Thesis Paper On

Exact Solutions of Fractional Differential Equations by using New Generalized (G'/G)-Expansion Method

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Abstract

In this thesis, Exact Solutions of fractional differential equations by using (G'/G)-Expansion Method the nonlinear partial fractional differential equations are renewed to the nonlinear ordinary differential equations by using the fractional complex transformation. We apply the extended (G'/G)-expansion method to generate travelling wave solutions to the time and space fractional derivative nonlinear KdV equation. The obtained solutions reveal that the extended (G'/G)-expansion method is very efficient and competent mathematical tool for generating abundant solutions and can be used world class of nonlinear evolution fractional order equations.

Dedication

To my beloved Parents and Mathematics enthusiasts all over the world.

Declaration

This is to certify that the work presented in this thesis is the outcome of the investigation carried out by the author under the supervision of Dr. Hasibun Naher, Associate Professor, Department of Mathematics, BRAC University, 66, Mohakhali, Dhaka-1212. It has not been submitted anywhere for the award of any kind of degree or diploma and also cited properly.

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First of all,I remember to Almighty ALLAH for giving me an opportunity along with physical and mental energies to complete my Bachelors (BSc. in Mathemtics) degree smoothly.I would like to express all my venerate and wholeheartedness to Him as he has made me able to finish this thesis successfully.

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I would also like to recognize my accountability to all other respectable faculties of the Department of Mathematics and Natural Sciences in BRAC University.

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Contents

Chapter 1

INTRODUCTION

1.1 Mathematical Introduction

The affair of differential equation constructs a comprehensive and very significant wing of modern mathematics. Form the preceding days of calculus the subject has been a sector of innumerous hypothetical analysis and applied implementations. Leibnez and Newton[26,27] were the first Scientist and Mathematician who first commenced with the invention of calculus. In the year 1671 English physicist, Newton had listed three types of differential equations which were published in the year 1736, they are:

$$\frac{dy}{dx} = f(x) \tag{1.1}$$

$$\frac{dy}{dx} = f(x, y) \tag{1.2}$$

and

$$x_1 \frac{\delta y}{\delta x_1} + x_2 \frac{\delta y}{\delta x_2} = y. \tag{1.3}$$

In 1676 Newton solved the equations mentioned above and others affixed to these by utilizing infinite series and calculated about the non-uniqueness of the results. Jacob Bernoulli proposed the Bernoulli differential equation in 1695 which was an ordinary differential equation(ODE)[28] mentioned in the following structure:

$$p' + A(x)p = B(x)p^n. \tag{1.4}$$

In the following year Leibneiz gathered solutions by simplifying (1.4).

1.1.1 Differential Equation

One of the most incomparable successes of calculus is its ability to capture continuous motion mathematically and giving us the opportunity to analyze that motion instantaneously[9]. A prodigious volume of real world event involving moving quantities like the speed of a missile[1], the floating exchange rate [2], the voltage of an electrical signal[3-5], the intensity of an earthquake[6], the growth rate of population of species[7], the number of bacterial growth in a medium[8] and many more.Differential equations infiltrate the science and let us use it as a tool by which we can try to bring out the laws of motion of nature in an abridged mathematical language. We have heard a lot about differential equation for radioactive decay in nuclear physics[10-11] but there are also many other numerous differential equations like Newton's law of cooling in thermodynamics[12], the Navier-Stokes equations in general relativity[13-14], the Blacke-Scloles equation in finance[15], the heat equation in thermodynamics[16-17], the Cauchy-Riemann equations in complex analysis[18-19], Schdinger equation in quantum mechanics[20], the wave equation[21], the Lotka-Volterra equation in population dynamics[22], Maxwell's equations in electromagnetism [23], Laplace's equation and Poisson's equation[24], Einstein's field equation in general relativity[25] and many more.

Differential equation: When derivatives of one or more dependent variable with respect to one or more dependent variable in any equation is defined as differential equation (DE). For example, an expression as follows -

$$\frac{d}{dx}(ab) = \frac{da}{dx} + \frac{db}{dx} = 0 \tag{1.5}$$

The equation 1.5 is an example of differential equation.

1.1.2 Classifications of Differential Equation

Differential equation is classified into two forms, such as **Ordinary Differential Equations (ODEs)**and **Partial Differential Equation (PDEs)**.

Ordinary Partial Differential Equations (ODE) - When ordinary derivatives of one or more dependent variable with respect to only one independent variable in any differential equation is called ordinary differential equation (ODE). For example, such an expression given below -

$$\frac{dz}{dx} + \frac{dp}{dx} = 0 \tag{1.6}$$

The equation 1.6 is one of the form of ordinary differential equation where, for example: $y = x + cosx$ and $z = x^3 + 5x$.

Partial Differential Equation (PDE) -When one or more partial derivatives of one or more dependent variable with respect to more than one independent variable in any differential equation is called partial differential equation (PDE). For example, an expression given below -

$$\frac{dp}{dx} + \frac{dp}{dy} = 0 \tag{1.7}$$

The equation 1.7 is one of the form of partial differential equation where, for example $z = y^6 + sinx$.

In this thesis we will deal with the section of partial differential equation, over here we will be talking about the catagorizations of the partial differential equations only. Partial differential equation is classified into two forms such as, **Linear Partial Differential Equations (LPDEs)** and **Non-Linear Partial Differential Equations (NLPDEs)**.

Linear Partial Differential Equations (LPDEs) - When the power of the dependent variable and each partial derivative contained in the equation is one, and the coefficients of each variable as well as the coefficients of each partial derivative are constants or independent variables in partial differential equation is said to be linear partial differential equation. For example, an expression as follows -

$$y^m + b_{m-1}(x)y^{m-1} + ... + b_1(x)y' + b_0(x)y = 0 \tag{1.8}$$

The equation 1.8 is the one of the form of a linear partial differential equation.

Non-Linear Partial Differential Equations (NLPDEs) - When any of the condition for being linear is not satisfied then the equation is called non-linear. An expression is giveb below-

$$\frac{dp}{dt} = F(p) \tag{1.9}$$

The equation 1.9 is an example of non-linear partial differential equation.

Homogeneous Partial Differential Equations (HPDEs) - If all the term of the P.D.E. consists the dependent variable x or one of its derivatives then only it is known as homogeneous partial differential equation. to support the definition, an expression is given below-

$$y^m + b_{m-1}(x)y^{m-1} + ... + b_1(x)y' + b_0(x)y = 0 \tag{1.10}$$

The equation 1.10 is a type of homogeneous partial differential equation.

Non-homogeneous Partial Differential Equations (IPDEs) - If any of the condition for being homogeneous is not satisfied in a differential equation it is said to be non-homogeneous. To support the definition, an expression is given below –

$$y^m + b_{m-1}(x)y^{(m-1)} + ... + b_1(x)y' + b_0(x)y = c \tag{1.11}$$

The equation 1.11 is the form of non-homogeneous partial differential equation.

Non-linear Evolution Equations (NLEEs) - A NLPDE, as a single independent variable such as time t expressed in terms of space variable $l(x,t)$is called nonlinear evolution equation (NLEE). For example: the KdV equation .

Fractional differential equations (FDEs) – There are three important definitions of fractional differential equations, they are the Riemann-Liouville, the Grunwald-Letnikove and the M. Caputo definition. In this work we will use the Riemann-Liouville definition.
The Riemann–Liouville type fractional derivative of order $\beta > 0$ of a function $g : (0, \infty) \to R$ is defined by

$$\mathbf{D}^\beta g(t) = \frac{d^m}{dt^m}\frac{1}{\Gamma(m-\beta)}\int_0^t (t-\tau)^{m-\beta-1} g(\tau)d\tau \tag{1.12}$$

where $m = [\beta] + 1$ and $[\beta]$ is the integer part of β.

In this paper we will be talking about the analytical solution of the non-linear KdV equation with time space fractional derivatives.

1.2 Waves

According to physics, waves are a disturbances that propagate energy through a medium. the energy propagation depends on the synergy between the particles that make up the medium. Although there is no net motion of particles, particles move as the waves pass. This means, once a wave has passed the particles return to their original position. As a result, energy, not matter, is propagated by waves. There are three main types of waves they are Mechanical wave, Electromagnetic wave and Matter wave. Mechanical Waves act as the propagation of a disturbance through a material medium due to the repeated periodic motion of the particles of the medium about their mean positions, the disturbance being handed over from one particle to the next. Electromagnetic Waves are the disturbance, which does not require any material medium for its propagation and can travel even through vacuum. They are caused due to varying electric and magnetic fields, and lastly Matter Waves are the waves produced in electrons and particles.

The fairest wave generation equation could be written in the following form:

$$l_{tt} = k^2 l_{xx} \tag{1.13}$$

where the amplitude of the wave is represented by $l(x,t)$, and k represents the speed of the wave. This equation has the general d'Alembert' solution and that is -
$l(x,t) = p(x - kt) + q(x - kt)$ where p and q are arbitrary constraints denoting the right and left propagation accordingly and these two individual waves move without changing their oneness.

1.3 Dispersion

Dispersion of water waves generally refers to frequency dispersion, which means that waves of different wavelengths travel at different phase speeds. Wave dispersion in water waves refers to the property that longer waves have lower frequencies and travel faster. The dispersive waves are waves in which the wave velocity varies with the wave number. Dispersive effects usually give a relationship between the frequency and the wave speed. If wave speed varies with wave number, in which case the different wave number components will have different speeds, then the phenomenon known as wave dispersion. Therefore, the way they interfere with one another will change with time. So the shape of the disturbance will change.

1.4 Dissipation

A wave that loses amplitude, due to loss of energy over time, is called a dissipative wave.

1.5 Travelling waves

When a vibrating source disturbs the atoms wave formed or particles of medium (water, air, spring, string etc.) that travels continuously along the direction of wave motion with speed and without altering its shape is called traveling waves. Particles get displaced from their rest position temporarily but there is a force that acts on the particle when the medium is disturbed , causing them to get back to its original form or position. Without any net motion in the medium this disturbance travels down through the system and through this disturbance energy and momentum is shifed from one end to another . To understand the occurrence of travelling wave solutions, mathematical dealings have been used to describe the travelling wave function in the form of $l(x,t) = f(x-kt)$ where $l(x,t)$ represents the wave distressed movement along the direction of x when $k > 0$or $k < 0$ accordingly. These are obtained when a NLEE is reduced to ODE, taking $l(x,t) = l(\xi)$ where $\xi = x - kt$ and k is the speed of the wave where they can be solved by using suitable method. There has been a lot of development for obtaining travelling wave solutions during the past few decades and they appear in various forms and few of them are:

1.5.1 Soliton

According to Wazwaz[29],property with elastic scattering are called Solitons which are a form of solitary waves. Even after colliding with each other they tend to keep their original form and speed. they are seen in various physical phenomena and have been appeared as the results of a extensive group of weakly nonlinear dispersive partial differential equations describing systems of physic. The dissimilarity between solitary waves and solitons has become obfuscated in physical term. Soliton-like solutions of nonlinear evolution equations are Solitary waves that explains wave processes in dissipative and dispersive environment. It is usually called as single soliton solution like solitary wave, but when more than one soliton appear in a solution they are called solitons. A nonlinear partial differential equation that shows the following possessions are solutions of a soliton :
(i) the solution should substantiate a wave of stable form;
(ii) the solution either decays exponentially to zero like the solitons provided by the KdV equation, or converges to a constant at infinity such as the solitons given by the Sine-Gordon equation, meaning the solution is localized;
(iii) by holding its own character the soliton intercats with different solitons. One basic expression of a solitary wave solution is of the form-

$$l(x,t) = f(x-kt), \tag{1.14}$$

where k is the speed of wave circulation. The wave travels in the negative direction for $k < 0$ and For $k > 0$, the wave travels in the positive direction. The solutions of nonlinear equations can be in the form of $sech^2$, $sech$, $arctan \exp^{\alpha(x-kt)}$ functions. Different methods had been introduced to acquire solitons.

1.5.2 Solitary Waves

Solitary waves were first seedn by John Scott Russell in the year 1834. He observed a large projection of water slowly traveling on the Edinburgh-Glasgow canal (Scotland) without any change of its shape. He named he bulge of water as "great wave of translation", and it was traveling along the channel of water for a long period of time while still preserving its shape. The finding is described here in Scott Russell's own words: "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called Wave of translation." [29, 30]. Russel got inspired from this surprising discovery to conduct physical laboratory experiments so that he could highlight his observance and study those solitary waves. He empirically derived the relation in the following form: $k^2 = g(p+h)$ where k is the solitary wave speed, h is the maximum amplitude above the water surface, p is the finite depth and g is the acceleration of the gravity. This single humped wave of bulge of water is now called solitary waves or solitons. The solitons–localized, highly stable waves that retain its identity (shape and speed), upon interaction–was discovered experimentally by Russell[29].

A solitary wave is a localized wave which propagates without any temporal evolution in shape or size when viewed in the reference frame.The envelope of the wave has one global peak and decays far away from the peak. These waves arise in many circumstances, including the elevation of the surface of water and the intensity of light in optical fibers. They have finite amplitude and propagate with constant speed and constant shape.

Chapter 2

LITERATURE SURVEY

2.1 Analytical Methods for linear and non-linear fractional differential equations

The area of differential equation is vast enough to discuss about. The fractional calculus is a current research topic in applied sciences such as applied mathematics, physics, mathematical biology, economy, demography, engineering, geophysics, medicine, bio-engineering and mathematical biology all from the sources [49-68]. The rule of fractional derivative is not unique till date. The definition of fractional derivative is given by many authors. The commonly used definition is the Riemann-Liouvellie (R-L) definition [31]. Other useful definition includes Caputo definition of fractional derivative(1967) [32]. The solution and its interpretation of the fractional differential equations is a rising field of Applied Mathematics. Mathematical perspectivs of fractional differential equations and methods of their results were discussed by many authors: Iteration method[42], the series method[43], the Fourier transform technique in [44, 45]. To solve the linear and non-linear differential equations recently used methods are Adomain decomposition method [34-36], Variational Iteration Method [47], Differential transform method [41], Homotopy Perturbation Method [37-40], Predictor-Corrector method [48], Jumarie's left handed modification of R-L fractional derivative is useful to avoid non-zero fractional derivative of a constant functions [33] and Jumarie Derivative in Term of Mittag-Leffler Function[46].

2.2 Expansion Method

Over the last few decades, an extensive research has been going on to find distinctive solutions of NLEEs which are used as representations in order to describe many salient and tricky physical phenomena in diverse fields of science. Some exact solution of NLEEs by using huge range of upgraded and productive methods were introduced by different group of scientists who has fabricated , for example, the exp-function method [76],inverse scattering transformation [73], the sub-ODE method [74,75] etc. For creating more new applications of it in furtherance to understand the nonlinear phenomena better. as prescribed beforehand , no such united method have yet been manifested to work out with this type of NLEEs. The (G'/G) Expansion method, originated by Wang is one of the strongest and finest methods to solve nonlinear problems. To demonstrate and construct travelling wave solutions of non-identical type of NLEEs.Differential equation is executed in (G'/G) expansion method where a second order linear ordinary that is

$$G'' + G'\lambda + G\mu = 0 \tag{2.1}$$

where λ and $\mu$are arbitrary constants. To show the usefulness of the (G'/G) expansion method many researchers have carried out several investigations i.e. Zhang has extended the (G'/G) expansion method and named it improved (G'/G) expansion method. The difference between the original and extended (G'/G) expansion method is that, In original method-

$$l(\xi) = \sum_{j=0}^{m} a_j (G'/G)^j \tag{2.2}$$

where $a_m \neq 0$, on the other hand
In Zhang's method-

$$l(\xi) = \sum_{j=-m}^{m} a_j (G'/G)^j \tag{2.3}$$

where $a_m \neq 0$ or $a_{-m} \neq 0$ but both cannot be zero simultaneously.
The extended (G'/G) expansion method to achieve travelling wave solution of the Whithsm Broer-Kaup-like method and couple Hirota-Satsuma KdV equations are acquainted with Guo and Zhou in the form

$$l(\xi) = \sum_{j=1}^{m} \{a_j (G'/G)^j + b_j (G'/G)^{j-1} \sqrt{\sigma(1 + \frac{1}{\mu}(G'/G)^2}\} \tag{2.4}$$

Here we are going to explain the basic (G'/G) expansion method to find travelling wave solutions of nonlinear evolution equation. Assumimmg that the nonlinear equation in two independent variables are x and t, is in the form of-

$$P = (l, l_t, l_x, l_{tt}, l_{xt}, l_{xt}, ... = 0) \tag{2.5}$$

P is a polynomial in $l = l(x, t)$ and its several partial derivatives in which the highest order derivatives and nonlinear terms are affiliated. In the following steps we will show the main steps of the (G'/G) expansion method.
Step-1: By connecting the independent variables x and t into one variable $\xi = x - kt$ we presume that

$$l(x, t) = l(\xi), \xi = x - kt \tag{2.6}$$

The travelling wave variable (2.2) let us insert eq. (2.1) to an ODE for $l = l(\xi$

$$P(l, -kl', l', k^2 l'', -kl'', l'', ...) = 0 \tag{2.7}$$

Step-2: Supposing that the solution of ODE (2.3) could be shown by a polynomial in (G'/G) as below:

$$l(\xi) = \beta_m \left(\frac{G'}{G}\right)^m + ... \tag{2.8}$$

Where $G = G(\xi)$ convinces the second order LODE in the form

$$G'' + \lambda G' + \mu G = 0, \tag{2.9}$$

$a_m, ..., \lambda$ are constants to be determined later,$\beta_m \neq 0$, the unwritten part in (2.4) is also a polynomial in $(\frac{G'}{G})$, the degree of which is generally equal to or less than $m - 1$, but the positive integer m can be seen by considering the homogeneous balance between the highest order derivatives and nonlinear terms in ODE (2.3).
Step-3: By replacing (2.4) into Eq. (2.3) and using second order LODE(2.5), gathering every terms with the same order of $(\frac{G'}{G})$ altogether, the left hand side of Eq. (2.3) is changed into another polynomial in $(\frac{G'}{G})$. Calculating each co-efficient of this polynomial to zero, produces a set of algebraic equations for $\beta_m, ..., k, \lambda$ and μ.
Step-4: Supposing that the constraints $\beta_m, ..., k, \lambda$ and μ can be obtained by equating the algebraic equations in Step 3, general results of the second order LODE(2.5) are in the mean time have been recognized for us, then substituting $\alpha_m, ..., k$ and the normal solutions of Eq.(2.5) into Eq.(2.4) we will get more travelling wave solutions of the nonlinear evolution equation (2.1).

Chapter 3

METHODOLOGY

3.1 New Generalized (G'/G) Expansion Method with Non-Linear Auxiliary Equation:

As it is known that (G'/G) -expansion method is one of the simplest and most powerful method for obtaining travelling wave solutions of NLFDEs and so far its application have been used in various ways to solve nonlinear evolution problems. By considering the following nonlinear partial fractional differential equation:

$$P(l, D_1^{\alpha} l, D_x^{\beta} l, D_y^{\gamma} l, D_z^{\delta} l, D_t^{\alpha} D_t^{\alpha} l, D_t^{\alpha} D_x^{\beta} l, D_x^{\beta} D_x^{\beta} l, D_x^{\beta} D_y^{\gamma} l, D_y^{\gamma} D_y^{\gamma} l, ... = 0, 0 < \alpha, \beta, \gamma, \delta < 1), \tag{3.1}$$

Where l is an unknown function, P is a polynomial of l and its partial fractional derivatives in which the highest order derivatives and nonlinear terms are involved. The most important algorithms of the method as below:

Step 1. Li and He [19, 20] proposed a fractional complex transform to convert fractional differential equations into an ordinary differential equations (ODEs) so all analytical methods which are devoted to the advanced calculus can be can be easily applied to the fractional calculus. By using the traveling wave variable

$$l(x, y, z, t) = l(\xi), \xi = \frac{Kx^{\beta}}{\Gamma(\beta+1)} + \frac{Ny^{\gamma}}{\Gamma(\gamma+1)} + \frac{Mz^{\delta}}{\Gamma(\delta+1)} + \frac{Lt^{\alpha}}{\Gamma(\alpha+1)}, \tag{3.2}$$

where K, L, M and N are nonzero arbitrary constants and we can rewrite Eq.3.1 into an ODE of $l = l(\xi)$ in the form:

$$Q(l, l', l'', l''', ...) = 0. \tag{3.3}$$

If possible, we should integrate Eq.(3.3) term by term one or more times. The integral constant may be zero, for simplicity.

Step 2. Suppose that the travelling wave solution of Eq. (3.3) can be expressed as follows:

$$l(\xi) = \sum_{j=0}^{m} a_j [\Psi(\xi)]^j + \sum_{j-1}^{m} b_j [\Psi(\xi)]^{-j}, \tag{3.4}$$

where $\Psi(\xi) = [d + \varphi(\xi)]$ and $\varphi(\xi)$ is:

$$\varphi(\xi) = (G'(\xi)/G(\xi)). \tag{3.5}$$

Here a_m or b_m may be zero, but both of them cannot be zero at the same time, $a_j(j = 0, 1, 2, ..., m)$, $b_j(j = 1, 2, ..., m)$ and d are arbitrary constants to be determined later and $G = G(\xi)$ satisfies the second order nonlinear ODE:

$$AGG'' - BGG' - C(G')^2 - EG^2 = 0, \tag{3.6}$$

where A, B, C and E are real parameters.

Step 3. To determine the positive integer m, taking the homogeneous balance between the highest order nonlinear terms and the highest order derivatives appearing in Eq.3.3.

Step 4. Substituting Eq.(3.4) and Eq.(3.6) along with Eq.(3.5) and Eq.(3.3) with the value of m obtained in Step 3 and yields polynomials in $(d + \varphi(\xi))^m$ where $(m = 0, 1, 2, ...)$ and $(d + \varphi(\xi))^{-m}$ $(m = 1, 2, 3, ...)$. Then, each coefficient of the resulted polynomials to be zero, yields a set of algebraic equations for $a_j(j = 0, 1, 2, ..., m)$, $b_j(j = 1, 2, ..., m)$, $K, L, M$and d.

Step 5. Suppose that the value of the constants can be found by solving the algebraic equations which are obtained in step 5. Substituting the values of d and the general solution of Eq.(3.6) into Eq.(3.4). We can obtain a variety of exact travelling wave solutions of Eq.(3.1).

Step 6. From the general solution of Eq. (3.6), we find the following form,

Family 1. Hyperbolic function:

When $B \neq 0$, $\lambda = A - C$ and $\Omega = B^2 + 4E(A - C) > 0$, and C_1, C_2 are arbitrary constants.

$$\Phi(\xi) = (\frac{G'}{G}) = \frac{B}{\lambda} + \frac{\sqrt{\Omega}}{2\lambda} \frac{C_1 sinh(\frac{\sqrt{\Omega}}{2\lambda}\xi) + C_2 cosh(\frac{\sqrt{\Omega}}{2\lambda}\xi)}{C_1 cosh(\frac{\sqrt{\Omega}}{2\lambda}\xi) + C_2 sinh(\frac{\sqrt{\Omega}}{2\lambda}\xi)} \tag{3.7}$$

Family 2. Trigonometric function:

When $B \neq 0$, $\lambda = A - C$ and $\Omega = B^2 + 4E(A - C) < 0$, and C_1, C_2 are arbitrary constants.

$$\Phi(\xi) = (\frac{G'}{G}) = \frac{B}{\lambda} + \frac{\sqrt{-\Omega}}{2\lambda}\frac{C_1 sinh(\frac{\sqrt{-\Omega}}{2\lambda}\xi) + C_2 cosh(\frac{\sqrt{-\Omega}}{2\lambda}\xi)}{C_1 cosh(\frac{\sqrt{-\Omega}}{2\lambda}\xi) + C_2 sinh(\frac{\sqrt{-\Omega}}{2\lambda}\xi)} \tag{3.8}$$

Family 3. Rational form:

When $B \neq 0$, $\lambda = A - C$ and $\Omega = B^2 + 4E(A - C) = 0$, and C_1, C_2 are arbitrary constants.

$$\Phi(\xi) = (\frac{G'}{G}) = \frac{B}{\lambda} + \frac{C_2}{C_1 + C_2\xi} \tag{3.9}$$

Family 4. Hyperbolic form:

When $B = 0$, $\lambda = A - C$ and $\Delta = \lambda E > 0$, and C_1, C_2 are arbitrary constants.

$$\Phi(\xi) = (\frac{G'}{G}) = \frac{\sqrt{\Delta}}{\lambda}\frac{C_1 sinh(\frac{\sqrt{\Delta}}{\lambda}\xi) + C_2 cosh(\frac{\sqrt{\Delta}}{\lambda}\xi)}{C_1 cosh(\frac{\sqrt{\Delta}}{\lambda}\xi) + C_2 sinh(\frac{\sqrt{\Delta}}{\lambda}\xi)} \tag{3.10}$$

Family 5. Trigonometric form:

When $B = 0$, $\lambda = A - C$ and $\Delta = \lambda E < 0$, and C_1, C_2 are arbitrary constants.

$$\Phi(\xi) = (\frac{G'}{G}) = \frac{\sqrt{-\Delta}}{\lambda}\frac{-C_1 sinh(\frac{\sqrt{-\Delta}}{\lambda}\xi) + C_2 cosh(\frac{\sqrt{-\Delta}}{\lambda}\xi)}{C_1 cosh(\frac{\sqrt{-\Delta}}{\lambda}\xi) + C_2 sinh(\frac{\sqrt{-\Delta}}{\lambda}\xi)} \tag{3.11}$$

Chapter 4

Application

4.1 Nonlinear Korteweg-de Vries(KdV) equation with time and space fractional derivatives

4.1.1 Kdv Equation:

Complicated Nonlinear features are generally acquired by the physical phenomena and processes that take place in nature.For the real processes this leads to nonlinear mathematical models . There is much interest in the practical issues involved, as well as the development of methods to investigate the associated nonlinear mathematical problems including nonlinear wave propagation. The development of the inverse scattering method for the Korteweg-de Vries (KdV) equation and the subsequent interest in soliton theory is an early example of the latter. Now a days soliton theory has been applied in many wings of science. The ubiquitous KdV equation in dimensionless variables reads

$$l_t + nll_x + l_{xxx} = 0. \tag{4.1}$$

Korteweg de Vries derived the KdV equation to describe shallow water waves of long wavelength and small amplitude. It is a nonlinear evolution equation that represents a various of important finite amplitude dispersive wave occurence. This equation has also been used to describe a number of important physical phenomena such as acoustic waves in a harmonic crystal and ion-acoustic waves in plasmas. As mentioned earlier, this equation is the easiest nonlinear equation embodying two effects: nonlinearity represented by ll_x, and linear dispersion represented by l_{xxx}. Nonlinearity of ll_x tends to localize the wave whereas dispersion spreads the wave out. The fine balance between the weak nonlinearity of ll_x and the linear dispersion of l_{xxx} defines the formulation of solitons consisting of single bulged waves. Result of the fine balance between the two effects of nonlinearity and dispersion is the stability of solitons. This equation is the pioneer of model equations that gives soliton solutions which characterize solitary waves that decrease monotonically at infinity.

This equation models a variety of nonlinear wave phenomena such as shallow water waves, acoustic waves in a harmonic crystal, and ion-acoustic waves in plasmas. The KdV equation is completely integrable and gives rise to multiple soliton solutions. The inverse scattering method and the Backlund transformation method and various

other methods were used as well were studied The KdV equation has beed stuied by these.
We have substituted the wave variable $\xi = x - kt$, k is the speed of wave into Eq.(4.1) and integrating once to obtain

$$-kl + \frac{n}{2}l^2 + l'' = 0. \tag{4.2}$$

4.1.2 Application method

Let us consider the non-linear Korteweg-de-Vries(KdV) equation with time and space fractional derivatives[77]:

$$\frac{\delta^\alpha l}{\delta t^\alpha} + al\frac{\delta^\beta l}{\delta x^\beta} + \frac{\delta^{3\beta} l}{\delta x^{3\beta}} = 0, t > 0, 0 < \alpha, \beta \leq 1. \tag{4.3}$$

By using the complex transformation Eq.(3.2), Eq.(4.3) transformed into an ordinary differential equation. Then integrating twice, we obtain:

$$\frac{1}{2}Ll^2 + \frac{a}{6}Kl^3 + \frac{K^3}{2}(l')^2 + p_1 l + p_2 = 0 \tag{4.4}$$

where p_1 and p_2 are integral constants. Considering the homogeneous balance between the highest order non-linear terms and the highest order derivatives in Eq.(4.4) we have,

$$l(\xi) = a_0 + a_1\Psi(\xi) + a_2[\Psi(\xi)]^2 + b_1[\Psi(\xi)]^{-1} + b_2[\Psi(\xi)]^{-2}. \tag{4.5}$$

4.2 System of equations

$(\frac{G'}{G} + d)^6$:
$12K^3a_2^2C^2 + 12K^3a_2^2A^2 + \alpha KA^2a_3^2 - 24K^3a_2^2CA = 0$

$(\frac{G'}{G} + d)^5$:
$96K^3a_2^2AdC - 24K^3a_2^2AB + 12K^3a_2A^2a_1 - 48K^3a_2^2C^2d - 48K^3a_2^2A^2d + 24K^3a_2^2CB + 3\alpha KA^2a_1a_2^2 - 24K^3a_2Ca_1A + 12K^3a_2C^2a_1 = 0$

$(\frac{G'}{G} + d)^4$:
$-48K^3a_2A^2da_1 + 12K^3a_2^2B^2 + 3K^3a_1^2C^2 + 3LA^2a_2^2 + 72K^3a_2^2AdB - 72K^3a_2^2CdB + 3\alpha KA^2a_1^2a_2 + 72K^3a_2^2C^2d^2 + 3\alpha KA^2a_0a_2^2 - 24K^3a_2^2AE + 3K^3a_1^2A^2 + 96K^3a_2Ada_1C - 48K^3a_2C^2da_1 - 24K^3a_2Aa_1B - 6K^3a_1^2CA - 144K^3a_2^2Ad^2C + 24K^3a_2^2CE + 72K^3a_2^2A^2d^2 + 24K^3a_2Ca_1B = 0$

$(\frac{G'}{G} + d)^3$:
$6LA^2a_1a_2 + \alpha KA^2a_1^3 - 48K^3a_2^2A^2d^3 - 48K^3a_2^2C^2d^3 - 24K^3a_2^2B^2d - 12K^3a_1^2A^2d - 12K^3a_1^2C^2d - 12K^3a_2C^2b_1 - 12K^3a_2A^2b_1 + 24K^3a_2^2BE + 12K^3a_2B^2a_1 + 6K^3a_1^2CB - 6K^3a_1^2AB + 3\alpha KA^2a_2^2b_1 + 96K^3a_2^2Ad^3C - 72K^3a_2^2Ad^2B + 72K^3a_2A^2d^2a_1 + 48K^3a_2^2AdE + 72K^3a_2^2Cd^2B + 72K^3a_2C^2d^2a_1 - 48K^3a_2^2CdE + 24K^3a_1^2AdC + 6\alpha KA^2a_0a_1a_2$
$-144K^3a_2Ad^2a_1C + 72K^3a_2Ada_1B - 72K^3a_2Cda_1B + 24K^3a_2Ca_1E + 24K^3a_2Cb_1A - 24K^3a_2Aa_1E = 0$

$(\frac{G'}{G}+d)^2:$
$12K^3a_2^2E^2+6p_1A^2a_2+3K^3a_1^2B^2+3LA^2a_1^2+6LA^2a_0a_2+12K^3a_2^2C^2d^4+12K^3a_2^2B^2d^2+12K^3a_2^2A^2d^4+18K^3a_1^2A^2d^2+18K^3a_1^2C^2d^2-24K^3a_2C^2b_2-24K^3a_2A^2b_2+6K^3a_1^2CE-6K^3a_1C^2b_1-6K^3a_1^2AE-6K^3a_1A^2b_1+3\alpha KA^2a_0^2a_2+3\alpha KA^2a_0a_1^2+3\alpha KA^2a_2^2b_2-48K^3a_2A^2d^3a_1+48K^3a_2A^2db_1-48K^3a_2C^2d^3a_1+48K^3a_2C^2db_1-24K^3a_2^2Cd^3B-24K^3a_2^2Cd^4A+24K^3a_2^2Cd^2E+24K^3a_2^2Bd^3A-24K^3a_2^2BdE-24K^3a_2B^2da_1-24K^3a_2^2Ad^2E-36K^3a_1^2Ad^2C+18K^3a_1^2AdB-18K^3a_1^2CdB+6\alpha KA^2a_1a_2b_1+96K^3a_2Ad^3a_1C-72K^3a_2Ad^2a_1B+48K^3a_2Ada_1E-96K^3a_2Adb_1C+72K^3a_2Cd^2a_1B-48K^3a_2Cda_1E+12K^3a_1Cb_1A-24K^3a_2Cb_1B+48K^3a_2Cb_2A+24K^3a_2Ab_1B+24K^3a_2Ba_1E=0$

$(\frac{G'}{G}+d)^1:$
$6p_1A^2a_1+6LA^2a_0a_1+6LA^2a_2b_1-12K^3a_1^2A^2d^3-12K^3a_1^2C^2d^3-6K^3a_1^2B^2d-12K^3a_2B^2b_1-12K^3a_1c^2b_2-12K^3a_1A^2b_2+12K63a_2E^2a_1+6K^3a_1^2BE+3\alpha KA^2a_0^2a_1+3\alpha KA^2a_1^2b_1-72K^3a_2A^2d^2b_1+96K^3a_2A^2db_2-72K^3a_2C^2d_2b_1+96K^3a_2C^2db_2+12K^3a_2C^2d^4a_1+12K^3a_2B^2d^2a_1+12K^3a_2A^2d^4a_1+24K^3a_1^2Ad^3C-18K^3a_1^2Ad^2B+12K^3a_1AdE+24K^3a_1A^2db_1+18K^3a_1^2Cd^2B-12K^3a_1^2CdE+24K^3a_1C^2db_1+6\alpha KA^2a_0a_2b_1+6\alpha KA^2a_1a_2b_2+144K^3a_2Ad^2b_1C-72K^3a_2Ab_1B-192K^3a_2Adb_2C+72K^3a_2Bd^3a_1A-24k^3a_2Cd^3a_1B-24K_3a_2Cd^4a_1A+24K^3a_2Cd^2a_1E+24K^3a_2Bd^3a_1A-24K^3a_2Bda_1E-24K^3a_2Ad^2a_1E-48K^3a_1Adb_1C-12K^3a_1Cb_1B+24K^3a_1Cb_2A+12K^3a_1Ab_1B-24K^3a_2Cb_1E-48K^3a_2Cb_2B+24K^3a_2Ab_1E+48K^3a_2Ab_2B=0$

$(\frac{G'}{G}+d)^0:$
$6p_1A^2a_0+3K^3a_1^2E^2+3LA^2a_0^2+3K^3b_1^2C^2+3K^3b_1^2A^2+6LA^2a_1b_1+6LA^2a_2b_2+\alpha KA^2a_0^3+3K^3a_1^2C^2d^4+3K^3a_1^2B^2d^2+3K^3a_1^2A^2d^4-24K^3a_2B^2b_2-6K^3a_1B^2b_1-6K^3b_1^2CA+3\alpha KA^2a_1^2b_2+3\alpha KA^2a_2b_1^2+48K^3a_2A^2d^3b_1-144K^3a_2A^2d^2b_2+48K^3a_2C^2d^3b_1-144K^3a_2C^2d^2b_2+24K^3a_2B^2db_1-36K^3a_1A^2d^2b_1+48K^3a_1A^2db_2-36K^3a_1C^2d^2b_1+48K^3a_1C^2db_2-6K^3a_1^2Cd^3B-6K^3a_1^2Cd^4A+6K^3a_1^2Cd^2E+6K^3a_1^2Bd^3A-6K^3a_1^2BdE-6K^3a_1^2Ad^2E+6p_2A^2+6\alpha KA^2a_0a_1b_1+6\alpha KA^2a_0a_2b_2-96K^3a_2Ad^3b_1C+72K^3a_2Ad^2b_1B+288K^3a_2Ad^2b_2C-48K^3a_2Adb_1E-144K^3a_2Adb_2B-72K^3a_2Cd^2b_1B+48K^3a_2Cdb_1E+144K^3a_2Cdb_2B+72K^3a_1Ad^2b_1C-36K^3a_1Adb_1B-96K^3a_1Adb_2C-12K^3a_1Cb_1E-24K^3a_1Cb_2B+12K^3a_1Ab_1E+24K^3a_1Ab_2B-48K^3a_2Cb_2E+48K^3a_2Ab_2E-24K^3a_2Bb_1E+36K^3a_1Cdb_1B=0$

$(\frac{G'}{G}+d)^{-1}:$
$6p_1A^2b_1+6LA^2a_0b_1+6LA^2a_1b_2-12K^3b_1^2A^2d-12K^3b_1^2C^2d-12K^3a_2E^2b_1-12K^3a_1b_2+6K^3b_1^2CB+12K^3b_1C^2b_2-6K^3b_1^2AB+12K^3b_1A^2b_2+3\alpha KA^2a_0^2b_1+3\alpha KA^2a_1b_1^2+96K^3a_2A^2d^3b_2+96K^3a_2C^2d^3b_2-12K^3a_2C^2d^4b_1-12K^3a_2B^2d^2b_1+48K^3a_2B^2db_2-12K^3a_2A^2d^4b_1+24K^3a_1A^2d^3b_1-72K^3a_1A^2d^2b_2+24K^3a_1C^2d^3b_1-72K^3a_1C^2d^2b_2+12K^3a_1B^2db_1+24K^3b_1^2AdC+6\alpha KA^2a_0a_1b_2+6\alpha KA^2a_2b_1b_2-192K^3a_2Ad^3b_2C+144K^3a_2Ad^2b_2B-96K^3a_2Adb_2E-144K^3a_2Cd^2b_2B+96K^3a_2Cdb_2E+24K^3a_2Cd^3b_1B+24K^3a_2Cd^4b_1A-24K^3a_2Cd^2b_1E-24K^3a_2Bd^3b_1A+24K^3a_2Bdb_1E+24K^3a_2Ad^2b_1E-48K^3a_1Ad^3b_1C+36K^3a_1Ad^2b_1B+144K^3a_1Ad^2b_2C-24K^3a_1Adb_1E-72K^3a_1Adb_2B-24K^3a_1Cb_2E+2424K^3a_Ab_2E-12K^3a_1Bb_1E-24K^3b_1Cb_2A-48K^3a_2Bb_2E-36K^3a_1Cd^2b_1B+24K^3a_1Cdb_1E+72K^3a_1Cdb_2B=0$

$(\frac{G'}{G}+d)^{-2}:$
$(12K^3b_2^2C^2+12K^3b_2^2A^2+3K^3b_1^2A^2+3K^3b_1^2+6p_1A^2b_2+6LA^2a_0b_2+18K^3b_1^2A^2d^2+$

$18K^3b_1^2C^2d^2 - 24K^3a_2E^2b_2 - 6K^3a_1E^2b_1 + 6K^3b_1^2CE - 6K^3b_1^2AE - 24K^3b_2^2CA + 3\alpha KA^2a_0^2b_2 + 3\alpha KA^2a_0b_1^2 + 3\alpha KA^2a_2b_2^2 - 24K^3a_2C^2d^4b_2 - 24K^3a_2B^2d^2b_2 - 24K^3a_2A^2d^4b_2 + 48K^3a_1A^2d^3b_2 + 48K^3a_1C^2d^3b_2 - 6K^3a_1C^2d^4b_1 - 6K^3a_1B^2d^2b_1 + 24K^3a_1B^2db_2 - 6K^3a_1A^2d^4b_1 - 36K^3b_1^2Ad^2C + 18K^3b_1^2AdB - 48K^3b_1A^2db_2 - 18K^3b_1^2CdB - 48K^3b_1C^2db_2 + 6\alpha KA^2a_1b_1b_2 + 48K^3a_2Cd^3b_2B + 48K^3a_2Cd^4b_2A - 48K^3a_2Cd^2b_2E - 48K^3a_2Bd^3b_2A + 48K^3a_2Bdb_2E + 48K^3a_2Ad^2b_2E - 96K^3a_1Ad^3b_2C + 72K^3a_1Ad^2b_2B - 24K^3a_1Bb_2E + 24K^3b_1Cb_2B - 24K^3b_1Ab_2B - 48K^3a_1Adb_2E - 72K^3a_1Cd^2b_2B + 48K^3a_1Cdb_2E + 12K^3a_1Cd^3b_1B + 12K^3a_1Cd^4b_1A - 12K^3a_1Cd^2b_1E - 12K^3a_1Bd^3b_1A + 12K^3a_1Bdb_1E + 12K^3a_1Ad^2b_1E + 96K^3b_1Adb_2C = 0$

$(\frac{G'}{G} + d)^{-3}$:
$6LA^2b_1b_2 + \alpha KA^2b_1^3 - 12K^3b_1^2A^2d^3 - 12K^3b_1^2C^2d^3 - 6K^3b_1^2B^2d - 48K^3b_2^2A^2d - 48K^3b_2^2C^2d - 12K^3a_1E^2b_2 + 6K^3b_1^2BE + 12K^3b_1B^2b_2 + 24K^3b_2^2CB - 24K^3b_2^2AB + 3\alpha KA^2a_1b_2^2 + 96K^3b_2^2AdC - 12K^3a_1C^2d^4b_2 - 12K^3a_1B^2d^2b_2 - 12K^3a_1A^2d^4b_2 + 24K^3b_1^2Ad^3C - 18K^3b_1^2Ad^2B + 72K^3b_1A^{d2}b_2 + 12K^3b_1^2AdE + 18K^3b_1^2Cd^2B + 72K^2b_1C^2d^2b_2 - 12K^3b_1^2CdE + 6\alpha KA^2a_0b_1b_2 + 24K^3b_1Cb_2E - 24K^3b_1b_2E + 24K^3a_1Cd^3b_2B + 24K^3a_1Cd^4b_2A - 24K^3a_1Cd^2b_2E - 24K^3a_1Bd^3b_2A + 24K^3a_1Bdb_2E + 24K^3a_1Ad^2b_2E - 144K^3b_1Ad^2b_2C + 72K^3b_1Adb_2B - 72K^3b_1Cdb_2B = 0$

$(\frac{G'}{G} + d)^{-4}$:
$3K^3b_1^2E^2 + 12K^3b_2^2B^2 + 3LA^2b_2^2 + 3K^3b_1^2C^2d^4 + 3K^3b_1^2B^2d^2 + 3K^3b_1^2A^2d^4 + 72K^3b_2^2A^2d^2 + 72K^3b_2^2C^2d^2 + 24K^3b_2^2CE - 24K^3b_2^2AE + 3\alpha KA^2a_0b_2^2 + 3\alpha KA^2b_1^2b_2 - 72K^3b_2^2CdB - 6K^3b_1^2Ad^2E - 144K^3b_2^2Ad^2C + 72K^3b_2^2AdB - 48K^3b_1A^2d^3b_2 - 48K^3b_1C^2d^3b_2 - 6K^3b_1^2Cd^3B - 6K^3b_1^2Cd^4A + 6K^3b_1^2Cd^2E + 6K^3b_1^2Bd^3A - 6K^3b_1^2BdE - 24K^3b_1B^2db_2 + 24K^3b_1Bb_2E + 96K^3b_1Ad^3b_2C - 72K^3b_1Ad^2b_2B + 48K^3b_1Adb_2E + 72K^3b_1Cd^2b_2B - 48K^3b_1Cdb_2E = 0$

$(\frac{G'}{G} + d)^{-5}$:
$-24K^3b_2^2B^2d + 12K^3b_1A^2d^4b_2 + 12K^3b_1E^2b_2 + 72K^3b_2^2Cd^2B + 12K^3b_1C^2d^4b_2 + 96K^3b_2^2Ad^3C - 48K^3b_2^2A^2d^3 + 3\alpha KA^2b_1b_2^2 + 24K^3b_1Bd^3b_2A - 24K^3b_1Ad^2b_2E - 24K^3b_1Cd^4b_2A + 24K^3b_2^2BE - 48K^3b_2^2CdE - 24K^3b_1Cd^3b_2B + 12K^3b_1B^2d^2b_2 + 24K^3b_1Cd^2b_2E - 48K^3b_2^2C^2d^3 - 24K^3b_1Bdb_2E + 48K^3b_2^2AdE - 72K^3b_2^2Ad^2B = 0$

$(\frac{G'}{G} + d)^{-6}$:
$-24K^3b_2^2BdE - 24K^3b_2^2Ad^2E - 24K^3b_2^2Cd^3B + 12K^3b_2^2E^2 + \alpha KA^2b_2^3 + 24K^3b_2^2Bd^3A + 12K^3b_2^2A^2d^4 - 24K^3b_2^2Cd^4A + 24K^3b_2^2Cd^2E + 12K^3b_2^2B^2d^2 + 12K^3b_2^2C^2d^4 = 0$

4.3 Cases of equations

Case 1

$$L = \frac{-K}{A^2}\{12K^2d^2\lambda^2 + 12K^2Bd\lambda - 8K^2E\lambda + a_0aA^2 + K^2B^2\},$$

$$a_1 = \frac{12K^2}{aA^2}(2d\lambda^2 + B\lambda),$$

$$d = d,$$
$$a_2 = \frac{-12K^2\lambda^2}{aA^2},$$
$$b_1 = 0,$$
$$b_2 = 0,$$
$$p_1 = \frac{K}{2aA^4}(F_1 + ... + F_6),$$
$$p_2 = \frac{-K}{6a^2A^6}(S_1 + ...S_{16}),$$
$$\lambda = A - C,$$

$$F_1 = 144K^4d^4(A^4 + C^4) + a^2a_0^2A^4 + aa_0K^2A^2(16CE + 24C^2d^2 - 48ACd^2),$$

$$F_2 = 576K^4d^2ACE(A - C) - 24K^4B^2E(A - C) - 96K^4ACE^2 + 24aa_0dK^2A^3B,$$

$$F_3 = 2aa_0K^2A^2B^2 - 864K^4d^3ABC(A-C) - 192dK^4BE(A^2 - C^2) - 336d^2K^4AB^2C,$$

$$F_4 = 24aa_0d^2K^2A^4 - 576K^4d^4AC(A^2-C^2) - 192d^2K^4A^3E - 288d^3K^4BC^3 + 864d^4K^4A^2C^2,$$

$$F_5 = 24dK^4B^3(A - C) + 168K^4d^2B^2(A^2 + C^2) + 192d^2K^4C^3E + 228d^3K^4BA^3,$$

$$F_6 = 384dK^4ABCE - 16aa_0K^2A^3E + 48K^4E^2(A^2 + C^2) - 24aa_0dK^2A^2BC,$$

$$S_1 = 432K^6B^2C^2E^2 + a^3a_0^3A^6 + 432d^4K^4A^2C^2 - -288aa_0K^4A^3CE^2 - 72aa_0K^3A^3CE^2$$
$$-72aa_0K^4A^3B^2E - 576aa_0K^4A^3B^2E,$$

$$S_2 = 2592aa_0d^4K^4A^4C^2 - 1728aa_0d^4K^4A^3C(A^2+C^2) + 846aa_0d^3K^4A^5B + 504aa_0d^2K^4A^4B^2$$
$$+36a^2a_0^2d^2K^2A^4C^2,$$

$$S_3 = 72aa_0dK^4A^3B^3 + 24a^2a_0^2K^2A^4CE - 72a^2a_0^2d^2K^2A^5C + 36a^2a_0^2dK^2A^5B + 144aa_0K^4A^2C^2E^2,$$

$$S_4 = 1152aa_0dK^4A^3BCE - 576aa_0dK^4A62BC^2E + 432aa_0d^4K^4a^6 + 36a^2a_0^2d^2K^2A^6$$
$$-24a^2a_0^2K^2A^5E + 3a^2a_0^2K^2A^4B^2,$$

$$S_5 = 144aa_0K^4A^4E^2 + 12960d^2K^6AB^2CE(A-C) - 41472d^3K^6A^2BC^2E + 5184dK^6ABC^2E^2$$
$$+27648d^3K^6ABC^3E,$$

$$S_6 = 27648d^3K^6A^3BCE - 518dK^6A^2BCE^2 + 1728dK^6A^2BCE^2 + 4320d^2K^6B^2C^3E$$
$$-6912dK^6BC^4E - 1728dK^6BC^3E^2,$$

$$S_7 = 34560d^4K^6A^2C^3E - 17280d^4K^6AC^4E - 6912d^2K^6AC^3E^2 - 22464d^4K^6AB^2C^3$$
$$-864dK^6B^3C^2E - 51840d^5K^6A^2BC^3,$$

$$S_8 = 25920d^5K^6ABC^4 - 34560d^4K^6A^3C^2E + 10368d^2K^6A^2C^2E^2 + 33696d^4K^6A^2B^2C^2$$
$$+7776d^3K^6AB^3C^2 + 51840d^5K^6A^3BC^2,$$

$$S_9 = 17280d^4K^6A^4CE - 6912d^2K^6A^3CE^2 - 864d^2K^6AB^4C - 22464d^4K^6A^3B^2C$$
$$-7776d^3K^6A^2B^3C - 864K^6AB^2CE^2,$$

$$S_{10} = 1728dK^6A^3BE^2 - 25920d^5K^6BC - 4320d^2K^6A^3B^2 - 6912d^3K^6A^4BE - 864dK^6A^2B^3E$$
$$+5616d^4K^6B^2C^4,$$

$$S_{11} = 3456d^4K^6C^5E - 25920d^3K^6B^3C^3 - 5184d^5K^6BC^5 + 1728d^2K^6C^4E^2 + 25920d^6K^6A^4C^2$$
$$-10368d^6K^6AC^5,$$

$$S_{12} = 432d^2K^6B^4C^2 + 25920d^6K^6A^2C^4 - 34560d^6K^6A^3C^3 - 10368d^6K^6A^5C - 3456d^4K^6A^5E$$
$$+1728d^2K^6A^4E^2,$$

$$S_{13} = 432d^2K^6A^2B^4 + 5616d^4K^6A^4B^2 + 2592d^3K^6A^3B^3 + 432K^6A^2B^2E^2 + 5184d^5K^6A^5B$$
$$+1728d^6K^6C^6,$$

$$S_{14} = 1728d^6K^6A^6 + 72aa_0K^4A^2B^2CE + 1728aa_0d^2K^4A^3CE(A-C) + 2592aa_0d^3K^4A^3BC^2$$
$$-2592aa_0d^3K^4A^4BC,$$

$$S_{15} = 567aa_0d^2K^4A^2C^3E - 576aa_0dK^4A^4BE - 1008aa_0d^2K^4A^3B^2C - 864aa_0d^3K^4A^2BC^3,$$

$$S_{16} = 504aa_0d^2K^4A^2B^2C^2 72aa_0dK^4A^2B^3C - 36a^2a_0^2dK^2A^4BC$$

Where,

$$\Psi(\xi) = d + \phi(\xi)$$

and A, B, C, d, E are free constraints. (4.6)

Case 2

$$L=\frac{-K}{A^2}\{12K^2d^2(A-C)^2+12K^2Bd(A-C)-8K^2E(A-C)+a_0aA^2+K^2B^2\},$$

$$a_1=0,$$

$$d=d,$$

$$a_2=0,$$

$$b_1=\frac{12K^2}{aA^2}\{2d^3(A-C)^2+3d^2B(A-C)-2dE(A-C)+dB^2-BE\},$$

$$b_2=\frac{-12K^2}{aA^2}\{d^4(A-C)^2+2d^3B(A-C)-2d^2E(A-C)+d^2B^2-2dBE+E^2\},$$

$$p_1=\frac{K}{2aA^4}(q_1+...+q_6),$$

$$p_2=\frac{-K}{6a^2A^6}(h_1+...h_{15}),$$

$$q_1=144K^4d^4(A^4+C^4)+a^2a_0^2A^4+aa_0K^2A^2(16CE+24C^2d^2-48ACd^2),$$

$$q_2=24aa_0K^2dA^3B-96K^4ACE^2-24K^4B^2E(A-C)+576d^2K^4ACE(A-C),$$

$$q_3=2aa_0K^2A^2B^2-864d^3K^4ABC(A-C)-192dK^4BE(A^2+C^2)-336d^2K^4AB^2C,$$

$$q_4=24aa_0d^2K^2A^4-192d^3K^4A^3E-228d^3K^4BC^3+864d^4K^4A^2C^2-576d^4K^4AC(A^2+C^2),$$

$$q_5=192d^2K^4C^3E+288d^3K^4BA^3+168d^2K^4B^2(A^2+C^2)+24dK^4B^3(A-C),$$

$$q_6=384dK^4ABCE-16aa_0K^2A^3E+48K^4E^2(A^2+C^2)-24aa_0K^2A^2BC,$$

$$h_1=432K^6B^2C^2E^2+a^3a_0^3A^6+432aa_0d^2K^4A^2C^4-aa_0K^4A^3E(228EC+72B^2)$$
$$-576a_0d^2K^4A^5E,$$

$$h_2=aa_0d^4K^4A^3C\{2592AC-1728C(C^2+A^2)\}+aa_0d^2K^4A^4B(864dA+504AB)$$
$$+36a^2a_0^2d^2K^2A^4C^2,$$

$$h_3=72aa_0dK^4A^3B^3+a^2a_0^2K^2A^4(24CE-72d^2AC+36dAB)+aa_0K^4A^2EC(144EC+1152dAB),$$

$$h_4=aa_0K^4A^4(144E^2+432d^4A^2)-576aa_0dK^4A^2BC^2E+a^2a_0^2K^2A^4(36d^2A^2-24AE+3B^2),$$

$$h_5=d^2K^6A^2BCE(27648dA+12960B-41472dC)+dK^6ABC^2E(27648d^2C+5184E-12960dB),$$

$$h_6 = dK^6ABCE(1728B^2-5184AE)-d^3K^6C^4E(6912B+17280dA)+dK^6BC^3E(4320dB-1728E),$$

$$h_7 = d^4K^6AC^3(34560AE-22464B^2-51840dAB+25920d^2BC)-dK^6C^2E(6912dACE-864B^3),$$

$$h_8 = d^4K^6A^2C(33696B^2C+51840dABC+17280A^2E)+d^2K^6AC^2(10368AE^2+7776ddB^3 -34560d^2A^2E),$$

$$h_9 = -d^4K^6A^3BC(22464B+25920dA)-d^2K^6AC(6912A^2E^2+864B^4+7776dAB^3)-864K^6AB^2CE^2,$$

$$h_{10} = d^3K^6B^2C^3(5616dC+2592B)-d^2K^6A^3BE(4320B+6912dA)+dK^6A^2BE(1728AE-864B^2),$$

$$h_{11} = d^6K^6AC(25920A^3C-10368C^4)+d^4K^6C^5(3456E-5184dB)+d^2K^6C^2(1728C^2E^2+432B^4),$$

$$h_{12} = d^2K^6A^2(1728A^2E^2+432B^4)-d^6K^6A^3C(34560C^2+10368A^2)+d^4K^6A^2(25920C^4E^2 -3456A^3E),$$

$$h_{13} = 1728d^6K^6(A^6+C^6)+d^3K^6A^3B(5616AB+2592B^2+5184d^2A^2)+432K^6A^2B^2E^2,$$

$$h_{14} = 72aa_0K^4A^2B^2CE+1728aa_0d^2K^4A^3CE(A-C)-2592aa_0d^3K^4A^3BC(A-C) -576aa_0dK^4A^4BE,$$

$$h_{15} = 576aa_0d^2K^4A^2C^3E-1008aa_0d^2K^4A^3B^2C-864aa_0d^3K^4A^2BC^3+504aa_0d^2K^4A^2B^2C^2 -72aa_0dK^4A^2B^3C-36a^2a_0^2dKA^4BC.$$

Where,

$$\Psi(\xi) = d + \phi(\xi)$$

and A, B, C, d, E are free constraints. (4.7)

Case 3

$$L = \frac{-K}{A^2}\{aa_0A^2 - 2K^2B^2 - 8K^2E(A-C)\}$$

,

$$a_1 = 0,$$
$$d = \frac{-B}{2(A-C)},$$

$$a_2 = \frac{-12K^2(A-C)^2}{aA^2},$$
$$b_1 = 0,$$
$$b_2 = \frac{-3K^2\{16E^2(A-C) + 8B^2E\}}{4aA^2(A-C)},$$
$$p_1 = \frac{K}{2aA^4}(m_1 + m_2),$$
$$p_2 = \frac{-K}{6a^2A^6}(n_1 + n_2 + n_3),$$
$$m_1 = a^2a_0^2A^4 - 16aa_0K^2A^2E(A-C) - 96K^4B^2E(A-C),$$
$$m_2 = 384K^2ACE^2 - 192K^4A^2E^2 - 4aa_0K^2A^2B^2 - 12K^4B^4 - 192K^4C^2E^2,$$
$$n_1 = a^3a_0^3A^6 - 24a^2a_0^2K^2A^4E(A-C) - 576aa_0K^4A^2E^2(A-C)^2 - 6a^2a_0^2K^2A^4B^2,$$
$$n_2 = 13824K^6E^2(A-C)^3 - 288aa_0K^4A^2B^2E(A-C) - 36aa_0K^4A^2B^4,$$
$$n_3 = 10368K^6B^2E^2(A-C)^2 - 2592K^6B^4E(A-C) + 216K^6B^6$$

Where,
$$\Psi(\xi) = d + \phi(\xi)$$
and A, B, C, d, E are free constraints. (4.8)

Case 4

$$L = \frac{-K}{A^2}\{aa_0A^2 - 8K^2E(A-C)^2 - 2K^2B^2\},$$
$$a_1 = 0,$$
$$d = \frac{-B}{2(A-C)},$$
$$a_2 = 0,$$
$$b_1 = 0,$$
$$b_2 = \frac{-3K^2\{16E^2(A-C) + 8B^2E(A-C) + B^4\}}{4aA^2(A-C)^2},$$
$$p_1 = \frac{K}{2aA^4}(48K^4E^2(A-C)^2 + 24K^4B^2E(A-C) - 16a_0aK^2A^2E(A-C) + 3K^4B^4 + a^2a_0^2A^4 - 4a_0aK^2A^2B^2),$$
$$p_2 = \frac{-a_0K}{6aA^4}(144K^4E^2(A-C)^2 + 72K^4B^2E(A-C) - 24a_0aK^2A^2E(A-C) + 9K^4B^4 + a^2a_0^2A^4 - 6a_0aK^2A^2B^2),$$

Where,
$$\Psi(\xi) = d + \phi(\xi)$$
and A, B, C, d, E are free constraints. (4.9)

4.4 Solutions of equations

By using the above results of the KdV Equation we have found out the following wave solutions of the KdV Equation :

Hyperbolic form of the wave solutions:

Substituting Eq.(4.6) and Eq.(3.7) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l1_{110} = a_0 + \frac{12K^2(2d\lambda^2+B\lambda)\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{\sqrt{\Omega}C_2\coth\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)}{\lambda}\right)}{aA^2} - \frac{12K^2\lambda^2\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{\sqrt{\Omega}C_2\coth\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)}{\lambda}\right)^2}{aA^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l1_{120} = a_0 + \frac{12K^2(2d\lambda^2+B\lambda)\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{\sqrt{\Omega}C_1\tanh\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)}{\lambda}\right)}{aA^2} - \frac{12K^2\lambda^2\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{\sqrt{\Omega}C_1\tanh\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)}{\lambda}\right)^2}{aA^2}$$

Substituting Eq.(4.7) and Eq.(3.7) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l1_{210} = a_0 + \frac{12K^2\left\{2d^3\lambda^2+3Bd^2\lambda+B^2d-2Ed\lambda-BE\right\}}{aA^2\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{\sqrt{\Omega}C_2\coth\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)}{\lambda}\right)} - \frac{12K^2\left\{d^4\lambda^2+2Bd^3\lambda+B^2d^2-2Ed^2\lambda-2BEd+E^2\right\}}{aA^2\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{\sqrt{\Omega}C_2\coth\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)}{\lambda}\right)}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l1_{220} = a_0 + \frac{12K^2\left\{2d^3\lambda^2+3Bd^2\lambda+B^2d-2Ed\lambda-BE\right\}}{aA^2\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{\sqrt{\Omega}C_1\tanh\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)}{\lambda}\right)} - \frac{12K^2\left\{d^4\lambda^2+2Bd^3\lambda+B^2d^2-2Ed^2\lambda-2BEd+E^2\right\}}{aA^2\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{\sqrt{\Omega}C_1\tanh\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)}{\lambda}\right)}$$

Substituting Eq.(4.8) and Eq.(3.7) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l1_{310} = a_0 - \frac{3K^2\Omega C_2^2\coth\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)^2}{aA^2} + \frac{3K^2\left\{8B^2E+16E\lambda\right\}\lambda}{aA^2\Omega C_2^2\coth\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l1_{320} = a_0 - \frac{3K^2\Omega C_1^2\tanh\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)^2}{aA^2} + \frac{3K^2\left\{8B^2E+16E\lambda\right\}\lambda}{aA^2\Omega C_1^2\tanh\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)^2}$$

Substituting Eq.(4.9) and Eq.(3.7) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l1_{410} = a_0 - \frac{3K^2\left\{B^4+8B^2E\lambda+16E^2\lambda^2\right\}}{aA^2\Omega {C_2}^2\coth\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l1_{420} = a_0 - \frac{3K^2\left\{B^4+8B^2E\lambda+16E^2\lambda^2\right\}}{aA^2\Omega {C_1}^2\tanh\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)^2}$$

Trigonometric form of the wave solutions:

Substituting Eq.(4.6) and Eq.(3.8) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l2_{110} = a_0 + \frac{12K^2(2d\lambda^2+B\lambda)\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{C_2\coth\left(\frac{1}{2\lambda}\xi\right)}{\lambda}\right)}{aA^2} - \frac{12K^2\lambda^2\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{C_2\coth\left(\frac{1}{2\lambda}\xi\right)}{\lambda}\right)^2}{aA^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l2_{120} = a_0 + \frac{12K^2(2d\lambda^2+B\lambda)\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{C_1\tanh\left(\frac{1}{2\lambda}\xi\right)}{\lambda}\right)}{aA^2} - \frac{12K^2\lambda^2\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{C_1\tanh\left(\frac{1}{2\lambda}\xi\right)}{\lambda}\right)^2}{aA^2}$$

Substituting Eq.(4.7) and Eq.(3.8) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l2_{210} = a_0 + \frac{12K^2\left\{2d^3\lambda^2+3Bd^2\lambda+B^2d-2Ed\lambda-BE\right\}}{aA^2\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{\sqrt{\Omega}C_2\coth\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)}{\lambda}\right)} - \frac{12K^2\left\{d^4\lambda^2+2Bd^3\lambda+B^2d^2-2Ed^2\lambda-2BEd+E^2\right\}}{aA^2\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{\sqrt{\Omega}C_2\coth\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)}{\lambda}\right)^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l2_{220} = a_0 + \frac{12K^2\left\{2d^3\lambda^2+3Bd^2\lambda+B^2d-2Ed\lambda-BE\right\}}{aA^2\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{\sqrt{\Omega}C_1\tanh\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)}{\lambda}\right)} - \frac{12K^2\left\{d^4\lambda^2+2Bd^3\lambda+B^2d^2-2Ed^2\lambda-2BEd+E^2\right\}}{aA^2\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{\sqrt{\Omega}C_1\tanh\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)}{\lambda}\right)^2}$$

Substituting Eq.(4.8) and Eq.(3.8) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l2_{310} = a_0 - \frac{3K^2\Omega C_2^2\coth\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)^2}{aA^2} + \frac{3K^2\left\{8B^2E+16E\lambda\right\}\lambda}{aA^2\Omega C_2^2\coth\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l2_{320} = a_0 - \frac{3K^2\Omega C_1^2\tanh\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)^2}{aA^2} + \frac{3K^2\left\{8B^2E+16E\lambda\right\}\lambda}{aA^2\Omega C_1^2\tanh\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)^2}$$

Substituting Eq.(4.9) and Eq.(3.8) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l2_{410} = a_0 - \frac{3K^2\left\{8B^2E\lambda+16E^2\lambda^2+B^4\right\}}{aA^2\Omega C_2^2 \coth\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l2_{420} = a_0 - \frac{3K^2\left\{8B^2E\lambda+16E^2\lambda^2+B^4\right\}}{aA^2\Omega C_1^2 \tanh\left(\frac{1}{2}\frac{\sqrt{\Omega}}{\lambda}\xi\right)^2}$$

Rational form of the wave solutions:

Substituting Eq.(4.6) and Eq.(3.9) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l3_{110} = a_0 + \frac{12K^2(2d\lambda^2+B\lambda)\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{C_2}{C_2\xi}\right)}{aA^2} - \frac{12K^2\lambda^2\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{1}{2}\frac{C_2}{C_2\xi}\right)^2}{aA^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l3_{120} = a_0 + \frac{12K^2(2d\lambda^2+B\lambda)\left(d+\frac{1}{2}\frac{B}{\lambda}\right)}{aA^2} - \frac{12K^2\lambda^2\left(d+\frac{1}{2}\frac{B}{\lambda}\right)^2}{aA^2}$$

Substituting Eq.(4.7) and Eq.(3.9) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l3_{210} = a_0 + \frac{12K^2\left\{2d^3\lambda^2+3Bd^2\lambda+B^2d-2Ed\lambda-BE\right\}}{aA^2\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{C_2}{C_2\xi}\right)} - \frac{12K^2\left\{d^4\lambda^2+2Bd^3\lambda+B^2d^2-2Ed^2\lambda-2BEd+E^2\right\}}{aA^2\left(d+\frac{1}{2}\frac{B}{\lambda}+\frac{C_2}{C_2\xi}\right)^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l3_{220} = a_0 + \frac{12K^2\left\{2d^3\lambda^2+3Bd^2\lambda+B^2d-2Ed\lambda-BE\right\}}{aA^2\left(d+\frac{1}{2}\frac{B}{\lambda}\right)} - \frac{12K^2\left\{d^4\lambda^2+2Bd^3\lambda+B^2d^2-2Ed^2\lambda-2BEd+E^2\right\}}{aA^2\left(d+\frac{1}{2}\frac{B}{\lambda}\right)^2}$$

Substituting Eq.(4.8) and Eq.(3.9) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l3_{310} = a_0 - \frac{12K^2\lambda^2C_2^2}{aA^2C_2^2\xi} + \frac{3}{4}\frac{K^2\left\{8B^2E+16E\lambda\right\}C_2^2\xi}{aA^2\lambda C_2^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l3_{320} = \text{does not exist}$$

Substituting Eq.(4.9) and Eq.(3.9) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l3_{410} = a_0 - \frac{3}{4}\frac{K^2\left\{16E^2\lambda^2+8B^2E\lambda+B^4\right\}{C_2}^2\xi}{aA^2{C_2}^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l3_{420} = \text{does not exist}$$

Hyperbolic form of the wave solutions:

Substituting Eq.(4.6) and Eq.(3.10) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l4_{110} = a_0 + \frac{24K^2d\lambda^2\left(d+\frac{\sqrt{\Delta}C_2\coth\left(\frac{1}{2}\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)}{aA^2} - \frac{12K^2\lambda^2\left(d+\frac{\sqrt{\Delta}C_2\coth\left(\frac{1}{2}\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)^2}{aA^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l4_{120} = a_0 + \frac{24K^2d\lambda^2\left(d+\frac{\sqrt{\Delta}C_1\tanh\left(\frac{1}{2}\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)}{aA^2} - \frac{12K^2\lambda^2\left(d+\frac{\sqrt{\Delta}C_1\tanh\left(\frac{1}{2}\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)^2}{aA^2}$$

Substituting Eq.(4.7) and Eq.(3.10) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l4_{210} = a_0 + \frac{12K^2\left\{2d^3\lambda^2-2Ed\lambda\right\}}{aA^2\left(d+\frac{\sqrt{\Delta}C_2\coth\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)} - \frac{12K^2\left\{d^4\lambda^2-2Ed^2\lambda+E^2\right\}}{aA^2\left(d+\frac{\sqrt{\Delta}C_2\coth\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l4_{220} = a_0 + \frac{12K^2\left\{2d^3\lambda^2-2Ed\lambda\right\}}{aA^2\left(d+\frac{\sqrt{\Delta}C_1\tanh\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)} - \frac{12K^2\left\{d^4\lambda^2-2Ed^2\lambda+E^2\right\}}{aA^2\left(d+\frac{\sqrt{\Delta}C_1\tanh\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)^2}$$

Substituting Eq.(4.8) and Eq.(3.10) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l4_{310} = a_0 + \frac{12K^2\Delta{C_2}^2\coth\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)^2}{aA^2} - \frac{3}{4}\frac{K^2\left\{16E\lambda\right\}\lambda}{aA^2\Delta{C_2}^2\coth\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l4_{320} = a_0 + \frac{12K^2\Delta{C_1}^2\tanh\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)^2}{aA^2} - \frac{3}{4}\frac{K^2\left\{16E\lambda\right\}\lambda}{aA^2\Delta{C_1}^2\tanh\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)^2}$$

Substituting Eq.(4.9) and Eq.(3.10) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l4_{410} = a_0 - \frac{3}{4}\frac{K^2\left\{16E^2\lambda^2\right\}}{aA^2\Delta C_2{}^2\coth\left(\frac{1}{2}\frac{\sqrt{\Delta}}{\lambda}\xi\right)^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l4_{420} = a_0 - \frac{3}{4}\frac{K^2\left\{16E^2\lambda^2\right\}}{aA^2\Delta C_1{}^2\tanh\left(\frac{1}{2}\frac{\sqrt{\Delta}}{\lambda}\xi\right)^2}$$

Substituting Eq.(4.6) and Eq.(3.11) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l5_{110} = a_0 + \frac{24K^2d\lambda^2\left(d+\frac{\sqrt{\Delta}C_2\coth\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)}{aA^2} - \frac{12K^2\lambda^2\left(d+\frac{\sqrt{\Delta}C_2\coth\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)^2}{aA^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l5_{120} = a_0 + \frac{24K^2d\lambda^2\left(d+\frac{\sqrt{\Delta}C_1\tanh\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)}{aA^2} - \frac{12K^2\lambda^2\left(d+\frac{\sqrt{\Delta}C_1\tanh\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)^2}{aA^2}$$

Substituting Eq.(4.7) and Eq.(3.11) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l5_{210} = a_0 + \frac{12K^2\left\{2d^3\lambda^2-2Ed\lambda\right\}}{aA^2\left(d+\frac{\sqrt{\Delta}C_2\coth\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)} - \frac{12K^2\left\{d^4\lambda^2-2Ed^2\lambda+E^2\right\}}{aA^2\left(d+\frac{\sqrt{\Delta}C_2\coth\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l5_{220} = a_0 + \frac{12K^2\left\{2d^3\lambda^2-2Ed\lambda\right\}}{aA^2\left(d+\frac{\sqrt{\Delta}C_1\tanh\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)} - \frac{12K^2\left\{d^4\lambda^2-2Ed^2\lambda+E^2\right\}}{aA^2\left(d+\frac{\sqrt{\Delta}C_1\tanh\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)}{\lambda}\right)^2}$$

Substituting Eq.(4.8) and Eq.(3.11) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l5_{310} = a_0 - \frac{12K^2\Delta C_2{}^2\coth\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)^2}{aA^2} - \frac{3}{4}\frac{K^2\left\{16E\lambda\right\}\lambda}{aA^2\Delta C_2{}^2\coth\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l5_{320} = a_0 - \frac{12K^2\Delta C_1{}^2\tanh\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)^2}{aA^2} - \frac{3}{4}\frac{K^2\left\{16E\lambda\right\}\lambda}{aA^2\Delta C_1{}^2\tanh\left(\frac{\sqrt{\Delta}}{\lambda}\xi\right)^2}$$

Substituting Eq.(4.9) and Eq.(3.11) into Eq.(4.5) and simplifying, we get following travelling wave solutions:

If $C_1 = 0$ and $C_2 \neq 0$

$$l5_{410} = a_0 - \frac{3}{4}\frac{K^2\left\{16E^2\lambda^2\right\}}{aA^2\Delta {C_2}^2 \coth\left(\frac{1}{2}\frac{\sqrt{\Delta}}{\lambda}\xi\right)^2}$$

If $C_1 \neq 0$ and $C_2 = 0$

$$l5_{420} = a_0 - \frac{3}{4}\frac{K^2\left\{16E^2\lambda^2\right\}}{aA^2\Delta {C_1}^2 \tanh\left(\frac{1}{2}\frac{\sqrt{\Delta}}{\lambda}\xi\right)^2}$$

4.5 Graphical Illustrations of KdV Equation:

We have taken some different values of constraints A,B,C,d,E,C_1 and C_2 for finding out the values of u and have found the graphical presentation of u in maple. For different types of family we have found different types of graph and we have found difference between the two graphs for the same equation when we have change the values of the arbitrary constants. With the help of computational software, Maple, we have plotted graphs of some traveling waves solutions listed below:

Considering the values of $A = 0.9, B = 0.8, C = 0.7, d = 0.6, E = 0.5, k = \frac{-5}{\sqrt{6}}$, $x = -25..25, t = -25..25$ and $\xi = x - kt$, we get the following figure:

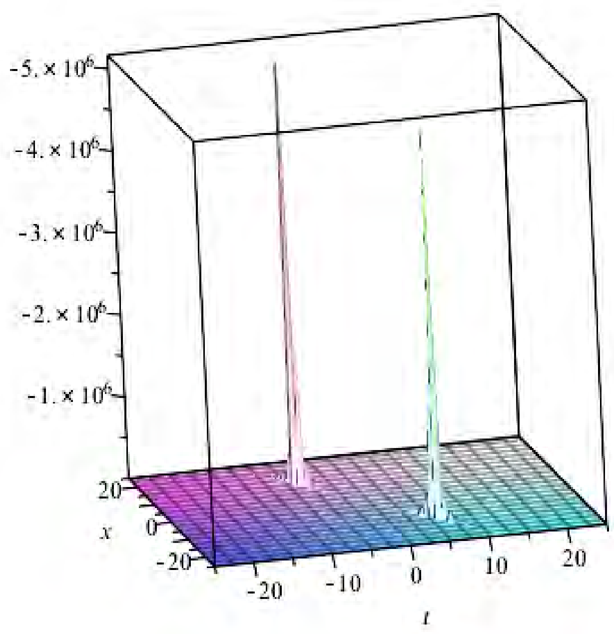

figure 1: 3D graph of equation $l1_{110}$

Considering the values of $A = 1.0, B = 0.8, C = 0.7, d = 0.6, E = 0.5, k = \frac{-5}{\sqrt{6}}$, $x = -25..25, t = -25..25$ and $\xi = x - kt$, we get the following figure:

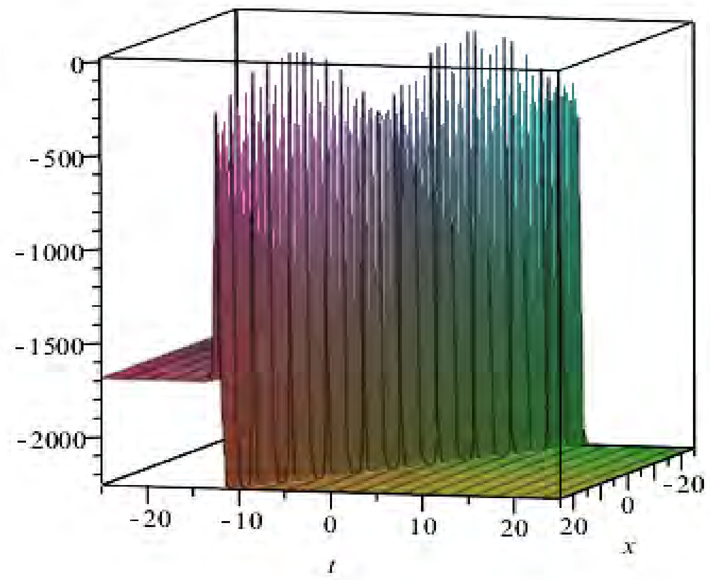

figure 2: 3D graph of equation l_{120}

Considering the values of $A = 0.9, B = 0.8, C = 0.7, d = 0.6, E = 0.5, k = \frac{-5}{\sqrt{6}}$, $x = -25..25, t = -25..25$ and $\xi = x - kt$, we get the following figure:

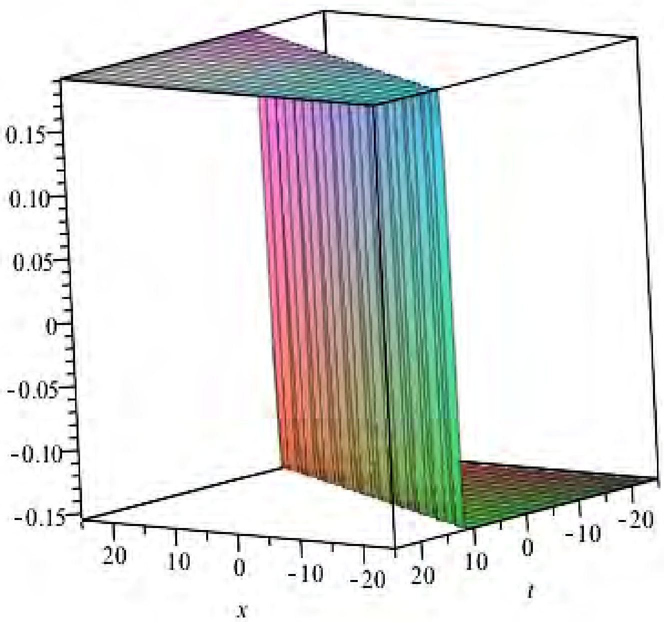

figure 3: 3D graph of equation $l1_{210}$

Considering the values of $A = 1.0, B = 0.8, C = 0.7, d = 0.6, E = 0.5, k = \frac{-5}{\sqrt{6}}$, $x = -5..5, t = -5..5$ and $\xi = x - kt$, we get the following figure:

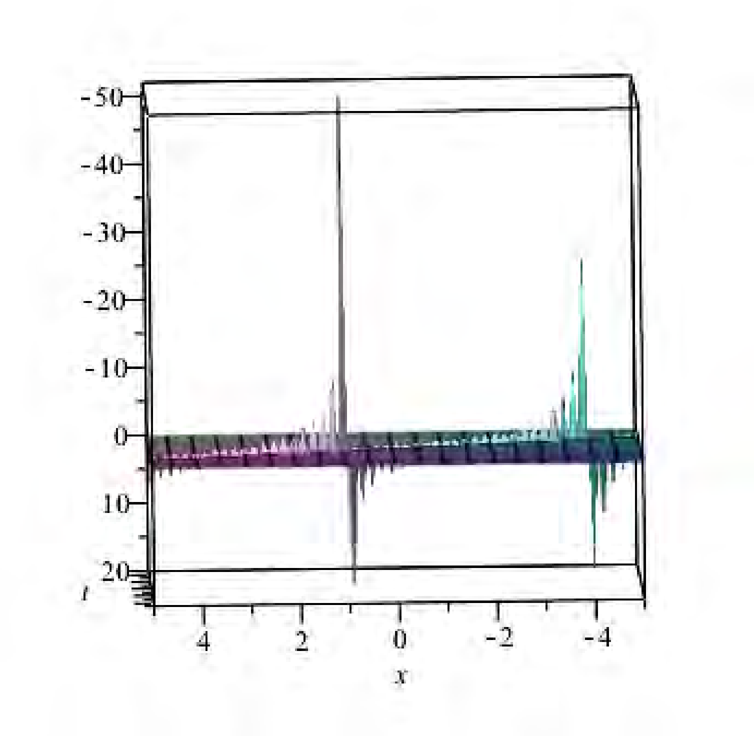

figure 4: 3D graph of equation $l1_{220}$

Considering the values of $A = 1.0, B = 0.8, C = 0.7, d = \frac{-B}{2(A-C)}, E = 0.5, k = \frac{-5}{\sqrt{6}}$, $x = -25..25, t = -25..25$ and $\xi = x - kt$, we get the following figure:

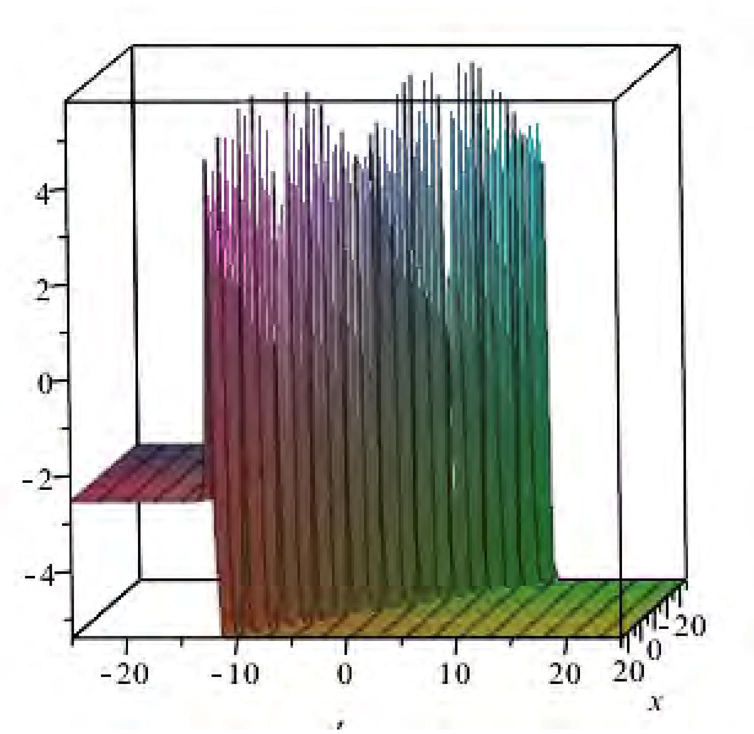

figure 5: 3D graph of equation $l1_{320}$

Considering the values of $A = 1.0, B = 0.8, C = .7, d = \frac{-B}{2(A-C)}, E = 0.5, k = \frac{-5}{\sqrt{6}}$, $x = -25..25, t = -25..25$ and $\xi = x - kt$, we get the following figure:

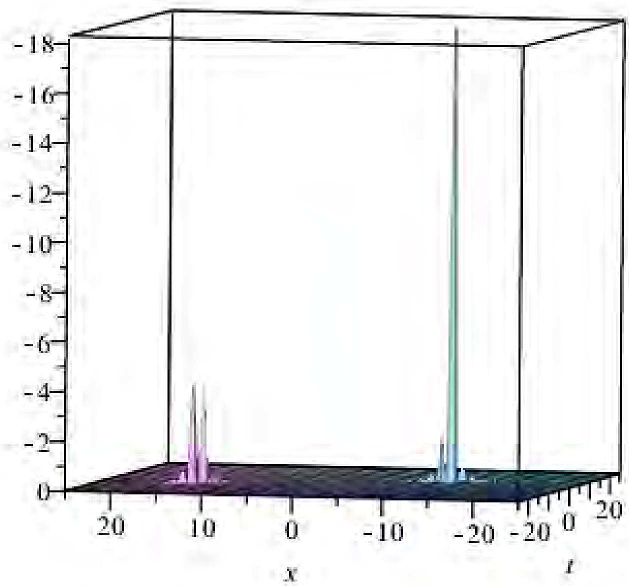

figure 6: 3D graph of equation $l1_{420}$

Considering the values of $A = 1.0, B = 0.8, C = 5, d = 0.6, E = 0.5, k = \frac{-1}{\sqrt{6}}$, and $\xi = x - kt$, we get the following figure:

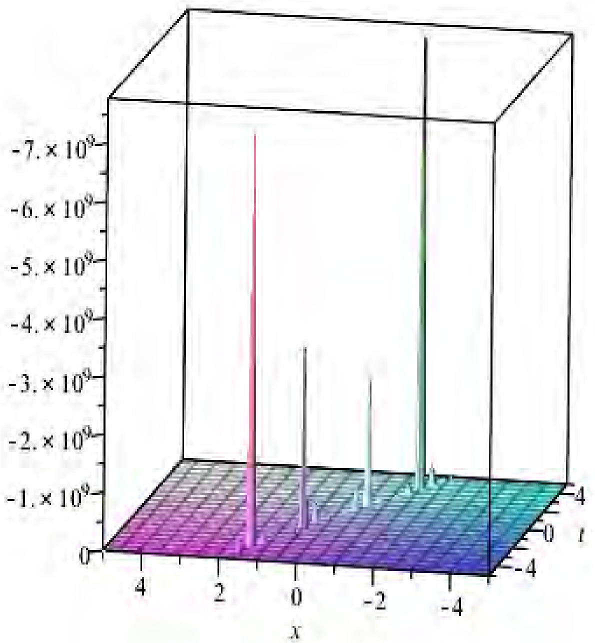

figure 7: 3D graph of equation $l2_{110}$

Considering the values of $A = 1.0, B = 0.8, k = 0.7, d = 0.6, E = 0.5, k = \frac{-5}{\sqrt{6}}$, $x = -5..5, t = -5..5$ and $\xi = x - kt$, we get the following figure:

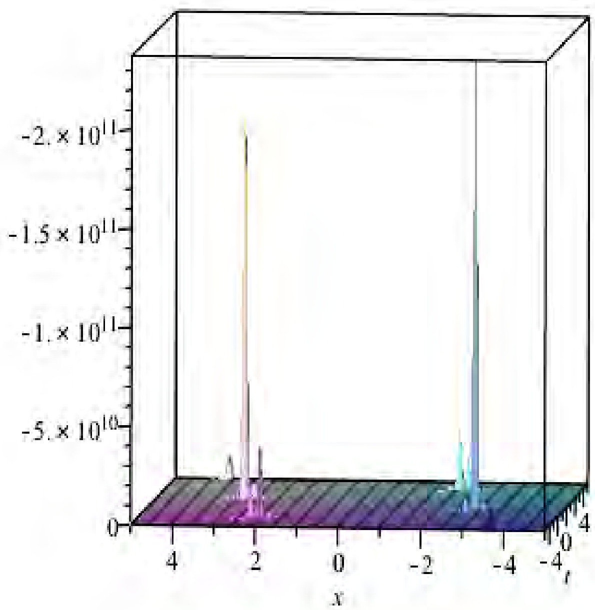

figure 8: 3D graph of equation $l2_{120}$

Considering the values of $A = 1.0, B = 0.8, C = 5, d = \frac{-B}{2(A-C)}, E = 0.5, k = \frac{-5}{\sqrt{6}}$, $x = -25..25, t = -25..25$ and $\xi = x - kt$, we get the following figure:

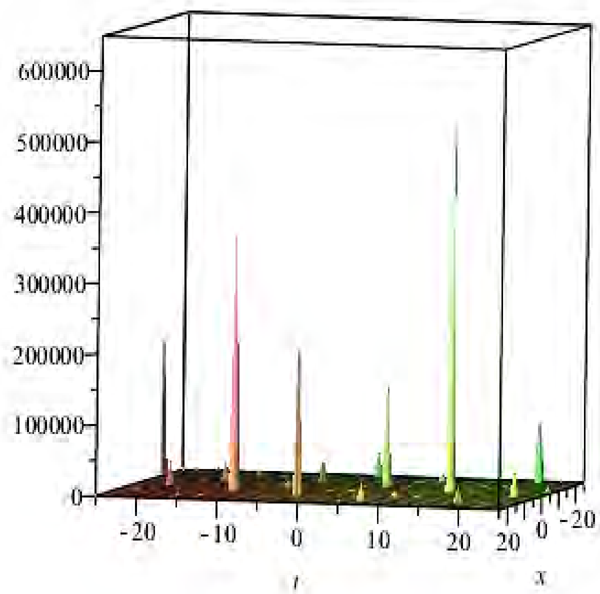

figure 9: 3D graph of equation $l2_{410}$

Considering the values of $A = 2.0, B = 2.0, C = 1.0, d = 0.6, E = -1, k = 9.0$, $x = -25..25, t = -25..25$ and $\xi = x - kt$, we get the following figure:

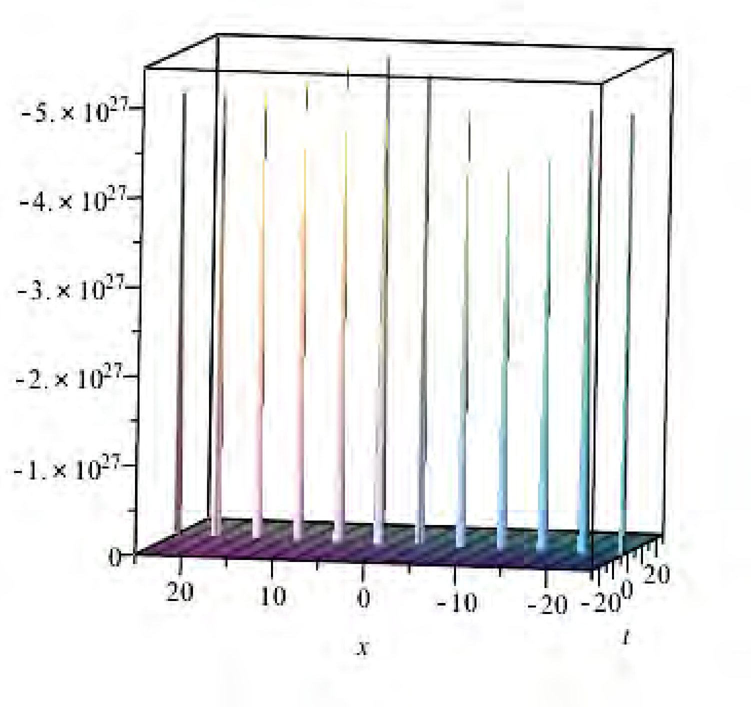

figure 10: 3D graph of equation $l3_{110}$

Considering the values of $A = 2.0, B = 2.0, C = 1.0, d = 0.6, E = -1.0, k = \frac{-5}{\sqrt{6}}$, $x = -25..25, t = -25..25$ and $\xi = x - kt$, we get the following figure:

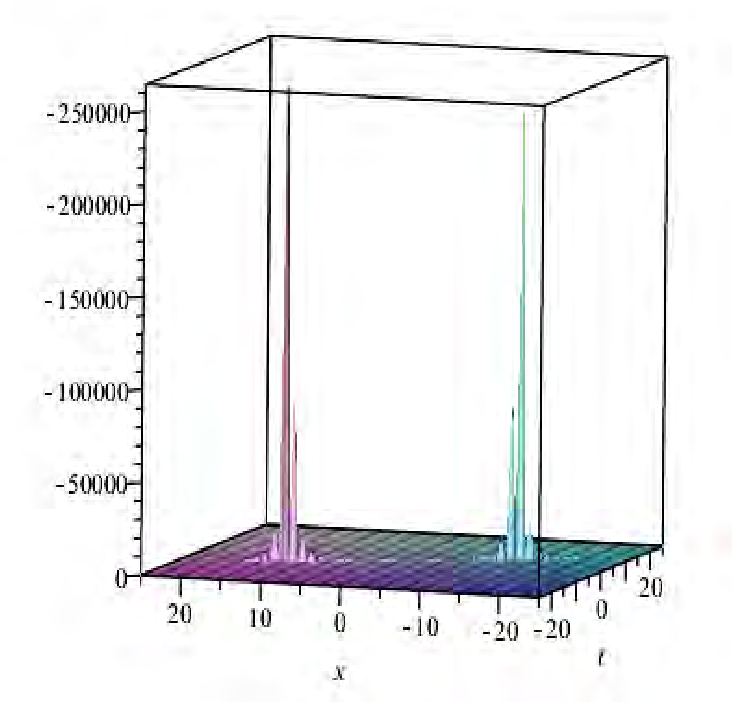

figure 11: 3D graph of equation $l3_{110}$

Considering the values of $A = 1.0, B = 0.0, C = 0.7, d = 0.0, E = 0.5, k = \frac{-5}{\sqrt{6}}$, $x = -5..5, t = -5..5$ and $\xi = x - kt$, we get the following figure:

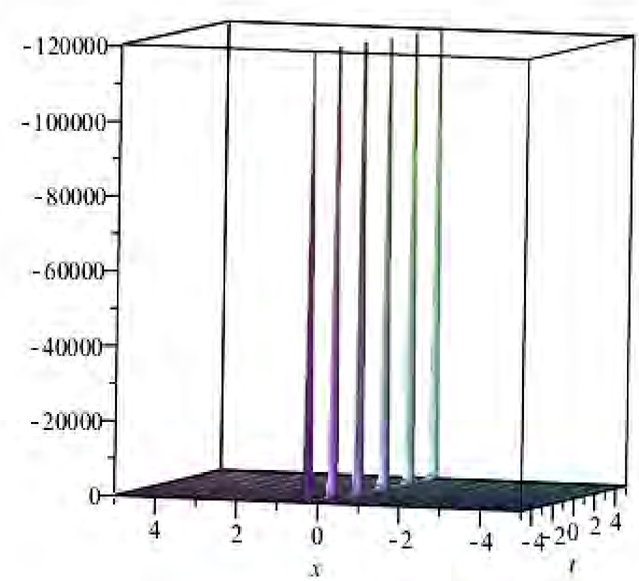

figure 12: 3D graph of equation $l4_{420}$

Considering the values of $A = 1.0, B = 0.0, C = 0.7, d = 0.6, E = 0.5, k = \frac{-5}{\sqrt{6}}$, $x = -25..25, t = -25..25$ and $\xi = x - kt$, we get the following figure:

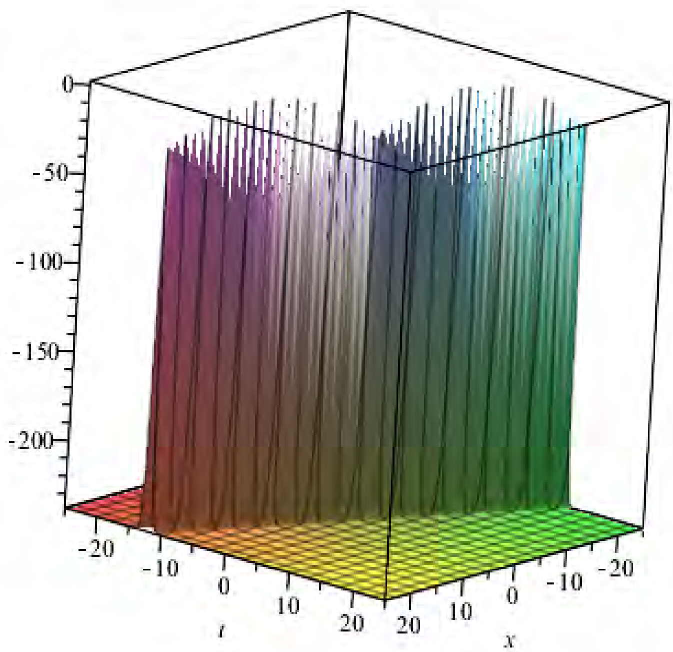

figure 13: 3D graph of equation $l5_{120}$

Considering the values of $A = 1.0, B = 0.0, C = 0.7, d = 0.0, E = 0.5, k = -0.5$, $x = -25..25, t = -25..25$ and $\xi = x - kt$, we get the following figure:

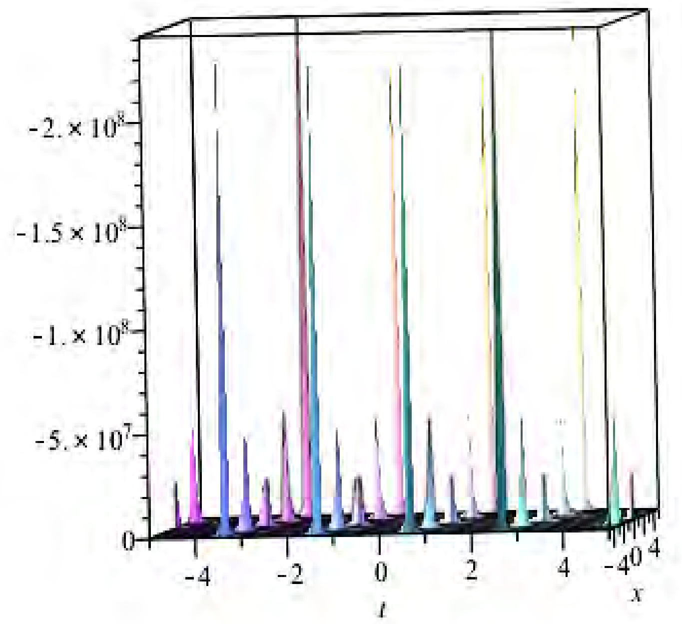

figure 14: 3D graph of equation $l5_{310}$

Considering the values of $A = 1.0, B = 0.0, C = 0.7, d = 0.0, E = 0.5, k = -0.5$, $x = -5..5, t = -5..5$ and $\xi = x - kt$, we get the following figure:

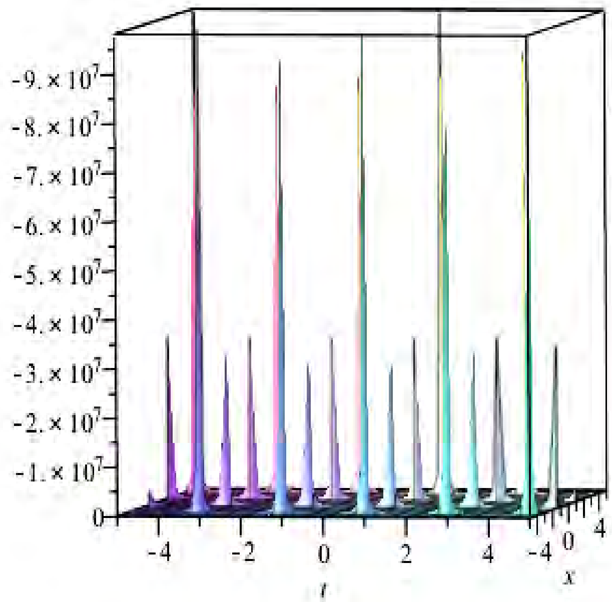

figure 15: 3D graph of equation $l5_{320}$

Considering the values of $A = 1.0, B = 0.0, C = 0.7, d = 0.0, E = 0.5, k = -0.5$, $x = -5..5, t = -5..5$ and $\xi = x - kt$, we get the following figure:

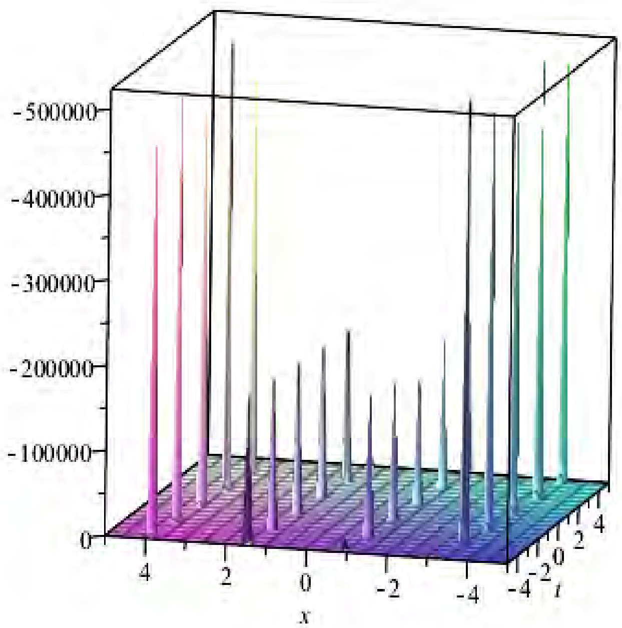

figure 16: 3D graph of equation $l5_{420}$

Chapter 5

Conclusion

In this thesis, to find the travelling wave solutins for the KdV equation we successfully used the (G'/G)-expansion method to solve fractional nonlinear partial differential equations. This method is dependable and efficient also provides new solutions. Now, briefly summarizing the results in this thesis. First of all, for solving nonlinear fractional differential equations the fractional complex transform is extremely simple but effective. Secondly, the (G'/G)-expansion method for nonlinear fractional differential equations has its own pros: direct, fundamental, succinct; and it can be used for many other nonlinear equations. The auxiliary equation used in the method, which involves many arbitrary parameters can take any real values and then the nonlinear P.D.E. produces various new solutions. By using this method we have successfully found five types of solutions in terms of hyperbolic, trigonometric and rational functions.To smooth the computation of the complex systems of algebraic equations the solutions can be investigated with the help of symbolic computational software like the Maple or Matlab .

Chapter 6

Future Work

With many real parameters, which are straightforward and precise the method used in this thesis provides many new and more plentiful general and explicit travelling wave solutions . Simultaneously it also discloses more of the important insight mechanism of the complex physical phenomena of NLFDEs.This method could be used to solve different and all sorts of NLFDEs that arises frequently in mathematical physics, engineering sciences and in many scientific real time application fields. Additionally the solutions that we have found from the given results can be used for solving purpose in the application of plasma physics such as tsunami, typhoon or cyclones that include water bodies where nonlinear problems are involved.

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