Generalized Inverse And It's Applications To The Solutions of System of Linear Equations And Semigroup

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By

Muhtadi Nasrullah Shiham

ID: 12216002

Department of Mathematics and Natural Sciences

BRAC University

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Declaration

I do hereby declare that the thesis titled "Generalized Inverse And It's Applications To The Solutions of System of Linear Equations And Semigroup" is submitted to the Department of Mathematics and Natural Sciences of BRAC University in partial fulfilment of the Bachelor of Science in Mathematics. This is my original work and has not been submitted elsewhere for the award of any other degree or any other publication. Every work that has been used as reference for this work has been cited properly.

Date: 27th September, 2018

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Candidate

Name: Muhtadi Nasrullah Shiham

ID: 12216002

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Certified

Mohammad Maruf Ahmed

Assistant Professor

Department of Mathematics and Natural Sciences

BRAC University

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Abstract

There are a number of versatile generalizations of the usual inverse matrix, referred to in this thesis as generalized inverse matrices. The definitions and properties of some of the common generalized inverse matrices are described, including methods for constructing them.

A number of applications are discussed, including their use in solving consistent systems of linear equations which do not have the same number of equations as variables, or which have a singular coefficient determinant. A certain type of generalized inverse is shown to give the least-squares solution of an inconsistent system of linear equations. Other applications are to systems of nonlinear equations, to integer solutions of systems of equations and to linear programming.

The purpose of the thesis is to show that a singular or $m \times n$ matrix has a generalized inverse (g-inverse). A matrix A^{-} is said to be a generalized inverse if it fulfils the condition $AA^{-}A = A$. The raw canonical system is used to find the generalized inverse. So we will be using theorems, examples and programming language to prove and solve G-inverse. This thesis will also consist of semigroups, contour integration and applications which is related generalized inverse.

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Chapter-1

Introduction

It is not easy to find the inverse of a matrix if it is singular or rectangular.

Let's consider a system of linear equations

$$Ax = b$$

Where the matrix A has to be a square and has to be a non-zero determinant.

If A is an $n \times n$ non-singular matrix, then the linear equation given above has a solution which is unique given by

$$x = A^{-1}b$$

However, in certain cases the matrix A is not a square matrix or a singular matrix; this is where the solution of linear equation is inconsistent. In such cases there is a theory to solve the system of linear equation. The method is known as generalized inverses of matrices. It is also known as Pseudo-inverse, Moore-Penrose inverse or simply g-inverse.

1.1. Definition

[1] Let the matrix A be $m \times n$ matrix, in order to be a g-inverse matrix has to have a rank of $R(A) = r \leq \min(m, n)$. Then we denote generalized inverse of A by A^{-} in the form of $n \times m$ matrix. The solution will be consistent for the set of linear equation $Ax = b$ for the condition

$$x = A^{-}b.$$

A matrix A^{-} satisfying $AA^{-}A = A$ coincides with A^{-1} when A^{-1} exists.

1.2. Existence of a generalized inverse

Theorem 1.1. [1] A generalized inverse A^- exists if and only if $AA^-A = A$.

Proof.

Suppose that A^- exists. The equations $Ax = b$are consistent for any arbitrary vector v such that $b = Av$. Since $R(A\colon b) = R(A\colon Av) = R(A)$ that is, b lies in the in the column space of A. Then we have $AA^-b = b = Ax$ so that $Ax = AA^-b = AA^-Av$ since $b = Av$. $\therefore A = AA^-A$ since v is a solution of $Ax = b$. Conversely, suppose that $A = AA^-A$ and $Ax = b$ are a consistent system. Then there exist a vectorxv^* such that $Av^* = b$, that is,

$AA^-Av^* = b$

Or, $AA^-b = b = A(A^-b)$ implying the existence of a matrix A^- such that $A^-b = x$.

Hence the theorem is proved.

1.3. Different Conditions of G-Inverse

[2] Let A be a matrix over a complex field $C.$ When the matrix is defined on an real field such equations are getable:

1. $AXA = A$
2. $XAX = X$
3. $(XA)^* = XA$
4. $(AX)^* = AX$

Where $'*'$ is conjugate transpose.

X is a g-inverse if it fulfils the equation (i) and denoted by

$$X = A^-$$

a) If the equations (i) and (ii) is fulfilled then X is a **reflexive g-inverse** denoted by $X = A^r$.

b) If the equations (i), (ii) and (iii) is fulfilled then X is a **left weak g-inverse** denoted by $X = A^w$.

c) If the equations (i), (ii) and (iv) is fulfilled then X is a **right weak g-inverse** denoted by $X = A^n$

d) If the equations (i), (ii), (iii) and (iv) is fulfilled then X is **Pseudo-inverse** or **Moore-Penrose g-inverse** denoted by $X = A^+$.

1.4. The Method of Performing Generalized Inverse

Theorem 1.2. [1] Let A be a $m \times n$ matrix and $A = [a_{ij}]$.

Where $i = 1,2, \dots, m$ and $j = 1,2, \dots, n$.

We assume the rank of A is r and can be partitioned in a way such that the minor

$r \times r$ is non-singular and that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

And the order and the rank of $A_{11}$is r.

Hence the g-inverse of A is shown by

$$A^{-} = \begin{bmatrix} {A_{11}}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

The null matrices of are to make A^{-} an order of $n \times m$.

Proof.

To show that A^{-} is g-inverse of A.

Let us prove it by $AA^{-}A = A$

Where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

And

$$A^{-} = \begin{bmatrix} {A_{11}}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

Solving left hand side of the equation

$$\therefore \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} {A_{11}}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}{A_{11}}^{-1} & 0 \\ A_{21}{A_{11}}^{-1} & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$= \begin{bmatrix} I_r & 0 \\ A_{21}{A_{11}}^{-1} & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Where I_r is identity matrix of r order

$$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21}{A_{11}}^{-1}A_{11} & A_{21}{A_{11}}^{-1}A_{12} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{21}{A_{11}}^{-1}A_{12} \end{bmatrix} \ldots\ldots(1)$$

Now we can partition A and can write it in this way

$$[A_{21} \quad A_{22}] = B[A_{11} \quad A_{12}] = [BA_{11} \quad BA_{12}]$$

From this we can write

$$A_{21} = BA_{11} \text{ and } A_{22} = BA_{12}$$

So $B = A_{21}{A_{11}}^{-1}$

$$\therefore A_{22} = A_{21}{A_{11}}^{-1}A_{12}$$

Let us put the value of A_{22} in (1)

$$AA^{-}A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = A$$

Hence $A^{-} = \begin{bmatrix} {A_{11}}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ is a g-inverse of A with $n \times m$ dimension.

1.5. Example of finding a G-inverse matrix

Let us consider a matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 7 & 0 & 5 \\ 2 & 4 & 6 & 8 \end{bmatrix}$

The matrix A is rectangular and if we apply the Gauss elimination method then the rank of the matrix is 2. Hence we need to find the pseudo inverse of the matrix A.

So we are going to partition the matrix A which will form like

$$A = \left[\begin{array}{cc:cc} 1 & 2 & 3 & 4 \\ 3 & 7 & 0 & 5 \\ \hline 2 & 4 & 6 & 8 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$\text{Where } A_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$$

Here $|A_{11}| = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = 7 - 6 = 1 \neq 0$

$\therefore$ It is a non-singular matrix and hence ${A_{11}}^{-1}$ exists.

Now let us find the inverse of A_{11} using raw canonical form

$$(A_{11}|I) = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 7 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -3 & 1 \end{array}\right) \text{ when } R_2{}' = R_2 - 3R_1$$

$$= \left(\begin{array}{cc|cc} 1 & 0 & 7 & -2 \\ 0 & 1 & -3 & 1 \end{array}\right) \text{ when } R_1{}' = R_1 - 2R_2$$

$$= \left(I \middle| A_{11}{}^{-1}\right)$$

∴ The pseudo inverse of the matrix A is:

$$A^{-} = \begin{bmatrix} A_{11}{}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^{-} = \begin{bmatrix} 7 & -2 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1.6. Algorithm and Code for G-inverse

Algorithm. [1] Let A be a $m \times n$ matrix of rank r; then the generalized inverse can be computed by the following steps:

i. Compute $B = A^T A$

ii. Let $C_1 = I$ of order r.

iii. Compute $C_{i+1} = I\left(\frac{1}{i}\right) tr(C_i B) - C_i B$ for $i = 1,2, \dots, r-1$

iv. Compute $IC_r A^T / tr(C_r B)$ and this is $\overline{A}$. Also $C_{r+1}B = 0$, but $tr(C_r B) \neq 0$.

Code.

Example. Let A be a matrix $= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}$

According to the algorithm and using the Python software the **Code** will be like

```
import numpy as np
import sys
# muliplication of matrix system
def multiply(AT,A):
    m,n=AT.shape
    p,q=A.shape
    if n!=p:
        print ' Matrix product not possible'
        sys.exit(0)
    B=np.zeros((m,q))
    for i in range(m):
        for j in range(q):
            sum=0
            for k in range(n):
                sum=AT[i,k]*A[k,j]+sum
            B[i,j]=sum

    B=np.matrix(B)
    return B
#trace of a matrix system
def trace(O):
        m,n=O.shape
        add=0
        for i in range(m):
            for j in range(n):
                if i==j:
                    add=O[i,j]+add
        return add
```

```
if __name__=="__main__":

    A=np.matrix('1 0 -2; 0 1 -1; -1 1 1; 2 -1 2')
    AT=np.matrix('1 0 -1 2; 0 1 1 -1; -2 -1 1 2')

    # multiplies A transpose with A giving B

    B=multiply(AT,A)

    # Computing C1, C2 and C3 since its a 3 by 3 matrix

    Cfirst=np.matrix('1 0 0; 0 1 0; 0 0 1')
    I=Cfirst

    M=multiply(Cfirst,B)
    X=trace(M)
    Csecond=(I*X)-M

    N=multiply(Csecond,B)
    Y=trace(N)
    Cthird=(.5*I*Y)-N

    # calculates generalised inverse of A

    P=multiply(Cthird,B)
    Z=trace(P)

    R=multiply(Cthird,AT)

    ginverseA=(3*R)/float(Z)

    print ginverseA
```

The output of the **Code** is.

```
In [1]: %run "E:\BRAC-Shiham\MNS\MAT400\generalised_inverse.py"
[[ 0.26666667  0.33333333  0.06666667  0.4       ]
 [ 0.13333333  0.66666667  0.53333333  0.2       ]
 [-0.2         0.          0.2         0.2       ]]
```

Chapter-2

Representation of G-Inverse In Contour Integration

Definition. Contour integration is the process of calculating the values of a contour integral around a given contour in the complex plane. As a result of a truly amazing property of holomorphic functions, such integrals can be computed easily simply by summing the values of the complex residues inside the contour.

Theorem 2.1. [3] If A is any $m \times n$ matrix such that $(AA^*)^{-1}$ exists, then

$$A^+ = \frac{1}{2\pi i}\oint A^*(AA^* - Iz)^{-1}\frac{1}{z}dz$$

where the integral sign is a closed contour containing non-zero eigenvalues of AA^* but not containing the zero eigenvalue of AA^* in or on the closed contour integral.

Proof. Let

$$X = \frac{1}{2\pi i}\oint A^*(AA^* - Iz)^{-1}\frac{1}{z}dz$$

Then we have to show that X satisfies the following four conditions:

1. $AXA = A$
2. $XAX = X$
3. $(AX)^* = AX$
4. $(XA)^* = XA$

Now, let us prove the conditions.

1. $AXA = A$

$$\therefore AXA = \frac{1}{2\pi i}\oint AA^*(AA^* - Iz)^{-1}A\frac{1}{z}dz$$

$$= \frac{1}{2\pi i}\oint\{(AA^*)^{-1}\}^{-1}(AA^* - Iz)^{-1}A\frac{1}{z}dz$$

$$= \frac{1}{2\pi i}\oint\{(AA^* - Iz)(AA^*)^{-1}\}^{-1}A\frac{1}{z}dz$$

$$= \frac{1}{2\pi i}\oint\{I - z(AA^*)^{-1}\}^{-1}A\frac{1}{z}dz$$

$$= \frac{1}{2\pi i}\oint\frac{f(z)}{(z - z_0)}dz$$

Where $z_0 = 0$ and $f(z) = \{I - z(AA^*)^{-1}\}^{-1}A$

$$= f(z_0) = f(0)$$

$$= (I - 0)^{-1}A = A$$

$\therefore AXA = A$

2. $XAX = X$

$$\therefore XAX = \frac{1}{2\pi i}\oint A^*(AA^* - Iz)^{-1}\frac{1}{z}dz\, A\frac{1}{2\pi i}\oint A^*(AA^* - Iz)^{-1}\frac{1}{z}dz$$

$$= \frac{1}{2\pi i}\oint A^*(AA^* - Iz)^{-1}\frac{1}{z}dz\, I$$

$$= XI = X$$

$\therefore XAX = X$

3. $(AX)^* = AX$

$$\therefore AX = A\frac{1}{2\pi i}\oint A^*(AA^* - Iz)^{-1}\frac{1}{z}dz$$

$$= I$$

$\therefore (AX)^* = I^* = I = AX$

4. $(XA)^* = XA$

$$\therefore XA = \frac{1}{2\pi i}\oint A^*(AA^* - Iz)^{-1}\frac{1}{z}Adz$$

$$= \frac{1}{2\pi i}\oint \frac{f(z)}{(z - z_0)}dz$$

where $z_0 = 0$ and $f(z) = A^*(AA^* - Iz)^{-1}A$

$= f(z_0)$ (using Cauchy's Integral formula)

$= A^*(AA^*)^{-1}A$ (which is Hermitian)

$\therefore (XA)^* = XA$

Thus X satisfies the four condition of Moore-Penrose g-inverse.

Hence $X = A^+$.

So we have

$$A^+ = \frac{1}{2\pi i}\oint A^*(AA^* - Iz)^{-1}\frac{1}{z}dz$$

Let us use a complex matrix as an example:

$$A = \begin{pmatrix} 5 & -3 \\ 2i & 0 \end{pmatrix}$$

Let us use the python software to find A^* and AA^*.

Code.

```
import numpy as np
import sys
A=np.matrix('5. -3.; 0.+2.j 0.')
B=A.getH() # B is the conjugate tranpose of A
C=A*B
print "The conjugate tranpose of A is:", B
print "The product of A with it's conjugate transpose is:", C
```

And the output of the **Code** is.

```
In [6]: %run "E:/BRAC-Shiham/MNS/MAT400/thesiswork2.py"
The conjugate tranpose of A is: [[ 5.-0.j  0.-2.j]
 [-3.-0.j  0.-0.j]]
The product of A with it's conjugate transpose is: [[ 34. +0.j   0.-10.j]
 [  0.+10.j   4. +0.j]]
```

Here the first matrix is A^* and the second matrix is AA^*.

Now let us find the determinant of AA^* to see if any inverse exists.

$$\therefore \det[AA^*] = (34 \times 4) - (10i \times -10i) = 36 \neq 0$$

Therefore, $(AA^*)^{-1}$ exists and $(AA^*)^{-1} = \begin{pmatrix} 1/9 & 5i/18 \\ -5i/18 & 17/18 \end{pmatrix}$.

$$A^+ = \frac{1}{2\pi i} \oint A^*(AA^* - Iz)^{-1} \frac{1}{z} dz$$

$$= \frac{1}{2\pi i} \oint \frac{A^*(AA^* - Iz)^{-1}}{z} dz$$

$$= \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0} dz$$

where $z_0 = 0$ and $f(z) = A^*(AA^* - Iz)^{-1}$

$$= f(0) = A^*(AA^*)^{-1}$$

$$= \begin{pmatrix} 5 & -2i \\ -3 & 0 \end{pmatrix} \begin{pmatrix} 1/9 & 5i/18 \\ -5i/18 & 17/18 \end{pmatrix}$$

$$A^+ = A^*(AA^*)^{-1}$$

Using the calculator and manual calculation we get

$$A^+ = \begin{pmatrix} 0 & -i/2 \\ -1/3 & -5i/6 \end{pmatrix}$$

Now we have to verify it using the four conditions:

1. $AA^+A = A$
2. $A^+AA^+ = A^+$
3. $(AA^+)^* = AA^+$
4. $(A^+A)^* = A^+A$

1.

$$AA^+A = \begin{pmatrix} 5 & -3 \\ 2i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i/2 \\ -1/3 & -5i/6 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ 2i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ 2i & 0 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ 2i & 0 \end{pmatrix} = A$$

$\therefore AA^+A = A$

2.

$$A^+AA^+ = \begin{pmatrix} 0 & {}^{-i}/_{2} \\ {}^{-1}/_{3} & {}^{-5i}/_{6} \end{pmatrix}\begin{pmatrix} 5 & -3 \\ 2i & 0 \end{pmatrix}\begin{pmatrix} 0 & {}^{-i}/_{2} \\ {}^{-1}/_{3} & {}^{-5i}/_{6} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & {}^{-i}/_{2} \\ {}^{-1}/_{3} & {}^{-5i}/_{6} \end{pmatrix} = \begin{pmatrix} 0 & {}^{-i}/_{2} \\ {}^{-1}/_{3} & {}^{-5i}/_{6} \end{pmatrix} = A^+$$

$\therefore A^+AA^+ = A^+$

3.

$$AA^+ = \begin{pmatrix} 5 & -3 \\ 2i & 0 \end{pmatrix}\begin{pmatrix} 0 & {}^{-i}/_{2} \\ {}^{-1}/_{3} & {}^{-5i}/_{6} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore (AA^+)^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = AA^+$$

$\therefore (AA^+)^* = AA^+$

4.

$$A^+A = \begin{pmatrix} 0 & {}^{-i}/_{2} \\ {}^{-1}/_{3} & {}^{-5i}/_{6} \end{pmatrix}\begin{pmatrix} 5 & -3 \\ 2i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore (A^+A)^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A^+A$$

$\therefore (A^+A)^* = A^+A$

Hence all four conditions are fulfilled.

Theorem 2.2. [3] The Moore-Penrose g-inverse of a $m \times n$ matrix A of complex numbers is given by the formula

$$A^{+} = \int_{0}^{\infty} e^{-A^{*}At} A^{*}\, dt$$

Proof. Let

$$X = \int_{0}^{\infty} e^{-A^{*}At} A^{*}\, dt$$

then we have to show that X satisfies the fours conditions of M-P generalized inverse.

1.

$$AXA = A \int_{0}^{\infty} e^{-A^{*}At} A^{*}A\, dt$$

$$= -A[e^{-\infty} - e^{0}]$$

$$= -A\left[\frac{1}{e^{\infty}} - I\right] = A$$

$\therefore AXA = A$

2.

$$XAX = \int_{0}^{\infty} e^{-A^*At}A^*\,dtA\int_{0}^{\infty} e^{-A^*At}A^*\,dt$$

$$= -[e^{-\infty} - e^{0}]\int_{0}^{\infty} e^{-A^*At}A^*\,dt$$

$$= \int_{0}^{\infty} e^{-A^*At}A^*\,dt = X$$

$\therefore XAX = X$

3.

$$AX = A\int_{0}^{\infty} e^{-A^*At}A^*\,dt$$

$$= -[e^{-\infty} - e^{0}] = I$$

$\therefore (AX)^* = AX$

4.

$$XA = \int_0^\infty e^{-A^*At} A^* \, dtA$$

$$= -[e^{-\infty} - e^0] = I$$

$\therefore (XA)^* = XA$

Hence X satisfies the four conditions of M-P generalized inverse.

So,

$$A^+ = \int_0^\infty e^{-A^*At} A^* \, dt$$

Chapter-3

Applications to the System of Linear Equations

Theorem 3.1. [5] Let A be $m \times n$ order matrix and G be the g-inverse of A and let

$$H = GA.$$

A general solution of consistent non homogeneous equation $AX = Y$ is

$$X = GY + (H - I)Z$$

Where Y and Z are both arbitrary vectors.

Proof. Let us take a matrix

$$A = \begin{pmatrix} 2 & 0 & -4 \\ 0 & 2 & -2 \\ -2 & 2 & 2 \\ 4 & -2 & 4 \end{pmatrix}$$

Using the Python Generalized Inverse code we are going to find g-inverse of A (shown in part 1.6.). Hence we get:

$$G = \frac{1}{30}\begin{pmatrix} 4 & 5 & 1 & 6 \\ 2 & 10 & 8 & 3 \\ -3 & 0 & 3 & 3 \end{pmatrix}$$

In order to prove the theorem let us take two arbitrary vectors:

$$Y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 15 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 14 \\ 6 \\ 5 \end{pmatrix}$$

We know $H = GA$

$$H = \frac{1}{30}\begin{pmatrix} 4 & 5 & 1 & 6 \\ 2 & 10 & 8 & 3 \\ -3 & 0 & 3 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 & -4 \\ 0 & 2 & -2 \\ -2 & 2 & 2 \\ 4 & -2 & 4 \end{pmatrix}$$

If we multiply the matrices H becomes an identity matrix.

$$\therefore H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Now we are going to find X

$$X = GY + (H - I)Z$$

$$= \frac{1}{30}\begin{pmatrix} 4 & 5 & 1 & 6 \\ 2 & 10 & 8 & 3 \\ -3 & 0 & 3 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 15 \end{pmatrix} + \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \times \begin{pmatrix} 14 \\ 6 \\ 5 \end{pmatrix}$$

$$\therefore X = \begin{pmatrix} 3 \\ 1.5 \\ 1.5 \end{pmatrix}$$

Now we are going to find Y using this value of X. If the value of Y matches with the arbitrary vector Y then the theorem is proved.

$$Y = AX$$

$$= \begin{pmatrix} 2 & 0 & -4 \\ 0 & 2 & -2 \\ -2 & 2 & 2 \\ 4 & -2 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 1.5 \\ 1.5 \end{pmatrix}$$

$$\therefore Y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 15 \end{pmatrix}$$

Code.

```python
import numpy as np
import sys
# muliplication of matrix system
def multiply(AT,A):
    m,n=AT.shape
    p,q=A.shape
    if n!=p:
        print ' Matrix product not possible'
        sys.exit(0)
    B=np.zeros((m,q))
    for i in range(m):
        for j in range(q):
            sum=0
            for k in range(n):
                sum=AT[i,k]*A[k,j]+sum
            B[i,j]=sum

    B=np.matrix(B)
    return B
#trace of a matrix system
def trace(O):
        m,n=O.shape
        add=0
        for i in range(m):
            for j in range(n):
                if i==j:
                    add=O[i,j]+add
        return add



if __name__=="__main__":

    A = np.array([[2, 0, -4], [0, 2, -2], [-2, 2, 2], [4, -2, 4]])
    AT = A.transpose()

    # multiplies A transpose with A giving B

    B=multiply(AT,A)

    # Computing C1, C2 and C3 since its a 3 by 3 matrix
    # I is the identity matrix

    Cfirst=np.matrix('1 0 0; 0 1 0; 0 0 1')
    I=Cfirst

    M=multiply(Cfirst,B)
    Exam=trace(M)
    Csecond=(I*Exam)-M

    N=multiply(Csecond,B)
    Dark=trace(N)
    Cthird=(.5*I*Dark)-N

    # calculates generalised inverse of A

    P=multiply(Cthird,B)
    logic=trace(P)

    uber=multiply(Cthird,AT)

    G=(3*uber)/float(logic)#here G is the g-inverse

    print "The g-inverse is:" ,G
```

```python
#now we are proving the theorem
H=G*A
# If we multiply G with A it gives us the identity element hence
#which is one the properties of generalized inverses
H=I
# Now let us consider 2 arbritrary vector Y and Z and assume any values

Z=np.matrix('14; 6; 5')
Y=np.matrix('0; 0; 0; 15')

X=G*Y+(H-I)*Z
print "The matrix X is:" ,X

rat=A*X
print "The matrix Y is:" ,rat
#if rat is equal to Y then theorem is proved
```

The output of this **Code** is.

```
In [2]: %run "E:\BRAC-Shiham\MNS\MAT400\homogeneous theorem.py"
The g-inverse is: [[ 0.13333333  0.16666667  0.03333333  0.2        ]
 [ 0.06666667  0.33333333  0.26666667  0.1        ]
 [-0.1         0.          0.1         0.1        ]]
The matrix X is: [[ 3. ]
 [ 1.5]
 [ 1.5]]
The matrix Y is: [[  0.]
 [  0.]
 [  0.]
 [ 15.]]
```

The code and output also proves the theorem above.

Example 3.1. We are going to use numerical computation to solve $Ax = b$ where matrix noninvertible and singular and we have an Algorithm to solve the value of x as shown in the next page

Algorithm for the generalized inverse and solution of $\boldsymbol{Ax = b}$

- **Assume $\boldsymbol{A}$ is $\boldsymbol{m \times n}$ matrix and a column vector $\boldsymbol{b \in R^m}$**
- **Choose any non-singular sub-matrix $\boldsymbol{H}$ of dimension $\boldsymbol{r}$,**
- **Find $(\boldsymbol{H^{-1}})^{\boldsymbol{T}}$,**
- **Replace the elements of sub-matrix $\boldsymbol{H}$ in the original matrix $\boldsymbol{A}$ by elements of $(\boldsymbol{H^{-1}})^{\boldsymbol{T}}$,**
- **Replace all other elements by zeros to get a new matrix $\overline{\boldsymbol{A}}$,**
- **The generalized matrix $\boldsymbol{G} = (\overline{\boldsymbol{A}})^{\boldsymbol{T}}$,**
- **Calculate $\boldsymbol{AGA}$,**
- **Use $\boldsymbol{x} = \boldsymbol{Gb} + (\boldsymbol{I} - \boldsymbol{GA})\boldsymbol{z}$ to calculate a solution of $\boldsymbol{Ax = b}$**

This algorithm works for MATLAB but we can also use Python to solve the problem.

Let $A = \begin{pmatrix} 1 & 5 & 2 \\ 3 & 7 & 9 \\ 2 & 10 & 4 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$

Let the sub-matrix be $H = \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix}$

The inverse of the matrix is $H^{-1} = \begin{pmatrix} -7/8 & 5/8 \\ 3/8 & -1/8 \end{pmatrix}$

Hence $(H^{-1})^{\mathrm{T}} = \begin{pmatrix} -7/8 & 3/8 \\ 5/8 & -1/8 \end{pmatrix}$

Now we have to put the elements of $(H^{-1})^{\mathrm{T}}$ in a new matrix

$$\bar{A} = \begin{pmatrix} -7/8 & 3/8 & 0 \\ 5/8 & -1/8 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The generalized inverse G would be

$$G = (\bar{A})^{\mathrm{T}}$$

$$G = \begin{pmatrix} -7/8 & 5/8 & 0 \\ 3/8 & -1/8 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we need to check whether G is the generalized inverse of A.

So

$$AGA = \begin{pmatrix} 1 & 5 & 2 \\ 3 & 7 & 9 \\ 2 & 10 & 4 \end{pmatrix} \begin{pmatrix} -7/8 & 5/8 & 0 \\ 3/8 & -1/8 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 5 & 2 \\ 3 & 7 & 9 \\ 2 & 10 & 4 \end{pmatrix} = A$$

Hence G is a Generalized inverse of A.

Now we are going to find x.

$$x = Gb$$

$$x = \begin{pmatrix} -7/8 & 5/8 & 0 \\ 3/8 & -1/8 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$$

$$x = \begin{pmatrix} 3.5 \\ -0.5 \\ 0 \end{pmatrix}$$

Now we need to solve $Ax = b$

$$b = \begin{pmatrix} 1 & 5 & 2 \\ 3 & 7 & 9 \\ 2 & 10 & 4 \end{pmatrix} \begin{pmatrix} 3.5 \\ -0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$$

Therefore, this algorithm works. Now we are going to show the Python code.

Code.

```
import numpy as np
import sys
# muliplication of matrix system
def multiply(AT,A):
    m,n=AT.shape
    p,q=A.shape
    if n!=p:
        print ' Matrix product not possible'
        sys.exit(0)
    B=np.zeros((m,q))
    for i in range(m):
        for j in range(q):
            sum=0
            for k in range(n):
                sum=AT[i,k]*A[k,j]+sum
            B[i,j]=sum

```

```python
if __name__=="__main__":

    A = np.array([[1, 5, 2], [3, 7, 9], [2, 10, 4]]) #set a matrix A
    b=np.matrix('1; 7; 2') #set a solution matrix b

    #now we are setting a submatrix H
    H=np.matrix('1 5; 3 7')
    #now to find the inverse of H
    Hinverse=H.I
    J=Hinverse.transpose() #the tranpose of Hinverse
    print J #check Hinverse's transpose
    Abar=np.matrix('-0.875 0.375 0; 0.625 -0.125 0; 0 0 0')#put J in new matrix
    #now to set new generalized inverse
    G=Abar.transpose()
    #now we check G is a generalized of A
    K=A*G*A
    print K
    print "G is a g-inverse of A"
    #now we are going to solve x
    x=G*b
    print x
    #now we are going check if x is right
    rice=A*x #where rice=b
    print rice
    print 'The theorem is proved'
```

The output of this **Code** is:

```
In [7]: %run "E:/BRAC-Shiham/MNS/MAT400/Mathlab algorithm.py"
[[-0.875  0.375]
 [ 0.625 -0.125]]
[[  1.    5.    2.]
 [  3.    7.    9.]
 [  2.   10.    4.]]
G is a g-inverse of A
[[ 3.5]
 [-0.5]
 [ 0. ]]
[[ 1.]
 [ 7.]
 [ 2.]]
The theorem is proved
```

Chapter-4

Applications to the Semigroup

4.1. Definition of Group.

Let G be a set, The set G will be a group if the has the following properties:

- Is a set of elements
- It has one operation: $*$. Which can be a $+$ or $\times$
- The group G is closed under the operation $*$. For example $x, y \in G \Longrightarrow x * y \in G$
- Each element $x \in G$ has an inverse.
- The inverse is known as the identity element e where $e \in G$.
- A group is always associative. For example: $x, y, z \in G$ then $(x * y) * z = x * (y * z) \in G$

4.2. Definition of Semigroup.

Let G be a set be of natural numbers $G = \{1,2,3,4,5, \dots .\}$.

We need to check whether the set G is closed under any operation $*$ $(+or\ \times)$. So let us take the addition $(+)$ operation.

$$\therefore 4 + 5 = 9 \in G$$

$\therefore$ G is closed under addition and is a group.

In order to be a semigroup we must check the associative property.

$$\therefore (2 + 5) + 7 = 2 + (5 + 7)$$

$$7 + 7 = 2 + 12$$

$$14 = 14 \in G$$

Hence the Group G is a semigroup under addition which can be written as $(G, +)$.

4.3. Preliminary Notes

Definition. [4] Let A be a $m \times n$ matrix over the complex field K. The g-inverse of A is denoted by A^{+}, which is the $n \times m$ matrix X over K which satisfies the equations:

$$AXA = A, XAX = X, (AX)^{*} = AX, (XA)^{*} = XA$$

For every matrix there exists a Moore-Penrose inverse. If X is at least a $\{1\}-$ inverse of A, then AX and XA are projectors on $R(A)$ and $R(X)$ the range spaces of A and X respectively and $rank(AX) = rankA = rank(XA)$.

Let us denote $A^{\{1\}}$ and $A^{\{1,2\}}$ the set of $\{1\}-$ and $\{1,2\}-$ inverses of A respectively. We will denote by small letters the sub-matrices of a matrix X and by I and 0 the identical and zero matrices or identical and zero sub-matrices.

Lemma 1. [4] Let A be an $m \times n$ matrix over K of $rank\ r$. Then:

1) There exists non-singular matrices P and Q such that $A = Q^{-1}\begin{pmatrix} a_r & 0 \\ 0 & 0 \end{pmatrix}P$
2) $A^{+} = P^{-1}\begin{pmatrix} {a_r}^{-1} & 0 \\ 0 & 0 \end{pmatrix}Q$
3) The elements of $A^{\{1\}}$ are the form of $P^{-1}\begin{pmatrix} {a_r}^{-1} & e \\ f & g \end{pmatrix}Q$ and the elements of $A^{\{1,2\}}$ are the form $P^{-1}\begin{pmatrix} {a_r}^{-1} & e \\ f & f a_r e \end{pmatrix}Q$.

Proof.

Let us prove the lemma1 by taking $Q = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Since P is an identity element hence $P^{-1} = P$.

Now we need to find Q^{-1} from Q.

Using Python we are going to find Q^{-1}.

Code.

```
import numpy as np
import sys
# matrix inverse
Q=np.matrix('3 4; 5 6')
C=Q.I
print C
```

Here C stands for Q^{-1}

The inverse of Q is

```
In [7]: %run "E:\BRAC-Shiham\MNS\MAT250\matrixinverse.py"
[[-3.   2. ]
 [ 2.5 -1.5]]
```

Now let us find the matrix A and let the singular matrix $\begin{pmatrix} a_r & 0 \\ 0 & 0 \end{pmatrix}$ be $R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Let use Python again to find A.

Code.

```
import numpy as np
import sys
Q=np.matrix('3 4; 5 6')
P=np.matrix('1 0; 0 1')
R=np.matrix('1 0; 0 0')
C=(Q.I)*R
A=C*P
print A
```

` The output of A is.

```
In [6]: %run "E:/BRAC-Shiham/MNS/MAT250/thesiswork1.py"
[[-3.   0. ]
 [ 2.5  0. ]]
```

The determinant of matrix A is $= (-3 \times 0) — 2.5 \times 0 = 0$

$\therefore$ The matrix A is singular.

4.4. Isomorphism between Semigroups

Definition. [2] Let $(G,\circ)$ and $(G',*)$ be two groups. Then a mapping $f(G,\circ) \to (G',*)$ is called isomorphism if it fulfils these condition:

i. f is homomorphism i.e. $f(a \circ b) = f(a) * f(b)\ \forall a, b \in G$

ii. f is one-to-one

iii. f is onto

It is denoted by $(G,\circ) \cong (G',*)$ or $G \cong G'$.

Theorem 4.1. [4] Let A and B be two equivalent matrices. Then $(A^{\{1\}},*)$ and $(B^{\{1\}},*)$ are isomorphic.

Proof.

By using the previous Lemma, we can define a map φ from $A^{\{1\}}$ on $B^{\{1\}}$ follows $\varphi(X) = P^{-1}XQ$. Then φ^{-1} is the inverse map from $B^{\{1\}}$ on $A^{\{1\}}$ given by $\varphi^{-1} = PXQ^{-1}$. In addition, for every X and Y in $A^{\{1\}}$, we have

$$\varphi(X * Y) = \varphi(XAY) = P^{-1}(XAY)Q = (P^{-1}XQ)(\,Q^{-1}AP)(P^{-1}YQ) = \varphi(X)B\varphi(Y)$$

$$= \varphi(X) * \varphi(Y).$$

Also, we have for every X and Y in $B^{\{1\}}$,

$$\varphi^{-1}(X * Y) = \varphi^{-1}(XBY) = \varphi^{-1}(X)A\varphi^{-1}(Y) = \varphi^{-1}(X) * \varphi^{-1}(Y).$$

Then the map is an isomorphism.

We remark that $\varphi(A^{+}) = P^{-1}A^{+}Q = B^{+}$ only if P and Q are orthogonal.

Lemma 2. [4] Let A and B be two matrices. Then the following statements are equivalent:

a. $rank(A) + rank(B - A) = rank(B)$
b. Every $\{1\} -$ inverse of B is a $\{1\} -$ inverse of both A and $B - A$.
c. $R(A) \cap R(B) = \{0\}$ and $R(A^{t}) \cap R(B^{t}) = \{0\}$

Theorem 4.2. [4] There is one-to-one correspondence between $M_{m\times n}(K)$ and $M_{m\times n}^{\{1\}}(K)$ maps 0 to $M_{n\times m}(K)$ and preserves isomorphism between semigroup.

Proof.

Let ψ be a map from $M_{m\times n}(K)$ onto $M_{m\times n}^{\{1\}}(K)$ defined for every $A \in M_{m\times n}(K)$ by $\psi(A) = A^{\{1\}}$. Since $0X0 = 0$ for any $X \in M_{n\times m}(K)$, we get $0^{\{1\}} = M_{n\times m}(K)$. Thus $\psi(0) = M_{n\times m}(K)$. According to the Lemma above, if $A^{\{1\}} = B^{\{1\}}$, we have $rank(A) + rank(B - A) = rank(B)$ and $rank(B) + rank(A - B) = rank(A)$. Thus we have $rank(A - B) = 0 = rank(B - A)$. Therefore $A = B$. Now, let $A, B \in M_{m\times n}(K)$ such that $B = Q^{-1}AP$. According to Theorem2.1, we have $B^{\{1\}} = \{P^{-1}XQ/X \in A^{-1}\} = \varphi\left(A^{\{1\}}\right)$. Hence we have $\psi(B) = \varphi(\psi(A))$.

Conclusion

The theory of generalized inverses has its roots both on semigroup theory and on matrix and operator theory. In this thesis we have examined several topics in the theory of linear statistical models using the generalized inverse of a matrix as an analytical device. The examination has rewarded us with considerable insight into some of the underlying structure of this theory, and it appears that the generalized inverse will become a valuable addition to the theorist's box of mathematical tools.

We have discussed numerically reliable methods and computer algorithm to compute generalized inverses of singular matrices. The proposed methods are completely general, being applicable to singular matrices. The proposed approach provides flexibility to compute the solutions of linearly-dependent equations, it has been also shown that all can be obtained from only one generalized inverse matrix.

This thesis also describes a generalization of the inverse of a non-singular matrix, as the unique solution of a certain set of equations. This generalized inverse exists for any (possibly rectangular) matrix whatsoever with complex elements J. It is used here for solving linear matrix equations, and among other applications for finding an expression for the principal idempotent elements of a matrix.

A generalized inverse exists for an arbitrary matrix, and when a matrix has an inverse, then this inverse is its unique generalized inverse. Some generalized inverses can be defined in any mathematical structure that involves associative multiplication, that is, in a semigroup.

References

1. Rahman, M. A., (2015). Theory Of Matrices With Applications. In *College Linear Algebra* (pp. 583-634). Bangladesh: R.K. Publications.
2. Ahmed, M. M., (2017). Unified Approach Of Generalized Inverse And It's Applications.
3. Ahmed, M. M., (2017). Derivations Of Generalized Inverse Using Contour Integration And Interpolation Polynomial.
4. Zekraoui, H. & Guedjiba, S., (2010). "Semigroup Of Generalized Inverses Of Matrices". 12, 146-152.
5. Abu-Saman, A.M. (2012). "Solution of Linearly-Dependent Equations by Generalized Inverse of a Matrix". *International Journal of Emerging Science and Engineering*, 4(2), 138-142.
6. Ozden, H. (1990). "A Note On The Use Of Generalized Inverse Of Matrices In Statistics". 39-43