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A SIMPLE APPROACH TO COUNTABILITY

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We draw attention to a simple principle, which can be used to prove many of the usual and important theorems on countabilty of sets. We formulate it as the

Countability Lemma. Suppose to each element of the set A there is assigned, by some definite rule, a unique natural number in such a manner that to each $n \in$ **N** there corresponds at most a finite number of elements of the set A. Then A is countable.

The principle embodied in this lemma is not new, but we have not seen it expressly formulated anywhere. Needless to say, **N** denotes the set of natural numbers (which naturally begins with 1). We recall the basic definitions

A set S is said to be **equinumerous** with a set T if there exists a bijective mapping from S onto T. Many authors say S is *equivalent* to T, but we prefer not to overuse the adjective "equivalent".
A set S is called **finite** if it is either empty or it is nonempty and there exists a natural number *n* such that S is equinumerous with $\{1, 2, 3,..., n\}$, the set of the first *n* natural numbers. If no such *n* exists, then S is called an **infinite** set.
An infinite set is called **denumerable** if it is **equinumerous** with **N.**
A set is called **countable** if it is either finite or denumerable. Some authors use the term countable instead of denumerable; for our "countable" they have to say "at most countable".
The elements of a countable set S can be written down as a finite or an infinite sequcence

$$a_1,\ a_2, a_3, ...$$

where repetitions are allowed. We call this an **enumeration** of S.

Proof of the Countability Lemma. Let $k(n)$ denote the number of elements of A which correspond to *n.* We denote by a_1, $a_2, \dots, a_{k(1)}$ the $k(1)$ elements of A corresponding to $n = 1$, provided $k(1) > 0$; then denote by $a_{k(1)+1}, a_{k(1)+2}, \dots, a_{k(1)+k(2)}$ the $k(2)$ elements of A corresponding to $n = 2$, provided $k(2) > 0$; and so on. This gives us an enumeration of all the elements of A

Theorem 1. Every subset of a countable set is countable.

Proof. Suppose $a_1,\ a_2, a_3, ...$ is an enumeration of the countable set A and B is any nonempty subset of A. If, for some $n \in$ **N,** the element a_n belongs to B, then we assign the natural number *n* to it. For each $n \in$ **N** let $k(n)$ denote the number of elements among $a_1,\ a_2,\ ..., a_n$, which belong to the subset B. Then $0 \le k(n) \le n$. Therefore, B is countable by the Countability Lemma.

Theorem 2. The set **Q** of rational numbers is countable.

Proof. To $0 \in$ **Q** we assign the natural number 1, and to each nonzero rational number $\frac{r}{s}$ in reduced form (where $r, s \in$ **Z** are coprime and $s \neq 0$) we assign the natural number
$n = |r| + |s| \ge 2$. Then to each $n \in$ **N** there corresponds a finite number of rational numbers, because $|r|$ and $|s|$ are natural numbers and $|a| = \pm a$. Therefore, **Q** is countable by the Countability Lemma.

Theorem 3. The Cartesian product of two countable sets is countable.

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Proof. Suppose A, B are countable sets and $a_1, a_2, a_3, \ldots$ is an enumeration of A and $b_1, b_2, b_3, \ldots$ is an enumeration of B. Then

$$A \times B = \{(a_i, b_j) : i \in N,\ j \in N\}.$$

To the element (a_i, b_j) we assign the natural number $n = i + j$. Then to each natural number $n \geq 2$ there correspond $n - 1$ elements of $A \times B$, viz. (a_1, b_{n-1}), (a_2, b_{n-2}), …, (a_{n-1}, b_1) Therefore $A \times B$ is countable by the Countability Lemma.

Corollary. The Cartesian product of a finite family of countable sets is countable.

Remark. This result does not extend to a denumerable family of denumerable sets, as shown by the example $\mathbf{N}^{\mathbf{N}}$**.** It is the set of all sequences of natural numbers**,** which is known to be uncountable.

Theorem 4. The union of a countable family of countable sets is countable.

Proof. Without lost of generality, we can denote a countable family of sets by A_1, A_2, A_3, …. Suppose a_{i1}, a_{i2}, a_{i3}, … is an enumeration of A_i. Then

$$\bigcup_{i=1}^{\infty} A_i = \{ a_{ij} : i \in \mathbf{N}, j \in \mathbf{N} \}.$$

To the element a_{ij} we assign the natural number $n = i + j$. Then to each natural number $n \geq 2$ there correspond at most $n - 1$ distinct elements of A. Therefore A is countable by the Countability Lemma.

Remark. The union of a denumerable family of denumerable sets is denumerable, even when the sets in the family are pairwise disjoint; whereas the Cartesian product of denumerable family of denumerable sets is non-denumerable (uncountable), even when all the sets in the family are the same. This is one of the many surprises of transfinite set theory, " the paradise created by Cantor from which no one can drive us out" (Hilbert).

Theorem 5. The family of all subsets of a countable set having a fixed and finite number of elements is countable.

Proof. Suppose A is countable and $a_1, a_2, a_3, \ldots$ is an enumeration of A. Then every *k*-element subsets of A has the form

$$\{ a_{i_1}, a_{i_2}, \ldots\ldots, a_{i_k} \} \quad \text{where}$$

$i_1 < i_2 < \ldots\ldots < i_k$.

To this subset we assign the natural number $n = i_1 + i_2 + \ldots\ldots + i_k$. Then to each natural number n there corresponds only a finite number of possibilities for $i_1, i_2, \ldots\ldots, i_k$, that is, only a finite number of k- element subsets of A. Therefore A is countable by the Countability Lemma.

Theorem 6. The family of all finite subsets of a countable set is countable.

Proof. Suppose A is countable. Denote by A_n the family of all *n*-element subsets of A. Each A_n is countable. The family of all finite subsets of A is the union of all A_n, $n \in \mathbf{N}$**;** as such it is countable by Theorem 4.

A real or complex number is called **algebraic** if it satisfies a polynomial equation with integer coefficient; that is, if it is a root of an equation of the form

(1)

$$a_0 x^k + a_1 x^{k-1} + \ldots + a_k = 0,$$

where $a_0, a_1, \ldots, a_k$ are integers and $n \in \mathbf{N}$.

In particular every rational number is algebraic; so are many irrational numbers. A real number which is not algebraic, is called **transcendental**. It is known that *e* and π, the two most important numbers in Mathematics, are transcendental.

Theorem 7. The set of all algebraic numbers, real and complex, is countable.

Proof. If x is algebraic and satisfies the equation (1), the to *x* we assign the natural number $n = k + |a_0| + |a_1| + \ldots + |a_k|$. Then to each $\boldsymbol{n} \in \mathbf{N}$ there corresponds a finite number of choices for $k, a_0, a_1, \ldots, a_k$; that is, a finite number of equations of the form (1). Each of these equations has at most k distinct roots in the system of complex numbers (according to the Fundamental Theorem of Algebra), each of which is an algebraic number. Therefore, to each $\boldsymbol{n} \in \mathbf{N}$ there corresponds a finite number of algebraic numbers. Therefore, the set of all algebraic numbers is countable by the Countability Lemma.

Theorem 8. The set of real transcendental numbers is uncountable.

Proof. **R** is the (disjoint) union of the set of real algebraic numbers, which is countable, and the set of real transcendental numbers. If the latter set were countable, **R** would be countable.

This existence theorem ranks among the most amazing instances of the power of mathematical reasoning. We have established the existence of uncountably many real transcendental numbers, without needing to know a single specific transcendental number.

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