauge problem arises with the perturbation and we try to find a way to overcome this problem. Then we go through the perturbation of matters and their perturbed linearised equations.

3.1 Perturbed metric

The flat FRW metric in terms of conformal time(τ) is

$$ds^2 = a^2(\tau)\left(-d\tau^2 + \delta_{ij}dx^i dx^j\right). \tag{3.1}$$

Let's assume a small perturbation around the FRW metric, which can be written as

$$ds^2 = a^2(\tau)\left(-(1+2A)d\tau^2 + 2B_i dx^i d\tau + (\delta_{ij} + h_{ij})dx^i dx^j\right). \tag{3.2}$$

Here, A, B_i and h_{ij} are space and time functions. To compute the scalar, vector and tensor quantities individually, we need to adopt a Scalar-Vector-Tensor(**SVT**) decomposition. SVT decomposition allows us to split a 3-vector into a vector that is divergenceless and the gradient of a scalar.

$$B_i = \partial_i B + \mathbf{B}_i, \tag{3.3}$$

Here, $(\partial_i B)$ is a scalar and $(\mathbf{B}_i)$ is a vector with $(\partial^i \mathbf{B}_i = 0)$. And any symmetric rank-2 tensor can be written as

$$h_{ij} = 2C\delta_{ij} + 2\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E + (\partial_i\mathbf{E}_j + \partial_j\mathbf{E}_i) + 2\mathbf{E}_{ij}. \tag{3.4}$$

The tensor perturbation is traceless and transverse. Therefore, $\partial^i E_{ij} = \delta^{ij}E_{ij} = 0$. The metric has $(4+4+2)$ SVT degrees of freedom. Here,

- A, B, C and E are the scalars.
- $\mathbf{B}_i$ and $\mathbf{E}_i$ are the vectors and
- $\mathbf{E}_{ij}$ is the tensor.

If we consider only the scalars then equation 3.2 takes the form

$$ds^2 = a^2(\tau)\Bigg(-(1+2A)d\tau^2 + 2\partial_i B dx^i d\tau + \left[(1+2C)\delta_{ij} + 2\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E\right] dx^i dx^j\Bigg). \tag{3.5}$$

3.2 The Gauge Problem

There are some problems in the metric perturbations in equation 3.2. Metric perturbations depend on what coordinates we are choosing. Changing the coordinates can change the perturbation variables and may form fake perturbations. This problem of appearing fictitious perturbations in our metric is called the **Gauge problem**. For example, if we change our spatial coordinates, x^i to $\tilde{x}^i$; where, $\tilde{x}^i = x^i + \xi^i$ and $dx^i = d\tilde{x}^i - \partial_\tau \xi^i d\tau - \partial_k \xi^i dx^k$. Then the metric takes the form

$$ds^2 = a^2(\tau)\left(-d\tau^2 + 2\xi_i' d\tilde{x}^i d\tau + (\delta_{ij} + 2\partial_{(i}\xi_{j)})d\tilde{x}^i d\tilde{x}^j\right). \tag{3.6}$$

Here, ξ_i' and $\partial_{(i}\xi_{j)}$ are the fake perturbations. This fake perturbations are called *gauge modes* and these modes can be eliminated by changing the coordinates back to the previous ones. Likewise, changing the time slicing, $\tau \to \tau + \xi^0$, changes the pressure as

$$\rho(\tau + \xi^0) = \rho(\tau) + \xi^0 \rho'(\tau) \tag{3.7}$$

Here, $\xi^0\rho'(\tau)$ is a fake density perturbation. Therefore, the gauge problem indicates that we can create fake perturbations by changing the coordinates.

3.3 Gauge Transformations

Now, let's assume the following gauge transformation

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu, \qquad \text{where, } \xi^0 \equiv T, \quad \text{and } \xi^i \equiv \partial^i L. \tag{3.8}$$

Here, we have only contemplated the perturbations which are scalars. In terms of these gauge transformations, the metric can be written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \tilde{g}_{\alpha\beta} d\tilde{x}^\alpha d\tilde{x}^\beta, \tag{3.9}$$

where,

$$g_{\mu\nu} = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\tilde{g}_{\alpha\beta}. \tag{3.10}$$

Then, the diagonal (00) component of the metric can be written as

$$g_{00} = \left(\frac{\partial \tilde{x}^0}{\partial x^0}\right)^2 \tilde{g}_{00}. \tag{3.11}$$

Applying the gauge transformations(equation 3.8) into the metric(equation 3.2), we get

$$\begin{aligned} a^2(\tau)(1+2A) &= (1+T')^2 a^2(\tau+T)(1+2\tilde{A}) \\ &= (1+2T'+...)(a(\tau)+a'(T)+...)^2(1+2\tilde{A}) \\ &= a^2(\tau)(1+2\mathcal{H}T+2T'+2\tilde{A}+...), \end{aligned} \tag{3.12}$$

Here, the Hubble parameter in conformal time, $\mathcal{H} \equiv a'/a$. Therefore, the metric perturbation A transforms as

$$A \mapsto \tilde{A} = A - T' - \mathcal{H}T. \tag{3.13}$$

Similarly, the other metric perturbations transform as

$$B \mapsto \tilde{B} = B + T - L', \tag{3.14}$$

$$C \mapsto \tilde{C} = C - \mathcal{H}T - \frac{1}{3}\nabla^2 L, \tag{3.15}$$

$$E \mapsto \tilde{E} = E - L. \tag{3.16}$$

3.4 Gauge Fixing

A way of fixing the gauge problem is to define metric perturbations which is invariant under change of coordinates. For doing that, we need to introduce some gauge-invariant variables which are called the *Bardeen potentials*. They are

$$\Psi \equiv A + \mathcal{H}(B - E') + (B - E')', \tag{3.17}$$

$$\Phi \equiv -C - \mathcal{H}(B - E') + \frac{1}{3}\nabla^2 E. \tag{3.18}$$

These potentials eliminate the chance of producing any fictitious perturbations since they remain same under coordinate transformation. In Newtonian gauge, we consider $\Psi = A$ and $\Phi = -C$. If we let, $B = E = 0$, then the metric becomes

$$ds^2 = a^2\left(-(1+2\Psi)d\tau^2 + (1-2\Phi)\delta_{ij}dx^i dx^j\right). \tag{3.19}$$

3.5 Perturbed Matter

The energy-momentum tensor of a perfect fluid is

$$\bar{T}^\mu_\nu = (\bar{\rho} + \bar{P})\bar{U}^\mu \bar{U}_\nu + \bar{P}\delta^\mu_\nu, \tag{3.20}$$

We introduce small perturbations, $T^\mu_\nu = \bar{T}^\mu_\nu + \delta T^\mu_\nu$ and write the perturbed energy-momentum tensor as

$$\delta T^\mu_\nu = (\delta\rho + \delta P)\bar{U}^\mu \bar{U}_\nu + (\bar{\rho} + \bar{P})(\delta U^\mu \bar{U}_\nu + \bar{U}^\mu \delta U_\nu) + \delta P \delta^\mu_\nu + \Pi^\mu_\nu, \tag{3.21}$$

Here, Π^μ_ν is the anisotropic stress which is mostly negligible. In the four-velocity, perturbation can induce energy flux, T^0_j, and momentum density, T^i_0, which can not be vanished. We need to compute the perturbed four-velocity in equation 3.2. First of all, to derive δU^μ, we consider $g_{\mu\nu}U^\mu U^\nu = -1$. This implies

$$\delta g_{\mu\nu}\bar{U}^\mu\bar{U}^\nu + 2\bar{U}_\mu\delta U^\mu = 0, \tag{3.22}$$

Here, considering $\bar{U}^\mu = a^{-1}\delta^0_\mu$ and $\delta g_{00} = 2a^2 A$, we can figure out that, $\delta U^0 = -Aa^{-1}$. Therefore, $\delta U^i \equiv v^i/a$, where the *coordinate velocity*, $v^i \equiv dx^i/d\tau$. Thus

$$U^\mu = a^{-1}[1 - A, v^i]. \tag{3.23}$$

Similarly, we can find that

$$U_\mu = a[-(1 + A), (v_i + B_i)]. \tag{3.24}$$

Applying the value of equation 3.23 and equation 3.24 in equation 3.21, we get

$$\begin{aligned} \delta T^0_0 &= -\delta\rho, \\ \delta T^0_i &= (\bar{\rho} + \bar{P})v_i + B_i), \\ \delta T^i_0 &= -(\bar{\rho} + \bar{P})v^i \equiv -q^i \rightarrow \text{(3-momentum density)}, \\ \text{and} \quad \delta T^i_j &= \delta P\delta^i_j + \Pi^i_j. \end{aligned}$$

If we consider a universe which is multi-component, then the total energy-momentum tensor becomes, $T_{\mu\nu} = \sum_I T^I_{\mu\nu}$ and hence

$$\begin{aligned} \delta\rho &= \sum_I \delta\rho_I\,, \\ \delta P &= \sum_I \delta P_I\,, \\ q^i &= \sum_I q^i_I\,, \\ \text{and} \quad \Pi &= \sum_I \Pi_I\,. \end{aligned}$$

The perturbations only add the density, pressure and anisotropic stress but the velocities do not add up. Under gauge transformation, the stress-energy tensor becomes

$$T^\mu_\nu = \frac{\partial x^\mu}{\partial\tilde{x}^\alpha}\frac{\partial\tilde{x}^\beta}{\partial x^\nu}\tilde{T}^\alpha_\beta\,. \tag{3.25}$$

From this, we get

$$\begin{aligned} \delta\rho &\mapsto \delta\rho - T\bar{\rho}'\,, \\ \delta P &\mapsto \delta P - T\bar{P}'\,, \\ q_i &\mapsto q_i + (\bar{\rho} + \bar{P})L'_i\,, \\ v_i &\mapsto v_i + L'_i\,, \\ \text{and} \quad \Pi_{ij} &\mapsto \Pi_{ij}\,. \end{aligned}$$

From the metric, we can forme various quantities that are invariant under gauge transformation. One useful gauge-invariant quantity is

$$\bar{\rho}\Delta \equiv \delta\rho + \bar{\rho}'(v + B). \tag{3.26}$$

Here, Δ= Comoving-gauge density perturbation.
There are two useful matter gauges worth mentioning. These are:

- Uniform density - Setting the total density perturbation equal to zero, $\delta\rho \equiv 0$
- Comoving - Vanishing the scalar momentum density, $q \equiv 0$

Since, we have perturbed all our quantities, Now we can talk about the *equations of motion* for the metric and matter perturbations.

3.6 Linearised Evolution Equations

In Newtonian gauge, the metric tensor takes the form

$$g_{\mu\nu} = a^2 \begin{pmatrix} -(1+2\Psi) & 0 \\ 0 & (1-2\Phi)\delta_{ij} \end{pmatrix}, \tag{3.27}$$

and

$$g^{\mu\nu} = \frac{1}{a^2} \begin{pmatrix} -(1-2\Psi) & 0 \\ 0 & (1+2\Phi)\delta^{ij} \end{pmatrix}. \tag{3.28}$$

The perturbed connection coefficients for the metric tensor(equation 3.27) is defined as

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\lambda}(\partial_\nu g_{\lambda\rho} + \partial_\rho g_{\lambda\nu} - \partial_\lambda g_{\nu\rho}). \tag{3.29}$$

We can calculate the perturbed connection coefficients by substituting equation 3.27 and equation 3.28 into equation 3.29.
For, Γ^0_{00}

$$\begin{aligned}\Gamma^0_{00} &= \frac{1}{2} g^{00}\Big(2\partial_0 g_{00} - \partial_0 g_{00}\Big), \\ &= \frac{1}{2} g^{00}\partial_0 g_{00}\,, \\ &= \frac{1}{2a^2}\Big(1-2\Psi\Big)\partial_0\Big[a^2(1+2\Psi)\Big], \\ &= \mathcal{H} + \Psi'. \end{aligned} \tag{3.30}$$

Similarly,

$$\Gamma^0_{0i} = \partial_i \Psi\,, \tag{3.31}$$

$$\Gamma^i_{00} = \delta^{ij}\partial_j \Psi\,, \tag{3.32}$$

$$\Gamma^0_{ij} = \mathcal{H}\delta_{ij} - \Big[\Phi' + 2\mathcal{H}(\Phi+\Psi)\Big]\delta_{ij}\,, \tag{3.33}$$

$$\Gamma^i_{0j} = \mathcal{H}\delta^i_j - \Phi'\delta^i_j\,, \tag{3.34}$$

$$\text{and} \quad \Gamma^i_{jk} = -2\delta^i_{(j}\partial_{k)}\Phi + \delta_{jk}\delta^{il}\partial_l\Phi\,. \tag{3.35}$$

3.7 Conservation Equations

The conservation law for the stress tensor states that [11]

$$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\mu\alpha} T^\alpha_\nu - \Gamma^\alpha_{\mu\nu} T^\mu_\alpha = 0 \,. \tag{3.36}$$

Now, if we set the ν component of equation 3.36 to zero, then

$$\partial_0 T^0_0 + \partial_i T^i_0 + \Gamma^\mu_{\mu 0} T^0_0 + \Gamma^\mu_{\mu i} T^i_0 - \Gamma^0_{00} T^0_0 - \Gamma^0_{i0} T^i_0 - \Gamma^i_{00} T^0_i - \Gamma^i_{j0} T^j_i = 0 \,. \tag{3.37}$$

Implementing the values of connection coefficients and the perturbed stress-energy tensor into equation 3.37 gives

$$\partial_0(\bar{\rho} + \delta p) + \partial_i q^i + (\mathcal{H} + \Psi' + 3\mathcal{H} - 3\Phi')(\bar{\rho} + \delta\rho) - (\mathcal{H} + \Psi')(\bar{\rho} + \delta\rho) \\ - (\mathcal{H} + \Phi')\delta^i_j \left[-(\bar{P} + \delta P)\delta^j_i \right] = 0 \,, \tag{3.38}$$

which gives

$$\bar{\rho}' + \partial_0 \delta\rho + \partial_i q^i + 3\mathcal{H}(\bar{\rho} + \delta\rho) - 3\bar{\rho}\Phi' + 3\mathcal{H}(\bar{P} + \delta P) - 3\bar{P}\Phi' = 0 \,. \tag{3.39}$$

The zeroth-order of equation 3.39 is

$$\bar{\rho}' = -3\mathcal{H}(\bar{\rho} + \bar{P}) \,. \tag{3.40}$$

And the first-order of equation 3.39 is

$$\partial_\eta \delta\rho = -3\mathcal{H}(\delta\rho + \delta P) + 3\Phi'(\bar{\rho} + \bar{P}) - \nabla.\mathbf{q} \,. \tag{3.41}$$

Here, equation 3.40 is the equation for *energy conservation* in homogeneous region and equation 3.41 is the *continuity equation* in density perturbation.
Now, setting the ν component of equation 3.36 to i gives

$$\partial_0 T^0_i + \partial_j T^j_i + \Gamma^\mu_{\mu 0} T^0_i + \Gamma^\mu_{\mu j} T^j_i - \Gamma^0_{0i} T^0_0 - \Gamma^0_{ji} T^j_0 - \Gamma^j_{0i} T^0_j - \Gamma^j_{ki} T^k_j = 0 \,. \tag{3.42}$$

Implementing the values of connection coefficients and the perturbed stress-energy tensor into equation 3.42 gives

$$-\partial_0 q_i + \partial_j \left[-(\bar{P} + \delta P)\delta^j_i - \Pi^j_i \right] - 4\mathcal{H} q_i - (\partial_j \Psi - 3\partial_j \Phi)\bar{P}\partial^j_i \\ - \partial_i \Psi \bar{\rho} - \mathcal{H}\partial_{ji} q^j + \mathcal{H}\partial^j_i q_j + (-2\delta^j_{(i}\partial_{k)}\Phi + \delta_{ki}\delta^{jl}\partial_l \Phi)\bar{P}\delta^k_j = 0 \,, \tag{3.43}$$

which leads to

$$\partial_\eta q_i = -4\mathcal{H} q_i - (\bar{\rho} + \bar{P})\partial_i \Psi - \partial_i \delta P - \partial^j \Pi_{ij} \,. \tag{3.44}$$

Equation 3.44 is the *Euler equation* for a viscous fluid.

3.8 Perturbed Einstein Equations

So far, we have computed the linearised equations for perturbed matter. Now, we focus on the perturbed *Einstein equations*. The Einstein tensor states that

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{3.45}$$

Here, $R_{\mu\nu}$= Ricci Tensor, and R= Ricci scalar.
In terms of the Christoffel symbols, the Ricci tensor can be written as

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} \,. \tag{3.46}$$

We need to compute the Ricci tensor and Ricci scalar in terms of perturbed connection coefficients. This part is computed in the **Appendix** section.
The 00-component of the Einstein tensor is,

$$\begin{aligned} G_{00} =& R_{00} - \frac{1}{2} g_{00} R \,, \\ =& - 3\mathcal{H}' + \nabla^2\Psi + 3\mathcal{H}(\Phi' + \Psi') + 3\Phi'' + 3(1+2\Psi)(\mathcal{H}' + \mathcal{H}^2) \\ & - \frac{1}{2}\Big[2\nabla^2\Psi - 4\nabla^2\Phi + 12(\mathcal{H}' + \mathcal{H}^2)\Psi + 6\Phi'' + 6\mathcal{H}(\Psi' + 3\Phi')\Big] \,, \\ =& 3\mathcal{H}^2 + 2\nabla^2\Phi - 6\mathcal{H}\Phi' \,. \end{aligned} \tag{3.47}$$

In Newtonian gauge, $g_{0i} = 0$, therefore only R_{0i} survives for the $0i$-component. Thus

$$G_{0i} = 2\partial_i(\Phi' + \mathcal{H}\Psi) \,, \tag{3.48}$$

and the ij-component of the Einstein tensor is

$$\begin{aligned} G_{ij} =& R_{ij} - \frac{1}{2} g_{ij} R \,, \\ =& \Big[\mathcal{H}' + 2\mathcal{H}^2 - \Phi'' + \nabla^2\Phi - 2(\mathcal{H}' + 2\mathcal{H}^2)(\Phi + \Psi) - \mathcal{H}\Psi' - 5\mathcal{H}\Phi'\Big]\delta_{ij} \\ & + \partial_i\partial_j(\Phi - \Psi) - 3(1-2\Phi)(\mathcal{H}' + \mathcal{H}^2)\delta_{ij} \\ & + \frac{1}{2}\Big[2\nabla^2\Psi - 4\nabla^2\Phi + 12(\mathcal{H}' + \mathcal{H}^2)\Psi + 6\Phi'' + 6\mathcal{H}(\Psi' + 3\Phi')\Big]\delta_{ij} \,, \\ =& -(2\mathcal{H}' + \mathcal{H}^2)\delta_{ij} + \\ & \Big[\nabla^2(\Psi - \Phi) + 2\Phi'' + 2(2\mathcal{H}' + \mathcal{H}^2)(\Phi + \Psi) + 2\mathcal{H}\Psi' + 4\mathcal{H}\Phi'\Big]\delta_{ij} \\ & + \partial_i\partial_j(\Phi - \Psi) \,. \end{aligned} \tag{3.49}$$

From the equation 3.49, only considering the trace free portion gives

$$\partial_{\langle i}\partial_{j\rangle}(\Phi - \Psi) = 0 \Rightarrow \Phi = \Psi \,. \tag{3.50}$$

For the 00-component, the Einstein tensor is, $G_{00} = 8\pi G T_{00}$. Therefore, from equation 3.47, we get

$$\begin{aligned} 3\mathcal{H}^2 + 2\nabla^2\Phi - 6\mathcal{H}\Phi' &= 8\pi G g_{0\mu} T^\mu_0 \,, \\ &= 8\pi G(g_{00}T^0_0 + g_{0i}T^i_0) \,, \\ &= 8\pi G a^2(1+2\Phi)(\bar{\rho} + \delta\rho) \,, \\ &= 8\pi G a^2\bar{\rho}(1 + 2\Phi + \delta) \,. \end{aligned} \tag{3.51}$$

In equation 3.51, the zeroth-order part is

$$3\mathcal{H}^2 = 8\pi G a^2 \bar{\rho} \,. \tag{3.52}$$

This is the **Friedman equation**. Again, from equation 3.51, the first-order part is

$$\begin{aligned} \nabla^2\Phi &= 4\pi G a^2 \bar{\rho}\delta + 8\pi G a^2 \bar{\rho}\Phi + 3\mathcal{H}\Phi' \,, \\ &= 4\pi G a^2 \bar{\rho}\delta + 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) \,. \end{aligned} \tag{3.53}$$

Now, for the $0i$-component, the Einstein tensor is, $G_{0i} = 8\pi G T_{0i}$, where

$$T_{0i} = g_{0\mu} T^{\mu}_{i} = g_{00} T^{0}_{i} = \bar{g}_{00} T^{0}_{i} = -a^2 q_i \,. \tag{3.54}$$

From equation 3.48, we can write

$$\partial_i(\Phi' + \mathcal{H}\Phi) = -4\pi G a^2 q_i \,. \tag{3.55}$$

Here, $q_i = (\bar{\rho} + \bar{P})\partial_i \upsilon$ and integrating the equation 3.55, we get

$$\Phi' + \mathcal{H}\Phi = -4\pi G a^2 (\bar{\rho} + \bar{P})\upsilon = -4\pi G a^2 q \,. \tag{3.56}$$

substituting equation 3.56 into equation 3.53 gives,

$$\nabla^2\Phi = 4\pi G a^2 \bar{\rho}\Delta \,, \tag{3.57}$$

here, $\bar{\rho}\Delta \equiv \bar{\rho}\delta - 3\mathcal{H}(\bar{\rho} + \bar{P})\upsilon$. Equation 3.57 is the **Poisson equation**. So far, we have discussed the linearised evolution equations, now we shall discuss the different types of perturbations.

3.9 Adiabatic Perturbations

Adiabatic perturbations can be defined as a quantum fluctuations in the field ϕ, which can be represented as a local time perturbation $\delta\tau$. Adiabatic density perturbations are defined as

$$\delta\rho_I(\tau, \boldsymbol{x}) \equiv \bar{\rho}_I(\tau + \delta\tau(\boldsymbol{x})) - \bar{\rho}_I(\tau) = \bar{\rho}'_I \delta\tau(\boldsymbol{x}) \,, \tag{3.58}$$

where,

$$\delta\tau = \frac{\delta\rho_I}{\bar{\rho}'_I} = \frac{\delta\rho_J}{\bar{\rho}'_J} \,. \tag{3.59}$$

From the continuity equation, we can write

$$\bar{\rho}'_I \propto (1 + w)\rho_I \Rightarrow \frac{\delta_I}{1 + w_I} = \frac{\delta_J}{1 + w_J} \,. \tag{3.60}$$

As shown in figure 3.1, in adiabatic perturbations, different fluids can have different amplitudes but they maintain same profile. Thus, we can write the total density perturbation as

$$\delta\rho = \bar{\rho}\delta = \sum_I \bar{\rho}_I \delta_I \,. \tag{3.61}$$

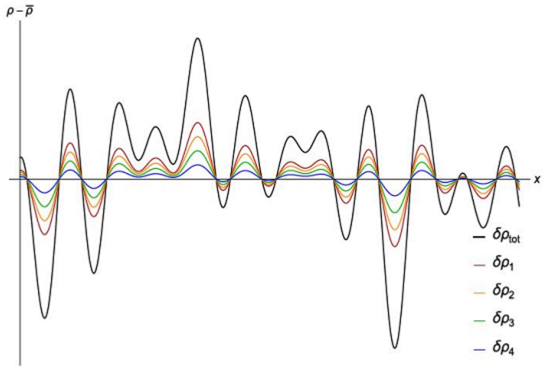

Figure 3.1: A representation of adiabatic perturbations.

3.10 Curvature Perturbations

In adiabatic perturbations, we have proved that $\delta_I \propto \delta$. Now, if we consider the density gauge equals to zero then all we have left with is the curvature perturbations. From adiabatic perturbations, we can write that

$$\delta\tau = \frac{\delta\rho}{\bar{\rho}'} \, . \tag{3.62}$$

Now, if we set the density gauge, $\delta\rho = 0$, then $\delta\tau = \delta\rho = o$ and the perturbation becomes curvature perturbation. The 3-D Ricci scalar for this

$$a^2 R^{(3)} = 4\nabla^2(-C + \frac{1}{3}\nabla^2 E) \, . \tag{3.63}$$

Now, if we denote the constant density curvature perturbation to ξ then

$$\xi = \left[-C + \frac{1}{3}\nabla^2 E \right]_{\delta\rho = o} \tag{3.64}$$

Now, applying a gauge transformation on equation 3.64, we get

$$\begin{aligned} \tilde{\xi} &= -\tilde{C} + \frac{1}{3}\nabla^2 \tilde{E} \, , \\ &= -C + \mathcal{H}T + \frac{1}{3}\nabla^2 L + \frac{1}{3}\nabla^2 (E - L) \, , \\ &= \xi + \mathcal{H}T \, . \end{aligned} \tag{3.65}$$

Here, $\tilde{\delta\rho} = \delta\rho - T\bar{\rho}'$. Therefore, we can make our gauge invariant density curvature perturbation as

$$\xi \equiv -C + \frac{1}{3}\nabla^2 E + \mathcal{H}\frac{\delta\rho}{\bar{\rho}'} \, . \tag{3.66}$$

Chapter 4

Time Dependent Orbifolds

So far, we have discussed the dynamics and geometry of FRW cosmologies. We have also talked about the cosmological perturbation theory. Now, we shall examine the geometry of different types of time-dependent orbifolds. Orbifold is the generalized forms of manifold and we shall consider our manifold space to be flat 3-dimensional Minkowski space, $\mathbb{M}^3$. Moreover, we shall discuss about the single particle wave functions of these different types of time-dependent orbifolds.

4.1 Orbifold classification and generalities

First, we start with a killing vector field k, which is on a manifold $\mathcal{M}$ and has isometry group G. This killing vector can be represented as [10]

$$P \sim e^{nk} P \; , \tag{4.1}$$

Here, $n \in \mathbb{Z}$ and e^k produces a discrete subgroup $\Gamma \subset G$. Now, conjugation by G of this killing vector also represents the same orbifold, which is

$$k \to h^{-1} k h \; . \tag{4.2}$$

Now, to discuss about orbifolds, here we consider our manifold space to be flat 3-dimensional Minkowski space, $\mathbb{M}^3$. To introduce different types of orbifolds, we shall discuss about the killing vectors of this 3-dimensional space. First, consider the Minkowski coordinates X^0, X^1, X^2 and the coordinates of the light-cones are

$$x^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^1) \; . \tag{4.3}$$

The general form of a killing vector is

$$k = 2\pi i(\alpha^a p_a + \beta^{ab} J_{ab}) \; , \tag{4.4}$$

where, the generators of the Poincare groups are

$$\begin{aligned} iJ_{ab} &= X_a \partial_b - X_b \partial_a \; , \\ \text{and} \quad iP_a &= \partial_a \; . \end{aligned} \tag{4.5}$$

Also, the dual form of β^{ab} is

$$\beta^{ab} = \epsilon^{abc} \beta_c \; . \tag{4.6}$$

Now, conjugating the killing vector gives the following transformation by rotation

$$\alpha^a \to \alpha^a + \beta^{ab}\omega_b \,,$$
$$\beta^{ab} \to \beta^{ab} \,. \tag{4.7}$$

From, equation 4.6 and equation 4.7, it appears that under conjugation, $\alpha^a\beta_a$ and $\beta^a\beta_a$ are invariant. Now, we shall consider $\beta_a \neq 0$ or we shall only have translation orbifold. Considering what sign β^2 gives, we can have three types of orbifolds. They are

- Elliptic when, $\beta^2 < 0$.
- Hyperbolic when, $\beta^2 > 0$.
- Parabolic when, $\beta^2 = 0$.

Now, for hyperbolic orbifolds, if we go through a Lorentz transformation, $\beta_2 = \Delta$ and $\beta_\pm = 0$, the killing vector can be written as

$$k = 2\pi i(\Delta J_{+-} + RP_2) \,, \qquad \text{where, } \alpha^2 = R \,. \tag{4.8}$$

This killing vector is created by a boost along one direction and a translation along the diagonal direction [2]. When, $R \neq 0$, this type of orbifold is called **shifted-boost orbifold** and when, $R = 0$, this is called **boost orbifold** [6]. For the parabolic orbifolds, $\beta_- = \Delta$ and $\alpha^- = R$, the killing vector is defined as

$$k = 2\pi i(\Delta J_{+2} + RP_-) \,. \tag{4.9}$$

When, $R \neq 0$, this type of orbifold is called ***O*-plane orbifold** and when, $R = 0$, this is called **null-boost orbifold**. So, we have introduced different types of orbifolds. Now, we shall discuss each type of orbifolds more deeply and their quotient space geometry. To do so, we shall first change our coordinate system, therefore, the killing vector can have trivial solutions. Then we shall apply the Kaluza-Klein ansatz, which reads

$$ds_3^2 = ds^2 + \Phi^2((dz + A)^2 \,, \tag{4.10}$$

Now, after classifying the orbifolds, our primary focus should be to figure out how a single particle wave functions propagates on the covering space of the three-dimensional orbifold.To construct our single particle wave function, we shall use the Klein-Gordon equation

$$\Box\psi = m^2\psi \,. \tag{4.11}$$

From this we can rely on the point that, for ψ to be invariant under a discrete group Γ, the Klein-Gordon equation must satisfy

$$\psi(X) = \psi(e^{nk}X) \,. \tag{4.12}$$

Then, we choose a set of basis for the functions, which is

$$k\psi_n = 2\pi i n\psi_n \,. \tag{4.13}$$

Now, to construct single particle wave functions, we write the plane wave

$$\phi_p(X) = e^{ip.X} \,, \tag{4.14}$$

In terms of continuous isometry, e^{sk}, this follows

$$\phi_p(e^{sk}X) = \phi_{e^{sk}p}(X)e^{i\phi(p.s)} . \tag{4.15}$$

If we consider, $p^2 + m^2 = 0$, the function becomes

$$\psi_p(X) = \sum_n \phi_p(E^{nk}X) . \tag{4.16}$$

We can write this sum in terms of Fourier transformation. So, the single particle wave functions become

$$\begin{aligned} \psi_{p,n}(X) &= \int ds \phi_p(e^{sk}X)e^{-2\pi ins} , \\ &= \int ds \phi_{e^{sk}p}(X)e^{i\varphi(p,s)-2\pi ins} . \end{aligned} \tag{4.17}$$

This is also the integral form of the wave functions. Now we shall discuss more generally about the different types of time-dependent orbifolds and their single particle wave functions.

4.2 Shifted-boost orbifold

For shifted-boost orbifold, the killing vector is [2]

$$k = 2\pi i(\Delta J_{+-} + RP_2) . \tag{4.18}$$

From Lorentz algebra, the orbifolds identifications are

$$\begin{aligned} X^{\pm} &\sim e^{\pm 2\pi\Delta}X^{\pm} , \\ \text{and} \quad X &\sim X + 2\pi R . \end{aligned} \tag{4.19}$$

From figure 4.1, we have specified three different space-time regions for shifted-boost orbifold in the $X^{\pm}$-plane. Region I_{in} is the past light-cones and I_{out} is the future light-cones; regions II_L and II_R are between the light-cones and the $k^2 = 0$ surface and finally, we introduces regions III_L and III_R. In the first two regions k is space-like and the last region k is time-like. Now, under coordinate transformation the killing vector becomes trivial, which is

$$\begin{aligned} X^{\pm} &= y^{\pm}e^{\pm Ez} , \\ \text{and} \quad X &= z . \end{aligned} \tag{4.20}$$

Now, we shall write the metric in terms of Kaluza-Klein form. To do so, we need to fix our coordinates as

$$y^{\pm} = \frac{t}{\sqrt{2}}e^{\pm Ey} , \tag{4.21}$$

From this, we can get the Kaluza-Klein fields as

$$ds^2 = -dt^2 + \frac{(Et)^2}{\Phi^2}dy^2 , \tag{4.22}$$

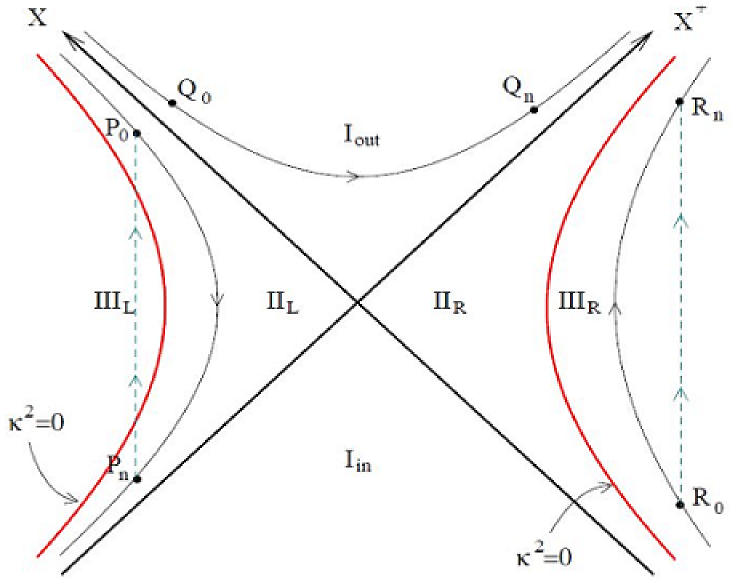

Figure 4.1: Three different regions for a shifted-boost orbifold in $X^{\pm}$-plane

where, $\Phi^2 = 1 + (Et)^2$ and $A = (1 - \Phi^{-2})dy$. Now, for $(Et) << 1$, equation 4.22 becomes Milne metric in two-dimensional space and it is defined as a horizon. In region I_{in}, we get $t < 0$, and the Milne metric contracts towards a future horizon. On the other-hand, in region I_{out}, we get $t > 0$, and the Milne metric expands from a past horizon. To answer the question of what is happening outside the horizon we first define our coordinate transformation that covers the second and third regions, which is

$$y^{\pm} = \pm \frac{x}{\sqrt{2}} e^{\pm Ew} . \tag{4.23}$$

And this leads to

$$ds^2 = -\frac{(Ex)^2}{\Phi^2} dw^2 + dx^2 , \tag{4.24}$$

where, $\Phi^2 = 1 - (Ex)^2$ and $A = (1 - \Phi^{-2})dw$. Now, for $(Ex) << 1$, equation 4.24 is the Rindler metric. Therefore, in region II at $x = 0$, there is a horizon , that is similar as a black hole horizon. If we go further away from the horizon, at $Ex = 1$ we get a curvature singularity.

Now, we shall describe the single particle wave functions on the shifted-boost orbifold. We can write the wave functions as

$$\psi_{p,n} = f(t) e^{i(py + (n/R)z)} . \tag{4.25}$$

Here, we introduce a Bessel function $f(t)$, which has ν imaginary order. Then, the Klein-Gordon equation is as follows

$$\left[t^2 \frac{d^2}{dt^2} + t \frac{d}{dt} + (\omega t)^2 - \nu^2 \right] f(t) = 0 , \tag{4.26}$$

where, $\omega^2 = m^2 + (p - \frac{n}{R})^2$ and $\nu = i\frac{p}{E}$. In region I, a complete basis of this wave functions is given by

$$\psi^{\pm}_{p,n} = J_{\pm\nu}(\omega |t|) e^{i(py + (n/R)z)} . \tag{4.27}$$

Also, for region II, the basis of the wave functions is

$$\psi^{\pm}_{p,n} = J_{\pm\nu}(i\omega|x|)e^{i(p\omega+(n/R)z)} \,. \tag{4.28}$$

We can write this wave functions as a superposition of the plane waves. It is given by [7]

$$e^{i(py+(n/R)z)} \int d\sigma e^{(\pm i\omega t \cosh\sigma - i\frac{p}{E}\sigma)} \,,$$

This can be represented as

$$H^{(1,2)}_{\nu}(x) = \pm\frac{1}{\pi i}e^{\mp\frac{i\pi\nu}{2}} \int d\sigma e^{(\pm ix\cosh\sigma - \nu\sigma)} \,. \tag{4.29}$$

Here, $H^{(1,2)}_{\nu}$ is the Hankel functions, which is constructed from the Bessel functions.

4.3 Boost orbifold

In light-cones coordinates, the space–time points for the boost orbifold are identified as

$$X^{\pm} \sim e^{\pm 2\pi\Delta}X^{\pm} \,, \tag{4.30}$$

Here, the X–direction have no importance at all. Figure 4.2 [10] shows the fundamental regions for the boost orbifold.

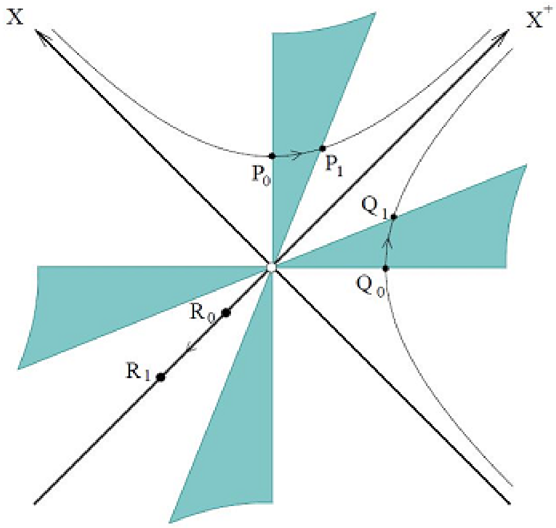

Figure 4.2: The fundamental regions of a boost orbifold.

The geodesic path square of the images are, $8\sinh^2(n\pi\Delta)X^+X^-$ and from this we can find on both left and right portions, there are time–like curves(CTC's) which are closed, and generally known as the whiskers. The coordinate transformation is

$$X^{\pm} = \frac{t}{\sqrt{2}}e^{\pm\Delta z} \,,$$
$$\text{and} \quad X = y \,.$$

The Kaluza–Klein form of this metric is

$$ds_3^2 = -dt^2 + dy^2 + (\Delta t)^2 dz^2 \,. \tag{4.31}$$

Now, we shall discuss about the single particle wave functions on boost orbifold. The integral form of this is [8]

$$e^{ikX} \int dse^{i\left(\pm\frac{\omega}{\sqrt{2}}e^s X^+ \pm \frac{\omega}{\sqrt{2}}e^{-s}X^- - \frac{n}{\Delta}s\right)} \,, \tag{4.32}$$

here, $\omega^2 = m^2 + k^2$. In the $X\pm$–plane the functions are Bessel functions which has imaginary order $\nu = i\frac{n}{\Delta}$. In the Milne wedge we can write the functions

$$J_{\pm i\frac{n}{\Delta}}(\omega|t|)e^{i(ky+(n/R)z)} \,. \tag{4.33}$$

4.4 O-plane orbifold

For the O–plane orbifold, the Killing vector is

$$k = 2\pi i(\Delta J_{+2} + RP_-) \,. \tag{4.34}$$

If we consider the action of e^k, the space–time points for the O–plane orbifold are

$$X^- \sim X^- + 2\pi R \,, \tag{4.35}$$

$$X^+ \sim X^+ - (2\pi\Delta)X + \frac{1}{2}(2\pi\Delta)^2 X^- + \frac{1}{6}(2\pi)^3 R\Delta^2 \,, \tag{4.36}$$

$$\text{and} \quad X \sim X - (2\pi\Delta)X^- - \frac{1}{2}(2\pi)^2 R\Delta \,. \tag{4.37}$$

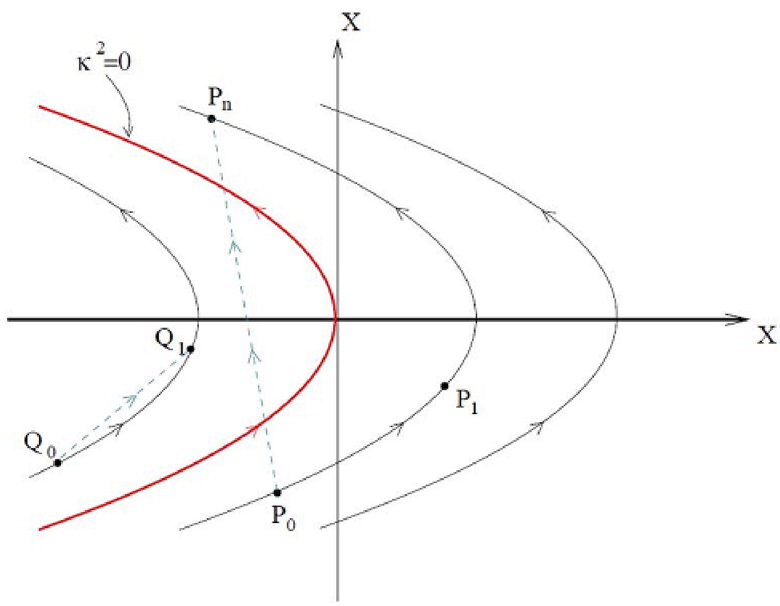

Figure 4.3: For k, the different orbital lines on the O-plane orbifold.

From figure 4.3, the space–time is divided into space–like and time–like regions. The geodesic path difference square in between the images are given by

$$2E(2\pi Rn)^2\left(X+\frac{1}{2}E(X^-)^2-\frac{1}{12}E(2\pi Rn)^2\right).$$

Now, under the coordinate transformations

$$\begin{aligned} X^- &= y^- \ , \\ X^+ &= y^+ - Eyy^- + \frac{E^2}{6}(y^-)^3 \ , \\ \text{and} \quad X &= y - \frac{E}{2}(y^-)^2 \ . \end{aligned}$$

Then, the metric becomes

$$ds_3^2 = -2dy^+dy^- + 2Ey(dy^-) + dy^2 \ . \tag{4.38}$$

and the Kaluza-Klein form of this metric is

$$ds_3^2 = \frac{(dy^+)^2}{2Ey} + dy^2 + 2Ey\left(dy^- - \frac{dy^+}{2Ey}\right)^2 \ . \tag{4.39}$$

Finally, the single particle wave functions on O–plane orbifold is given by

$$\psi_{p_+,n} = f(y)e^{i(p_+y^+ + (n/R)y^-)} \ . \tag{4.40}$$

Here, the function $f(y)$ maintains the differential equation form, which is

$$\frac{d^2f}{d\omega^2} = \omega f \ .$$

Here, the solutions to this function are Airy functions, $Ai(\omega)$ and $Bi(\omega)$. Considering the normalizable solution, we can write the wave functions as

$$\psi_{p_+,n} \propto Ai(\omega)e^{i(p_+y^+ + (n/R)y^-)} \ . \tag{4.41}$$

The integral form of the Airy function is defined as

$$Ai(\omega) = \frac{1}{2\pi}\int dt e^{i(\omega t + \frac{t^3}{3})} \ . \tag{4.42}$$

which leads to

$$\psi_{p_+,n} \propto e^{i(p_+y^+ + \frac{n}{R}y^-)}\int ds e^{i(y+\frac{n}{ERp_+}-\frac{m^2}{2Ep_+^2})s-\frac{i}{6}\frac{s^3}{Ep_+^2}} \ . \tag{4.43}$$

In terms of normalization, this becomes

$$\psi_{p_+,n} = \frac{1}{\sqrt{|p_+|}}\int dp\phi_{p_+,p}(X)e^{\frac{i}{E}\left(\frac{np}{Rp_+}-\frac{m^2p}{2p_+^2}-\frac{p^3}{6p_+^2}\right)} \ . \tag{4.44}$$

4.5 Null-boost orbifold

If we set, $R = 0$ in O–plane orbifold we get null-boost orbifold [9]. Same as the O–plane orbifold, the space-time points for null-boost orbifold are identified as

$$X^- \sim X^- \,, \tag{4.45}$$

$$X^+ \sim X^+ - (2\pi\Delta)X + \frac{1}{2}(2\pi\Delta)^2 X^- \,, \tag{4.46}$$

$$\text{and} \quad X \sim X - (2\pi\Delta)X^- \,. \tag{4.47}$$

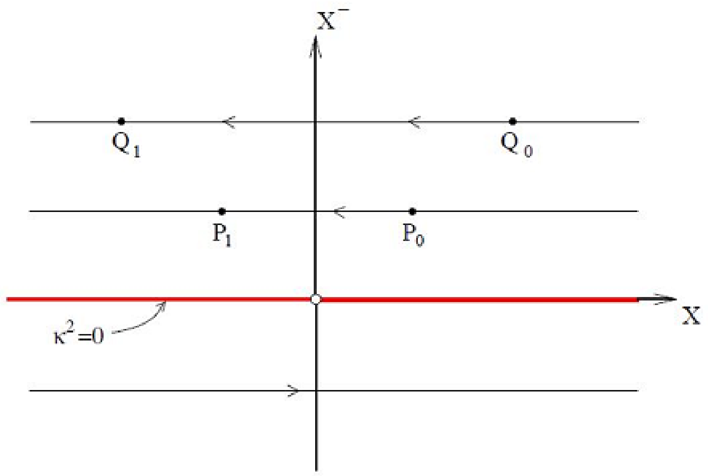

Figure 4.4: For k, different orbital lines on the null-boost orbifold.

From figure 4.4, The Killing vector of null-boost orbifold is $k = 2\pi i\Delta J_{+2}$. This is space–like excluding the point at $X^- = 0$. At $X^- = 0$, it is null.
The geodesic path difference square between images is

$$(2\pi\Delta n X^-)^2 \,,$$

Under the following coordinate transformations

$$X^- = y^- \,,$$

$$X^+ = y^+ + \frac{\Delta^2}{2} z^2 y^- \,,$$

$$\text{and} \quad X = \Delta z y^- \,.$$

our metric becomes

$$ds_3^2 = -2dy^-dy^+ + (\Delta y^-)^2 dz^2 \,. \tag{4.48}$$

The integral form of the wave functions can be obtained from the integral form of the O–plane orbifold by setting, $X^- \to 0$,

$$\lim_{X^- \to 0} \psi_{p+,n} = 2\pi\sqrt{\frac{i}{|p_+|}}\,\delta\left(X + \frac{n}{\Delta p_+}\right) e^{ip+X^+} \,. \tag{4.49}$$

Chapter 5

Strings on Orbifolds-Stability

In the previous chapter we have talked about the geometry and single particle wave functions of different time-dependent orbifolds. In this chapter, we shall talk about the particle interactions. One way to talk about the interactions is to discuss the large black holes formation. Another way of discussing interactions is to compute the tree level amplitudes. We shall conclude this chapter by going through the calculations of One-loop amplitudes.

5.1 Formation of large black–holes

If we consider small perturbations then most of the time–dependent orbifolds become unstable[5]. This argument is provided by general relativity, however, string theory does not hold this argument. Now, to state this argument, consider a particle on the orbifold, which interacts with infinite collection of particles within the orbifold geometry. If the interactions form a black hole, then we get an indicator that a black hole is generated in the orbifold quotient space. The condition of happening this is, the Schwarzchild radius has to be greater than the impact parameter b of the particle, which has center of mass energy ε,

$$G\varepsilon > b^{D-3}. \tag{5.1}$$

Here, $D=$ the dimension of space–time. If we consider a world–line which has null geodesic then

$$X^a(\lambda) = p^a\lambda + C^a. \tag{5.2}$$

Here, $p^a=$ momentum and $C^a=$ point on the geodesic. The world line of the n-th image geodesic is

$$X^a_n(\lambda) = p^a_n\lambda + C^a_n. \tag{5.3}$$

Here, with the orbifold action e^{nk}, p^a_n and C^a_n are the images. The impact parameter b is,

$$b^2 = Y^2 - \frac{2(p.Y)(p_n.Y)}{p.p_n}, \tag{5.4}$$

and the center of mass energy, ε is,

$$\varepsilon^2 = -2p.p_n, \tag{5.5}$$

where $Y = C - C_n$. Now, under these standards, we can find out which orbifolds are stable. First we start with the O-plane orbifold. We can define p^a and C^a by

$$p = \begin{pmatrix} p^+ \\ p^- \\ p \\ \overrightarrow{p}_\perp \end{pmatrix},$$

and

$$C = \begin{pmatrix} C^+ \\ 0 \\ C \\ \overrightarrow{C}_\perp \end{pmatrix},$$

So, for the n-th image particle the momentum is

$$p_n = \begin{pmatrix} p^+ - \beta p + \dfrac{\beta^2}{2} p^- \\ p^- \\ p - \beta p^- \\ \overrightarrow{p}_\perp \end{pmatrix}.$$

If we consider our n to be large then b and ε become

$$b \simeq \frac{2}{3} R\Delta(\pi n)^2, \qquad \varepsilon \simeq |p^-| 2\pi\Delta n.$$

According to our standards, O-plane orbifold is stable. For null-boost orbifold, the impact parameter becomes $b \simeq 2C$ if we set $R = 0$. That means we get unstable conditions for the null-boost orbifold. Similarly, we can find out that boost and shifted-boost orbifolds are unstable too. Here one important thing to notice is that, the stability argument only holds in three-dimension, where we get topological gravitational interactions.

5.2 Backreaction in three–dimensions

If we ant to understand the three dimensional orbifold geometry which has small perturbations then it is important to discuss the two–dimensional dilaton gravity. Here, we discuss about the backreaction problem for the null–boost and the shifted–boost orbifolds.
Now, the general form of the three-dimensional metric is

$$ds_3^2 = ds_2^2 + \Phi^2 (dz + A)^2 \,. \tag{5.6}$$

here, $\partial_z =$ the Killing direction. And the three–dimensional Hilbert action is

$$\int d^2x \sqrt{-g} \big(\Phi R - \frac{1}{2} \Phi^3 F^2 \big) \,. \tag{5.7}$$

Here, we introduce $\Phi^3 \star F$, which is constant. If we scale down the z, A and Φ^{-1}, we can amend the constant to our desired value thus $\star F = 2/\Phi^3$. For the O-plane and shift–boost orbifolds this is achievable. If we consider, $F \neq 0$, the equations of motion becomes

$$\int d^2x(\Phi R - V(\Phi)), \tag{5.8}$$

where, $V(\Phi) = \frac{2}{\Phi^3}$. Now, we are able to follow the orbifold geometry in two dimensional dilaton gravity. If we put on the matter portion, then the general form of the action is

$$S_{2D}(g, \Phi) + S_M(g, \Phi, \text{Matter}). \tag{5.9}$$

Then the equations of motion are

$$\begin{aligned} 2\nabla_a \nabla_b \Phi &= g_{ab}(2\Box\Phi + V) - \tau_{ab} \, , \\ \text{and} \quad R &= \frac{dV}{d\Phi} + \rho \, , \end{aligned} \tag{5.10}$$

where,

$$\begin{aligned} \tau_{ab} &= -\frac{2\partial S_M}{\sqrt{-g}\partial g^{ab}} \, , \\ \text{and} \quad \rho &= -\frac{1}{\sqrt{-g}} \frac{\partial S_M}{\partial \Phi} \, . \end{aligned}$$

The stress energy tensor τ_{ab} in terms of dilaton current ρ is

$$\nabla^a \tau_{ab} + \rho \nabla_b \Phi = 0 \, . \tag{5.11}$$

Now, in dilaton gravity, we define $J(\Phi)$ as [4]

$$J = \int V d\Phi \, , \tag{5.12}$$

and considering the function

$$C = (\nabla\Phi)^2 + J(\Phi) \, , \tag{5.13}$$

also, for the vector field we define

$$k^a = \frac{2}{\sqrt{g}} \epsilon^{ab} \nabla_b \Phi \, . \tag{5.14}$$

Now, if we consider the solutions for vacuum then, $\tau_a b$ and ρ becomes zero and our function C becomes a constant and k becomes a Killing vector. For vacuum solution, the equations of motion is given by

$$\nabla_a C = -\tau_{ab} \nabla^b \Phi + \nabla_a \Phi(\tau_{bc} g^{bc}) \, . \tag{5.15}$$

For the coordinates $z^{\pm}$, the metric becomes

$$- dz^+ dz^- e^{\Omega} \tag{5.16}$$

Then, $k\pm = \mp\nabla \pm \Phi$ and our killing vector become $\nabla + \nabla + \Phi = \nabla - \nabla - \Phi = 0$. This is true when $\tau_{ab} = 0$. Finally, these equations are similar as

$$\partial_- k^+ = \partial_+ k^- = 0 \, .$$

Now, analyzing the geometry

$$\tau_{\pm} = \rho = 0 \,,$$
$$\text{and} \quad \partial - \tau_{++} = \partial_{+}\tau_{--} = 0 \,.$$

We can describe the effect of these matters by shock wave [1], in this case our stress energy tensor becomes

$$\tau_{--}(z^-) = \epsilon\delta(z^- - z_0^-) \,. \tag{5.17}$$

For, $\tau_{--} = 2(\nabla - \eta)^2 > 0$ our equation of motion becomes

$$\nabla_{-}C = 2\tau_{--}\nabla_{+}\Phi e^{-\Omega} = \tau_{--}k^-, \tag{5.18}$$

Now, if we want to focus on the orbifold, the coupling of matter in the region II becomes

$$ds_2^2 = -dt^2\Big(\frac{E^2x^2}{1 - E^2x^2}\Big) + dx^2 \,,$$
$$\text{and} \quad \sqrt{E}\Phi = \sqrt{1 - E^2x^2} \,. \tag{5.19}$$

Here, to have normalized potential $2\Phi^{-3}$, which is canonical, we rescale the field Φ from previous section. As in vacuum solution, we get the same solutions for shock wave but with a different constant term, which is

$$E' = E - \epsilon k^- \,,$$

here, ϵ is getter than zero and the direction of the killing vector is along the shock wave. If we consider the vacuum, our killing vector is a function of z^+, but in matter the killing vector becomes

$$\partial_{-}k^+ = e^{-\Omega}\tau_{--} \,.$$

Here, along the horizon, Ω is a constant and our function k^+ changes position along the shock wave. This solves the changing position for the horizon.

5.3 The Tree–level Amplitudes

Now, for the orbifolds with dimensions $(\mathbb{M}^3/e^k) \times \mathbb{T}^{D-3}$, we want to calculate the tree-level amplitudes. Using the inheritance principle, we limit our focus to external states. Under the orbifold action, these states are invariant.
Here, the n–point amplitude is given by

$$\delta^3\Big(\sum_i \vec{p}_i\Big) A(\vec{p}_1, ..., \vec{p}_n) \,,$$

where, $\vec{p}_i$ is the momenta. Now, considering the momenta $\vec{p}_{i\perp}$ which is along the torus directions $\mathbb{T}^{D-3}$. The mass in D-dimension is given by

$$m_i^2 = M_i^2 + (\vec{p}_{i\perp})^2. \tag{5.20}$$

From, the inheritance principle, we get the following expression

$$\frac{1}{\sqrt{|\Pi_i p_{i+}|}} \int dp_i.....dp_n \delta\Big(\sum_i p_{i+}\Big)\delta\Big(\sum_i p_i\Big)\delta\Big(\sum_i p_{i-}\Big)e^{i\varphi(p_i)}\mathcal{A}(\vec{p}_i) \,. \tag{5.21}$$

From the on–shell condition, we get the momenta

$$p_- = \frac{p^2 + m^2}{2p_+}$$

Then, the amplitude $\varphi(p_i)$ becomes

$$\varphi(p_i) = \frac{1}{E}\sum_i \Big(\frac{p_i n_i}{R p_{i+}} - \frac{p_i m_i^2}{2p_{i+}^2} - \frac{p_i^3}{6p_i^2}\Big) . \tag{5.22}$$

Here, we have counted the amplitude twice due to isometries. To solve this, we define the following momenta transformation

$$\begin{aligned} p'_{i+} &= p_{i+} , \\ p'_i &= p_i + \beta p_{i+} , \\ \text{and} \quad p'_{i-} &= p_{i-} + \beta p_i + \frac{1}{2}\beta^2 p_{i+} , \end{aligned}$$

where, the action of the isometry, $\beta \in R$. using the condition of $\sum_i \overrightarrow{p}_i = 0$, we prove that

$$\varphi(p'_i) = \varphi(p_i) + \frac{\beta}{ER}\sum_i n_i . \tag{5.23}$$

Under the conservation of the charge n_i, the phase $\varphi(p_i)$ becomes invariant. Now, from the integral form of the inheritance principle, we get

$$|\sum_i c_i p_{i+}| \int d\beta \delta\Big(\sum_i c_i p'_i\Big), \tag{5.24}$$

Here, considering $\sum_i c_i p_i \neq 0$, we change the variables of the momenta and get the delta function

$$\delta\Big(\sum_i c_i p_i\Big),$$

Then, the normalization becomes

$$\Big|\sum_i c_i p_{i+}\Big| \int d\beta e^{i\frac{\beta}{ER}\sum_i n_i} \to 2\pi ER\Big|\sum_i c_i p_{i+}\Big|\delta_{\sum_i n_i} \tag{5.25}$$

Using the Kronecker symbol instead of the Dirac delta function, we can write

$$\begin{aligned} \mathcal{A}(p_{i+}, n_i) =& (2\pi ER)\delta_{\sum_i n_i}\delta(\sum_i p_{i+})\frac{|\sum_i c_i p_{i+}|}{\sqrt{\Pi_i p_{i+}}} \\ & \int dp_1 ... dp_n \delta(\sum_i p_{i+})\delta(\sum_i p_{i-})\delta(\sum_i c_i p_i) e^{i\varphi(p_i)}\mathcal{A}(\overrightarrow{p}_i) . \end{aligned}$$

5.4 The three–point amplitude

If we consider particles 1 and 2 are coming towards and particle 3 is going outwards, then $p_1+, p_2+ > 0$ and $p_3+ < 0$. Here, we are considering $\mathcal{A} = 1$. For the gauge $p_3 = 0$, the amplitude becomes

$$\sqrt{\Big|\frac{p_{3+}}{p_{1+}p_{2+}}\Big|} \int dp_1 dp_2 dp_3 \delta(\sum_i p_i)\delta(\sum_i p_{i-})\delta(p_3) e^{i\varphi(p_i)} , \tag{5.26}$$

Considering, $p_1 = -p_2$ and $p_3 = 0$, we get

$$2\sqrt{\frac{|p_{3+}|}{|p_{1+}p_{2+}|}} \int dp_1 \delta(4\alpha + p_1^2(\mu_{12})^{-1}) e^{i\varphi(p_i)} \,, \tag{5.27}$$

In equation 5.27,

$$\mu_{12} = \frac{p_{1+}p_{2+}}{p_{1+} + p_{2+}} \,,$$
$$\text{and} \quad \alpha = \frac{m_i^2}{4p_{i+}}.$$

If $\alpha > 0$, the amplitude becomes zero. Thus, the general form of this in terms of φ is

$$2\frac{\sqrt{\mu_{12}}}{\bar{p}} \theta(-\alpha) \cos\varphi(\bar{p}, -\bar{p}, 0) \,. \tag{5.28}$$

Here, $\bar{p} = \sqrt{-4\alpha\mu_{12}}$.

5.5 The four–point amplitude

If we consider particles 1, 2 are coming towards and particles 3 and 4 are going outwards, then $p_{1+}, p_{2+} > 0$ and $p_{3+}, p_{4+} < 0$. For the gauge $p_1 + p_2 = 0$, the amplitude becomes

$$\frac{p_{1+} + p_{2+}}{\sqrt{p_{1+}p_{2+}p_{3+}p_{4+}}} \int dp_1 \ldots.. dp_n \delta(\sum_i p_i) \delta(\sum_i p_{i-}) \delta(p_1 + p_2) e^{i\varphi(p_i)} \mathcal{A}(s,t), \tag{5.29}$$

Here, we introduce the Mandelstam variables, which are

$$s = -(\overrightarrow{p}_1 + \overrightarrow{p}_2)^2 + s_\perp \,,$$
$$\text{and} \quad t = -(\overrightarrow{p}_1 + \overrightarrow{p}_3)^2 + t_\perp \,.$$

where, $s_\perp = -(\overrightarrow{p}_{1\perp} + \overrightarrow{p}_{2\perp})^2$ and $t_\perp = -(\overrightarrow{p}_{1\perp} + \overrightarrow{p}_{3\perp})^2$. Same as from the three-point amplitude, for four-point amplitude we introduce μ_{12} and μ_{34} as

$$\mu_{12} = \frac{p_{1+}p_{2+}}{p_{1+} + p_{2+}} \,,$$
$$\text{and} \quad \mu_{34} = \frac{p_{3+}p_{4+}}{p_{3+} + p_{4+}} \,.$$

Also, we introduce

$$\alpha = \sum_i \frac{m_i^2}{4p_{i+}}$$

Now, we can write the amplitude in terms of integral form as

$$\int dq d\tilde{q} \delta(q\tilde{q} - \alpha) e^{i\varphi} \mathcal{A} \,. \tag{5.30}$$

where, the momenta p_i are as follows

$$p_1 = -p_2 = \sqrt{\mu_{12}}(q - \tilde{q}) \,,$$
$$\text{and} \quad p_3 = -p_4 = -\sqrt{\mu_{34}}(q + \tilde{q}) \,.$$

Now, if we consider with the following terms,

$$n_1 + n_3 = p_{1+} + p_{3+} = 0 \tag{5.31}$$

For this case, $\alpha = 0$, and $\mu_{12} = \mu_{34}$. Similarly from the previous section, the Mandelstam variables for four-points are

$$\begin{aligned} s(q) &= s_\perp + \big(m^2(\mu_{12})^{-1} + q^2\big)(p_{1+} + p_{2+})\,, \\ \text{and} \quad t(q) &= t_\perp\,. \end{aligned}$$

Combining everything, we get the amplitude for four points,

$$\int \frac{dq}{|q|}\mathcal{A}(s(q), t_\perp)\,. \tag{5.32}$$

We can write this in the form of

$$\mathcal{A} \sim G\frac{s^J}{-t}\,,$$

where, G= coupling and J= spin of the coupled particle.

5.6 Eikonal Resummation

For the massless particle, we consider

$$\overrightarrow{p}_{i\perp} = M^2 = m^2 = 0$$

Here, we introduce

$$\delta = \frac{1}{2\sqrt{p_{1+}p_{2+}}}(p_{1+} + p_{3+})$$

and the Mandelstam variables are

$$\begin{aligned} \alpha' s &= \alpha' q^2(p_{1+} + p_{2+}) = \lambda^2\,, \\ \text{and} \quad \alpha' t &\simeq -\alpha' s\partial^2 = -\lambda^2\delta^2\,. \end{aligned}$$

The amplitude is

$$\phi(\lambda) \simeq \frac{\delta}{E\alpha'^{3/2}p_{1+}p_{2+}}\Big[-\frac{n}{R}(p_{1+} + p_{2+})\alpha'\lambda + \frac{1}{6}\lambda^3)\Big]\,. \tag{5.33}$$

Here,

$$\frac{-t}{s} \simeq \delta^2$$

Also, the integral form of the amplitude is

$$\mathcal{A} \sim G\frac{s^2}{-t} \sim \frac{G}{\alpha'}\frac{\lambda^2}{\delta^2}$$

In three dimension this becomes

$$\mathcal{A} \sim -G\frac{s^2}{t + (2\pi G s)^2}$$

where,

$$(2\pi Gs)^2 >> -t \,,$$
$$\text{and} \quad \lambda >> \lambda_e \,.$$

and the amplitude follows

$$\mathcal{A} \sim \frac{-1}{2\pi^2}\frac{1}{G}$$

Finally,the amplitude for the orbifold is

$$2\frac{G}{\alpha'}\frac{1}{\delta^2}\int_0^{\lambda} d\lambda\lambda - \frac{1}{2\pi^2}\frac{1}{G}\int_{\lambda_e}^{\lambda_t}\frac{d\lambda}{\lambda} \sim \frac{1}{(2\pi)^2}\frac{1}{G}\Big[1 + 2ln\big(\frac{\sqrt{\alpha'\delta^2}}{2\pi G}\big)\Big]\,. \tag{5.34}$$

5.7 One–loop Amplitudes

Now, we compute the partition function in bosonic string theory which is known as one-loop amplitude computation. Mathematically these computations are possible, but their physical interpretation is still not clear. They generates divergences which are not still understood and might create a problem in perturbation theory. In this section, we shall concentrate on the shifted boost orbifold for calculating the one-loop amplitudes. For, shifted-boost orbifold we know that

$$X^{\pm} \sim e^{\pm 2\pi\Delta}X^{\pm}\,,$$
$$\text{and} \quad X \sim X + 2\pi R\,. \tag{5.35}$$

Here we define a modified constraint which acts on the total momentum P, which maintain $e^{2\pi i(RP+\Delta J)} = 1$[3]. We can write this as,

$$P = \frac{1}{R}(n - \Delta J),$$

The momenta for X is

$$p_{L,R} = P \pm \frac{wR}{2}.$$

For the fields $X\pm(z,\tilde{z})$, the modes are

$$X^{\pm}(z,\tilde{z}) = i\sum_n\Big(\frac{1}{n \pm iv}\frac{a_n^{\pm}}{z^{n\pm iv}} + \frac{1}{n \mp iv}\frac{\tilde{a}_n^{\pm}}{\tilde{z}^{n\mp iv}}\Big),$$

where, $\nu = w\Delta$ and we can write the commutative relations as

$$[a_m^{\pm}, a_n^{\mp}] = -(m \pm iv)\delta_{m+n},$$

$$[\tilde{a}_m^{\pm}, \tilde{a}_n^{\mp}] = -(m \mp iv)\delta_{m+n}.$$

Here, the hermitian conditions are defined as $(a_m^{\pm})^{\dagger} = a_{-m}^{\pm}$, $(\tilde{a}_m^{\pm})^{\dagger} = \tilde{a}_{-m}^{\pm}$.

For zero–mode, the relations become

$$[a_0^{\pm}, a_0^{\mp}] = \pm iv,$$

$$[\tilde{a}_0^{\pm}, \tilde{a}_0^{\mp}] = \mp iv.$$

If we want to quantize the above relations, we can write the following combinations

$$a_0^{\pm} = P^{\pm} \pm \frac{v}{2} x^{\pm},$$

$$\tilde{a}_0^{\pm} = P^{\pm} \mp \frac{v}{2} x^{\pm}.$$

Now, to discuss about this relations interms of the orbifold geometry, we introduce Virasoro generators, which are represented as

$$L_0 = ... + \frac{1}{2} iv(1 - iv) - \sum_{n \geq 1} a_n^+ a_n^- - \sum_{n \geq 0} a_n^- a_n^+,$$

$$\tilde{L}_0 = ... + \frac{1}{2} iv(1 - iv) - \sum_{n \geq 1} \tilde{a}_n^+ \tilde{a}_n^- - \sum_{n \geq 0} \tilde{a}_n^- \tilde{a}_n^+.$$

Finally, the partition function Z becomes

$$Z = (q\bar{q})^{-\frac{1}{8}} \sum \frac{R}{\sqrt{2_\tau 2}} \sum_{w,w'} exp\Big[- \frac{\pi R^2}{2_\tau 2} T\bar{T} - 2\pi\tau_2 \Delta^2 w^2 \Big]$$

$$\times q^{(1/2)iv} Tr_L e^{2\pi i T \Delta J_L q^L}$$

$$\times \bar{q}^{(1/2)iv} Tr_R e^{2\pi i \bar{T} \Delta J_R \bar{q}^{\tilde{L}}},$$

Now, we introduce a constant term c, which is

$$c = e^{2\pi i(\Delta T)} = q^{iv} e^{2\pi w' \Delta},$$

Then, the partition function is

$$Z = \frac{R}{\sqrt{\alpha' \tau_2}} \sum_{w,w'}^{\prime} e^{-(\pi R^2/\alpha')(T\bar{T}/\tau_2) - 2\pi\tau_2 \Delta^2 w^2} \Big| \theta_1(i\Delta T|\tau) \Big|^{-2}. \qquad (5.36)$$

Chapter 6

Conclusion

In this thesis, we have discussed the physics of time-dependent orbifolds and their geometry. We have focused on the orbifolds of three-dimensional Minkowski space. Discussing about the geometry of these orbifolds and their interactions gave us the opportunity to discuss such problems like large black hole formation, particle production on the orbifolds and one-loop calculations. The solutions to these problems are essential to understand quantum gravity in a more well defined way. In future, development in this field will definitely helps us understand the evolution of the universe in more clear ways.

Bibliography

[1] C. Callan, S. Giddings, J. Harvey, and A. Strominger, "Evanescent black holes", *Phys.Rev.D45:1005-1009*, 1992. DOI: 10.1103/PhysRevD.45.R1005.

[2] L. Cornalba and M. S. Costa, "A new cosmological scenario in string theory", *Phys.Rev. D66 (2002) 066001*, p. 28, 2002. DOI: 10.1103/PhysRevD.66.066001.

[3] B. Craps, D. Kutasov, and G. Rajesh, "String propagation in the presence of cosmological singularities", *JHEP 0206 (2002) 053*, 2002. DOI: 10.1088/1126-6708/2002/06/053.

[4] D. Grumiller, W. Kummer, and D. Vassilevich, "Dilaton gravity in two dimensions", *Phys.Rept.369:327-430*, 2002. DOI: 10.1016/S0370-1573(02)00267-3.

[5] G. T. Horowitz and J. Polchinski, "Instability of spacelike and null orbifold singularities", *Phys.Rev.D66:103512*, p. 16, 2002. DOI: 10.1103/PhysRevD.66.103512.

[6] J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt, and N. Turok, "From big crunch to big bang", *Phys.Rev.D65:086007,2002*, 2002. DOI: 10.1103/PhysRevD.65.086007.

[7] H. Liu, G. Moore, and N. Seiberg, "Strings in time-dependent orbifolds", *JHEP 0210:031,2002*, p. 32, 2002. DOI: 10.1088/1126-6708/2002/10/031.

[8] N. A. Nekrasov, "Milne universe, tachyons, and quantum group", *Surveys High Energ.Phys. 17 (2002) 115-124*, 2002. DOI: 10.1080/0142241021000054176.

[9] J. Simon, "The geometry of null rotation identifications", *JHEP 0206 (2002) 001*, 2002. DOI: 10.1088/1126-6708/2002/06/001.

[10] L. Cornalba and M. S. Costa, "Time-dependent orbifolds and string cosmology", *Fortsch.Phys.52:145-199,2004*, vol. V2, p. 74, 2005. DOI: 10.1002/prop.200310123.

[11] D. Baumann, *Cosmology.* [Online]. Available: http://cosmology.amsterdam/education/cosmology/.

[12] J. Fergusson, *Part iii cosmology.* [Online]. Available: http://www.damtp.cam.ac.uk/user/jjvk2/notes/cosmology.pdf.

Perturbed Ricci Tensor and Ricci Scalar

In terms of the connection, the Ricci tensor is

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} \tag{6.1}$$

The 00-component is,

$$\begin{aligned} R_{00} &= \partial_i \Gamma^i_{00} - \partial_0 \Gamma^i_{0i} + \Gamma^\alpha_{00}\Gamma^i_{\alpha i} - \Gamma^\alpha_{0i}\Gamma^i_{0\alpha} \\ &= \partial_i \Gamma^i_{00} - \partial_0 \Gamma^i_{0i} + \Gamma^0_{00}\Gamma^i_{0i} + \Gamma^j_{00}\Gamma^i_{ji} - \Gamma^0_{0i}\Gamma^i_{00} - \Gamma^j_{0i}\Gamma^i_{0j} \\ &= \nabla^2\Psi - 3\partial_0(\mathcal{H} - \Phi') + 3(\mathcal{H} + \Psi')(\mathcal{H} - \Phi') - (\mathcal{H} - \Phi')^2\delta^j_i\delta^i_j \\ &= -3\mathcal{H}' + \nabla^2\Psi + 3\mathcal{H}(\Phi' + \Psi') + 3\Phi'' \end{aligned} \tag{6.2}$$

The $0i$-component is,

$$R_{0i} = 2\partial_i\Phi' + 2\mathcal{H}\partial_i\Psi \tag{6.3}$$

The ij-component is,

$$\begin{aligned} R_{ij} = [\mathcal{H}' + 2\mathcal{H}^2 - \Phi'' + \nabla^2\Phi - 2(\mathcal{H}' + 2\mathcal{H}^2)(\Phi + \Psi) - H\Psi' - 5\mathcal{H}\Phi']\delta_{ij} \\ + \partial_i\partial_j(\Phi - \Psi) \end{aligned} \tag{6.4}$$

Now, for Ricci scalar

$$R = g^{00}R_{00} + 2g^{0i}R_{0i} + g^{ij}R_{ij} \tag{6.5}$$

Which leads to

$$\begin{aligned} a^2R =& (1 - 2\Psi)R_{00} - (1 + 2\Phi)\delta^{ij}R_{ij} \\ =& (1 - 2\Psi)[-3\mathcal{H}' + \nabla^2\Psi + 3\mathcal{H}(\Phi' + \Psi') + 3\Phi''] - 3(1 + 2\Phi) \\ & [\mathcal{H}' + 2\mathcal{H}^2 - \Phi'' + \nabla^2\Phi - 2(\mathcal{H}' + 2\mathcal{H}^2)(\Phi + \Psi) - \mathcal{H}\Psi' - 5\mathcal{H}\Phi'] \\ & - (1 + 2\Phi)\nabla^2(\Phi - \Psi) \\ =& -6(\mathcal{H}' + \mathcal{H}^2) + 2\nabla^2\Psi - 4\nabla^2\Phi + 12(\mathcal{H}' + \mathcal{H}^2)\Psi + 6\Phi'' \\ & + 6\mathcal{H}(\Psi' + 3\Phi') \quad \text{[Removing the non-linear terms]} \end{aligned} \tag{6.6}$$

Overleaf: GitHub for LaTeX projects

This Project was developed using Overleaf(https://www.overleaf.com/), an online LaTeX editor that allows real-time collaboration and online compiling of projects to PDF format. In comparison to other LaTeX editors, Overleaf is a server-based application, which is accessed through a web browser.